Characterizing bipartite states with vanishing basis-dependent correlations

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Since both coherence and quantum correlations arise from the superposition principle and can be regarded as resources in quantum information tasks, it is of significance to investigate the interplay between them from different perspectives. In this work we focus on the basis-dependent correlations in a bipartite state defined by the coherence difference between global state and local state relative to a local basis and characterize bipartite states with vanishing basis-dependent correlations. Using the relative entropy of coherence, the structure of such states has been determined by Yadin *et al.* [Phys. Rev. X **6**, 041028 (2016)], which we call block-diagonal product states here. The first result of this work is to demonstrate that the set of block-diagonal product states can also be characterized by the property of possessing vanishing basis-dependent correlations using the coherence measure based on skew information. As a by-product of this result, we describe the structure of quantum ensembles saturating the convexity inequality in the resource theory of coherence using the coherence measure based on skew information, which may be of independent interest. Next, we characterize the set of bipartite states with vanishing basis-dependent correlations using the l_1 norm of coherence, and show that the set of block-diagonal product states is a subset of it. Furthermore, we provide an operational interpretation of block-diagonal product states in an interference model. Finally, we compare the amount of basis-dependent correlations using the three mentioned coherence measures through several examples such as Werner states, isotropic states, Bell-diagonal states, and a family of classical-quantum states.

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I. INTRODUCTION

One of the most important issues in quantum information theory is to characterize the valuable resources in quantum information tasks that enable them to outperform their classical counterparts. Among these resources, coherence and quantum correlations are two related ones, because they both arise from the superposition principle, a fundamental feature of quantum mechanics. Traditionally, coherence is relative to a reference basis of the system Hilbert space. A quantum state with vanishing off-diagonal elements in this basis is called incoherent, while any other states are coherent relative to the basis. With the development of quantum information theory, the resource theory of coherence has been established [1–5], several coherence measures have been proposed from different perspectives [1,6–15], and the role of coherence has been investigated in quantum algorithms [16–21], quantum metrology [22–25], quantum phase transitions [26–30], and quantum biology [31,32]. Moreover, the notion of coherence relative to an orthonormal basis (von Neumann measurement) has been extended to coherence relative to Lüders measurements (projective measurements), to POVMs, and to quantum channels, together with the resource theory of these generalized coherence [7,33–40].

Understanding correlations in composite quantum systems has been a long-term issue since the establishment of quantum theory [41–63]. Several types of correlations have been proposed such as Bell nonlocality [42,43], steering [41,44,48,62], entanglement [45,53], and quantum discord [46,47,54,58]. Recently, much effort has been made to connect coherence with correlations [64–94]. Notice that the definitions of correlations are basis-independent, while the notion of coherence is basis-dependent. Most of the works relating coherence with correlations are to get rid of the dependence on the basis for coherence by optimization or taking minimum or maximum over all bases and then to connect the basis-independent coherence with correlations. In contrast, Yadin *et al.* proposed an alternative viewpoint of investigating the basis-dependent correlations and linking the basis-dependent correlations with the traditional coherence in Ref. [75].

In this work, we will follow the latter viewpoint to investigate the basis-dependent correlations and aim to characterize bipartite states with vanishing basis-dependent correlations. To be precise, given a local reference basis for a bipartite system, it is natural to consider the coherence of both global state and local state relative to the local basis. In general, the coherence of global state should be no less than the coherence of local state relative to the local basis due to the correlations contained in the bipartite system. Because this kind of correlations is relative to the specific local basis, we call it basis-dependent correlations. Obviously, bipartite states with nonvanishing basis-dependent correlations can provide

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more coherence resource relative to the local basis when using global state instead of local state. A natural question arises: What is the structure of such bipartite states? Or, equivalently, what is the structure of bipartite states with vanishing basis-dependent correlations? In Ref. [75], Yadin *et al.* characterized bipartite states with vanishing basis-dependent correlations using the relative entropy of coherence. Here we will use the coherence measure based on skew information and the l_1 norm of coherence to answer this question.

The rest of this work is arranged as follows. In Sec. II we give some preliminaries including the resource theory of coherence, the definitions of basis-independent and basis-dependent discord, and the definition of block-diagonal product states. The main results of this work are put in Sec. III. After introducing the motivation of this work, we determine an explicit structure of quantum ensembles saturating the convexity inequality in the resource theory of coherence, and then apply this result to prove that the set of block-diagonal product states can be characterized by the property of possessing vanishing basis-dependent correlations using the coherence measure based on skew information. At the end of Sec. III, we characterize bipartite states with vanishing basis-dependent correlations using the l_1 norm of coherence. We further provide an operational interpretation of block-diagonal product states in an interference model in Sec. IV, and calculate the basis-dependent correlations using three coherence measures mentioned here for four example states in Sec. V. Finally, we conclude with discussions in Sec. VI. The proofs for Propositions 1–4 are located in the Appendixes.

II. PRELIMINARIES

In this section, we give some preliminaries that we will use later to present our main results, including the resource theory of coherence, the definitions of three block coherence measures, the definitions of basis-independent discord and basis-dependent discord, and the definition of block-diagonal product states.

A. Resource theory of coherence

The original coherence in quantum information theory is relative to an orthonormal basis. Given a basis $\{|j\rangle : j = 1, 2, \dots, d\}$ of d-dimensional Hilbert space $\mathcal{H}$, any quantum state ρ, mathematically described by a unit-trace positive semidefinite operator on $\mathcal{H}$, can be expressed as $\rho = \sum_{j,j'} \langle j|\rho|j'\rangle |j\rangle\langle j'|$ in this basis. If $\langle i|\rho|j'\rangle = 0$ for any $j \neq j'$, then ρ is called incoherent. Otherwise, it is coherent and can be regarded to have the resource of coherence relative to the basis $\{|j\rangle\}$ [1]. Throughout this work, we do not distinguish between the basis $\{|j\rangle\}$ and its corresponding von Neumann measurement $\Pi = \{|j\rangle\langle j|\}$. According to the resource theory of coherence [1], a coherence measure $C(\rho, \Pi)$ of a quantum state ρ relative to Π should have the following properties:

(1) Faithfulness: $C(\rho, \Pi) \geqslant 0$, and $C(\rho, \Pi) = 0$ if and only if $\rho \in \mathcal{I}$, where $\mathcal{I}$ denotes the set of incoherent states.

