Preservation of entanglement in local noisy channels

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(Received 2 May 2024; accepted 25 July 2024; published 13 August 2024)

Entanglement subject to noise cannot be shielded against decaying. But, in case of many noisy channels, the degradation can be partially prevented by using local unitary operations. We consider the effect of local noise on shared quantum states and evaluate the amount of entanglement that can be preserved from deterioration. The amount of saved entanglement not only depends on the strength of the channel but also on the type of the channel, and in particular, it always vanishes for the depolarizing channel. The main motive of this work is to analyze the reason behind this dependency of saved entanglement by inspecting properties of the corresponding channels. In this context, we quantify and explore the biasnesses of channels towards the different states on which they act. We postulate that all biasness measures must vanish for depolarizing and unitary channels, and subsequently introduce a few measures of biasness. We also consider the entanglement capacities of channels. We observe that the joint behavior of the biasness quantifiers and the entanglement capacity explains the nature of saved entanglement. Furthermore, we find a pair of upper bounds on saved entanglement which are noticed to imitate the graphical nature of the latter.

DOI: 10.1103/PhysRevA.110.022419

I. INTRODUCTION

Entanglement [1–3] plays a crucial role in quantum information science. It is used as a resource in many quantum information tasks such as teleportation [4], quantum dense coding [5], quantum computation [6], entanglement-based quantum cryptography [7,8], and so on. But it is a very delicate characteristic of shared quantum systems. Realistic systems are subject to noise, and entanglement loss is typically unavoidable therein. Entanglement transformation, and in particular its disappearance, in presence of different classes of noise have been studied by various scientists [9–13]. Experiments along this line are explored, e.g., Refs. [14–16].

Numerous theoretical [17–31] as well as experimental [32–35] researches have been carried out on prevention or reduction of entanglement loss or regeneration of entanglement. In recent years, it has been realized that though local unitaries cannot change entanglement of any state, its operation can restrain or delay entanglement degradation, even when it is applied only on a single party. If a local unitary is operated afore the system is exposed to noise, the effect of the noise on the system may get reduced compared with the case without the application of any unitary [36–39]. This can be exemplified through analyzing multipartite graph states in presence of the local dephasing channel [37], and bipartite maximally entangled states after transforming through local amplitude damping and local dephasing channels [39]. Since local unitary operations are easy to implement, this method of preservation of entanglement is conceivably an experimentally friendly and low cost process [14,15].

Focusing on bipartite states, successful protection of entanglement from local noise acting identically on the two parties, through the help of local unitary operations, depends on the type and strength of the channels. We name the amount of entanglement that can be saved by applying the optimal local unitary on a single party as "saved entanglement" (SE). We observe that entanglement cannot be saved by using this method in the case of $\mathbb{SU}(d)$-covariant local channels, e.g., the local depolarizing channel, whereas it is possible to protect a finite amount of entanglement in the presence of a local amplitude damping, bit flip, phase flip, or bit-phase flip channel. We find that the saved entanglement is exactly the same for local bit flip, phase flip, and bit-phase flip noise. In this work, we try to explore the reason behind the disparate behavior of saved entanglement for distinct channels.

Since local unitaries just rotate the states locally, if the resulting state is more robust to a noise, the reason must be the noise's partiality towards a set of states having a particular *direction*. This fact motivates us to investigate the property of "biasness" of channels, which describes the channel's bias toward a bunch of states. We make several observations in this direction, which lead us to state postulates that must be satisfied by a quantifier of biasness of a quantum channel. We subsequently introduce three such quantifiers. We also consider the entanglement capacity of quantum channels. We show that the nature of the SE can be explained by composing the behaviors of biasness and entanglement capacity. In particular, by considering two-qubit states and some paradigmatic noisy channels, viz. amplitude damping, bit flip (or phase flip or bit-phase flip) acting locally and identically on both of the qubits, we observe that at low noise strengths, SE monotonically increases with biasness of the local channel, whereas it decreases monotonically with entanglement capacity at higher noise strengths. Here, since unitaries can simply be inverted, we consider unitary channels as noiseless channels, and define "strength of a channel" as the minimum distance of the channel from unitary channels.

We also present two upper bounds on the saved entanglement. These bounds are seen to mimic the nature of SE.

The rest of the paper is organized as follows: In Sec. II, we briefly recapitulate definitions of some well-known quantities, which will be needed in the rest of the paper, such as $\mathbb{SU}(d)$-covariant channels, l_1 norm, distance between two channels, concurrence, etc. Saved entanglement and entanglement capacity are defined in Sec. III. To find our way towards defining eligible measures of biasness, we make a few observations about saved entanglement of channels in the same section. Based on these observations, we define various appropriate biasness quantifiers in Sec. IV. We determine two bounds on the saved entanglement in Sec. V. Different well-known examples of noise are considered in Sec. VI, and their behavior with respect to the biasness measures as well as entanglement capacity and saved entanglement are obtained and discussed. We present the concluding remarks in Sec. VII.

II. PREREQUISITES

In this section we briefly discuss some basic tools which will be used later.

Quantum channels transform a state, ρ, acting on a Hilbert space, $\mathcal{H}$, to another state, ρ', acting on the same or different Hilbert space, $\mathcal{H}'$. Operation of a quantum channel can be described using a completely positive trace-preserving (CPTP) map, Λ, and the transformation can be denoted as $\Lambda : \rho \to \Lambda(\rho)$. Corresponding to every CPTP map there exists a set of Kraus operators, $\{K_i\}_i$, satisfying $\sum_i K_i^\dagger K_i = I_d$, such that the transformation $\Lambda(\rho)$ can be expressed as $\Lambda(\rho) = \sum_i K_i \rho K_i^\dagger$ [40]. Here, I_d is the identity operator on $\mathcal{H}$.

Notations. In general, we denote density matrices, unitaries, and noisy channels acting on the Hilbert space, $\mathcal{H}$, of dimension d, by ρ, U, and Λ, respectively, unless specified otherwise. Identity channels and the channels which transform any state to a maximally mixed state is represented by Λ_0 and Λ_m, respectively. The set of rank-one states, density matrices, unitary operators with determinant 1, and all matrices acting on $\mathcal{H}$ are denoted as $\mathcal{P}(\mathcal{H})$, $\mathcal{S}(\mathcal{H})$, $\mathcal{U}(\mathcal{H})$, and $\mathcal{M}(\mathcal{H})$, respectively. In case of composite Hilbert spaces, $\mathcal{H} \otimes \mathcal{H}$, of dimension $d \times d$, we use the same notations but in bold symbols, i.e., the density matrices, unitaries, and channels are expressed as $\boldsymbol{\rho}$, $\boldsymbol{U}$, and $\boldsymbol{\Lambda}$.

Definition 1. Covariant channels [41–46]. Let G be a compact group. $\forall\, g \in G$, $g \to U_g$ represents a continuous unitary representation of G on $\mathcal{H}$. A quantum channel $\Lambda : \mathcal{H} \to \mathcal{H}$ is said to be covariant with respect to the representation if

$$\Lambda(U_g X U_g^\dagger) = U_g \Lambda(X) U_g^\dagger \tag{1}$$

holds for all $X \in \mathcal{M}(\mathcal{H})$ and $g \in G$. In this work, we only focus on $\mathbb{SU}(d)$-covariant channels that are defined below.

