


Nonstabilizerness of permutationally invariant systems

G. Passarelli ^{1,*}, R. Fazio ^{2,1} and P. Lucignano ¹

¹*Dipartimento di Fisica, Università di Napoli “Federico II,” 80126 Napoli, Italy*

²*The Abdus Salam International Center for Theoretical Physics, 34151 Trieste, Italy*

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Typical measures of nonstabilizerness of a system of N qubits require computing 4^N expectation values, one for each Pauli string in the Pauli group, over a state of dimension 2^N . For permutationally invariant systems, this exponential overhead can be reduced to just $O(N^3)$ expectation values on a state with a dimension $O(N)$. We exploit this simplification to study the nonstabilizerness phase transitions of systems with hundreds of qubits.

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I. INTRODUCTION

Entanglement, by itself, is not sufficient to achieve quantum advantage. States with low entanglement can be represented with polynomial effort using tensor networks [1–3], but there also exist highly entangled states that can be efficiently manipulated using classical computers [4,5]. On a quantum computer, any N -qubit evolution can be arbitrarily approximated using a combination of four types of unitary operators, the two-qubit controlled-NOT (CNOT) gate and the one-qubit Hadamard H , $\pi/2$ phase shift S , and $\pi/4$ phase shift T gates [6]. The subset $\{H, S, \text{CNOT}\}$ is nonuniversal and generates the Clifford group \mathcal{C}_N . States that can be built by only applying gates in \mathcal{C}_N on the computational basis states are known as stabilizers and they admit an efficient classical representation regardless of their entanglement [4,7,8].

Adding the T gate to the ones in \mathcal{C}_N allows for universal quantum computation. States that are built using unitaries from the Clifford + T group have an extra resource known as nonstabilizerness, or, colloquially, quantum magic [9–14], which renders them classically intractable when their entanglement is beyond the capabilities of tensor networks. This nontrivial interplay between entanglement and quantum magic is believed to be the key to unlock quantum advantage. For this reason, quantifying the amount of nonstabilizerness has become a central pursuit, not only in quantum information theory, but also in related areas such as condensed-matter and statistical physics [15].

While there exist many universally accepted entanglement measures [6,16], measures of nonstabilizerness are still under scrutiny. In general, the nonstabilizerness of a state can be measured as its distance from the set of stabilizers [11,17–26], but quantities based on this definition are often difficult to compute. More appealing options are based on the analysis of the properties of the expectation values of all Pauli strings over the quantum state, the so-called Pauli spectrum. In fact, the Pauli spectrum of stabilizer states has precise properties [8] and any deviation is a symptom of nonstabilizerness [27].

Among these measures, we cite the stabilizer nullity [28], the average entanglement-spectrum flatness over Clifford orbits [29,30], and the stabilizer k -Rényi entropies (SREs) [31].

Evaluating the above quantities, however, is extremely challenging, and the investigation of their properties so far has been limited to systems of $N \sim 10$ qubits [31,32] even in the case of integrable systems [33,34], with only few exceptions [35–38]. In fact, the dimension of the Hilbert space of a system of N spins grows exponentially as $D = 2^N$, and the number of Pauli strings $\hat{P} = \hat{\sigma}_1^{\alpha_1} \otimes \cdots \otimes \hat{\sigma}_N^{\alpha_N}$, with $\alpha_j \in \{0, x, y, z\}$ in the Pauli group \mathcal{P}_N , grows as $D^2 = 4^N$. This exponential complexity in both the state dimension and the number of expectation values makes it impossible to compute the nonstabilizerness of large systems. This is especially relevant since many-body systems have been shown to host a plethora of exotic entanglement-related phenomena [39–42] and recent works started investigating these types of phenomena also in magic [31,32,37,43,44]. It is therefore essential to find systems where the nonstabilizerness can be computed efficiently for large N . An important step forward in this direction has been achieved when the quantum state can be efficiently represented by tensor networks [35–38]. It would also be desirable to find nontrivial many-body systems that admit an exact solution. In this work we focus on systems that are symmetric under the permutation group [45,46]. In this case one can exploit the symmetry to reduce the interesting portion of the Hilbert space to a more manageable subspace, allowing the study of much larger systems than normally accessible. For permutationally invariant systems, it is possible to compute the nonstabilizerness measures based on the Pauli spectrum with only a polynomial effort rather than an exponential one [47,48]. This allows studying the nonstabilizerness phase diagram for this broad class of systems for large N .

This paper is organized as follows. In Sec. II we review the definition of nonstabilizerness and some of its measures. In Sec. III we show that permutational invariance allows us to simplify the evaluation of these quantities and explicitly construct the representation of the symmetric Pauli group. In Sec. IV we apply our technique to the study of the Lipkin-Meshkov-Glick model [49] and, in Sec. V, of the p -spin model. In Sec. VI we apply mean-field theory to the two

*Contact author: gianluca.passarelli@unina.it

models studied. In Sec. VII we summarize and draw our conclusion.

II. NONSTABILIZERNESS MEASURES

The stabilizer Rényi entropies are extensively studied as probes of quantum magic. Entropies with Rényi index $k < 2$ are nonmonotone under measurements in the computational basis followed by conditioned Clifford operations [50], which is why we will focus on the 2-Rényi entropy, the simplest one without this issue [51], and refer to it as SRE. For pure states, the SRE is defined as

$$\mathcal{M}_2(|\psi\rangle) = -\ln \left(\sum_{\hat{P} \in \mathcal{P}_N} (\langle \psi | \hat{P} | \psi \rangle^2 / D)^2 \right) - \ln D. \quad (1)$$

We recall that Clifford operators map each string of Pauli operators to a single string of Pauli operators. Given a pure-state density matrix $\hat{\rho} = |\psi\rangle\langle\psi|$, the state $|\psi\rangle$ is a stabilizer if and only if $\hat{\rho}$ has 2^N nonzero components over the strings of the Pauli group while the rest are zero [8]. Therefore, the SRE measures the entropy of the probability distribution $\Pi_P = \text{Tr}(\hat{P}\hat{\rho})^2/D$, shifted by that of a stabilizer state. As shown in Ref. [52], it is a good nonstabilizerlessness measure and satisfies the bound $0 \leq \mathcal{M}_2 < N \ln 2$, with $\mathcal{M}_2 = 0$ if and only if $|\psi\rangle$ is a stabilizer.

The stabilizer nullity

$$\nu(|\psi\rangle) = N - \log_2(|\text{Stab } |\psi\rangle|) \quad (2)$$

is another measure of nonstabilizerlessness that has the important property of being a provable nonstabilizerlessness monotone after measurements of Pauli operators [28]. Here $\text{Stab } |\psi\rangle$ denotes the subset of Pauli strings \hat{P} such that $\hat{P}|\psi\rangle = |\psi\rangle$. Stabilizer states are such that $|\text{Stab } |\psi\rangle| = 2^N$; hence their nullity is zero by definition. States with finite nonstabilizerlessness have $|\text{Stab } |\psi\rangle| = 2^M$ for some $M < N$ and a nonzero nullity. States that are stabilized only by the identity have maximum nullity $\nu = N$. The stabilizer nullity is related to the stabilizer k -Rényi entropies \mathcal{M}_k by the limit

$$\nu = \lim_{k \rightarrow \infty} (k-1)\mathcal{M}_k \quad (3)$$

and can also be computed from the Pauli spectrum [see Eq. (16)]. The number of 1's in the Pauli spectrum is equal to $|\text{Stab } |\psi\rangle|$ [28].