(2) Monotonicity: $C(\rho, \Pi) \geqslant C(\mathcal{E}(\rho), \Pi)$ and $C(\rho, \Pi) \geqslant \sum_k p_k C(E_k \rho E_k^\dagger / p_k, \Pi)$ for any incoherent operation $\mathcal{E}$ which is defined as a quantum channel $\mathcal{E}(\rho) = \sum_k E_k \rho E_k^\dagger$ with the set of Kraus operators $\{E_k\}$ satisfying $\sum_k E_k^\dagger E_k = \mathbf{1}$ and $E_k \mathcal{I} E_k^\dagger \subset \mathcal{I}$ for each k. Here $p_k = \mathrm{tr} E_k \rho E_k^\dagger$, $\mathbf{1}$ is the identity operator on $\mathcal{H}$.

(3) Convexity: It holds that

$$C\left(\sum_k \lambda_k \rho_k, \Pi\right) \leqslant \sum_k \lambda_k C(\rho_k, \Pi), \tag{1}$$

for any ensemble $\{(\lambda_k, \rho_k)\}$ with ρ_k some quantum states on $\mathcal{H}$ and $\{\lambda_k\}$ a probability distribution.

The notion of coherence relative to a basis has been generalized to a Lüders measurement which is a generalization of von Neumann measurement (i.e., rank-1 Lüders measurement). For a Lüders measurement $\Pi_{\mathrm{L}} = \{P_l : l = 1, 2, \dots, m\}$ on $\mathcal{H}$ with P_l orthogonal projections satisfying $\sum_{l=1}^m P_l = \mathbf{1}$ and $m \leqslant d$,

$$\Pi_{\mathrm{L}}(\rho) = \sum_{l=1}^m P_l \rho P_l.$$

If a quantum state ρ satisfies $\rho = \Pi_{\mathrm{L}}(\rho)$, then it is said to be incoherent relative to Π_{L}. Otherwise, it has block coherence [7,35–38,40]. When $m = d$, each P_l is rank-1 and Π_{L} is a von Neumann measurement. Then the block coherence reduces to the original coherence.

Among several commonly used block coherence measures, we mainly focus on the relative entropy of coherence, the coherence measure based on skew information, and the l_1 norm of coherence. For a quantum state ρ and a Lüders measurement $\Pi_{\mathrm{L}} = \{P_l : l = 1, 2, \dots, m\}$, recall that the three block coherence measures are defined as follows:

(1) The relative entropy of coherence [1,6,7,95]

$$C_{\mathrm{r}}(\rho, \Pi_{\mathrm{L}}) := S(\Pi_{\mathrm{L}}(\rho)) - S(\rho), \tag{2}$$

with $S(\rho) := -\mathrm{tr}\rho \ln \rho$ being von Neumann entropy of a quantum state ρ.

(2) The coherence measure based on skew information [12,33,34]

$$I(\rho, \Pi_{\mathrm{L}}) := \sum_l I(\rho, P_l), \tag{3}$$

where $I(\rho, K) := -\mathrm{tr}[\sqrt{\rho}, K]^2/2$ is the Wigner-Yanase skew information of ρ with respect to an observable K [96], $[A, B]$ denotes the commutator of A and B.

(3) The l_1 norm of coherence [36]

$$C_{l_1}(\rho, \Pi_{\mathrm{L}}) := \sum_{l \neq l'} \|P_l \rho P_{l'}\|_{\mathrm{tr}}, \tag{4}$$

where $\|A\|_{\mathrm{tr}} := \mathrm{tr}\sqrt{A^\dagger A}$ denotes the trace norm of the operator A.

Note that when $m = d$ all the three block coherence measures reduce to the corresponding coherence measures relative to a basis. In addition, we remark that the notion of coherence relative to Lüders measurements has been further generalized to POVMs [35–38] and quantum channels [33,34,39].

B. From basis-independent to basis-dependent discord

Considering a bipartite quantum state ρ^{ab} on $\mathcal{H}^a \otimes \mathcal{H}^b$, if ρ^{ab} is a product state, i.e., $\rho^{ab} = \rho^a \otimes \rho^b$ with $\rho_a = \mathrm{tr}_b \rho^{ab}$ and $\rho_b = \mathrm{tr}_a \rho^{ab}$, then there are no correlations in ρ^{ab} and the two subsystems a and b are independent. Otherwise, there exist correlations between a and b. Two of the most studied types of correlations are entanglement and quantum discord. In 1989 Werner introduced the mathematical definition of entanglement: Any bipartite state that cannot be expressed as a mixture of product states such as $\sum_k p_k\rho_k^a\otimes\rho_k^b$ is called an entangled state, and otherwise it is separable [45]. Entanglement was often recognized as the only kind of quantum correlations until the introduction of quantum discord in 2001 by Ollivier and Zurek [46] and by Henderson and Vedral [47], independently. The original definition of quantum discord in Ref. [46] is

$$\delta(\rho^{ab}) := \min_{\Pi^a}\delta(\rho^{ab}|\Pi^a) \tag{5}$$

with

$$\delta(\rho^{ab}|\Pi^a) := I(\rho^{ab}) - I(\Pi^a\otimes\mathbf{1}^b(\rho^{ab})). \tag{6}$$

Here $I(\rho^{ab}) := S(\rho^a)+S(\rho^b)-S(\rho^{ab})$ denotes quantum mutual information of ρ^{ab}, the minimum is taken over all von Neumann measurements $\Pi^a=\{\Pi_j^a=|j\rangle\langle j|\}$ on $\mathcal{H}^a$, and $\Pi^a\otimes\mathbf{1}^b=\{\Pi_j^a\otimes\mathbf{1}^b\}$ is a Lüders measurement on $\mathcal{H}^a\otimes\mathcal{H}^b$ such that $\Pi^a\otimes\mathbf{1}^b(\rho^{ab})=\sum_j(\Pi_j^a\otimes\mathbf{1}^b)\rho^{ab}(\Pi_j^a\otimes\mathbf{1}^b)$ with $\mathbf{1}^b$ the identity operator on $\mathcal{H}^b$. Obviously, $\delta(\rho^{ab})$ is independent of the choice of the bases Π^a and thus is called the basis-independent discord of ρ^{ab}. In contrast, $\delta(\rho^{ab}|\Pi^a)$ is called the basis-dependent discord of ρ^{ab} relative to Π^a in Ref. [75].