Definition 2. $\mathbb{SU}(d)$*-covariant channels.* Any quantum channel $\Lambda : \mathcal{H} \to \mathcal{H}$ is said to be $\mathbb{SU}(d)$ covariant if the relation,

$$\Lambda(UXU^\dagger) = U\Lambda(X)U^\dagger, \tag{2}$$

is true for all $X \in \mathcal{M}(\mathcal{H})$ and $U \in \mathcal{U}(\mathcal{H})$.

An example of $\mathbb{SU}(d)$-covariant channels is the d-dimensional depolarizing channel [47], $\Lambda_{\rm DC}$. $\Lambda_{\rm DC}$ acts on a state, $\rho \in \mathcal{S}(\mathcal{H})$, in the following way:

$$\Lambda_{\rm DC}(\rho) = (1-p)\rho + \frac{p}{d^2-1}(dI_d - \rho), \tag{3}$$

where $p \in [0, 1]$ is a parameter which controls strength of the channel. Let us prove this statement, for completeness. Any single-qubit state can be represented as a point on or within the Bloch sphere. A unitary operator acting on the state just rotates the directed line (Bloch vector) joining the point from the center of the sphere, whereas the depolarizing channel shrinks the length of the vector keeping its direction fixed. Hence, the action of the unitary and the channel commutes with each other. Thus, a depolarizing channel acting on single-qubit states is a valid example of a $\mathbb{SU}(2)$-covariant channel. For proof that depolarizing channels acting on arbitrary dimensional states are also $\mathbb{SU}(d)$-covariant channels, see the Appendix. As the Choi states corresponding to $\mathbb{SU}(d)$-covariant channels are $U \otimes U^*$ invariant for all $U \in \mathcal{U}(\mathcal{H})$, it has been seen in previous literature that the depolarizing channel is the only $\mathbb{SU}(d)$-covariant channel in qudit systems [48,49]. Here U^* denotes complex conjugate of U. Therefore, all the $\mathbb{SU}(d)$-covariant channels ($\Lambda_{\rm CC}$) can be written as

$$\Lambda_{\rm CC} := (1-p)\rho + \frac{p}{d^2-1}(dI_d - \rho), \tag{4}$$

where $p \in [0, 1]$.

Definition 3. Unitarily equivalent channels. We say a pair of channels, Λ and Λ', is unitarily equivalent if there exist a fixed unitary, $\tilde{U}$, such that $\Lambda'(\rho) = \sum_i K_i \rho K_i^\dagger = \tilde{U} \sum_j M_j \rho M_j^\dagger \tilde{U}^\dagger = \tilde{U}\Lambda(\rho)\tilde{U}^\dagger$ for all ρ, where the set of Kraus operators, $\{M_j\}_j$ and $\{K_i\}_i$, describes action of the channels Λ and Λ', respectively.

For example, bit flip, phase flip, and bit-phase flip channels are unitarily equivalent to each other.

Definition 4. l_1 norm of matrix. The l_1 norm of a matrix A having m rows and n columns is defined as

$$||A||_1 := \max_{1\leqslant j\leqslant n} \sum_{i=1}^{m} |a_{ij}|, \tag{5}$$

where a_{ij} denotes the element of A situated at the intersection of the ith row and the jth column of A.

Similarly, the l_1-norm distance between two matrices of equal order, say A and B, can be defined as

$$||A - B||_1 := \max_{1\leqslant j\leqslant n} \sum_{i=1}^{m} |a_{ij} - b_{ij}|, \tag{6}$$

where $\{b_{ij}\}_{ij}$ are elements of B.

Next, we are going to define a measure of distance between two channels [50–52]. In this regard, let us first recapitulate the Choi–Jamiołkowski–Kraus–Sudarshan (CJKS) isomorphism [53–56]. Consider a "reference" Hilbert space $\mathcal{H}'$, having the same dimension d as $\mathcal{H}$. A maximally entangled state acting on the composite Hilbert-space $\mathcal{H} \otimes \mathcal{H}'$, can be defined as $|\boldsymbol{\phi}^+\rangle = \frac{1}{\sqrt{d}} \sum_{0\leqslant i<d} |ii\rangle$. Then according to the CJKS isomorphism the map $\Lambda \to \rho_\Lambda = I_d \otimes \Lambda(|\boldsymbol{\phi}^+\rangle\langle\boldsymbol{\phi}^+|)$ is bijective.

Definition 5. Distance between two channels [50]. A measure of the distance between two channels, Λ and Λ', acting on same dimensional states (say d), can be defined using the CJKS isomorphism as

$$\mathcal{D}(\Lambda\|\Lambda') = \mathcal{D}_s(\rho_\Lambda\|\rho_{\Lambda'}), \tag{7}$$

where $\mathcal{D}_s$ represents any measure of distance between two states. For numerical calculations, we use the l_1 norm as the distance measure $\mathcal{D}_s$.

Since a state affected by unitary channels can be easily rectified by operating the inverse of the unitary, in this paper, we do not consider unitary channels as noisy. In other words, we consider unitary channels to have zero noise strength. By being motivated from the definition of measure of ϵ *non-Markovianity* presented in Ref. [50], we define the strength of any arbitrary channel in the following manner:

Definition 6. Strength of a channel. We define the strength of a channel, Λ, as its minimum distance from the set of unitary channels, U, i.e.,

$$\mathrm{SC} = \min_{U\in\mathcal{U}(\mathcal{H})} \mathcal{D}(\Lambda\|U) = \min_{\rho_U} \mathcal{D}_s(\rho_\Lambda\|\rho_U), \tag{8}$$

where ρ_U and ρ_Λ denote the Choi states corresponding to the channels, U and Λ, respectively.

Definition 7. Entanglement capacity (EC). We define the maximum amount of entanglement that survives after the application of a local noisy channel, $\boldsymbol{\Lambda}$, on an arbitrary state, $\boldsymbol{\rho}$, i.e.,

$$\mathrm{EC} := \max_{\rho\in\mathcal{S}(\mathcal{H}\otimes\mathcal{H})} \mathcal{E}(\boldsymbol{\Lambda}(\boldsymbol{\rho})), \tag{9}$$

as entanglement capacity of the channel, $\boldsymbol{\Lambda}$.

The value of EC depends on the strength SC of the channel. It is usually a decreasing function of SC because as SC increases, the channel becomes more noisy and thus destroys the entanglement more.

Certain cases allow for a straightforward determination of the entanglement capacity (EC). For example, for the identity channel, Λ_0, the entanglement capacity is one, whereas, for $\boldsymbol{\Lambda}_m$, the entanglement capacity is zero.

For numerical calculations, we use the l_1 norm as the distance measure $\mathcal{D}_s$. Let us now move to a quantifier of entanglement. Precisely, we consider concurrence, which is an entanglement measure of bipartite quantum states of $\mathbb{C}^2\otimes\mathbb{C}^2$ [57–60]. The concurrence $\mathcal{C}$ of any two-qubit density operator, say $\boldsymbol{\rho}_2$, is given by

$$\mathcal{C}(\boldsymbol{\rho}_2) = \max\{0, \lambda_1-\lambda_2-\lambda_3-\lambda_4\}, \tag{10}$$

where $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are the eigenvalues of

$$\omega = \sqrt{\sqrt{\boldsymbol{\rho}_2}\tilde{\boldsymbol{\rho}}_2\sqrt{\boldsymbol{\rho}_2}}$$

in decreasing order, with $\tilde{\boldsymbol{\rho}}_2 = (\sigma_2\otimes\sigma_2)\boldsymbol{\rho}_2^*(\sigma_2\otimes\sigma_2)$. Here σ_2 is a Pauli matrix.

