Topological nonlocal operations on toroidal flux qubits

Adel Ali* and Alexey Belyanin†
Department of Physics and Astronomy, Texas A&M University, College Station, Texas 77843, USA

(Received 14 December 2023; revised 16 July 2024; accepted 18 July 2024; published 6 August 2024)

We propose a method to achieve coherent, field-free coupling between the toroidal flux qubit (tfluxon), which is isolated from the environment, and a charged particle on a quantum ring. The resulting nonlocal coupling is mediated only by the vector potential and can be used to control the tfluxon by acting on the quantum ring's degrees of freedom. Furthermore, the emergent coupling can facilitate coherent interaction between states of distant tfluxons mediated by an electron on a quantum ring. We demonstrate that the topological and nonlocal aspects of this system are of fundamental interest and can have important applications in quantum information, benefiting from the decoupling from the environment and the topological protection.

DOI: 10.1103/PhysRevA.110.022604

I. INTRODUCTION

Storing and processing quantum information requires two seemingly contradictory processes: complete isolation of a system from the environment and precise control of a system's degrees of freedom. One example of a well-isolated qubit candidate is a toroidal flux qubit or tfluxon, which is defined as a volume of enclosed magnetic flux lines whose outer surface is topologically equivalent to a torus. As in the case of standard flux qubits, the persistent toroidal current generating this magnetic flux can be in a superposition [1]. Tfluxons are expected to be better isolated from environmental noise such as flux noise, charge noise, $1/f$ noise, and external electromagnetic fields [2,3]. In [2] the authors proposed to control the tfluxon by coupling its dipolar toroidal moment to high-frequency electromagnetic fields. However, this brings back the associated noise.

Here we take another approach assuming that the high-frequency fields are filtered out and address the question of how to couple two different tfluxons. This raises the question of how to do operations on the quantum states of a system if it is isolated from external e.m. fields and the environment.

We propose solving this problem by nonlocally coupling the tfluxon with a charged particle on a quantum ring (QR) that encloses the flux lines of the tfluxon; see Fig. 1. The QR is a well-studied system implemented in many materials [4–8]. Any noise charge must close a loop around the tfluxon to get coupled to it. The environmental noise charge fluctuations do not affect the system unless they make coherently a trajectory that encloses the flux with a nonzero winding number. Yet, the control of the tfluxon quantum states is possible via the nonlocal Aharonov-Bohm (AB) coupling of the electron with the vector potential of the tfluxon.

The AB effect has been studied in a context where the gauge vector potential of a classical magnetic field affects nonlocally a charged particle's phase for scattered states and the energies of the bound states [9]. Furthermore, the AB interference effect is ubiquitous in superconducting circuits such as superconducting quantum-interference devices. In the proposed system, a quantized AB effect defines *interaction*. More precisely, for a QR threaded with a quantized toroidal magnetic flux, the electron and tfluxon states are coupled only via nonlocal quantized AB interaction. We demonstrate that this kind of coupling is fundamentally different from, e.g, the Jaynes-Cummings (JC) interaction in superconducting circuit QED or cavity QED. The detailed description of the model is presented in Sec. III. Some examples of the operations are given in Sec. IV. We show that the joint quantum state of the electron and tfluxon can be manipulated by external e.m. fields acting only on the electron degrees of freedom; hence we can deterministically control the tfluxon state despite its isolation from the environment.

In Sec. V we describe the dynamics and operations on two or more tfluxons coupled by QRs. We show that a QR coupled to two tfluxons that are separated in space and isolated from each other and the environment will result in an emergent interaction that inherits the nonlocal and topological aspects of the AB effect. This interaction can be used to coherently manipulate the state of the tfluxons and facilitate entanglement between them. The nonlocal coupling between the two tfluxons will result in the exchange of energy and can be used to create entanglement between them without direct or mediated local interactions. This reasoning is scalable to a chain of coupled tfluxons, where it simulates interesting spin chain systems with nonlocal coupling.

Finally, in Sec. VI we discuss possible experimental implementations of the proposed system and limitations stemming from decoherence.