The average entanglement-spectrum flatness along orbits of the Clifford group \mathcal{C}_N [30] is another recently proposed measure of nonstabilizerlessness. Compared to more intricate measures based on minimization of cost functions involving all stabilizer states, where symmetries are also equally beneficial [21], the entanglement-spectrum flatness shares with the SRE the fact that it is more easily computable, though, in general, with exponential effort. Moreover, it establishes a direct connection between entanglement response and nonstabilizerlessness.

Given the pure-state density matrix $\hat{\rho} = |\psi\rangle\langle\psi|$, we can consider a bipartition $A+B$ of the N -qubit system made up of two subsets with N_A and N_B qubits, respectively, with $N_A + N_B = N$. The reduced density matrix of subsystem A reads $\hat{\rho}_A = \text{Tr}_B(\hat{\rho})$, where Tr_B is the partial trace over B .

The entanglement-spectrum flatness is defined as $\mathcal{F}_A(|\psi\rangle) = \text{Tr}_A(\hat{\rho}_A^3) - \text{Tr}_A(\hat{\rho}_A^2)$.

Nonstabilizerlessness is related to the average spectrum flatness along the Clifford orbit $\hat{\Gamma}|\psi\rangle$ (for all $\hat{\Gamma} \in \mathcal{C}_N$), denoted by $\langle \mathcal{F}_A(\hat{\Gamma}|\psi\rangle) \rangle_{\mathcal{C}_N}$. It has been noted that states with finite nonstabilizerlessness have a nonflat average entanglement spectrum and vice versa. In Ref. [30] it was analytically proven that $\langle \mathcal{F}_A(\hat{\Gamma}|\psi\rangle) \rangle_{\mathcal{C}_N} = c(D, D_A)\mathcal{M}_{\text{lin}}(|\psi\rangle)$, where $\mathcal{M}_{\text{lin}} = 1 - D\|\Pi(|\psi\rangle)\|_2^2$ is the linear entropy, $\Pi(|\psi\rangle)$ is the vector of probabilities of the Pauli strings, $\|\cdot\|_2$ is the Euclidean norm, and $c(D, D_A)$ is a coefficient that depends on the size of the partitions. The linear entropy is related to the 2-Rényi entropy by the relation $\mathcal{M}_2 = -\ln(1 - \mathcal{M}_{\text{lin}})$; hence the average entanglement-spectrum flatness is directly related to the SRE, though it might be exponentially difficult to resolve due to the scaling of $c(D, D_A)$ with N .

Therefore, the average entanglement-spectrum flatness can be computed as

$$\langle \mathcal{F}_A \rangle = c(D, D_A)\mathcal{M}_{\text{lin}} \equiv \frac{(D^2 - D_A^2)(D_A^2 - 1)}{(D^2 - 1)(D_A + 2)D_A^2} \mathcal{M}_{\text{lin}}, \quad (4)$$

where $D = 2^N$ and D_A is the size of the partition A . For a balanced bipartition, $D_A = \sqrt{D}$ and $c(D, D_A) \sim D^{-1}$ for large N , which means detecting the average entanglement-spectrum flatness is exponentially hard in N .

In the next section we will show how permutational invariance allows the efficient computation of these nonstabilizerlessness measures.

III. PERMUTATIONAL INVARIANCE AND NONSTABILIZERNESS

A system is permutationally invariant when it is made up of N identical qubits and its Hamiltonian is only expressed in terms of collective spin operators $\hat{S}_\alpha^{(N)} = \sum_{j=1}^N \hat{\sigma}_j^\alpha / 2$. Its Hamiltonian remains the same after any relabeling of the particle indices. In this setting, the only states that matter are those that respect the permutation symmetry of the model and are built as a symmetric superposition of states with n excitations, with $n = 0, 1, \dots, N$. These states form a basis of the $(N+1)$ -dimensional symmetric subspace and are known as Dicke states [46]. They read

$$|N, n\rangle = \binom{N}{n}^{-1/2} \hat{S}[|0\rangle^{\otimes(N-n)} |1\rangle^{\otimes n}], \quad (5)$$

where \hat{S} is the symmetrizer operator. Many important states in quantum information can be expressed in the Dicke basis. For example, the Greenberger-Horne-Zeilinger (GHZ) state can be written as $|\text{GHZ}\rangle = (|N, 0\rangle + |N, N\rangle)/\sqrt{2} \equiv (|00\dots 0\rangle + |11\dots 1\rangle)/\sqrt{2}$, whereas the state $|N, 1\rangle$, with one excitation, corresponds to the $|W\rangle = (|00\dots 01\rangle + |00\dots 10\rangle + \dots + |10\dots 00\rangle)/\sqrt{N}$ state. Moreover, permutationally invariant systems have applications in quantum metrology and quantum error correction [53–56].

Because of permutation symmetry, the expectation value of a Pauli string on a symmetric state cannot depend on the order of the Pauli operators appearing in the string but only on the number of X , Y , Z , and I gates in the string [57]. Let us call these numbers N_x , N_y , N_z , and N_0 and group them together in

the quadruple $N = \{N_x, N_y, N_z, N_0\}$. Of course, one must have $N_x + N_y + N_z + N_0 = N$. Any summation over the elements of the Pauli group can then be split as follows for any function $f(\hat{P})$:

$$\sum_{\hat{P} \in \mathcal{P}_N} f(\hat{P}) = \sum_N g(N) f(\hat{P}(N)). \quad (6)$$

Here the operator $\hat{P}(N)$ is the permutationally symmetric representative and $g(N)$ is its degeneracy. The number of distinct representatives is given by [58]

$$\mathcal{D} = \binom{N+3}{3} = \frac{(N+1)(N+2)(N+3)}{6}, \quad (7)$$

while their degeneracy is given by the multinomial coefficient

$$g(N) = \binom{N}{N_x, N_y, N_z, N_0} = \frac{N!}{N_x! N_y! N_z! N_0!}. \quad (8)$$

It is possible to explicitly construct the matrix representation of the symmetric representative given a quadruple N , in the Dicke basis. We start by recalling the following property of Pauli matrices, which is a trivial consequence of Euler's formula:

$$\hat{\sigma}_j^{\alpha_j} = (-i) \exp[i(\pi/2)\hat{\sigma}_j^{\alpha_j}]. \quad (9)$$

Given a (nonsymmetric) Pauli string $\hat{P}(\alpha_1, \dots, \alpha_N) = \bigotimes_{j=1}^N \hat{\sigma}_j^{\alpha_j}$, we can rewrite it in the following alternative way, exploiting Eq. (9) and the fact that Pauli matrices acting on different particles commute with each other:

$$\begin{aligned} \hat{P}(\alpha_1, \dots, \alpha_N) &= (-i)^{N-N_0} \exp\left(i\frac{\pi}{2} \sum_{j:\alpha_j=x} \hat{\sigma}_j^x\right) \\ &\times \exp\left(i\frac{\pi}{2} \sum_{j:\alpha_j=y} \hat{\sigma}_j^y\right) \exp\left(i\frac{\pi}{2} \sum_{j:\alpha_j=z} \hat{\sigma}_j^z\right). \end{aligned} \quad (10)$$

We recognize, in the exponents, the components of the total spin operators built considering the particles on which the Pauli string acts with X , Y , or Z , respectively. The fundamental insight is that, for a permutationally invariant system, the specific lattice indices must not matter but only the number of times each gate appears in the Pauli string. This means that we can consider the total spin components of fully symmetric subsystems of N_x , N_y , and N_z particles and write the symmetric representative $\hat{P}(N)$ as

$$\hat{P}(N) = (-i)^{N-N_0} e^{i\pi \hat{S}_x^{(N_x)}} e^{i\pi \hat{S}_y^{(N_y)}} e^{i\pi \hat{S}_z^{(N_z)}}. \quad (11)$$