For all the numerical calculations ahead, we use concurrence as the measure of entanglement.

III. OBSERVATIONS

It is known that the action of local unitaries on a bipartite system cannot create or increase the entanglement content of that system. On the other hand, the shared entanglement can get reduced in presence of noise even if the noise acts on a single part of that two-party system. Interestingly, it may be possible to diminish the amount of reduction of entanglement if certain local unitaries are applied to the shared state before the action of the noise. However, if the shared state is pure two-qubit, and the noise acts locally only on one party, then such entanglement preservation using local unitaries is not possible [61]. In this paper, we want to explore the behavior of the amount of entanglement that can be secured from noise degradation by applying local unitaries before the operation of the noise, with respect to the properties of the noise. Therefore, we restrict ourselves to local noise which equivalently acts on both parties of bipartite system.

We first want to quantify the amount of entanglement that can be protected from the claws of noise. In this regard, we consider bipartite states $\boldsymbol{\rho}$ acting on a composite Hilbert space $\mathcal{H}\otimes\mathcal{H}$ and use local unitaries of the type $\boldsymbol{U} = I_d\otimes U$ to guard the entangled state from local noise, $\boldsymbol{\Lambda} = \Lambda\otimes\Lambda$.

Definition 8. Saved entanglement (SE). The maximum amount of entanglement that can be saved by the help of local unitaries of the form $\boldsymbol{U}$ $(=I_d\otimes U)$ from a noisy channel $\boldsymbol{\Lambda}$ $(=\Lambda\otimes\Lambda)$ with a fixed noise strength can be defined as

$$\mathrm{SE} := \max_{\rho\in\mathcal{S}(\mathcal{H}\otimes\mathcal{H})}\Big[\max_{U\in\mathcal{U}(\mathcal{H})} \mathcal{E}(\boldsymbol{\Lambda}(\boldsymbol{U}\boldsymbol{\rho}\boldsymbol{U}^\dagger)) - \mathcal{E}(\boldsymbol{\Lambda}(\boldsymbol{\rho}))\Big], \tag{11}$$

where $\mathcal{E}$ is any fixed measure of entanglement.

It is straightforward from the definition that SE is a property of the local noise $\boldsymbol{\Lambda}$, or, more precisely, of the individual single-party noise, Λ. For a trivial unitary, i.e., for $U = I_d$, the quantity $\mathcal{E}(\boldsymbol{\Lambda}(\boldsymbol{U}\boldsymbol{\rho}\boldsymbol{U}^\dagger)) - \mathcal{E}(\boldsymbol{\Lambda}(\boldsymbol{\rho}))$ is zero for all $\boldsymbol{\rho}\in\mathcal{S}(\mathcal{H}\otimes\mathcal{H})$. Since SE is defined as the maximum value of this quantity, maximized over all unitaries (and states), it is obvious that SE will always be greater than or equal to zero.

Although the mathematical definition of SE may look complicated, in case of certain noise, it is relatively straightforward to determine the corresponding saved entanglement (SE). Specifically, for both of the channels, Λ_0 and Λ_m, the saved entanglement is zero.

To get a deeper understanding about saved entanglement of noisy channels, let us discuss two theorems on SE.

Theorem 1. Saved entanglement is always zero for $\mathbb{SU}(d)$-covariant local channels.

Proof. Consider a bipartite system described by the composite Hilbert space $\mathcal{H}\otimes\mathcal{H}$ of dimension $d\times d$. Saved entanglement of a local channel, $\boldsymbol{\Lambda} = \Lambda\otimes\Lambda$, is

$$\mathrm{SE} = \max_{\rho\in\mathcal{S}(\mathcal{H}\otimes\mathcal{H})}\Big[\max_{U\in\mathcal{U}(\mathcal{H})} \mathcal{E}(\boldsymbol{\Lambda}(\boldsymbol{U}\boldsymbol{\rho}\boldsymbol{U}^\dagger)) - \mathcal{E}(\boldsymbol{\Lambda}(\boldsymbol{\rho}))\Big],$$

where $\boldsymbol{U} = I_d\otimes U$ is the local unitary used to save the entanglement. Let us assume that Λ is $\mathbb{SU}(d)$ covariant, i.e., $\Lambda(U\rho U^\dagger) = U\Lambda(\rho)U^\dagger$, for all $\rho\in\mathcal{S}(\mathcal{H})$. Then we can replace $\Lambda\otimes\Lambda[(I_d\otimes U)\boldsymbol{\rho}(I_d\otimes U^\dagger)]$ by $I_d\otimes U[\Lambda\otimes\Lambda(\boldsymbol{\rho})]I_d\otimes U^\dagger$. Thus we have

$$\begin{aligned}\mathrm{SE} &= \max_{\rho\in\mathcal{S}(\mathcal{H}\otimes\mathcal{H})}\Big[\max_{U\in\mathcal{U}(\mathcal{H})} \mathcal{E}(\boldsymbol{U}\boldsymbol{\Lambda}(\boldsymbol{\rho})\boldsymbol{U}^\dagger) - \mathcal{E}(\boldsymbol{\Lambda}(\boldsymbol{\rho}))\Big]\\ &= \max_{\rho\in\mathcal{S}(\mathcal{H}\otimes\mathcal{H})}[\mathcal{E}(\boldsymbol{\Lambda}(\boldsymbol{\rho})) - \mathcal{E}(\boldsymbol{\Lambda}(\boldsymbol{\rho}))] = 0.\end{aligned}$$

Here we have used the fact that entanglement remains unchanged under local unitary operations. ■

Remark 1. Since depolarizing channel is a $\mathbb{SU}(d)$-covariant channel, SE is always zero for any local depolarizing channel, $\Lambda_{\mathrm{DC}}\otimes\Lambda_{\mathrm{DC}}$.

Theorem 2. Unitarily equivalent channels have same saved entanglement (SE). *Proof.* Suppose the channels Λ and Λ' are unitarily equivalent, i.e., for a fixed unitary, say $\tilde{U}$, $\Lambda'(\rho) = \tilde{U}\Lambda(\rho)\tilde{U}^\dagger$ for all ρ. The SE of the channel

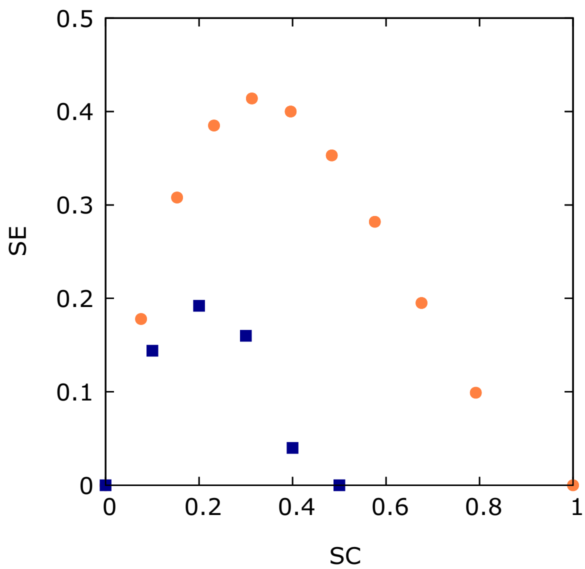

FIG. 1. Saved entanglements for paradigmatic channels. We exhibit the amounts of entanglement that can be protected, i.e., SEs, on the vertical axis, as functions of noise strength p (horizontal axis). The yellow and black points represent the values of SE for amplitude damping, bit flip channels respectively. The horizontal axis is dimensionless whereas the vertical axis is in ebits.