II. TOPOLOGICAL PROTECTION OF THE AB PHASE

Consider a QR of length $L = 2\pi r$ that surrounds a volume with a magnetic flux trapped inside, as sketched in

*Contact author: adel_ali@tamu.edu
†Contact author: belyanin@tamu.edu

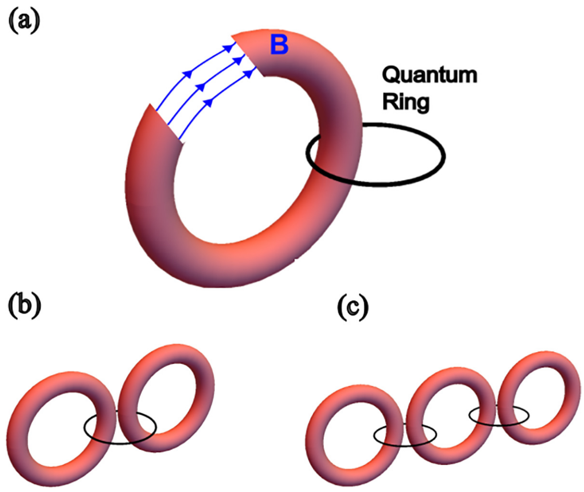

FIG. 1. (a) Schematic of a tfluxon coupled to an electron on a quantum ring. The magnetic field is confined inside the toroid and the electron on the ring is coupled to the vector potential only. (b) Two tfluxons coupled nonlocally through the electron on a quantum ring. (c) A chain of coupled tfluxons.

Fig. 1(a). The eigenvalues and eigenvectors of the system are invariant under length-preserving deformations of the charge path, deformations of the flux lines, or simultaneous deformations of the charge path and the closed flux lines [10]. In fact, in the semiclassical Bohr-Sommerfeld quantization condition,

$$\hbar \oint d\mathbf{r}\cdot\mathbf{k} - e\oint d\mathbf{r}\cdot\mathbf{A} = 2\pi\hbar n,$$

the second term is clearly invariant under the deformation of the closed flux lines. For length-preserving deformations, the first term is also invariant. Hence, the allowed momenta and energies are invariant and these systems have topological protection. Note that length-preserving shape deformations will not change the eigenvalues of energy as they depend on the length L of the charge path. Now consider a charge path enclosing two such fluxes as in Fig. 1(b). We notice that the second term depends only on the total flux enclosed, and therefore the system is invariant under changing the distribution of the magnetic field or changing the distance between the two toroids. This may address the design and fabrication challenges such as uniformity and reproducibility across qubits, disorder, and fabrication defects.

III. TFLUXON-ELECTRON COUPLING

The Hamiltonian for a massive charged particle in a field-free region outside of an infinite solenoid is given by

$$\hat{H} = \frac{1}{2m_e}(\hat{p}_\varphi - e\hat{A}_\varphi)^2, \quad A_\varphi = \frac{\Phi}{2\pi r}$$

and φ is an angular coordinate along the QR which changes from 0 to 2π. We quantize the enclosed flux Φ by demanding that $\Phi = \hat{n}\varphi_0$ where $\hat{n}$ is a number operator, and $\varphi_0 = h/2e$ is the flux quantum.

For all examples below, we will limit the tfluxon excitations to the two lowest states with n equal to 0 or 1 in order to have a two-state flux qubit, which is the simplest and most studied case for applications. In the "physical" basis $|n=0\rangle$, $|n=1\rangle$ the tfluxon Hamiltonian is $\Delta\sigma_x$ where Δ is the energy splitting between these two states and σ_x is a Pauli matrix. Note that it is often written as $\Delta\sigma_z$, after performing the Hadamard transformation [2]. Furthermore, we will start with the one-dimensional motion of a particle on a ring. The case of a finite-width two-dimensional annulus geometry is qualitatively the same, unless we want to utilize the energy levels related to transverse quantization which happens at higher-energy scale. The resulting tfluxon-electron Hamiltonian is

$$\hat{H} = \frac{1}{2m_e}\left(\hat{p}_\varphi - \frac{e\varphi_0\hat{n}}{2\pi r}\right)^2 + \Delta\sigma_x. \tag{1}$$