In order to apply Eq. (11) to a Dicke state, first we have to decompose the state to highlight its symmetric components. To do so, we can iteratively exploit the decomposition [59]

$$|N, m\rangle = \sum_{n_x=0}^{\min(N_x, n)} \sqrt{p_{n, n_x}} |N_x, n_x\rangle \otimes |N - N_x, n - n_x\rangle, \quad (12)$$

with $p_{n, n_x} = \binom{N_x}{n_x} \binom{N-N_x}{n-n_x} / \binom{N}{n}$, which after three applications leads to the representation

$$\begin{aligned} |N, m\rangle &= \sum_{n_x=0}^{N_x} \sum_{n_y=0}^{N_y} \sum_{n_z=0}^{N_z} \sqrt{\frac{\binom{N_x}{n_x} \binom{N_y}{n_y} \binom{N_z}{n_z} \binom{N_0}{n-n_x-n_y-n_z}}{\binom{N}{n}}} \\ &\times |N_x, n_x\rangle \otimes |N_y, n_y\rangle \otimes |N_z, n_z\rangle \\ &\otimes |N_0, n - n_x - n_y - n_z\rangle \Theta(n - n_x - n_y - n_z), \end{aligned} \quad (13)$$

where $\Theta(x)$ is the Heaviside step function [$\Theta(x < 0) = 0$, $\Theta(x \geq 0) = 1$]. At this point, we can use the following identities of the $(N_\alpha + 1)$ -dimensional irreducible representations of the rotation matrices [60] ($\alpha \in \{x, y, z\}$), which allow us to expand the three exponentials in Eq. (11) in the bases of their respective spaces:

$$e^{i\pi \hat{S}_x^{(N_x)}} = (i)^{N_x} \sum_{n_x''=0}^{N_x} |N_x, n_x''\rangle \langle N_x, N_x - n_x''|, \quad (14a)$$

$$e^{i\pi \hat{S}_y^{(N_y)}} = \sum_{n_y''=0}^{N_y} (-1)^{n_y''} |N_y, n_y''\rangle \langle N_y, N_y - n_y''|, \quad (14b)$$

$$e^{i\pi \hat{S}_z^{(N_z)}} = (i)^{N_z} \sum_{n_z''=0}^{N_z} (-1)^{n_z''} |N_z, n_z''\rangle \langle N_z, n_z''|. \quad (14c)$$

Putting everything together, we finally arrive at the matrix representation of $\hat{P}(N)$:

$$\begin{aligned} \langle N, m | \hat{P}(N) | N, n \rangle &= (-i)^{N_y} \sum_{n_x, n_y, n_z} (-1)^{n_y+n_z} \frac{\binom{N_x}{n_x} \binom{N_y}{n_y} \binom{N_z}{n_z} \binom{N_0}{n-n_x-n_y-n_z}}{\sqrt{\binom{N}{n} \binom{N}{m}}} \\ &\times \delta_{m, n+(N_x-2n_x)+(N_y-2n_y)} \Theta(n - n_x - n_y - n_z). \end{aligned} \quad (15)$$

Each of the $\mathcal{D} = O(N^3)$ symmetric Pauli strings is a sparse matrix with at most $O(N^2)$ nonzero matrix elements. According to Eq. (15), $O(N^2)$ operations are required to build each matrix element, since one of the three sums is canceled by the Kronecker δ . Therefore, with $O(N^7)$ easily parallelizable operations, it is possible to compute the full representation of the Pauli group in the Dicke basis and store it in files for later access. At that point, the representation can be reused: In order to read it from memory, only $O(N^5)$ operations are required since one no longer has to compute the double sum in Eq. (15). Thus, if $|\psi\rangle$ is a permutationally invariant state, instead of computing the 4^N elements of the Pauli spectrum, which is the set

$$\text{Spec} |\psi\rangle = \{|\langle \psi | \hat{P} | \psi \rangle|\}_{\hat{P} \in \mathcal{P}_N} \quad (16)$$

of 4^N real numbers $r_i \in [0, 1]$ given by the expectations of the Pauli strings on $|\psi\rangle$, one can just store the \mathcal{D} expectation values of the symmetric representatives $\hat{P}(N)$ and their degeneracy $g(N)$ for any quadruple N . Then the SRE can be computed using Eqs. (1) and (6).

Permutational invariance simplifies the calculation of other measures of nonstabilizerlessness, such as the entanglement-spectrum flatness [30] and the stabilizer nullity [28]. The

former is directly related to \mathcal{M}_{lin} and can be computed efficiently using the symmetric representation of the Pauli group, Eq. (15). Instead, the stabilizer nullity can be computed as

$$\nu(|\psi\rangle) = N - \log_2 \left(\sum_N^{|\langle \psi | \hat{P}(N) | \psi \rangle| = 1} g(N) \right), \quad (17)$$

where the sum is restricted to those N such that $|\langle \psi | \hat{P}(N) | \psi \rangle| = 1$, with $\text{poly}(N)$ operations.

This exponential simplification, arising from permutationally invariant state tomography [47,48], is what allows us to exactly study the nonstabilizerness of systems of hundreds of qubits.

IV. LIPKIN-MESHKOV-GLICK MODEL

Equipped with these tools, we study the nonstabilizerness in the spectrum of the Lipkin-Meshkov-Glick (LMG) model [49]. First, we focus on the ground state. We consider the Hamiltonian (in units of the exchange coupling J)

$$\hat{H}/J = -(1 - \xi)\hat{S}_z^{(N)} - (4\xi/N)(\hat{S}_x^{(N)})^2, \quad (18)$$

where $\xi \in [0, 1]$. The dimensionless transverse field, controlling the properties of the model, is $\gamma(\xi) \equiv (1 - \xi)/4\xi \in [0, \infty]$. This model features a second-order quantum phase transition at the critical value $\xi_c = 1/5$ ($\gamma_c = 1$): For $\gamma < \gamma_c$ the ground state (GS) is ferromagnetic (aligned along x), while for $\gamma > \gamma_c$ the state is paramagnetic. The order parameter describing this symmetry-breaking transition is the magnetization in the x direction m_x . The half-system entanglement entropy of the ground state $S_{N/2}$ displays a $\ln N$ divergence at the critical point of the mean-field (MF) transition, approaches the value $S_{N/2} = \ln 2$ in the zero-field limit, where the ground state is a GHZ state in the x direction, and becomes zero in the opposite limit, where the GS is the product state polarized along z [59].

While the ground-state properties of the SRE can be inferred by mean-field theory (see Sec. VI), states at finite energy density go beyond this description. The LMG Hamiltonian is made up of two terms: The first one has a $U(1)$ symmetry structure, whereas the second one has an $SO(2)$ symmetry. Together, they form a Hamiltonian with a $U(2)$ algebraic structure. When $\xi > \xi_c$ this model is known to have an excited-state quantum phase transition (ESQPT) at finite energy density [61]: For any value of $\xi_c \leq \xi \leq 1$ there exists a finite energy $E_{\text{ESQPT}}(\xi)$ around which the density of states is strongly peaked, i.e., the excited energy levels cluster around the separatrix line $E_{\text{ESQPT}}(\xi)$. In the following we will shift the Hamiltonian (18) by its ground-state energy $\hat{H}(\xi) \rightarrow \hat{H}(\xi) - E_{\text{GS}}(\xi)$. Then the analytical expression of $E_{\text{ESQPT}}(\xi)$ can be derived in the large- N limit with a semiclassical analysis and reads $E_{\text{ESQPT}}(\xi)/N = (1 - 5\xi)^2/16\xi$ for $\xi \geq \xi_c$. Some physical quantities, such as the longitudinal and transverse magnetization and the participation ratio in the x or z Dicke bases [61], are singular at the ESQPT and can thus be used as order parameters of the transition. Interestingly, the study of the ESQPT yields a detailed fingerprint of the properties of the full many-body spectrum that goes beyond the mean-field description of the transition itself.