It is shown that [46,54]

$$\delta(\rho^{ab})=0 \quad\Leftrightarrow\quad \rho^{ab}=\sum_\mu p_\mu|\mu\rangle\langle\mu|\otimes\rho_\mu^b, \tag{7}$$

for some orthonormal basis $\{|\mu\rangle\}$ of $\mathcal{H}^a$ and quantum states $\{\rho_\mu^b\}$ on $\mathcal{H}^b$ with $\{p_\mu\}$ a probability distribution. Such a bipartite state is called a classical-quantum state [50,97,98]. Yadin *et al.* proved that

$$\delta(\rho^{ab}|\Pi^a)=0 \quad\Leftrightarrow\quad \rho^{ab}=\sum_l p_l\rho_l^a\otimes\rho_l^b, \tag{8}$$

where all ρ_l^a are perfectly distinguishable by Π^a [75].

In the next subsection we will introduce an alternative definition for bipartite states satisfying Eq. (8) and call them block-diagonal product states relative to Π^a.

C. Block-diagonal product states

We follow the notations in the last subsection. Recall that a Lüders measurement $\Pi_{\mathrm{L}}^a=\{P_l^a : l=1,2,\dots,m\}$ on $\mathcal{H}^a$ with $d_a=\dim\mathcal{H}^a$ is called a coarse graining of Π^a if there exists a partition $\{I_l : l=1,2,\dots,m\}$ of the index set $I=\{1,2,\dots,d_a\}$ (i.e., $I_l\cap I_{l'}=\emptyset$ for $l\neq l'$ and $I=\cup_{l=1}^m I_l$) such that $P_l^a=\sum_{j\in I_l}\Pi_j^a$ for $l=1,2,\dots,m$. Correspondingly, $\mathcal{H}^a$ is decomposed into a direct-sum form of subspaces, i.e., $\mathcal{H}^a=\bigoplus_l P_l^a\mathcal{H}^a$ where $P_l^a\mathcal{H}^a=\mathrm{span}\{|j\rangle : j\in I_l\}$. From now on, we will denote by $S(\mathcal{H})$ the set of quantum states on Hilbert space $\mathcal{H}$. With these preparations, we introduce the definition of block-diagonal product states. Notice that states in this definition are equivalent to the states in Eq. (8), and it is easy to verify the equivalence.

Definition 1. A bipartite state ρ^{ab} on $\mathcal{H}^a\otimes\mathcal{H}^b$ is called a block-diagonal product state relative to a local von Neumann measurement $\Pi^a=\{\Pi_j^a=|j\rangle\langle j| : j=1,2,\dots,d_a\}$ on $\mathcal{H}^a$ if it can be represented as

$$\rho^{ab}=\bigoplus_{l=1}^m p_l\rho_l^a\otimes\rho_l^b \tag{9}$$

for some coarse graining $\Pi_{\mathrm{L}}^a=\{P_l^a=\sum_{j\in I_l}\Pi_j^a : l=1,2,\dots,m\}$ of Π^a, where $\rho_l^a\in S(P_l^a\mathcal{H}^a)$, $\rho_l^b\in S(\mathcal{H}^b)$, and $\{p_l\}$ is a probability distribution. In this case the reduced states for subsystems a and b are

$$\rho^a=\bigoplus_{l=1}^m p_l\rho_l^a,\quad \rho^b=\sum_{l=1}^m p_l\rho_l^b,$$

respectively. We remark that in Eq. (8), we regard ρ_l^a as a state on $\mathcal{H}^a$, while in Eq. (9), we regard ρ_l^a as a state on $P_l^a\mathcal{H}^a$. Due to the isomorphism between the internal and external direct sums, we do not distinguish between the sum and the direct sum in ρ^{ab}.

Let S_{Π^a} be the set of block-diagonal product states relative to Π^a,

$$S_{\Pi^a} := \{\text{Block-diagonal product states relative to } \Pi^a\}. \tag{10}$$

Two special types of block-diagonal product states relative to Π^a are as follows: (1) Any product state $\rho^a\otimes\rho^b$ is a block-diagonal product state relative to Π^a for $m=1$. (2) Any state represented as $\rho^{ab}=\sum_j p_j|j\rangle\langle j|\otimes\rho_j^b$ is a block-diagonal product state relative to Π^a for $m=d_a$. Such a state is called an incoherent-quantum state (relative to Π^a) in Ref. [76] and is a special type of classical-quantum states. A simple example for $m\neq 1, d_a$ as shown in [75] is

$$\tfrac{1}{2}|+\rangle\langle+|\otimes|0\rangle\langle 0|+\tfrac{1}{2}|2\rangle\langle 2|\otimes|1\rangle\langle 1|, \tag{11}$$

which is a block-diagonal product state on $\mathbb{C}^3\otimes\mathbb{C}^2$ relative to the local basis $\{|0\rangle,|1\rangle,|2\rangle\}$ of $\mathbb{C}^3$. Here $|+\rangle=(|0\rangle+|1\rangle)/\sqrt{2}$.

From the above observations, we know that the set of product states is contained in S_{Π^a},

$$\{\text{Product states}\}\subset S_{\Pi^a}. \tag{12}$$

Furthermore, substituting the spectral decompositions of $\rho_l^a=\sum_k q_{l,k}|\psi_{l,k}\rangle\langle\psi_{l,k}|$ into Eq. (9), we obtain that $\rho^{ab}=\sum_{l,k}p_lq_{l,k}|\psi_{l,k}\rangle\langle\psi_{l,k}|\otimes\rho_l^b$ and $\{|\psi_{l,k}\rangle\}$ constitutes an orthonormal basis of $\mathcal{H}^a$. Thus, any block-diagonal product state relative to Π^a is a classical-quantum state, which implies that

$$S_{\Pi^a}\subset\{\text{Classical-quantum states}\}. \tag{13}$$

On the converse, any classical-quantum state $\sum_\mu p_\mu|\mu\rangle\langle\mu|\otimes\rho_\mu^b$ can be seen as a block-diagonal product state relative to the local basis $\{|\mu\rangle\}$. However, given a local basis, an arbitrary classical-quantum state might not be a block-diagonal product state relative to the given local basis. For example, given a local basis $\{|0\rangle,|1\rangle\}$ of $\mathcal{H}^a$, $\rho^{ab}=\frac{1}{2}|+\rangle\langle+|\otimes|0\rangle\langle 0|+\frac{1}{2}|-\rangle\langle-|\otimes|1\rangle\langle 1|$ with $|\pm\rangle=(|0\rangle\pm|1\rangle)/\sqrt{2}$ is a classical-quantum state but not a block-diagonal product state relative to the given basis. In a word, the relation between S_{Π^a} and the