We can see from expanding the first term that the Hamiltonian has an interaction term $\hat{p}_\varphi \otimes \hat{n}$. Here we have a two-level system coupled to fermionic excitations of matter field replacing the bosonic (photonic) excitations of the JC model. The interaction term has the coefficient

$$g_a = \frac{\hbar e\varphi_0}{2\pi m_e r^2} = \frac{\hbar^2}{2m_e r^2}. \tag{2}$$

This constant sets the scale for both the energy of the electron's first excited state and the electron-tfluxon interaction. For $r \approx 1$ μm and $m_e \approx 0.01$ of free electron mass, $g_a/\hbar \approx 1$ GHz. The energy splitting between the quantized flux states Δ is also expected to be a few GHz, meaning that the system is typically in the ultrastrong-coupling regime.

To solve for the eigenstates, we take the basis of the Hilbert space to be $|\psi_{mn}\rangle = |m\rangle \otimes |n\rangle$ where

$$\langle\varphi \mid m\rangle = \frac{e^{-im\varphi}}{\sqrt{2\pi r}} \text{ and } \hat{n}|n\rangle = n|n\rangle.$$

Using the ansatz $|\psi_m\rangle = |m\rangle \otimes (\alpha|0\rangle + \beta|1\rangle)$ to solve the time-independent Schrödinger equation $\hat{H}|\psi_m\rangle = E_m|\psi_m\rangle$, we obtain

$$\begin{aligned}\hat{H}|\psi_m\rangle &= g_a|m\rangle \otimes \{(\alpha m^2 + \Delta\beta)|0\rangle \\ &\quad + [\beta(m-1)^2 + \Delta\alpha]|1\rangle\} \\ &= E_m[|m\rangle \otimes (\alpha|0\rangle + \beta|1\rangle)].\end{aligned}$$

Setting the energy scale $g_a \equiv 1$ so that Δ is now in units of g_a and simplifying, we arrive at the eigenvalue problem which can be written in the matrix form as

$$\begin{bmatrix} m^2 & \Delta \\ \Delta & (m-1)^2 \end{bmatrix}\begin{bmatrix}\alpha \\ \beta\end{bmatrix} = E_m\begin{bmatrix}\alpha \\ \beta\end{bmatrix}. \tag{3}$$

Solving it gives

$$E_m^{\pm} = \frac{1}{2}[1 - 2m + 2m^2 \pm \sqrt{(1-2m)^2 + 4\Delta^2}]. \tag{4}$$

The new energy levels combine the parabolic dispersion of the electron m^2 and the energy of the two-level system Δ. Changing the QR material will change the dispersion relation for the electron energy. For example, in the case of graphene one would get a linear scaling with m as in the harmonic oscillator.