Λ' is given by

$$\begin{aligned}\mathrm{SE}(\Lambda') &:= \max_{\rho\in\mathcal{S}(\mathcal{H}\otimes\mathcal{H})}[\max_{U\in\mathcal{U}(\mathcal{H})}\mathcal{E}(\boldsymbol{\Lambda}'(U\rho U^\dagger))-\mathcal{E}(\boldsymbol{\Lambda}'(\rho))]\\ &= \max_{\rho\in\mathcal{S}(\mathcal{H}\otimes\mathcal{H})}[\max_{U\in\mathcal{U}(\mathcal{H})}\mathcal{E}(\boldsymbol{\Lambda}(U\rho U^\dagger))-\mathcal{E}(\boldsymbol{\Lambda}(\rho))]\\ &= \mathrm{SE}(\Lambda). \qquad (12)\end{aligned}$$

The first equality reduces to the second equality because of the fact that the entanglement of a bipartite state remains invariant under the action of local unitary operations on the state. ■

Using Theorem 2 we deduct that the SE is the same for local bit flip, local phase flip, and local bit-phase flip channels for any fixed noise strength.

Following the same path of mathematical logic as Theorem III, it can be proved that the entanglement capacity of a channel is also the same for unitarily equivalent channels.

Restricting ourselves to the composite Hilbert space of dimension $2\otimes 2$ and numerically optimizing over the set of pure states only, we obtain the nature of saved entanglement for some specific channels, such as local amplitude damping and local bit flip channels. The operational form of all the above-mentioned channels is illustrated in Sec. VI. In Fig. 1, the behavior of SEs of amplitude damping and bit flip (which is exactly equal to the same of phase flip, and bit-phase flip channels) is exhibited as functions of the corresponding strength of the channel, SC. For the amplitude damping channel, the range of SC is from zero to one, while the entire range of SC for the bit flip channel is limited to 0.5. One can notice from the figure that although the saved entanglements are quantitatively distinct for different channels, they have qualitative similarities. In particular, they monotonically increase with noise strength up to a cutoff value after which they start to decrease with SC. The value of the SE at the cutoff depends on the form of the noise. Therefore, we can assert the following two observations that are evident in the figure.

Observation 1. Local amplitude damping channel [39] has nonzero SE for all noise strengths except for SC = 0 and SC = 1.

Observation 2. The quantitative behavior of SE of the bit flip channel is completely different from the same of the amplitude damping channel for nonzero SC. We aim to investigate the specific channel properties that give rise to these diverse characteristics of SE. Specifically, we try to examine if the distinct nature of SE in different channels can be explained through biasnesses and entanglement capacities of the corresponding channels. In this regard, in the following section, we introduce a few measures of biasness.

IV. MEASURES OF BIASNESS

When a channel affects individual states differently, we say the channel is biased. Biasness of a channel can play an important role in many quantum informational tasks. The following are some of the examples of tasks where biasness can be utilized:

(1) *Teleportation.* Perfect teleportation of an unknown state, say from Alice to Bob, requires a maximally entangled state shared between them. Even if Alice and Bob initially share a maximally entangled state, the presence of noise can lead to a reduction in the entanglement shared between them over time. By investigating the bias of the noise, it is possible to identify states that are less susceptible to the detrimental effects of noise. Then, we can locally rotate the state accordingly to transform it into a state that is more secure against noise. This transformation to a more noise-resistant state enhances the likelihood of successful teleportation despite the presence of noise. Due to the same reason, bias can also be useful in entanglement-based cryptography.

(2) *Quantum error correction.* Several error-correcting codes have been developed which can tolerate higher noise rates, particularly when the noise exhibits bias. These codes demonstrate promising performance in practical quantum error correction applications [62–67].

In this section, we discuss various methods of quantifying the biasness. Although deterministic reversible channels are biased, they are trivial in the sense that we can always invert them and get back the original state. Therefore, in this paper, we define the biasness of such channels as zero.

Any single-qubit system's state can be represented by a point on or inside the Bloch sphere. We know that the action of the depolarizing channel, Λ_{DC}, on the state will contract the length of the distance between the point and the center of the sphere. The amount of contraction does not depend on the direction of the Bloch vector, that is the vector connecting the point from the center of the sphere. Thus, we can say that the channel does not have any bias towards the direction of the Bloch vector. As, depolarizing channels are only $\mathbb{SU}(d)$-covariant channels, we can conclude that $\mathbb{SU}(d)$-covariant channels are unbiased channels. By unbiased channels, we mean the channels whose action does not depend on the direction of the initial state.

From these analyses, we state an intuitively satisfactory postulate for a function mapping noise channels to real numbers to be an acceptable measure of biasness.

Postulate. The value of the measure of biasness should be zero for all unitary and $\mathbb{SU}(d)$-covariant channels.

We note that the identity channel is a particular example of a depolarizing channel. In the following sections, we construct three kinds of biasness measures of channels Λ or $\boldsymbol{\Lambda}$, all of which satisfy the above-mentioned property. Let us mention here that while in this paper we be concerned with exclusively utilizing the biasness measures for characterizing and understanding saved entanglement, we believe that the concept of biasness and the measures thereof will have a wider applicability.

A. Distance from $\mathbb{SU}(d)$-covariant channel

As discussed earlier, $\mathbb{SU}(d)$-covariant channels are only depolarizing channels and depolarizing channels do not have any biasness towards any input state. This fact motivates us to introduce the following measure of biasness: the minimum distance of a channel from the set of unitarily equivalent channels of $\mathbb{SU}(d)$-covariant channel (DCC). To evaluate the distance between two channels we use the measure defined in the previous section, i.e., Eq. (7). Hence, the biasness DCC of a channel Λ can be mathematically expressed as

$$\mathrm{DCC}(\Lambda) := \min_{p,U\in\mathcal{U}(\mathcal{H})} \mathcal{D}(U\Lambda_{\mathrm{CC}}U^\dagger||\Lambda), \tag{13}$$

where p is a parameter of the $\mathbb{SU}(d)$-covariant channel [see Eq. (4)].

DCC, from the definition itself, is zero for $\mathbb{SU}(d)$-covariant channels which implies that it will also be zero for all depolarizing channels. Note that DCC will be same for all unitarily equivalent channels if a unitary invariant distance measure is considered as a measure of distance for the evaluation of DCC. In our case, DCC are exactly same for bit flip, phase flip, bit-phase flip channels for all considered strengths of the channel, SC.