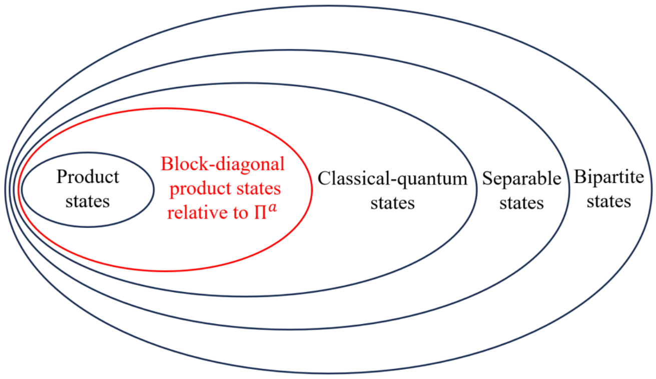

FIG. 1. Schematic illustration of the connections of block-diagonal product states relative to a local basis Π^a with other states: {Product states} $\subset$ {Block-diagonal product states relative to Π^a} $\subset$ {Classical-quantum states} $\subset$ {Separable states} $\subset$ {Bipartite states}.

set of classical-quantum states can be shown as follows:

$$\bigcup_{\Pi^a} S_{\Pi^a} = \{\text{Classical-quantum states}\}, \tag{14}$$

where the union is taken over all von Neumann measurements Π^a on $\mathcal{H}^a$.

We illustrate the connections between the set of block-diagonal product states relative to a specific local basis Π^a with the sets of product states, separable states, and classical-quantum states in Fig. 1. We remark that the set of block-diagonal product states is not convex. For example, considering a two-qubit system and a given local basis $\Pi^a = \{|0\rangle\langle 0|, |1\rangle\langle 1|\}$, $|+\rangle\langle +| \otimes |0\rangle\langle 0|$ and $|-\rangle\langle -| \otimes |1\rangle\langle 1|$ are two product states and thus belong to S_{Π^a}. However, their mixture $\rho^{ab} = \frac{1}{2}|+\rangle\langle +| \otimes |0\rangle\langle 0| + \frac{1}{2}|-\rangle\langle -| \otimes |1\rangle\langle 1| \notin S_{\Pi^a}$.

III. CHARACTERIZING BIPARTITE STATES WITH VANISHING BASIS-DEPENDENT CORRELATIONS

In this section, after introducing the notion of basis-dependent correlations and reviewing the result of Yadin *et al.* [75], we present our main results: (1) Determine the structure of quantum ensembles saturating the convexity inequality in the resource theory of coherence using the coherence measure based on skew information (Proposition 1). (2) Prove that the set of block-diagonal product states relative to a local basis can be characterized by the property of possessing vanishing basis-dependent correlations using the coherence measure based on skew information (Proposition 2), and then generalize the results to the cases when bilocal bases are given (Proposition 3). (3) Characterize the set of bipartite states with vanishing basis-dependent correlations when we use the l_1 norm of coherence, and show that the set of block-diagonal product states relative to a given local basis is a subset of it (Proposition 4).

A. Basis-dependent correlations

Consider a bipartite state ρ^{ab} on $\mathcal{H}^a \otimes \mathcal{H}^b$ with $d_a = \dim \mathcal{H}^a$ and a given local basis $\{|j\rangle : j = 1, 2, \dots, d_a\}$ of $\mathcal{H}^a$ which corresponds to a von Neumann measurement $\Pi^a := \{\Pi_j^a = |j\rangle\langle j|\}$ on $\mathcal{H}^a$. Then Π^a naturally induces a Lüders measurement $\Pi^a \otimes \mathbf{1}^b := \{\Pi_j^a \otimes \mathbf{1}^b\}$ on $\mathcal{H}^a \otimes \mathcal{H}^b$ such that $\Pi^a \otimes \mathbf{1}^b(\rho^{ab}) := \sum_j (\Pi_j^a \otimes \mathbf{1}^b)\rho^{ab}(\Pi_j^a \otimes \mathbf{1}^b)$.

Intuitively, the amount of the coherence in a bipartite state ρ^{ab} relative to $\Pi^a \otimes \mathbf{1}^b$ should be no less than that in the reduced state ρ^a relative to Π^a due to the existence of correlations in ρ^{ab}. Because of the dependence of such correlations on Π^a, we call them basis-dependent correlations. Namely, the basis-dependent correlations in ρ^{ab} (relative to Π^a) is defined as the coherence difference between the global state ρ^{ab} relative to $\Pi^a \otimes \mathbf{1}^b$ and the local state ρ^a relative to Π^a. For any block coherence measure $C(\rho, \Pi_{\rm L})$ of ρ relative to $\Pi_{\rm L}$ satisfying the following desirable property

$$C(\rho^{ab}, \Pi^a \otimes \mathbf{1}^b) \geqslant C(\rho^a, \Pi^a), \tag{15}$$

we can define a quantifier for basis-dependent correlations of ρ^{ab} relative to Π^a as

$$D_C(\rho^{ab}, \Pi^a) := C(\rho^{ab}, \Pi^a \otimes \mathbf{1}^b) - C(\rho^a, \Pi^a). \tag{16}$$

Here the subscript C of $D_C(\rho^{ab}, \Pi^a)$ represents which coherence measure $C(\rho, \Pi_{\rm L})$ is used. Actually, $D_C(\rho^{ab}, \Pi^a)$ is called the conditional coherence relative to Π^a in Ref. [99] and the generalized partial correlated coherence in Ref. [91].

On the one hand, because there are no correlations in a product state $\rho^a \otimes \rho^b$, the coherence in $\rho^a \otimes \rho^b$ relative to $\Pi^a \otimes \mathbf{1}^b$ arises only from the coherence in ρ^a relative to Π^a, and thus there is no difference between them, i.e., $C(\rho^a \otimes \rho^b, \Pi^a \otimes \mathbf{1}^b) = C(\rho^a, \Pi^a)$, which implies that $D_C(\rho^a \otimes \rho^b, \Pi^a) = 0$.