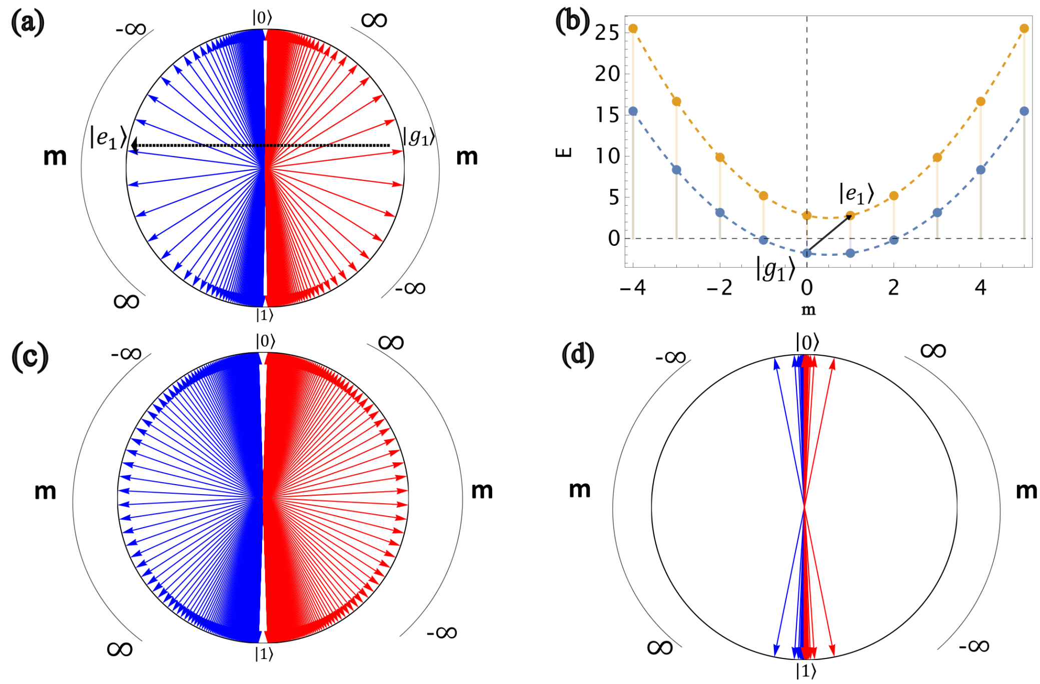

FIG. 2. (a) Plot of the eigenstates of the tfluxon for $\Delta = 5$ on a section of a Bloch sphere, where each arrow represents a state that has a different quantum number m. The red arrows on the right and blue arrows on the left correspond, respectively, to the lower and the upper energy bands shown in panel (b). The horizontal dotted arrow shows an example of an electric dipole allowed transition. (b) The energies $E^{\pm}$ of the states in panel (a) given by Eq. (4) as a function of quantum number m, with the transition from panel (a) shown with a solid black arrow. (c) Same as panel (a) but for $\Delta \gg 1$. (d) Same as panel (a) but for $\Delta \ll 1$. Both $E^{\pm}$ and Δ are in units of g_a and therefore dimensionless.

Note that the effective Hamiltonian matrix on the left-hand side of Eq. (3) can be written as $h_0 I + \vec{h}.\vec{\sigma}$ where $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$, $\vec{h} = (\Delta, 0, m - \frac{1}{2})$, and $h_0 = m^2 - m + 1$. In spin-$1/2$ terminology, Δ is the x component of the effective magnetic field, h_3 is its z component, and h_0 is a total-energy shift. So, changing the quantum number m (for example by the e.m. field) will result in changing the effective Hamiltonian of the tfluxon.

The eigenvectors that describe the tfluxon part of the eigenstates, namely $\alpha|0\rangle + \beta|1\rangle$, are shown as arrows on the Bloch sphere in Figs. 2(a), 2(c) and 2(d) for different values of Δ, whereas the eigenenergies from Eq. (4) are shown in Fig. 2(b). If $\Delta \ll 1$, the eigenvectors are concentrated around the poles of the Bloch sphere as in Fig. 2(d) because the z component will dominate. As Δ increases, the states initially condensed at the poles will start to spread out as shown in Fig. 2(a) for $\Delta = 5$ and Fig. 2(c) for $\Delta \gg 1$. In particular, the states with m of the same order as Δ will move towards the x axis of the Bloch sphere. One notices from Figs. 2(a) and 2(b) that even when $m = 0$, i.e., the electron has zero angular momentum as is the case for state $|g_1\rangle$ and therefore does not generate any magnetic field, the value of $h_3 = -1/2$ is still nonzero and the electron still affects the state of the tfluxon. Proving this in experiment would be a vivid demonstration of the "reality" of e.m. potentials and would close a loophole in the Tonomura experiment [11] where electrons move around magnets covered with superconducting films. Indeed, it was argued that the superconductor cannot perfectly shield from the e.m. field generated by fast-moving electrons, and therefore the observed phase shift could be due to the interaction energy resulting from the overlapping of the e.m. fields instead of the nonlocal AB effect due to potentials [12].

IV. NONLOCAL COUPLING AND OPERATIONS ON TFLUXONS

Can we use the gauge-field nonlocal coupling between the electron and the tfluxon to do quantum operations on the tfluxon(s)? The tfluxons are completely shielded from any e.m. field and we have access only to the electron on the QR degrees of freedom.