This measure of biasness can be more generalized by defining it as the “minimum distance from unbiased channels,” Λ_{UC}, which is given by

$$\mathrm{DUC}(\Lambda) := \min_{\Lambda_{UC}\in\mathbb{UC},U\in\mathcal{U}(\mathcal{H})} \mathcal{D}(U\Lambda_{UC}U^\dagger||\Lambda),$$

where $\mathbb{UC}$ is the set of all unbiased channels. This generalized measure, DUC, although geometrically more transparent, can be computationally more complicated. Therefore, whenever we numerically compute biasness of any channel we restrict ourselves to DCC instead of determining DUC. We anticipate that the results obtained will not qualitatively change if DUC is evaluated in place of DCC.

In the next section, we show that the biasness measure, DCC, of different channels, precisely, amplitude damping and bit flip is correlated with the amount of entanglement saved with the help of local unitaries.

B. Channel's dependence on state

To examine how the transformation of a state, by a channel, depends on the direction of the input state, we can define a channel's dependence on state (CDS). Let ρ and $\rho^\perp$ be two orthogonal pure qubit states. Then, the CDS of a channel, Λ, is defined as

$$\mathrm{CDS}(\Lambda) := \max_{\rho\in\mathcal{P}(\mathcal{H})}[F(\rho,\rho^\perp)] - \min_{\rho\in\mathcal{P}(\mathcal{H})}[F(\rho,\rho^\perp)], \tag{14}$$

where $F(\rho,\rho^\perp) = \max_{U\in\mathcal{U}(\mathcal{H})}\frac{1}{2}[\mathrm{Tr}(U^\dagger\Lambda(\rho)U\rho) + \mathrm{Tr}(U^\dagger\Lambda(\rho^\perp)U\rho^\perp)]$. Since for the identity channel, $\Lambda(\rho)\to\rho$, it is straightforward that for the identity channel, $\mathrm{CDS}=0$. For the $\mathbb{SU}(d)$-covariant channel, $F(\rho,\rho^\perp) = 2[1-\frac{d}{d+1}p]$, which is independent of ρ, for a given dimension, which implies $\mathrm{CDS}(\Lambda_{\mathrm{DC}})=0$ for all p. One can easily check CDS is zero for every unitary channels. Hence, CDS can be a contender for measuring biasness. It can be easily seen that CDS is the same for all unitarily equivalent channels.

C. Incovariance

Now, we discuss about the next quantifier of biasness, incovariance (IC).

Since $\mathbb{SU}(d)$-covariant local channels can never display a nonzero saved entanglement for any noise strength, local incovariance of channels (IC) can be a reason for the exhibition of nonzero saved entanglement and therefore can be another quantifier of biasness. To make the measure zero for unitary channels, we mathematically define IC in the following manner:

$$\mathrm{IC}(\Lambda) := \frac{1}{d^2}\sum_{i,j}\max_{U_2\in\mathcal{U}(\mathcal{H})}\min_{U_1\in\mathcal{U}(\mathcal{H})}||U_1\Lambda(U_2|i\rangle\langle j|U_2{}^\dagger)U_1^\dagger - U_2U_1\Lambda(|i\rangle\langle j|)U_1{}^\dagger U_2^\dagger||_1. \tag{15}$$

Here, although $U_{1(2)}|i\rangle\langle j|U_{1(2)}^\dagger$ are not states for $i\neq j$, we define the action of Λ on $U_{1(2)}|i\rangle\langle j|U_{1(2)}^\dagger$ as $\Lambda(U_{1(2)}|i\rangle\langle j|U_{1(2)}^\dagger) = \sum_l K_lU_{1(2)}|i\rangle\langle j|U_{1(2)}^\dagger K_l^\dagger$ where K_l are Kraus operators of Λ. IC is zero for any $\mathbb{SU}(d)$-covariant channel. Thus IC satisfies the postulate for being an acceptable measure of biasness. From the definition of IC, it can be easily proved that all unitarily equivalent channels will have same IC for all strengths of the channel. We analyze IC for different channels in the succeeding section.

V. BOUNDS ON SAVED ENTANGLEMENT

In this part, we introduce two bounds on the saved entanglement. Let us consider a bipartite state $\boldsymbol{\rho}$, and let the noise acting on the state be $\boldsymbol{\Lambda}$. Moreover, let us suppose that only the second party applies the unitary operator to protect entanglement. Therefore, the form of the applied local unitary is $\boldsymbol{U}\equiv I_d\otimes U$. Let $U_{\max}$ and $\boldsymbol{\rho}_{\max}$ be the unitary operator and the bipartite pure state, respectively, for which the optimization in Eq. (11) can be achieved. Then the saved entanglement of the channel $\boldsymbol{\Lambda}$ is

$$\mathrm{SE} := \mathcal{E}(\boldsymbol{\Lambda}(\boldsymbol{U}_{\max}\boldsymbol{\rho}_{\max}\boldsymbol{U}_{\max}^\dagger)) - \mathcal{E}(\boldsymbol{\Lambda}(\boldsymbol{\rho}_{\max})), \tag{16}$$

where $\boldsymbol{U}_{\max} = I_d\otimes U_{\max}$. As discussed in Sec. III, the above quantity is greater or equal to zero. Thus we have

$$\mathcal{E}(\boldsymbol{\Lambda}(\boldsymbol{U}_{\max}\boldsymbol{\rho}_{\max}\boldsymbol{U}_{\max}^\dagger)) \geqslant \mathcal{E}(\boldsymbol{\Lambda}(\rho_{\max})). \tag{17}$$

The operator $\boldsymbol{\Lambda}(\boldsymbol{U}_{\max}\boldsymbol{\rho}_{\max}\boldsymbol{U}_{\max}^\dagger)$ is a density matrix, and thus can be decomposed in terms of two density matrices in the following way:

$$\mathbf{\Lambda}(\boldsymbol{U}_{\max}\boldsymbol{\rho}_{\max}\boldsymbol{U}^{\dagger}_{\max}) = p_1(\mathbf{\Lambda}(\boldsymbol{\rho}_{\max})) + p_2\boldsymbol{\rho}', \tag{18}$$

where $p_1, p_2 \geqslant 0$ and $p_1 + p_2 = 1$. A trivial solution of the above equation is $p_1 = 0$, $p_2 = 1$, and $\boldsymbol{\rho}' = \mathbf{\Lambda}(\boldsymbol{U}_{\max}\boldsymbol{\rho}_{\max}\boldsymbol{U}^{\dagger}_{\max})$. But there can be multiple solutions.