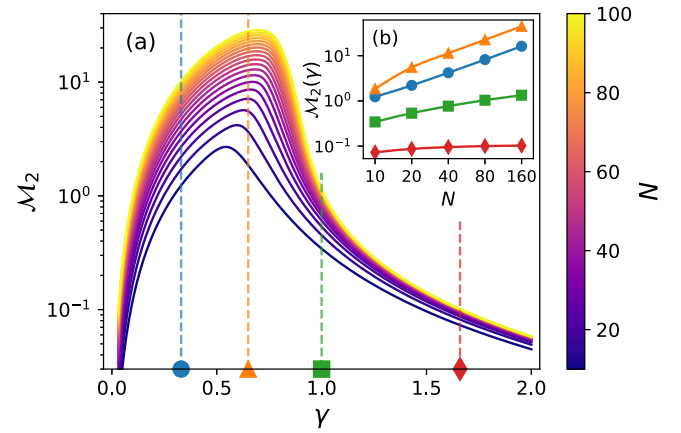


FIG. 1. (a) Stabilizer Rényi entropy \mathcal{M}_2 of the ground state of the LMG model as a function of the transverse field strength γ for several sizes (log-linear scale). (b) Scaling with N of the SRE for the values of the transverse field marked in Fig. 1(a) (log-log scale).

We will explore the properties of the nonstabilizerness of the whole energy spectrum at finite energy density in a numerically exact fashion. We will show that its study reveals a very rich phenomenology, not grasped by the analysis of entanglement.

A. Ground state

In Fig. 1(a) we report $\mathcal{M}_2(|\text{GS}(\gamma)\rangle)$ as a function of the transverse field strength for system sizes ranging from $N = 10$ to $N = 100$ with increments of five qubits between subsequent curves. We see that the ground-state SRE is zero both for $\gamma = 0$ and for large γ since both the GHZ state and the fully polarized state along the z axis are stabilizer states, while it develops a peak for $0 < \gamma < 1$ that grows with N . The analysis of the SRE in the thermodynamic limit tells us that the peak position tends to $\gamma^* = 1/\sqrt{2}$ and thus it does not approach the MF critical point, as opposed to the peak of entanglement [59,62–64].

We can clearly distinguish two different scalings of the SRE with N , depending on the strength of the transverse field. For $\gamma < \gamma_c$, \mathcal{M}_2 is extensive and the stabilizer density $m_2 = \mathcal{M}_2/N$ is constant. Adopting the same terminology used to describe entanglement phases, we identify this as a volume-law phase of magic. This result is surprising because it tells us that the ground state of the LMG model in the ferromagnetic phase cannot be prepared efficiently using Clifford gates, despite the fact that the large- N limit of this model is exactly reproduced by mean-field theory. Instead, when $\gamma > \gamma_c$ we observe that the SRE becomes independent of N , signaling the onset of what can be called an area-law phase of magic. Here the density of magic m_2 goes to zero with N , suggesting that large- N ground states of the LMG model in this phase are essentially stabilizer states [65]. However, we find that the stabilizer nullity is $\nu = 0$ only for $\gamma = 0$ (and eventually $\gamma = \infty$) and $\nu = N$ for all other values of the transverse field.

As shown in Fig. 1(b) for systems up to $N = 160$ spins, the SRE follows a power-law scaling of the form $\mathcal{M}_2 \sim N^\beta$ with $\beta \geq 0$. The scaling exponent vanishes in the area-law phase (see the red line with diamonds) and is close to one in the

TABLE I. Exponents of the scaling law $\mathcal{M}_2 \sim N^\beta$ and corresponding standard deviations for the LMG model.

γ	β	σ_β
0.33	0.973	0.002
0.65	0.988	0.001
1.00	0.401	0.006
1.66	0.054	0.003

volume-law phase (see the blue line with circles and orange line with triangles; see also Table I). At the transition (green line with squares), the scaling with N only looks subextensive due to finite-size effects, but mean-field theory tells us that the SRE is extensive (see Sec. VI). Near the critical point, due to finite-size effects, we also observe a superextensive region with an exponent $\beta > 1$, inconsistent with the mean-field predictions. The bound $\mathcal{M}_2 < N \ln 2$ is always satisfied [52].

The average flatness follows the behavior of \mathcal{M}_{lin} . For ease of visualization, we plot directly \mathcal{M}_{lin} in Fig. 2 for the GS of the LMG model. We can see that the flatness correctly detects the volume-law phase of magic, where \mathcal{M}_{lin} is close to one, while it becomes small in the area-law region or when $\gamma = 0$ and the ground state is a stabilizer. As shown in the inset, however, $\langle \mathcal{F}_A \rangle$ is more difficult to obtain for large N : Already for these sizes, it can go down below machine precision.

B. Excited states

We show the spectrum of the Hamiltonian (18) in Fig. 3(a) for $N = 256$. The separatrix line is marked with white dots and is clearly distinguishable by the peak in density of eigenstates. It has been shown that states exactly at the critical line are strongly localized around the ground state of the U(1) part of the Hamiltonian [61], i.e., the state with all spins pointing up. This is a stabilizer state with zero magic. Close to the critical line states are generally entangled and have finite magic. Therefore, it is reasonable to expect that both the entanglement entropy and the SRE would sharply decrease in correspondence with the ESQPT, and indeed this is what we observe in Figs. 3(b) and 3(c), where we show the entangle-

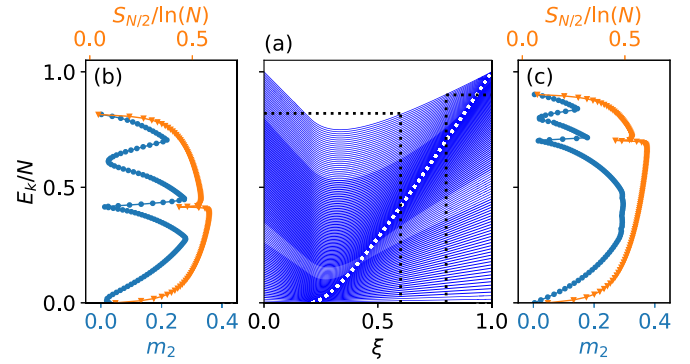


FIG. 3. (a) Spectrum of the Hamiltonian (18) for $N = 256$. (b) Entanglement entropy and density of magic for $\xi = 0.6$. (c) Same as (b) for $\xi = 0.8$.

ment entropy and the density of magic $m_2 = \mathcal{M}_2/N$ for all the eigenstates along the cuts $\xi = 0.6$ and 0.8 , respectively. Both these quantities sharply decrease when the cut intercepts the separatrix line, signaling the transition. Nevertheless, comparing the entanglement entropy and the SRE, we see that the latter has a richer structure, with cusps, local minima, and local maxima as a function of the energy density.