On the other hand, any incoherent-quantum state relative to Π^a (i.e., $\rho^{ab} = \sum_j p_j |j\rangle\langle j| \otimes \rho_j^b$) has no coherence relative to $\Pi^a \otimes \mathbf{1}^b$ because of the fact that $\rho^{ab} = \Pi^a \otimes \mathbf{1}^b(\rho^{ab})$ [7,35–38,40]. In the meanwhile, its reduced state $\rho^a = \sum_j p_j |j\rangle\langle j|$ has no coherence relative to Π^a either. Therefore, similar to the case of product states, the amount of coherence in $\rho^{ab} = \sum_j p_j |j\rangle\langle j| \otimes \rho_j^b$ relative to $\Pi^a \otimes \mathbf{1}^b$ is also the same as that of coherence in $\rho^a = \sum_j p_j |j\rangle\langle j|$ relative to Π^a in that both of them vanish, i.e., $C(\rho^{ab}, \Pi^a \otimes \mathbf{1}^b) = C(\rho^a, \Pi^a) = 0$. So, even though there exist correlations in $\rho^{ab} = \sum_j p_j |j\rangle\langle j| \otimes \rho_j^b$ when $\rho^{ab} \neq \rho^a \otimes \rho^b$, it has vanishing basis-dependent correlations relative to Π^a, i.e., $D_C(\rho^{ab}, \Pi^a) = 0$.

Now, a natural question arises: Given a local basis, are there any other types of bipartite states having an equal amount of coherence in the global state and the local state and thus possessing vanishing basis-dependent correlations relative to the given local basis?

Yadin *et al.* have proven that when we choose the relative entropy of coherence as the coherence measure, the answer is just the set of block-diagonal product states relative to the given local basis, which includes product states and incoherent-quantum states as two special cases [75]. To be precise, using the relative entropy of coherence $C_{\rm r}(\rho, \Pi_{\rm L})$ defined by Eq. (2), the quantifier of basis-dependent correlations becomes

$$D_{C_{\rm r}}(\rho^{ab}, \Pi^a) := C_{\rm r}(\rho^{ab}, \Pi^a \otimes \mathbf{1}^b) - C_{\rm r}(\rho^a, \Pi^a), \tag{17}$$

where $C_{\rm r}(\rho^{ab}, \Pi^a \otimes \mathbf{1}^b) = S(\Pi^a \otimes \mathbf{1}^b(\rho^{ab})) - S(\rho^{ab})$ and $C_{\rm r}(\rho^a, \Pi^a) = S(\Pi^a(\rho^a)) - S(\rho^a)$. Actually, Herbut has demonstrated that [6]

$$C_{\mathrm{r}}(\rho^{ab}, \Pi^a \otimes \mathbf{1}^b) - C_{\mathrm{r}}(\rho^a, \Pi^a) = \delta(\rho^{ab}|\Pi^a), \tag{18}$$

where $\delta(\rho^{ab}|\Pi^a)$ is the basis-dependent discord of ρ^{ab} relative to Π^a. Thus, combining Eqs. (17) and (18) implies that basis-dependent discord $\delta(\rho^{ab}|\Pi^a)$ is just the basis-dependent correlations $D_{C_{\mathrm{r}}}(\rho^{ab}, \Pi^a)$ in terms of the relative entropy of coherence. From Eq. (8), it follows that the set of block-diagonal product states relative to Π^a can be characterized by both the property of possessing vanishing basis-dependent correlations relative to Π^a in terms of the relative entropy of coherence and the property of possessing vanishing basis-dependent discord relative to Π^a, which can be shown as follows:

$$S_{\Pi^a} = \{\rho^{ab} : D_{C_{\mathrm{r}}}(\rho^{ab}, \Pi^a) = 0\} \tag{19}$$

$$= \{\rho^{ab} : \delta(\rho^{ab}|\Pi^a) = 0\}. \tag{20}$$

In this work we will prove the following two relations:

$$S_{\Pi^a} = \{\rho^{ab} : D_I(\rho^{ab}, \Pi^a) = 0\}, \tag{21}$$

$$S_{\Pi^a} \subset \{\rho^{ab} : D_{C_{l_1}}(\rho^{ab}, \Pi^a) = 0\}. \tag{22}$$

Equation (21) implies that the set of block-diagonal product states relative to Π^a can also be characterized by the property of possessing vanishing basis-dependent correlations relative to Π^a in terms of the coherence measure defined by Eq. (3), while Eq. (22) shows that to be a block-diagonal product state relative to Π^a is a sufficient but not necessary condition for having vanishing basis-dependent correlations relative to Π^a in terms of the l_1 norm of coherence defined by Eq. (4).

Here, in terms of the coherence measure defined by Eq. (3), the coherence in ρ^a relative to Π^a is quantified by $I(\rho^a, \Pi^a) = \sum_j I(\rho^a, \Pi^a_j)$, the coherence in ρ^{ab} relative to $\Pi^a \otimes \mathbf{1}^b$ is quantified by $I(\rho^{ab}, \Pi^a \otimes \mathbf{1}^b) = \sum_j I(\rho^{ab}, \Pi^a_j \otimes \mathbf{1}^b)$, and the basis-dependent correlations relative to Π^a is quantified by

$$D_I(\rho^{ab}, \Pi^a) := I(\rho^{ab}, \Pi^a \otimes \mathbf{1}^b) - I(\rho^a, \Pi^a). \tag{23}$$

In terms of the coherence measure defined by Eq. (4), the coherence in ρ^a relative to Π^a is quantified by $C_{l_1}(\rho^a, \Pi^a) = \sum_{j\neq j'} |\langle j|\rho^a|j'\rangle|$, the coherence in ρ^{ab} relative to $\Pi^a \otimes \mathbf{1}^b$ is quantified by $C_{l_1}(\rho^{ab}, \Pi^a \otimes \mathbf{1}^b) = \sum_{j\neq j'} \|\langle j|\rho^{ab}|j'\rangle\|_{\mathrm{tr}}$, and the basis-dependent correlations relative to Π^a is quantified by

$$D_{C_{l_1}}(\rho^{ab}, \Pi^a) := C_{l_1}(\rho^{ab}, \Pi^a \otimes \mathbf{1}^b) - C_{l_1}(\rho^a, \Pi^a). \tag{24}$$

B. Basis-dependent correlations via the coherence measure based on skew information

To establish Eq. (21), we need to prove the following result, which provides an explicit structure for a quantum ensemble $\{(\lambda_k, \rho_k)\}$ saturating the convexity inequality (1) in the resource theory of coherence when we use the coherence measure defined by Eq. (3). Without loss of generality, we assume that $\lambda_k > 0$ throughout this work.