Let us now restrict ourselves to the entanglement quantifiers which satisfy the convexity property [68]. Concurrence [57,58], relative entropy of entanglement [69–71], negativity [72] are some examples of such quantifiers. Using the convexity property and the expression given in Eq. (18), we can write

$$\begin{aligned}\mathcal{E}(\mathbf{\Lambda}(\boldsymbol{U}_{\max}\boldsymbol{\rho}_{\max}\boldsymbol{U}^{\dagger}_{\max})) &= \mathcal{E}(p_1[\mathbf{\Lambda}(\boldsymbol{\rho}_{\max})] + p_2\boldsymbol{\rho}') \\ &\leqslant p_1\mathcal{E}(\mathbf{\Lambda}(\boldsymbol{\rho}_{\max})) + p_2\mathcal{E}(\boldsymbol{\rho}'). \end{aligned} \tag{19}$$

Then an upper bound on the SE of channels can be determined as

$$\begin{aligned}\text{SE} &= \mathcal{E}(\mathbf{\Lambda}(\boldsymbol{U}_{\max}\boldsymbol{\rho}_{\max}\boldsymbol{U}^{\dagger}_{\max})) - \mathcal{E}(\mathbf{\Lambda}(\boldsymbol{\rho}_{\max})) \\ &\leqslant p_1\mathcal{E}(\mathbf{\Lambda}(\boldsymbol{\rho}_{\max})) + p_2\mathcal{E}(\boldsymbol{\rho}') - \mathcal{E}(\mathbf{\Lambda}(\boldsymbol{\rho}_{\max})) \\ &\leqslant p_2\mathcal{E}(\boldsymbol{\rho}') - (1-p_1)\mathcal{E}(\mathbf{\Lambda}(\boldsymbol{\rho}_{\max})) \\ &\leqslant p_2[\mathcal{E}(\boldsymbol{\rho}') - \mathcal{E}(\mathbf{\Lambda}(\boldsymbol{\rho}_{\max}))]. \end{aligned} \tag{20}$$

There may exist various decompositions of $\mathbf{\Lambda}(\boldsymbol{U}_{\max}\boldsymbol{\rho}_{\max}\boldsymbol{U}^{\dagger}_{\max})$ of the form provided in Eq. (18). Each of these decomposition may involve different p_2 and/or ρ'. Let us denote the set of all p_2 for each of which we can find a corresponding ρ' such that Eq. (18) is valid, as $\mathcal{P}$. To obtain a tighter upper bound for SE, we need to consider the minimum value of right-hand side of (20) minimized over all valid decompositions expressed in Eq. (18). Therefore the best bound we can find is

$$\text{SE} \leqslant \min_{p_2\in\mathcal{P}}[p_2[\mathcal{E}(\boldsymbol{\rho}') - \mathcal{E}(\mathbf{\Lambda}(\boldsymbol{\rho}_{\max}))]] \tag{21}$$

$$\leqslant \min_{p_2\in\mathcal{P}}[p_2\mathcal{E}(\boldsymbol{\rho}')]. \tag{22}$$

It is important to keep in mind that ρ' depends on the decomposition introduced in Eq. (18) and therefore it will change with $p_2 \in \mathcal{P}$. From inequalities (17) and (19), we see that $\boldsymbol{\rho}'$ cannot be separable, unless p_1 is one.

There can be numerous pairs of $\{\boldsymbol{U}_{\max}, \boldsymbol{\rho}_{\max}\}$ for which the optimization introduced in the definition of SE is achievable. All of the pairs $\{\boldsymbol{U}_{\max}, \boldsymbol{\rho}_{\max}\}$ will satisfy both the inequalities (21) and (22). Thus to get the tightest bound we have to minimize the right-hand side of those inequalities over the set of pairs $\{\boldsymbol{U}_{\max}, \boldsymbol{\rho}_{\max}\}$. Thus we define the following two quantities:

$$\text{EB1} := \min_{\boldsymbol{U}_{\max},\boldsymbol{\rho}_{\max},p_2\in\mathcal{P}}[p_2\mathcal{E}(\boldsymbol{\rho}') - p_2\mathcal{E}(\mathbf{\Lambda}(\boldsymbol{\rho}_{\max}))], \tag{23}$$

$$= \min_{\boldsymbol{U}_{\max},\boldsymbol{\rho}_{\max}} Q_1, \tag{24}$$

$$\text{EB2} := \min_{\boldsymbol{U}_{\max},\boldsymbol{\rho}_{\max},p_2\in\mathcal{P}} p_2\mathcal{E}(\boldsymbol{\rho}'), \tag{25}$$

$$= \min_{\boldsymbol{U}_{\max},\boldsymbol{\rho}_{\max}} Q_2, \tag{26}$$

as the bounds on SE, where $Q_1 := \min_{p_2\in\mathcal{P}}[p_2\mathcal{E}(\boldsymbol{\rho}') - p_2\mathcal{E}(\mathbf{\Lambda}(\boldsymbol{\rho}_{\max}))]$ and $Q_2 := \min_{p_2\in\mathcal{P}}[p_2\mathcal{E}(\boldsymbol{\rho}')]$.

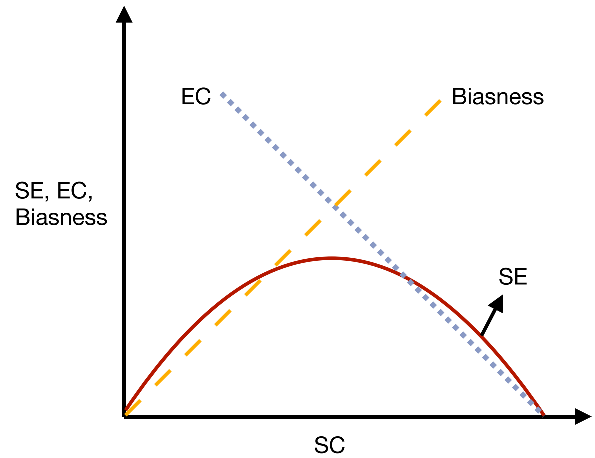

FIG. 2. Graphical nature of entanglement capacity, biasness, and saved entanglement. A schematic diagram is drawn to represent the behavior of the three distinct functions, viz. EC, SE, and a quantifier of biasness (vertical axis) with respect to noise strength (horizontal axis) of applied local noise. The robustness of entanglement against applied noise decreases with corresponding noise strength, whereas biasness of that noise towards individual input states increases with the strength. The amount of saved entanglement follows the nature of biasness at lower noise strengths and entanglement capacity at higher noise strengths. The quantities are qualitatively represented using yellow dashed line (biasness), black dotted line (EC), and red solid line (SE). SE and EC are in ebits, while other quantities are dimensionless.

In case of the identity and other depolarizing channels, the amount of saved entanglement is vanishing. Hence, in those cases, we can choose $\boldsymbol{U}_{\max}$ to be the identity matrix. Therefore, the solution of Eq. (18) for which the bounds given in inequality (21) and (22) are optimal is $\{p_1, p_2\} = \{1, 0\}$. Thus we see that EB1 and EB2 are zero for depolarizing channels.

These bounds, EB1 and EB2, hold for all channels, even for those acting on systems with higher dimensions. It is still unknown whether there is a direct relationship between the bounds and the biasness measures.

VI. BIASNESS AND ENTANGLEMENT CAPACITY AS AN ESCORT TO SAVED ENTANGLEMENT

In this section, we discuss how biasness and entanglement capacity are correlated with the behavior of entanglement saved against certain channels, viz. amplitude damping and bit flip. As we have mentioned earlier, SE and EC are the same for unitarily equivalent channels. One can easily check that the measures of biasness, CDS and IC, are also the same for them. For the DCC measure of biasness, the invariance is present for all the channels considered. Hence bit flip, phase flip, and bit-phase flip channels, being unitarily equivalent, have exactly the same SE, EC, and biasness. Therefore we not separately discuss phase flip and bit-phase flip channels.