Consider, as an example, the transitions between the electron m states caused by an external e.m. field and their effect on the tfluxon state. In the electric dipole approximation, the selection rules are $\Delta m = \pm 1$. A simple operation on the tfluxon would be to excite the joint system to a higher state, as shown in Fig. 2(b). This will result in rotating the tfluxon state on the Bloch sphere as indicated in Fig. 2(a). Note that each eigenstate of the system is a product state, so after the excitation, even if the ring were somehow "cut" (e.g., the electron removed from the ring), the x-basis measurement of the tfluxon would not change, as the tfluxon state would rotate around the x axis. Since the tfluxon is well isolated from the environment, it can be used to store quantum states. Injecting or removing the electrons from the QR could be used to turn the interaction on and off, and to control the strength of the interaction [13].

Another set of operations to manipulate the state of the tfluxon is to add a potential to the electron on the ring, for example, by applying a lateral electric field(s). Consider the lower two m states only, so that the Hamiltonian can be mapped to a two-qubit Ising model: $\hat{H}_I = (-\sigma_z^{(e)}\sigma_z^{(f)} + I) + \Delta\sigma_x^{(f)} + g_1(t)\sigma_x^{(e)} + g_2(t)\sigma_y^{(e)}$. Here superscripts (e) and (f) denote the operators acting only on electron and tfluxon degrees of freedom, respectively. Then geometric phase gates can be performed on the tfluxon [14] by adiabatically changing $g_{1,2}(t)$.

Creating entanglement between the electron state and the tfluxon has interesting implications in the context of nonlocal gauge effects. Indeed, consider an initial state of the system to be a product state of the electron and the tfluxon, so that there is no correlation between the measurements done on both. To illustrate the idea of nonlocal effects with the same simple model, we take the same Hamiltonian as in Eq. (1) but with an arbitrary coupling of the electron to an external field:

$$\hat{H} = \frac{1}{2m_e}\left(\hat{p}_\varphi^{(e)} - \frac{e\varphi_0\hat{n}^{(f)}}{2\pi r}\right)^2 + \Delta\sigma_x^{(f)} + \hat{H}_e, \quad (5)$$

where $H_e = g(t)\sigma_x^{(e)}$. The operator $\hat{H}_e$ is acting only on the electron subspace and can describe, e.g., the interaction of the electron dipole moment with an e.m. field. According to our scenario, at $t \leqslant 0$ we have $g = 0$ and the eigenstates of the Hamiltonian (5) are product states of the $|m\rangle$ and $|n\rangle$ basis states. At $t = 0$ the interaction term H_e is turned on. Accordingly, the time evolution will result in changing the reduced density matrix of the tfluxon. One can easily check numerically that the expectation value of the energy and von Neumann entanglement entropy of the tfluxon will increase due to the coupling. This implies that the exchange of information and energy between the two subsystems is nonlocally mediated by the gauge field potential only.

V. SCALING UP THE SYSTEM

The system can be scaled up to include more complicated operations. Consider a setup that involves two of such tfluxons, as in Fig. 1(b). The Hamiltonian is now

$$\hat{H} = \left(\hat{p}_\varphi - \frac{e\varphi_0\hat{n}_1}{2\pi r} + \frac{e\varphi_0\hat{n}_2}{2\pi r}\right)^2 + \Delta\sigma_x^{(f1)} + \Delta\sigma_x^{(f2)}. \quad (6)$$

Here $(f1)$ and $(f2)$ denote operators acting on the degrees of freedom of tfluxon 1 and 2. We assumed the configuration in which the two tfluxons have opposite directions, hence the opposite signs in parentheses in Eq. (6). For the eigenstates of the system, we use the ansatz

$$|\psi_m\rangle = |m\rangle \otimes (a_1|00\rangle + a_2|01\rangle + a_3|10\rangle + a_4|11\rangle),$$

and the effective Hamiltonian becomes

$$\begin{bmatrix} m^2 & \Delta & \Delta & 0 \\ \Delta & (m-1)^2 & 0 & \Delta \\ \Delta & 0 & (m-1)^2 & \Delta \\ 0 & \Delta & \Delta & m^2 \end{bmatrix}.$$

Its eigenenergies are $E_m^{(1)} = (m-1)^2$, $E_m^{(2)} = m^2$, and $E_m^{(3,4)} = \frac{1}{2}[1 - 2m + 2m^2 \pm \sqrt{(1-2m)^2 + 16\Delta^2}]$.