Proposition 1. For a von Neumann measurement $\Pi = \{\Pi_j = |j\rangle\langle j| : j = 1, 2, \ldots, d\}$ on $\mathcal{H}$ with $d = \dim \mathcal{H}$, an ensemble $\{(\lambda_k, \rho_k) : k = 1, 2, \ldots, n\}$ on $\mathcal{H}$ satisfies

$$I\left(\sum_{k=1}^n \lambda_k \rho_k, \Pi\right) = \sum_{k=1}^n \lambda_k I(\rho_k, \Pi) \tag{25}$$

if and only if each ρ_k can be represented as

$$\rho_k = \bigoplus_{l=1}^m p_{k,l}\sigma_l, \tag{26}$$

for some coarse graining $\Pi_{\mathrm{L}} = \{P_l = \sum_{j\in I_l} \Pi_j : l = 1, 2, \ldots, m\}$ of Π, where $\sigma_l \in S(P_l\mathcal{H})$ and for each k, $\{p_{k,l} : l = 1, 2, \ldots, m\}$ is a probability distribution.

See Appendix A for the proof.

Note that Lieb proved that the Wigner-Yanase skew information $I(\rho, K)$ of a quantum state ρ with respect to an observable K satisfies the convexity property [100]

$$I\left(\sum_k \lambda_k \rho_k, K\right) \leqslant \sum_k \lambda_k I(\rho_k, K). \tag{27}$$

Even though it is hard to derive when the saturation of the convexity inequality (27) happens in general, which may be heavily dependent on the structure of the observable K, we can characterize the structure of ensembles satisfying Eq. (25) as shown in Eq. (26).

To gain an intuitive understanding of ensembles $\{(\lambda_k, \rho_k)\}$ satisfying Eq. (25), let us see an example. Considering a qutrit system on $\mathbb{C}^3$ with a basis $\{|0\rangle, |1\rangle, |2\rangle\}$, it can be directly verified that the ensemble $\{(1/3, \rho_1), (2/3, \rho_2)\}$ with $\rho_1 = 1/2|+\rangle\langle+| + 1/2|2\rangle\langle 2|$, $\rho_2 = 1/4|+\rangle\langle+| + 3/4|2\rangle\langle 2|$ satisfies Eq. (25). The average state is $\rho = 1/3|+\rangle\langle+| + 2/3|2\rangle\langle 2|$, with $P_1 = |0\rangle\langle 0| + |1\rangle\langle 1|$, $P_2 = |2\rangle\langle 2|$, $\sigma_1 = |+\rangle\langle+|$, $\sigma_2 = |2\rangle\langle 2|$, $p_{1,1} = p_{1,2} = 1/2$, $p_{2,1} = 1/4$, $p_{2,2} = 3/4$.

Now, using Proposition 1 we can prove Eq. (21) as shown in the following result.

Proposition 2. For a local von Neumann measurement $\Pi^a = \{\Pi_j = |j\rangle\langle j| : j = 1, 2, \ldots, d_a\}$ on $\mathcal{H}^a$, a bipartite state ρ^{ab} on $\mathcal{H}^a \otimes \mathcal{H}^b$ satisfies

$$D_I(\rho^{ab}, \Pi^a) = 0 \tag{28}$$

if and only if it is a block-diagonal product state relative to Π^a, i.e., $\rho^{ab} \in S_{\Pi^a}$.

See Appendix B for the proof.

From Proposition 2 we know that block-diagonal product states can be identified as bipartite states that have vanishing basis-dependent correlations when we quantify the coherence by the coherence measure based on skew information.

Next, we extend the result in Proposition 2 to the cases when both local bases for $\mathcal{H}^a$ and $\mathcal{H}^b$ are given. Before doing that, we generalize the notion of block-diagonal product states relative to a unilocal basis to that relative to bilocal bases, and elucidate the relationship between block-diagonal product states relative to bilocal bases and classical-classical states. Recall that a classical-classical state is defined as $\rho^{ab} = \sum_{\mu\nu} p_{\mu\nu}|\mu\rangle\langle\mu| \otimes |\nu\rangle\langle\nu|$, where $\{p_{\mu\nu}\}$ is a bivariate probability distribution, and $\{|\mu\rangle : \mu = 1, 2, \ldots, d_a\}$ and $\{|\nu\rangle : \nu = 1, 2, \ldots, d_b\}$ are orthonormal bases of $\mathcal{H}^a$ and $\mathcal{H}^b$, respectively. Here $d_\alpha = \dim \mathcal{H}^\alpha$ for $\alpha = a, b$.

Definition 2. Given a von Neumann measurement $\Pi^a = \{\Pi^a_j = |j\rangle\langle j| : j = 1, 2, \dots, d_a\}$ on $\mathcal{H}^a$ and a von Neumann measurement $\Pi^b = \{\Pi^b_k = |k\rangle\langle k| : k = 1, 2, \dots, d_b\}$ on $\mathcal{H}^b$, a bipartite state ρ^{ab} on $\mathcal{H}^a \otimes \mathcal{H}^b$ is called a block-diagonal product state relative to both Π^a and Π^b if it can be represented as

$$\rho^{ab} = \bigoplus_{l=1}^{m} \bigoplus_{s=1}^{n} p_{l,s}\rho^a_l \otimes \rho^b_s, \tag{29}$$

for some coarse grainings $\Pi^a_{\rm L} = \{P^a_l = \sum_{j\in I^a_l} \Pi^a_j : l = 1, 2, \dots, m\}$ of Π^a and $\Pi^b_{\rm L} = \{P^b_s = \sum_{k\in I^b_s} \Pi^b_k : s = 1, 2, \dots, n\}$ of Π^b, where $\rho^a_l \in S(P^a_l\mathcal{H}^a)$, $\rho^b_s \in S(P^b_s\mathcal{H}^b)$, $\{p_{l,s}\}$ is a probability distribution, and $\{I^a_l\}$ and $\{I^b_s\}$ are some partitions of $I^a = \{1, 2, \dots, d_a\}$ and $I^b = \{1, 2, \dots, d_b\}$, respectively.

From the above definition and the spectral decompositions for each ρ^a_l and ρ^b_s, we know that on the one hand product states are block-diagonal product states relative to both Π^a and Π^b, and on the other hand, any block-diagonal product state relative to both Π^a and Π^b is a classical-classical state. Thus, block-diagonal product states relative to both Π^a and Π^b is an interpolation between product states and classical-classical states, similar to the relation between block-diagonal product states relative to Π^a and classical-quantum states.

The following proposition shows that block-diagonal product states relative to both Π^a and Π^b can be identified as bipartite states with vanishing basis-dependent correlations relative to both Π^a and Π^b.