To have an overall idea about the qualitative relation between SE, EC, and biasness of channels, in Fig. 2, we present a schematic diagram of the behavior of the functions. The saved entanglement, for all the considered noisy channels, shows a parabolic nature. It can be seen that the value of SE initially increases with noise strength up to a certain cutoff value. This can be explained through the nature of biasness of the corresponding channel which also is a monotonically increasing function of the same. Since biasness demonstrates the dependence of the channel on the initial state, it indicates that appropriately changing the initial state will alter the effect of the noise, resulting in less entanglement degradation. Thus, more the biasness, more is the possibility of securing entanglement. But after reaching the cutoff value, SE starts decreasing with noise strength. The reason behind this deterioration can be the effect of the intense noise on the initial states, which in turn immensely affects the entanglement of the states, making the states almost separable. That is, although the channel's impact on the states depends on the states themselves, but the outputs have one thing in common: poor entanglement. Thus the amount of saved entanglement, for smaller values of noise strength, follows the behavior of biasness, whereas for higher values of noise strength, it follows the nature of entanglement capacity. To grasp the characteristics in more detail, we discuss some exemplar noise models in the following subsections.

In the following subsections, we consider two-qubit systems and apply local noise of the form $\mathbf{\Lambda} = \Lambda \otimes \Lambda$, where Λ represents a typical noisy channel, for example, amplitude damping channel, bit flip channel, etc. To protect the entanglement, we consider local unitaries of the form $I_2 \otimes U$, where U is a single qubit unitary. To determine SE, we optimize over the set of pure states, $\mathcal{P}(\mathcal{H} \otimes \mathcal{H})$.

A. Amplitude damping channel

Let us first consider the amplitude damping channel, $\Lambda_{\rm AD}$. The Kraus operators of the channel is given by

$$K_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix}, \quad K_1 = \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix}.$$

Thus the corresponding map can be described as

$$\rho \to \Lambda_{\rm AD}(\rho) = \sum_{i=0}^{1} K_i \rho K_i^\dagger.$$

The amount of entanglement that can be saved using local unitaries, i.e., SE, and the biasness quantifiers of the channel, viz. DCC, CDS, and IC, are determined using Eqs. (11), (13), (14), and (15), respectively. We have used a numerical nonlinear optimizer to optimize the functionals. In Fig. 3, we plot SE using brown star points. It is the same curve that is plotted in Fig. 1 using yellow circular points. It can be seen that the value of SE increases with SC for smaller values of SC and then for SC > 0.3 it starts decreasing. We also plot the measures of biasness, i.e., DCC, CDS, IC, and entanglement capacity, EC, in the same figure, i.e., Fig. 3. We see that all biasness measures initially increase with SC and after reaching a maximum value start to decrease with the same whereas the entanglement capacity decreases with SC in the entire range of SC. It is clearly visible that, for lower values of SC, the nature of SE follows the behavior of biasness and for higher values of SC, it follows EC.

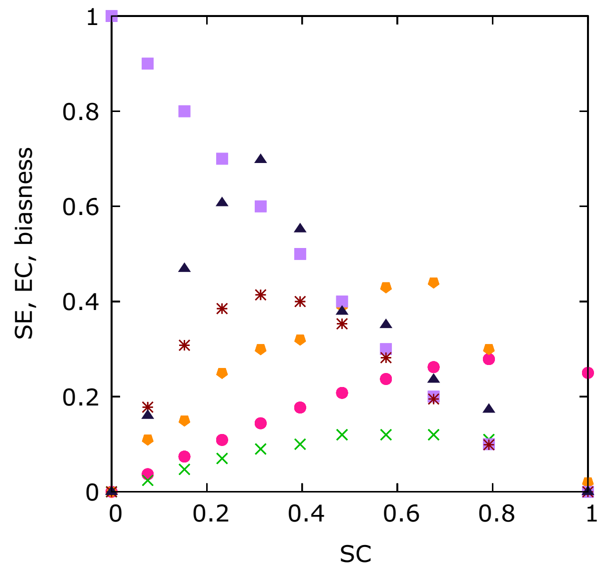

FIG. 3. Effect of local amplitude damping noise on bipartite pure states and biasness of the channel. We plot SE (brown stars), EC (violet squares), and the biasness measures, i.e., DCC (pink circles), CDS (green crosses), and IC (yellow pentagons) on the vertical axis against the corresponding noise strength, p, of applied local amplitude damping channel, represented on the horizontal axis. The bound on saved entanglement, upper bound on EB1 (black triangles) which is numerically equal to the upper bound on EB2, is also plotted on the same vertical axis. SE, EC, and the upper bound on EB1 are plotted using the dimension of ebits whereas other quantities are dimensionless.

Next we use the expressions of Q_1 and Q_2 presented in Eqs. (24) and (26), and instead of minimizing Q_1 and Q_2 over all $\boldsymbol{U}_{\rm max}$ and $\boldsymbol{\rho}_{\rm max}$ and determining the exact expressions of EB1 and EB2 we find only three different pairs of $\{\boldsymbol{U}_{\rm max}, \boldsymbol{\rho}_{\rm max}\}$ and minimize Q_1 and Q_2 over these three pairs to reduce numerical complexity. Hence the final quantities found by optimizing Q_1 and Q_2 over these three pairs provide upper bounds on EB1 and EB2 respectively. We also plot these upper bounds on EB1 and EB2 in the same figure. From the figure, it is evident that the determined upper bounds on EB1 or EB2 alone can reflect the behavior of the saved entanglement for all noise strengths.

Interestingly, numerically we have got the same values of the right-hand sides of inequalities (21) and (22), for each SC. Thus we can conclude that $\mathcal{C}(\Lambda(\boldsymbol{\rho}_{\rm max}))$ is zero for all SC of the channel.

B. Bit flip channel

Next we move to the bit flip noise, Λ_{BF}, in presence of which, the eigenstates of the σ_3 matrix (the Pauli matrix), that are $|0\rangle$ and $|1\rangle$, get exchanged with each other, with a finite probability $p/2$. This transformation can be mathematically expressed as

$$\rho \to \Lambda_{BF}(\rho) = \sum_{i=0}^{1} K_i \rho K_i^\dagger,$$

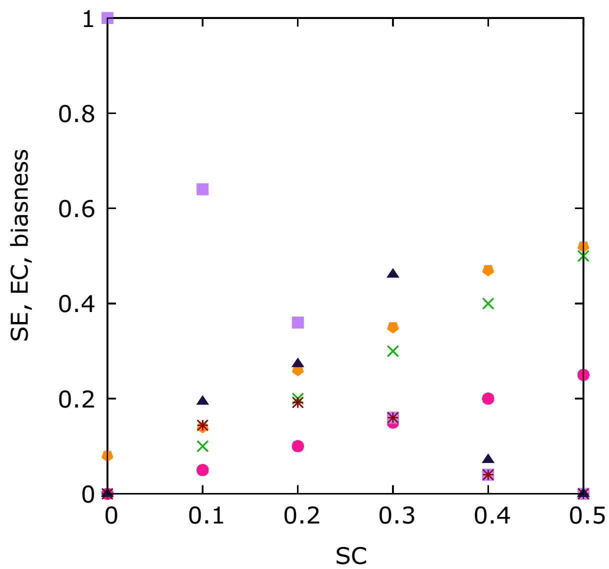

FIG. 4. Behavior of saved entanglement, entanglement capacity, and measured biasnesses for the bit flip channel. All considerations are the same as in Fig. 3 except the fact that here the noise under discussion is bit flip.