We focus on $\xi = 0.6$; any $\xi > \xi_c$ shows the same features. In Fig. 4(a) we show the SRE along this cut for different sizes. There are regions where \mathcal{M}_2 is extensive and other regions where it is not. We recall that in permutationally symmetric systems the maximum amount of bipartite entanglement entropy is $S_{\text{max}} = \log D = \log(N + 1)$, whereas the maximum amount of magic is not restricted by symmetry: The maximum SRE is associated with a flat Pauli spectrum (except for the projection on the identity operator), which is allowed by permutation symmetry. In Fig. 4(b) we plot the scaling of \mathcal{M}_2 with N for several energy densities, marked in Fig. 4(a) using the same symbols. We see indeed regions where the exponent of the scaling law $\mathcal{M}_2 \sim N^\alpha$ is close to $\alpha \sim 1$ and other regions where α becomes close to zero. This happens, in particular, around the critical energy density, where, additionally, the SRE has a cusp. In the thermodynamic limit,

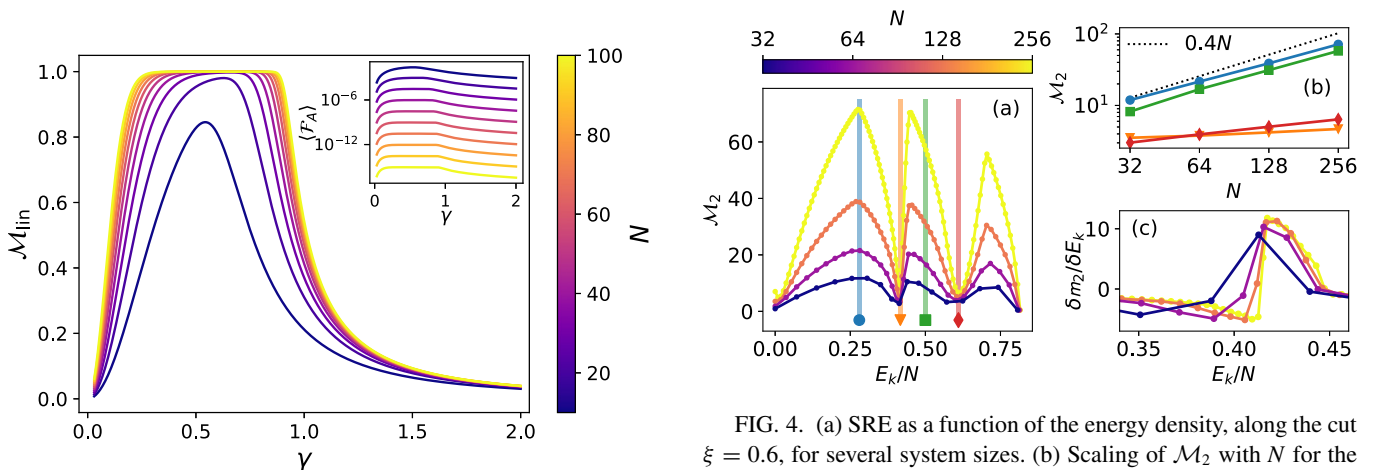


FIG. 4. (a) SRE as a function of the energy density, along the cut $\xi = 0.6$, for several system sizes. (b) Scaling of \mathcal{M}_2 with N for the energy densities marked in (a) with the same symbols and colors. (c) Numerical derivative of the SRE around the critical density for the same sizes as in (a).

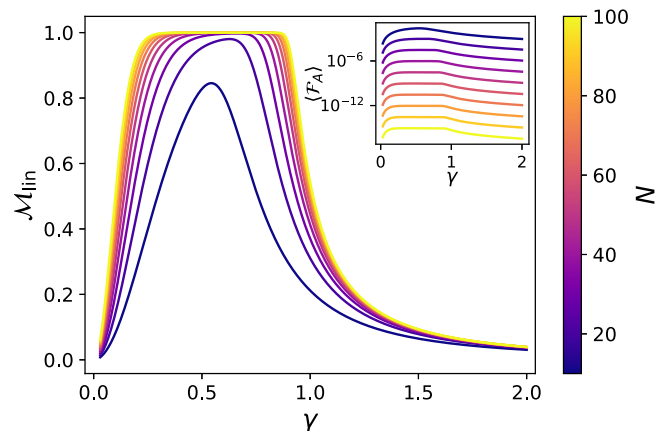


FIG. 2. Average entanglement-spectrum flatness of the ground state of the LMG model as a function of the transverse field strength.

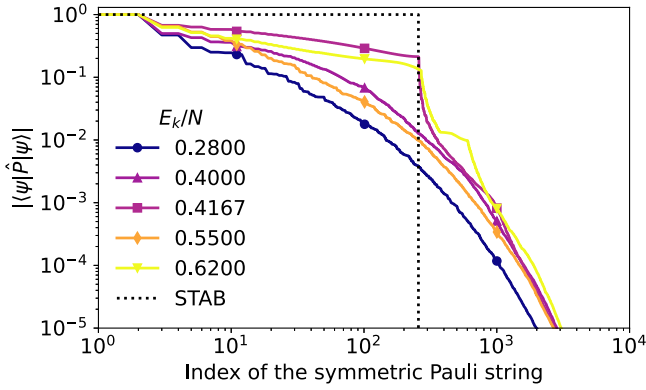


FIG. 5. Pauli spectrum of different eigenstates of the LMG model at finite energy densities ($N = 256$).

its first derivative has a finite jump at E_{ESQPT} as displayed in Fig. 4(c), where we plot the numerical derivative of the density of magic around the critical energy density and observe the development of a jump at the transition for large N .

In order to better understand the differences between excited states with low or high magic, it is interesting to analyze the Pauli spectrum [Eq. (16)] for different energy densities. For permutationally invariant systems, we only have to compute \mathcal{D} distinct expectation values with their degeneracies $g(N)$. In Fig. 5 we show the Pauli spectra of several eigenstates for $N = 256$ ($\mathcal{D} = 2\,862\,209$). In order to improve visualization, we sort the expectation values in decreasing order and only report one expectation value per symmetric representative without keeping track of its degeneracy. We observe that eigenstates in the volume-law phase of magic have a featureless Pauli spectrum. This has to be compared with the steplike spectra of states with minimal magic, which more closely resemble the spectrum of stabilizer states, which have nonzero expectation values (± 1) over exactly 2^N Pauli strings by definition. Notice, however, that in all cases the Pauli spectrum is significantly nonzero only for very few Pauli strings (compared to \mathcal{D}), which means one could easily combine the exponential reduction provided by permutation symmetry with the sampling techniques discussed in Refs. [36,50,66,67] for an even more efficient estimation of the nonstabilizerness.

V. FERROMAGNETIC p -SPIN MODEL

The LMG model can be generalized to systems with infinite-range p -body interactions with $p > 2$, giving rise to the ferromagnetic p -spin model [68,69]. Its Hamiltonian reads

$$\hat{H}/J = -\gamma \hat{S}_z^{(N)} - \frac{1}{2} \left(\frac{2}{N} \right)^{p-1} (\hat{S}_x^{(N)})^p, \quad (19)$$

from which one can reobtain the LMG model by setting $p = 2$. As opposed to the LMG, the p -spin model with $p > 2$ undergoes a first-order quantum phase transition in its ground state as a function of the transverse field γ , separating the paramagnetic from the ferromagnetic phase. The value of the critical field depends on p and has been derived using mean-field theory for all values of p [70]. The bipartite entanglement entropy obeys an area law everywhere and is discontinuous at the transition [71].