Proposition 3. Given von Neumann measurements Π^a on $\mathcal{H}^a$ and Π^b on $\mathcal{H}^b$, a bipartite state ρ^{ab} on $\mathcal{H}^a \otimes \mathcal{H}^b$ satisfies

$$D_I(\rho^{ab}, \Pi^a) = D_I(\rho^{ab}, \Pi^b) = 0 \tag{30}$$

if and only if it is a block-diagonal product state relative to both Π^a and Π^b. Here $D_I(\rho^{ab}, \Pi^b) := I(\rho^{ab}, \mathbf{1}^a \otimes \Pi^b) - I(\rho^b, \Pi^b)$ quantifies the basis-dependent correlations relative to Π^b in terms of the coherence measure defined by Eq. (3) with $I(\rho^{ab}, \mathbf{1}^a \otimes \Pi^b) = \sum_k I(\rho^{ab}, \mathbf{1}^a \otimes \Pi^b_k)$ and $I(\rho^b, \Pi^b) = \sum_k I(\rho^b, \Pi^b_k)$.

See Appendix C for the proof.

We remark that a similar argument shows that the conclusion in Proposition 3 also holds for the relative entropy of coherence. That is, a bipartite state ρ^{ab} satisfies

$$D_{C_{\rm r}}(\rho^{ab}, \Pi^a) = D_{C_{\rm r}}(\rho^{ab}, \Pi^b) = 0 \tag{31}$$

if and only if it is a block-diagonal product state relative to both Π^a and Π^b. Here $D_{C_{\rm r}}(\rho^{ab}, \Pi^b) := C_{\rm r}(\rho^{ab}, \mathbf{1}^a \otimes \Pi^b) - C_{\rm r}(\rho^b, \Pi^b)$ quantifies the basis-dependent correlations relative to both Π^b in terms of the relative entropy of coherence with $C_{\rm r}(\rho^{ab}, \mathbf{1}^a \otimes \Pi^b) = S(\mathbf{1}^a \otimes \Pi^b(\rho^{ab})) - S(\rho^{ab})$ and $C_{\rm r}(\rho^b, \Pi^b) = S[\Pi^b(\rho^b)] - S(\rho^b)$.

C. Basis-dependent correlations via the l_1 norm of coherence

In this subsection we provide an explicit structure for bipartite states satisfying $D_{C_{l_1}}(\rho^{ab}, \Pi^a) = 0$ and then demonstrate Eq. (22), showing that the set of block-diagonal product states relative to Π^a is a subset of such states.

Proposition 4. For a local von Neumann measurement $\Pi^a = \{\Pi^a_j = |j\rangle\langle j| : j = 1, 2, \dots, d_a\}$ on $\mathcal{H}^a$, a bipartite state ρ^{ab} on $\mathcal{H}^a \otimes \mathcal{H}^b$ satisfies

$$D_{C_{l_1}}(\rho^{ab}, \Pi^a) = 0 \tag{32}$$

if and only if for $j, j' = 1, 2, \dots, d_a$ with $j \neq j'$, each $\langle j|\rho^{ab}|j'\rangle$ satisfies

$$\langle j|\rho^{ab}|j'\rangle = e^{i\varphi_{jj'}}|\langle j|\rho^{ab}|j'\rangle|$$

for some $\varphi_{jj'} \in [0, 2\pi)$. Here $|A| := \sqrt{A^\dagger A}$ denotes the absolute value of operator A [101].

See Appendix D for the proof.

For a block-diagonal product state defined by Eq. (9), if there exists an l such that $j, j' \in I_l$, then

$$\langle j|\rho^{ab}|j'\rangle = p_l\langle j|\rho^a|j'\rangle\rho^b_l.$$

Otherwise, $\langle j|\rho^{ab}|j'\rangle = 0$. This implies that the set of block-diagonal product states relative to Π^a is contained in the set of bipartite states satisfying $D_{C_{l_1}}(\rho^{ab}, \Pi^a) = 0$. Yet the converse is not true. We will give an example (i.e., Example 4 in Sec. V) to show that there exist bipartite states that satisfy $D_{C_{l_1}}(\rho^{ab}, \Pi^a) = 0$ but do not belong to S_{Π^a}.

IV. INTERPRETING BLOCK-DIAGONAL PRODUCT STATES IN AN INTERFERENCE MODEL

Coherence is a fundamental resource in many quantum information tasks. Concerning a given local reference basis for a bipartite quantum system, a bipartite state might provide more coherence resources than the corresponding reduced state due to correlations. It is natural to ask which bipartite states contain correlations that contribute to an additional amount of coherence resources, i.e., basis-dependent correlations. Using the coherence measure defined by Eq. (3), Proposition 2 gives a complete answer to this question: Only those states that are not block-diagonal product states relative to the local basis can offer additional coherence resources. In other words, the correlations contained in block-diagonal product states relative to the local basis cannot be exploited to improve the coherence resources.

Now we give an interpretation of block-diagonal product states in an interference model. Consider a bipartite system described by $\mathcal{H}^a \otimes \mathcal{H}^b$ with $d_a = \dim \mathcal{H}^a$, for which only party a passes through a d_a-path interference with a phase shift in each interference path, as schematically depicted in Fig. 2. The interference paths can be described by the von Neumann measurement $\Pi^a = \{\Pi^a_j = |j\rangle\langle j| : j = 1, 2, \dots, d_a\}$ on $\mathcal{H}^a$ with each path undergoing a phase shift $e^{i\theta_j}$, and then the incorporation of the phase shifts $\{e^{i\theta_j} : j = 1, 2, \dots, d_a\}$ into the paths may be described by the unitary operator $U_{\boldsymbol\theta} = \sum_{j=1}^{d_a} e^{i\theta_j}\Pi^a_j$ on $\mathcal{H}^a$ with $\boldsymbol\theta = (\theta_1, \theta_2, \dots, \theta_{d_a}) \in [0, 2\pi)^{d_a}$. Let ρ^{ab} and ρ^a be the bipartite state on $\mathcal{H}^a \otimes \mathcal{H}^b$ and the reduced state on $\mathcal{H}^a$ after the beam splitter, respectively. Then the final bipartite state and reduced state on $\mathcal{H}^a$ passing through the d_a paths are $\rho^{ab}_{\boldsymbol\theta} = (U_{\boldsymbol\theta} \otimes \mathbf{1}^b)\rho^{ab}(U_{\boldsymbol\theta} \otimes \mathbf{1}^b)^\dagger$ and $\rho^a_{\boldsymbol\theta} = U_{\boldsymbol\theta}\rho^a U^\dagger_{\boldsymbol\theta}$, respectively.