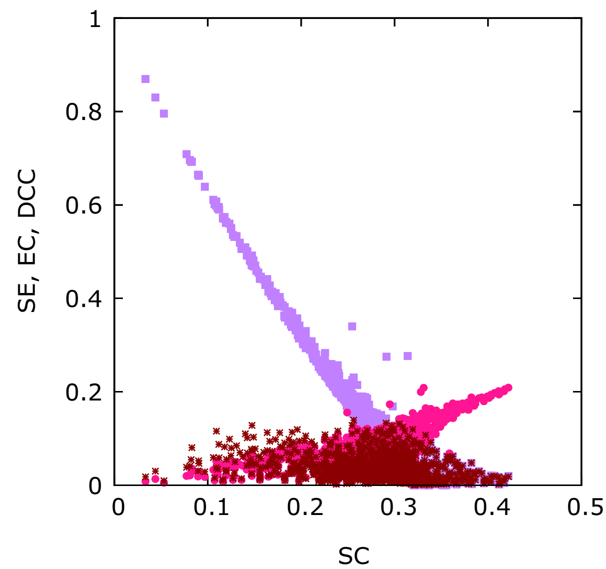

FIG. 5. Behavior of saved entanglement (SE), entanglement capacity (EC), and a quantifier of biasness (DCC) for the random Pauli channels. We plot SE, EC, and DCC for 10^3 randomly generated Pauli channels along the vertical axis using, respectively, brown stars, violet squares, and pink circles. The horizontal axis depicts the corresponding strength, SC, of those randomly selected channels. SE and EC are measured in ebits, whereas DCC and SC are dimensionless.

where

$$K_0 = \sqrt{1-p}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad K_1 = \sqrt{p}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Figure 4 portrays the behavior of the same functionals, as in Fig. 3 for the amplitude damping channel, viz. SE, DCC, CDS, IC, and EC for the bit flip channel, against the noise strength, SC. It is apparent from the figure that the quantifiers of biasness (DCC, CDS, and IC) and the amount of saved entanglement (SE) behave analogously within the range $0 \leqslant \text{SC} \leqslant 0.2$, that is, all of them increase with SC. After $\text{SC} = 0.2$, the values of the biasnesses continue to increase whereas the corresponding value of SE starts to decrease monotonically. Thus in this range, $0.2 \leqslant \text{SC} \leqslant 0.5$, the nature of SE and EC are alike. Hence we can argue that at first, SE increases because of the presence of biasness in the channel at low noise strength, and then its value starts to reduce at higher values of SC, because of corresponding low EC of the channel.

Instead of calculating EB1 and EB2 by minimizing over overall $\boldsymbol{U}_{\max}$, $\boldsymbol{\rho}_{\max}$, to reduce numerical complexity, we calculate the upper bounds on EB1 and EB2 by determining the expression needed to be minimized for only one pair of $\{\boldsymbol{U}_{\max}, \boldsymbol{\rho}_{\max}\}$.

We have calculated upper bounds on EB1 and EB2 for the bit flip channel by determining Q_1 and Q_2 for one pair of $\{\boldsymbol{U}_{\max}, \boldsymbol{\rho}_{\max}\}$. We see that the upper bound on EB1 of the bit flip channel again coincides with the upper bound on EB2 of the same channel for all considered values of the noise strength, p. We draw the upper bounds on EB1 or EB2 for different values of p in Fig. 4. We see that the upper bound on EB1 or EB2 can describe the behavior of SE for all noise strengths.

C. Random Pauli channels

Up to now, we have only provided examples of one-parameter channels to show that the channels' SE, EC, and biasness do follow the behavior illustrated in the schematic diagram, Fig. 2. Let us now investigate the nature of saved entanglement for certain multiparameter channels, viz. the Pauli channels. A Pauli channel, Λ_{PC}, transforms any state ρ into a new state $\Lambda_{\text{PC}}(\rho)$ in the following way:

$$\Lambda_{\text{PC}}(\rho) = \sum_{i=0}^{3} P_i \sigma_i \rho \sigma_i.$$

Here P_i are non-negative real numbers which satisfy $\sum_{i=0}^{3} P_i = 1$, σ_0 is the identity operator on the two-dimensional complex Hilbert space, and σ_1, σ_2, and σ_3 are the Pauli matrices. By exploring Pauli channels, we try to understand the qualitative relationship between saved entanglement, biasness, and entanglement capacity. In this regard, we select 10^3 sets of $\{P_0, P_1, P_2, P_3\}$ from the uniform distribution, keeping the constraint $\sum_{i=0}^{3} P_i = 1$ satisfied, and find the SC, SE, EC, and DCC of the corresponding 10^3 Pauli channels. In Fig. 5, we plot SE, EC, and DCC with respect to the strength of the randomly selected Pauli channels. As one can clearly notice from the figure, the qualitative nature of these channels' SE also shows the same behavior as presented in the schematic diagram, Fig. 2. In particular, biasness of the channels increases with SC whereas the EC decreases with SC. On the other hand, the SE has a parabolic nature, which at first increases with SC and after a point starts to decrease.

Although, for the random Pauli channels, we have depicted the behavior of a particular type of biasness, i.e., DCC, we have verified that the behavior would not change significantly if any other measures of biasness were considered.

VII. CONCLUSION

Although entanglement is an essential resource in many quantum tasks including teleportation, dense coding, and entanglement-based cryptography, it is a fragile characteristic of shared quantum systems. Various unavoidable noise tend to reduce entanglement of shared quantum systems. Preservation of entanglement from such impact of noise is of significant practical interest. It was observed that if certain local unitaries are applied on the entangled state before the system's interaction with noise, the entanglement can be partially protected. The amount of entanglement that can be saved in this way depends on the nature of the noise, and as an extreme example, the depolarizing channel's effect cannot be bypassed or diminished by utilizing local unitaries. In this work, we have tried to investigate the reason behind the partial protection provided by local unitaries.

We explored the phenomenon through two physical characteristics of quantum channels, viz. biasness, which we argue as being able to explain the nature of saved entanglement when the strength of the applied noise is low, and entanglement capacity, which we argue as explaining the behavior of saved entanglement for higher strengths of noise. We have also obtained two upper bounds on the saved entanglement, which we observed to represent the characteristics of the saved entanglement in the full range of noise strength.

ACKNOWLEDGMENTS

We acknowledge partial support from the Department of science and Technology, Government of India through the QuEST grant (grant number DST/ICPS/QUST/Theme3/2019/120). The research of K.S. was supported in part by the INFOSYS scholarship.

APPENDIX

Lemma 1. The depolarizing channel is an $\mathbb{SU}(d)$-covariant channel.

Proof. For the depolarizing channel expressed in Eq. (3),

$$\begin{aligned}\Lambda_{\rm DC}(U\rho U^\dagger) &= (1-p)U\rho U^\dagger + \frac{p}{d^2-1}(dI_d - U\rho U^\dagger),\\ &= (1-p)U\rho U^\dagger + \frac{p}{d^2-1}(dUU^\dagger - U\rho U^\dagger),\\ &= U\left((1-p)\rho + \frac{p}{d^2-1}(dI_d-\rho)\right)U^\dagger,\\ &= U(\Lambda_{\rm DC}(\rho))U^\dagger.\end{aligned}$$

This concludes the proof. ■

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