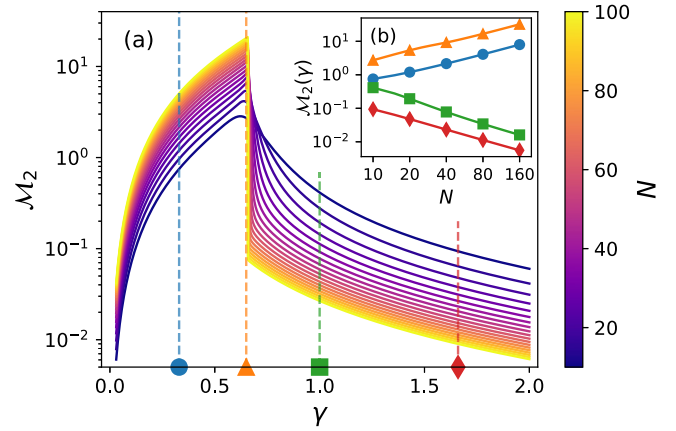


FIG. 6. (a) Stabilizer Rényi entropy \mathcal{M}_2 of the ground state of the p -spin model ($p = 3$) as a function of the transverse field strength γ for several sizes (log-linear scale). (b) Scaling with N of the SRE for the values of the transverse field marked in (a) (log-log scale).

We show the SRE of the ground state of the p -spin model with $p = 3$ in Fig. 6(a) as a function of γ . First, we see that, in the ferromagnetic phase, \mathcal{M}_2 grows with γ and N , until the critical point of the MF transition is reached. At that point, the SRE is discontinuous and suddenly jumps to a value close to zero. In the ferromagnetic phase, the SRE grows with N , scaling as $\mathcal{M}_2 \sim N^\beta$ with $\beta \approx 1$, i.e., it is extensive. By contrast, in the paramagnetic region, the SRE decreases with N : The scaling exponent β becomes negative, as shown in Fig. 6(b), for systems up to $N = 160$ spins, and in Table II. The negative exponent tells us that the next-order term in powers of $1/N$ around the mean-field nonstabilizerness density decreases as $1/N^2$, as opposed to the LMG model, where the next-order term scales as $1/N$ (discussed later). The stabilizer nullity is $\nu = 0$ only for $\gamma = 0$ (and eventually $\gamma = \infty$) and $\nu = N$ for all other values of the transverse field.

The average flatness follows \mathcal{M}_2 , as we show in Fig. 7. Here we see that the average flatness becomes discontinuous at the transition for large N and that the issue of detecting an exponentially small quantity highlighted in the previous section still remains.

Finally, in Fig. 8 we show the Pauli spectra of the GS of the LMG and of the p -spin model for the values of the transverse field marked in Figs. 1 and 6, for $N = 80$ ($\mathcal{D} = 91\,881$). In order to improve visualization, we sort the expectation values in decreasing order and only report one expectation value per symmetric representative without keeping track of its degeneracy. When $\gamma > \gamma_c$, we see that the ground state becomes close to a stabilizer state (which would be represented by a

TABLE II. Exponents of the scaling law $\mathcal{M}_2 \sim N^\beta$ and corresponding standard deviations for the p -spin model.

γ	β	σ_β
0.33	0.943	0.003
0.65	0.906	0.008
1.00	-1.138	0.008
1.66	-1.028	0.001

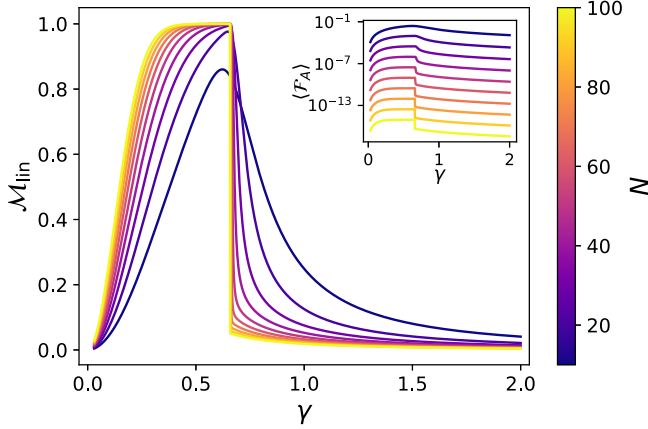


FIG. 7. Average entanglement-spectrum flatness of the ground state of the p -spin model ($p = 3$) as a function of the transverse field strength.

1-0 step function). Instead, when $\gamma \lesssim \gamma_c$, we see that the Pauli distribution develops a tail, which is significantly nonzero only for very few Pauli strings (compared to \mathcal{D}).

VI. MEAN-FIELD STABILIZER RÉNYI ENTROPY

For permutationally invariant systems, mean-field theory is exact in the thermodynamic limit. This allows deriving the analytical expression of the stabilizer Rényi entropy \mathcal{M}_2 of the ground states of the models considered for large N [65]. We report the general scheme in this section.

The system's Hamiltonian is only expressed in terms of collective magnetization operators $\hat{H} = \hat{H}(\{\hat{S}_\alpha\})$. We can express the Hamiltonian in terms of the magnetization components $\hat{m}_\alpha = \hat{S}_\alpha/S$ with $S = N/2$ and define the Hamiltonian density $\hat{h} = \hat{H}/N$. Then we note that $[\hat{m}_\alpha, \hat{m}_\beta] \sim O(1/N)$; therefore, in the thermodynamic limit, the magnetization components behave like classical variables. In this limit, correlations between spins are negligible and the system's ground state is a tensor product of single-particle states aligned along the same angles (θ, ϕ) , a so-called spin-coherent state:

$$|\theta, \phi\rangle = \bigotimes_{k=1}^N \left(\cos \frac{\theta}{2} |0\rangle_k + \sin \frac{\theta}{2} e^{i\phi} |1\rangle_k \right). \quad (20)$$

The expectation of \hat{h} over the spin-coherent state gives the variational semiclassical energy $h(\theta, \phi)$, whose minimum cor-

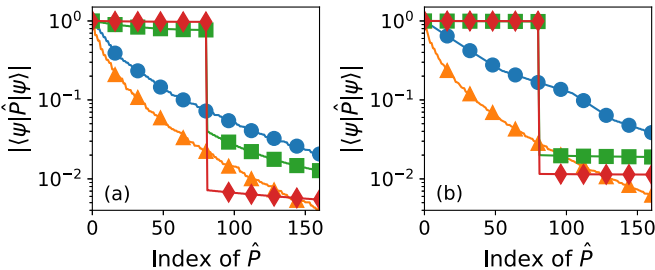


FIG. 8. Pauli spectrum for $N = 80$ ($\mathcal{D} = 91881$) for (a) the LMG model and (b) the p -spin model ($p = 3$). Symbols refer to the values of γ marked in Figs. 1 and 6.