They are plotted in Fig. 3. The salient feature of the system is that the tfluxon parts of the corresponding eigenstates are entangled states of the tfluxons. For example, $|\Phi^-\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}}$ is the tfluxon part of the eigenstate of the entire $E_m^{(1)}$ band, and $|\Psi^-\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$ is the same for the $E_m^{(2)}$ band. All the states of these bands are still orthogonal because they have different quantum numbers m. These are maximally entangled Bell states of the tfluxons. Therefore, a simple operation of exciting the system from the ground state $E_m^{(4)}$, which is a product state, to any state in the $E_m^{(1)}$ or $E_m^{(2)}$ bands, as shown in Fig. 3, will nonlocally create a maximally entangled state between the two tfluxons. Any known protocol to create entanglement involves either direct interaction between two quantum subsystems or entanglement swapping, which also requires, for example, that each of the entangled photons interacts locally with each subsystem. Here the entanglement creation does not require any local interaction between the fields of the tfluxons.

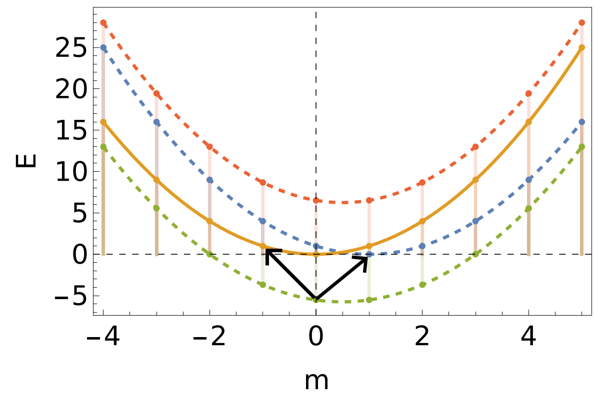

FIG. 3. The eigenenergies E of the two-tfluxon system plotted as a function of the quantum number m for $\Delta = 3$. From bottom to top the energies are $E^{(4)}$ (dashed green line), $E^{(2)}$ (solid orange line), $E^{(1)}$ (dashed blue line), and $E^{(3)}$ (dashed red line). Both E and Δ are in units of g_a and therefore dimensionless. The solid black arrows represent electric dipole allowed transitions from the ground state, which is a product state for the tfluxons $|n_1\rangle \otimes |n_2\rangle$, to excited states in E^1 and E^2 bands which are entangled Bell states.

The effective Hamiltonian given by the matrix above can be written as

$$\hat{H}_m = h_0 I + \vec{h}.\vec{\sigma}^{(f1)} + \vec{g}.\vec{\sigma}^{(f2)} + J_m\sigma_z^{(f1)}\sigma_z^{(f2)}, \quad (7)$$

where the nonzero coefficients are

$$h_0 = m^2, \quad h_1 = g_1 = \Delta, \quad h_3 = g_3 = m - 1/2,$$
$$J_m = 2m - 1.$$

The coupling coefficient J_m between the two initially noninteracting tfluxons emerged due to the coupling with the electron and describes electron-induced tfluxon-tfluxon interaction. Since J_m is a function of m, it can be tuned and changed between antiferromagnetic for $m > 0$ and ferromagnetic for $m < 0$. Moreover, the coupling coefficient does not change by deforming the QR or changing the distance between the two tfluxons as long as the electron path encircles both tfluxons. This is different from the usual situation when the interaction strength falls off with the distance between the interacting parts of the system. The interaction strength J_m is in units of g_a so the minimum value $J_m = 1$ is also in the GHz range.

Another interesting feature of the coupled QR-tfluxon models is that the energy can be transferred nonlocally

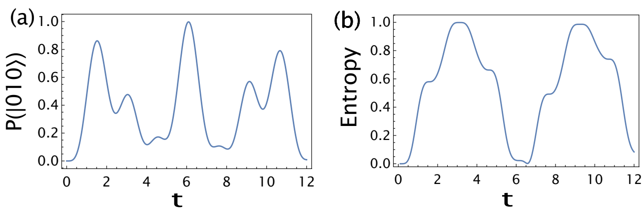

FIG. 4. Exchange of information and energy between the two tfluxons. (a) for $\Delta = 1$, the dimensionless probability for measuring $|010\rangle$ is plotted as function of time. Around $t = 6$ the excitation is completely transferred from tfluxon 2 to tfluxon 1. (b) The von Neumann entanglement entropy (in bits) of the tfluxon as a function of time. The time is normalized by $\hbar/g_a \approx 1$ ns.