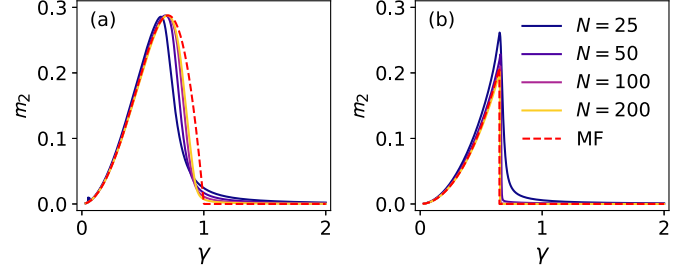


FIG. 9. Comparison between the nonstabilizer density m_2 evaluated with mean-field theory and with finite-size numerical simulations for the (a) LMG model and (b) p -spin model.

responds to the ground-state energy in the thermodynamic limit and whose optimal angles (θ^*, ϕ^*) allow writing the ground state as is Eq. (20). Then, since the ground state is a product state, we have that $\mathcal{M}_2 = Nm_2$, with

$$m_2 = -\ln \left(\frac{1 + \sin^4 \theta^* (\cos^4 \phi^* + \sin^4 \phi^*) + \cos^4 \theta^*}{2} \right). \quad (21)$$

For the LMG model [Eq. (18)] we observe that the Hamiltonian does not depend on \hat{S}_z , and thus we get $\phi^* = 0$ and write the semiclassical energy as

$$h(\theta) = -\frac{\gamma}{2} \cos \theta - \frac{1}{4} \sin^2 \theta. \quad (22)$$

When $\gamma < \gamma_c = 1$, this function is minimized by $\theta^* = 0$; instead, when $\gamma > \gamma_c$, the minimum is $\theta^* = \arccos \gamma$. The order parameter $m_x = \sin \theta^*$ is continuous at the transition. The mean-field SRE of Eq. (21), for $\gamma < \gamma_c$, reads

$$m_2 = -\ln(1 + \gamma^4 - \gamma^2) \quad (23)$$

and agrees with our numerical results [see Fig. 9(a)]. Its maximum value is found for $\gamma^* = 1/\sqrt{2}$.

For the p -spin model with $p = 3$ [Eq. (19)] the semiclassical energy reads

$$h(\theta) = -\frac{\gamma}{2} \cos \theta - \frac{1}{4} \sin^3 \theta \quad (24)$$

and can be minimized as before. Following Ref. [70], it can be shown that the order parameter is discontinuous at the transition and that the critical point is $\gamma_c = (\sqrt{3}/2)^3 \approx 0.6495$. The MF results agree with our large- N simulations [see Fig. 9(b)].

It is also interesting to study the next-order corrections to the mean-field density of magic for the two analyzed models. In general,

$$m_2(\gamma) = m_2^{(\text{MF})}(\gamma) + \epsilon(\gamma, N). \quad (25)$$

We numerically compute the difference ϵ and plot it in Figs. 10(a) and 10(b) as a function of the transverse field for several system sizes. We can see that in the ferromagnetic region $\gamma < \gamma_c$ the curves generally do decrease with N , with some oscillations. In the paramagnetic region $\gamma > \gamma_c$, we see that the plotted quantity decreases with N without oscillations. Moreover, we can clearly see that the scaling with N is different in the two cases [the curves are more spread out in Fig. 10(b) than in Fig. 10(a)]. By focusing on

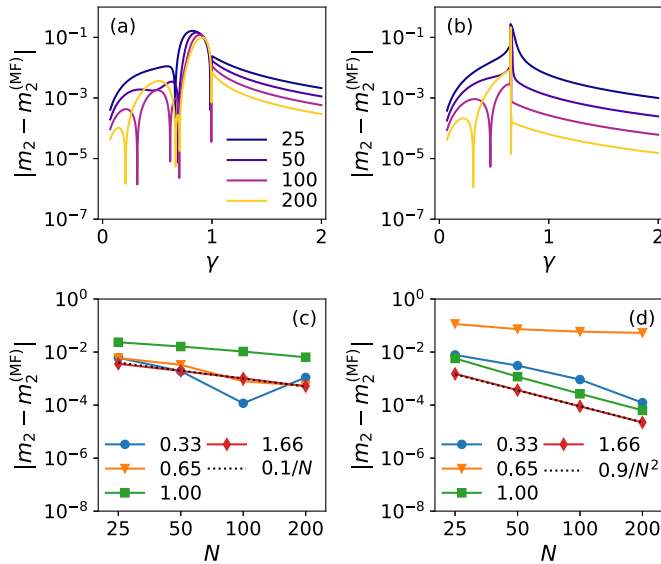


FIG. 10. Next-to-leading order of the density of nonstabilizer with respect to the mean-field solution as a function of (a) and (b) γ and (c) and (d) N for the two models analyzed.

some values of γ , we indeed observe different scalings with the system size. We plot ϵ as a function of N in Figs. 10(c) and 10(d). In the paramagnetic region, e.g., $\gamma = 1.66$, we find $\epsilon_{\text{LMG}} \approx 1/10N$ and $\epsilon_{p\text{-spin}} \approx 0.9/N^2$. While we do not have an analytical explanation of these scalings, they allow us to understand the two very different scalings of the SRE with N in the paramagnetic phase of the two models analyzed (Figs. 1 and 6).

VII. CONCLUSION

To summarize, exploiting the representation of the Pauli group in the Dicke basis, we studied the stabilizer Rényi

entropy and other nonstabilizer measures of permutationally symmetric systems, finding evidence of quantum magic phase transitions not detected in the entanglement that are related to their symmetry-breaking transitions. Above the critical point, quantum magic becomes zero in the thermodynamic limit. Below the critical point, the LMG and the p -spin models have a volume-law phase of magic. This result shows that it is possible to have a finite nonstabilizer density, despite the existence of a semiclassical limit. Possible future directions include the study of the nonstabilizer of permutationally invariant finite-temperature states, its dynamics after a quantum quench or annealing [34,36], and the relation between nonstabilizer and chaos [72–74] in infinite-range models.

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- [1] G. Vidal, *Phys. Rev. Lett.* **93**, 040502 (2004).
 - [2] D. Perez-Garcia, F. Verstraete, M. M. Wolf, and J. I. Cirac, *Quantum Inf. Comput.* **7**, 401 (2007).
 - [3] U. Schollwöck, *Ann. Phys. (NY)* **326**, 96 (2011).
 - [4] D. Gottesman, *Phys. Rev. A* **57**, 127 (1998).
 - [5] D. Gottesman, in *Proceedings of the 22nd International Colloquium on Group Theoretical Methods in Physics*, edited by S. P. Corney, R. Delbourgo, and P. D. Jarvis (International Press, Cambridge, 1998), pp. 32–43.
 - [6] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, New York, 2010).
 - [7] D. Gottesman, Stabilizer codes and quantum error correction, Ph.D. thesis, California Institute of Technology, 1997.
 - [8] S. Aaronson and D. Gottesman, *Phys. Rev. A* **70**, 052328 (2004).
 - [9] S. Bravyi and A. Kitaev, *Phys. Rev. A* **71**, 022316 (2005).
 - [10] M. Howard, J. Wallman, V. Veitch, and J. Emerson, *Nature (London)* **510**, 351 (2014).
 - [11] V. Veitch, S. A. H. Mousavian, D. Gottesman, and J. Emerson, *New J. Phys.* **16**, 013009 (2014).
 - [12] E. Chitambar and G. Gour, *Rev. Mod. Phys.* **91**, 025001 (2019).
 - [13] J. R. Seddon and E. T. Campbell, *Proc. R. Soc. A* **475**, 20190251 (2019).
 - [14] S. Zhou, Z.-C. Yang, A. Hamma, and C. Chamon, *SciPost Phys.* **9**, 087 (2020).
 - [15] Z.-W. Liu and A. Winter, *PRX Quantum* **3**, 020333 (2022).
 - [16] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, *Rev. Mod. Phys.* **81**, 865 (2009).
 - [17] M. Howard and E. Campbell, *Phys. Rev. Lett.* **118**, 090501 (2017).
 - [18] S. Bravyi and D. Gosset, *Phys. Rev. Lett.* **116**, 250501 (2016).
 - [19] S. Bravyi, G. Smith, and J. A. Smolin, *Phys. Rev. X* **6**, 021043 (2016).
 - [20] S. Bravyi, D. Browne, P. Calpin, E. Campbell, D. Gosset, and M. Howard, *Quantum* **3**, 181 (2019).
 - [21] M. Heinrich and D. Gross, *Quantum* **3**, 132 (2019).

- [22] X. Wang, M. M. Wilde, and Y. Su, *New J. Phys.* **21**, 103002 (2019).
- [23] X. Wang, M. M. Wilde, and Y. Su, *Phys. Rev. Lett.* **124**, 090505 (2020).
- [24] A. Heimendahl, F. Montealegre-Mora, F. Vallentin, and D. Gross, *Quantum* **5**, 400 (2021).
- [25] J. Jiang and X. Wang, *Phys. Rev. Appl.* **19**, 034052 (2023).
- [26] T. Haug and M. S. Kim, *PRX Quantum* **4**, 010301 (2023).
- [27] X. Turkeshi, A. Dymarsky, and P. Sierant, [arXiv:2312.11631](https://arxiv.org/abs/2312.11631).
- [28] M. Beverland, E. Campbell, M. Howard, and V. Kliuchnikov, *Quantum Sci. Technol.* **5**, 035009 (2020).
- [29] X. Turkeshi, M. Schirò, and P. Sierant, *Phys. Rev. A* **108**, 042408 (2023).
- [30] E. Tirrito, P. S. Tarabunga, G. Lami, T. Chanda, L. Leone, S. F. E. Oliviero, M. Dalmonte, M. Collura, and A. Hamma, *Phys. Rev. A* **109**, L040401 (2024).
- [31] L. Leone, S. F. E. Oliviero, G. Esposito, and A. Hamma, *Phys. Rev. A* **109**, 032403 (2024).
- [32] P. Niroula, C. D. White, Q. Wang, S. Johri, D. Zhu, C. Monroe, C. Noel, and M. J. Gullans, [arXiv:2304.10481](https://arxiv.org/abs/2304.10481).
- [33] S. F. E. Oliviero, L. Leone, and A. Hamma, *Phys. Rev. A* **106**, 042426 (2022).
- [34] D. Rattacaso, L. Leone, S. F. E. Oliviero, and A. Hamma, *Phys. Rev. A* **108**, 042407 (2023).
- [35] T. Haug and L. Piroli, *Phys. Rev. B* **107**, 035148 (2023).
- [36] G. Lami and M. Collura, *Phys. Rev. Lett.* **131**, 180401 (2023).
- [37] G. E. Fux, E. Tirrito, M. Dalmonte, and R. Fazio, [arXiv:2312.02039](https://arxiv.org/abs/2312.02039).
- [38] P. S. Tarabunga, E. Tirrito, M. C. Bañuls, and M. Dalmonte, *Phys. Rev. Lett.* **133**, 010601 (2024).
- [39] L. Amico, R. Fazio, A. Osterloh, and V. Vedral, *Rev. Mod. Phys.* **80**, 517 (2008).
- [40] N. Laflorencie, *Phys. Rep.* **646**, 1 (2016).
- [41] O. Lunt, J. Richter, and A. Pal, in *Entanglement in Spin Chains: From Theory to Quantum Technology Applications*, edited by A. Bayat, S. Bose, and H. Johannesson (Springer, Cham, 2022), pp. 251–284.
- [42] G. Passarelli, X. Turkeshi, A. Russomanno, P. Lucignano, M. Schirò, and R. Fazio, *Phys. Rev. Lett.* **132**, 163401 (2024).
- [43] M. Bejan, C. McLauchlan, and B. Béri, *PRX Quantum* **5**, 030332 (2024).
- [44] P. S. Tarabunga, *Quantum* **8**, 1413 (2024).
- [45] B. A. Chase and J. M. Geremia, *Phys. Rev. A* **78**, 052101 (2008).
- [46] N. Shammah, S. Ahmed, N. Lambert, S. De Liberato, and F. Nori, *Phys. Rev. A* **98**, 063815 (2018).
- [47] T. Moroder, P. Hyllus, G. Tóth, C. Schwemmer, A. Niggelbaum, S. Gaile, O. Gühne, and H. Weinfurter, *New J. Phys.* **14**, 105001 (2012).
- [48] L. Novo, T. Moroder, and O. Gühne, *Phys. Rev. A* **88**, 012305 (2013).
- [49] H. Lipkin, N. Meshkov, and A. Glick, *Nucl. Phys.* **62**, 188 (1965).
- [50] T. Haug and L. Piroli, *Quantum* **7**, 1092 (2023).
- [51] The investigation of this matter is still ongoing (see Ref. [50]).
- [52] L. Leone, S. F. E. Oliviero, and A. Hamma, *Phys. Rev. Lett.* **128**, 050402 (2022).
- [53] Y. Ouyang, *Phys. Rev. A* **90**, 062317 (2014).
- [54] Y. Ouyang and J. Fitzsimons, *Phys. Rev. A* **93**, 042340 (2016).
- [55] Y. Ouyang, in *Proceedings of the 2021 IEEE International Symposium on Information Theory, Melbourne, 2021* (IEEE, Piscataway, 2021).
- [56] Y. Ouyang and G. K. Brennen, [arXiv:2212.06285](https://arxiv.org/abs/2212.06285).
- [57] Y. Ouyang, N. Shettell, and D. Markham, *IEEE Trans. Inf. Theory* **68**, 1809 (2022).
- [58] There are N labels α_j to be separated in $k = 4$ sets by putting $k - 1 = 3$ separators among them. There are $(N + k - 1)! = (N + 3)!$ ways of doing that, and permutations of identical labels $N!$ and separators $(k - 1)! = 3!$ have to be excluded, hence Eq. (7).
- [59] J. I. Latorre, R. Orús, E. Rico, and J. Vidal, *Phys. Rev. A* **71**, 064101 (2005).
- [60] J. J. Sakurai, *Modern Quantum Mechanics* (Addison-Wesley, Reading, 1994).
- [61] L. F. Santos, M. Távora, and F. Pérez-Bernal, *Phys. Rev. A* **94**, 012113 (2016).
- [62] T. Barthel, S. Dusuel, and J. Vidal, *Phys. Rev. Lett.* **97**, 220402 (2006).
- [63] J. Vidal, S. Dusuel, and T. Barthel, *J. Stat. Mech.* (2007) P01015.
- [64] R. Orús, S. Dusuel, and J. Vidal, *Phys. Rev. Lett.* **101**, 025701 (2008).
- [65] P. S. Tarabunga and C. Castelnovo, *Quantum* **8**, 1347 (2024).
- [66] P. S. Tarabunga, E. Tirrito, T. Chanda, and M. Dalmonte, *PRX Quantum* **4**, 040317 (2023).
- [67] G. Lami and M. Collura, *Phys. Rev. Lett.* **133**, 010602 (2024).
- [68] B. Derrida, *Phys. Rev. B* **24**, 2613 (1981).
- [69] D. Gross and M. Mezard, *Nucl. Phys. B* **240**, 431 (1984).
- [70] V. Bapst and G. Semerjian, *J. Stat. Mech.* (2012) P06007.
- [71] M. Filippone, S. Dusuel, and J. Vidal, *Phys. Rev. A* **83**, 022327 (2011).
- [72] M. Hartmann, D. Poletti, M. Ivanchenko, S. Denisov, and P. Hänggi, *New J. Phys.* **19**, 083011 (2017).
- [73] L. Leone, S. F. E. Oliviero, Y. Zhou, and A. Hamma, *Quantum* **5**, 453 (2021).
- [74] G. Passarelli, P. Lucignano, D. Rossini, and A. Russomanno, [arXiv:2406.16585](https://arxiv.org/abs/2406.16585).