between the two tfluxons. To illustrate this, consider the initial state in the form $|\psi(0)\rangle = |m = 0\rangle \otimes |n_1 = 0\rangle \otimes |n_2 = 1\rangle$. The time evolution of this state will keep m fixed and results in rotation of the tfluxons part of the wave function by the effective Hamiltonian above. Accordingly, after some time τ, the state vector becomes $|\psi(\tau)\rangle = |0\rangle \otimes |1\rangle \otimes |0\rangle$. Figure 4(a) shows the probability of finding the system with $n_1 = 1$. This means that the flux (energy) initially in tfluxon 2 transfers to tfluxon 1. From an operational point of view this might be viewed as just swapping $|10\rangle$ to $|01\rangle$. However, in our case the two toroids are isolated from each other with no overlapping fields. Moreover, one can start from the state with $m = 0$, i.e., zero magnetic moment which rules out any effect of the magnetic field produced by the electron. Similarly to the case of one tfluxon, this would demonstrate the observable effects of the gauge potential while eliminating the possibility that the AB phase is acquired due to the interaction energy between the e.m. field of the moving particle and the external fields [12].

The models considered here can be straightforwardly scaled up or extended to include more operations. For example, more than one electron can be injected into a QR, which will lead to the occupation of several m states and excited energy bands in Fig. 3. Furthermore, a chain of the tfluxons can be created where each two consecutive tfluxons are coupled to each other, as shown in Fig. 1(c). Following the same procedure as in the case of two tfluxons, the effective Hamiltonian of the tfluxons on the chain can be written as

$$\hat{H}_m = \sum_i J_m \sigma_z^{(i)} \sigma_z^{(i+1)} + \Delta \sigma_x^{(i)} + h_m \sigma_z^{(i)} + h_0 I. \quad (8)$$

This is a transverse field Ising Hamiltonian with tunable parameters. Hence, this system can be used as a quantum simulator [15]. The two protocols discussed above can be applied for the chain setup, for example, by entangling the tfluxons on the chain pairwise consecutively by changing the m state of an electron in the corresponding QR.

VI. DISCUSSION AND CONCLUSIONS

Experimentally, the realization of the proposed qubit system in a solid-state platform seems possible by integration of two existing technologies: the ballistic QRs which allow observations of the AB oscillations [4–8,16–18] and three-dimensional (3D) integrated superconducting qubit technology, especially heterogeneous flip-chip 3D integration [19] and curved 3D nanoarchitectures [20] which allow fabrication of superconducting or magnetic nanotubes in the shape of a hollow cylinder with a broad range of diameters.

The size of the QR is limited by phase coherence length which in the above experiments was around several μm. In addition to that, a simple superconducting loop can be used with some qualitative differences from the QR such as equally spaced energy levels.

Obviously, the decoherence time will limit the number of operations discussed in this paper. The coherence time for the tfluxon is expected to outperform the micro- to millisecond coherence time of planar flux qubits due to the better decoupling from the environment. It will also depend on the dielectric material inside the toroid. Calculating the performance limits imposed by decoherence should be done for a specific design and choice of a material system, which is outside the scope of this paper. It is expected that the realization of a toroidal fluxon is going to be the most difficult technological part.

As the first experimental attempt, one could build a similar system on the basis of QRs coupled to standard planar flux qubits. Such a system would have the same dephasing timescale as a standard superconducting circuit. It would not have a toroidal topology protection from decoherence. Also, the magnetic fields would be directly threading the QR, thus weakening some of the above nonlocality arguments. However, one could still demonstrate the new type of tunable coupling and test the same operations within a less demanding design. In any scenario, we hope that our paper will attract the interest of various research communities and stimulate their interdisciplinary collaboration to implement this novel qubit system.

ACKNOWLEDGMENTS

This work has been supported in part by the Air Force Office for Scientific Research Grant No. FA9550-21-1-0272 and National Science Foundation Award No. 1936276.

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