In Ref. [102], it was shown that using the coherence measure based on skew information, $I(\rho^{ab}, U_{\boldsymbol\theta} \otimes \mathbf{1}^b)$ and $I(\rho^a, U_{\boldsymbol\theta})$ can be used to quantify the interference of ρ^{ab} and ρ^a with respect to $\Pi^a \otimes \mathbf{1}^b$ and Π^a, respectively. However, they depend on the phase shifts $\boldsymbol\theta$, which is consistent with the fact that

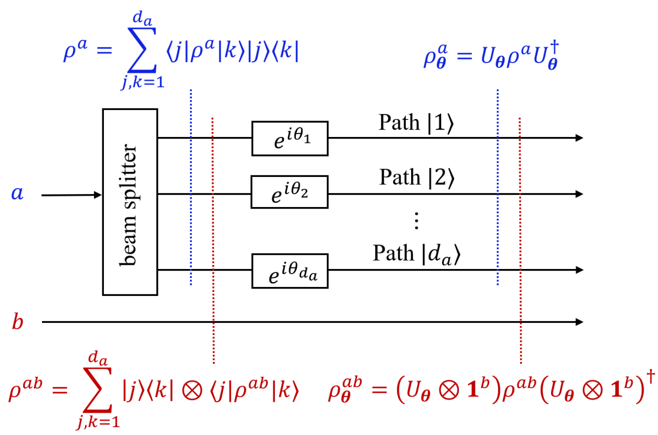

FIG. 2. Schematic illustration of n-path interference performed on subsystem a with phase shift $e^{i\theta_j}$ in path $|j\rangle$ where $U_{\boldsymbol{\theta}} = \sum_{j=1}^{d_a} e^{i\theta_j}\Pi^a_j$ with $\boldsymbol{\theta} = (\theta_1, \theta_2, \dots, \theta_{d_a}) \in [0, 2\pi)^{d_a}$.

the fringe visibility depends on the phase shifts. By taking the average over $\boldsymbol{\theta}$, intrinsic quantifiers of interference for ρ^{ab} and ρ^a with respect to the paths Π^a were obtained, and it was established that they coincide with the coherence of ρ^{ab} and ρ^a relative to $\Pi^a \otimes \mathbf{1}^b$ and Π^a, respectively,

$$I(\rho^{ab}, \Pi^a \otimes \mathbf{1}^b) = \int_{[0,2\pi)^{d_a}} I(\rho^{ab}, U_{\boldsymbol{\theta}} \otimes \mathbf{1}^b) d\boldsymbol{\theta},$$

$$I(\rho^a, \Pi^a) = \int_{[0,2\pi)^{d_a}} I(\rho^a, U_{\boldsymbol{\theta}}) d\boldsymbol{\theta},$$

where $d\boldsymbol{\theta} = d\theta_1 d\theta_2 \cdots d\theta_{d_a}$ is the normalized uniform measure on $[0, 2\pi)^{d_a}$. According to Proposition 2, block-diagonal product states relative to Π^a satisfy $I(\rho^{ab}, \Pi^a \otimes \mathbf{1}^b) = I(\rho^a, \Pi^a)$. In this sense, we get an interpretation of block-diagonal product states as shown in the following result.

Proposition 5. In the interference model described by Fig. 2, a bipartite state ρ^{ab} satisfies that the degree of the (average) interference of ρ^{ab} with respect to the paths is equal to that of ρ^a if and only if it is a block-diagonal product state relative to Π^a.

Note that the Wigner-Yanase skew information $I(\rho, K)$ of ρ with respect to an observable K is a special kind of generalized quantum Fisher information, and it is essentially equivalent to quantum Fisher information $F(\rho, K)$ in terms of the symmetric logarithm derivative in the sense that $I(\rho, K) \leqslant F(\rho, K) \leqslant 2I(\rho, K)$ [103]. So, it is reasonable to conjecture that if we replace the coherence measure based on skew information with the coherence measure based on quantum Fisher information in terms of the symmetric logarithm derivative, the set of block-diagonal product states relative to a given local basis still characterizes the structure of bipartite states with vanishing basis-dependent correlations. In this context, block-diagonal product states relative to a local basis have the following metrological interpretation.

In the above model, $\rho^{ab}_{\boldsymbol{\theta}}$ ($\rho^a_{\boldsymbol{\theta}}$) encodes the information of the parameters $\boldsymbol{\theta} = (\theta_1, \theta_2, \dots, \theta_{d_a})$. In order to extract the parameter information, one needs to perform quantum measurements on the output states. If only local measurements on $\mathcal{H}^a$ are allowed, the quantum Fisher information F^a_j of $\rho^a_{\boldsymbol{\theta}}$ about the parameter θ_j, which is shown to coincide with the quantum Fisher information $F(\rho^a, \Pi^a_j)$ of ρ^a with respect to the operator Π^a_j [104], i.e., $F(\rho^a, \Pi^a_j) = F^a_j$, sets an upper bound to the optimal precision of quantum parameter estimation [105–108]. Here F^a_j is defined as $F^a_j = \mathrm{tr}(\rho^a_{\boldsymbol{\theta}} L^a_j (L^a_j)^\dagger)/4$ with the symmetric logarithmic derivative L^a_j determined by $\partial_{\theta_j}\rho^a_{\boldsymbol{\theta}} = (\rho^a_{\boldsymbol{\theta}} L^a_j + L^a_j \rho^a_{\boldsymbol{\theta}})/2$, and $F(\rho^a, \Pi^a_j)$ is defined as $F(\rho^a, \Pi^a_j) = \mathrm{tr}(\rho^a \tilde{L}^a_j (\tilde{L}^a_j)^\dagger)/4$ with $\tilde{L}^a_j$ determined by $i[\rho^a, \Pi^a_j] = (\rho^a \tilde{L}^a_j + \tilde{L}^a_j \rho^a)/2$. Thus, the sum of quantum Fisher information F^a_j turns out to coincide with the coherence measure based on quantum Fisher information in terms of the symmetric logarithm derivative,

$$F(\rho^a, \Pi^a) = \sum_j F(\rho^a, \Pi^a_j) = \sum_j F^a_j. \tag{33}$$

Furthermore, it was proven that the coherence $F(\rho^a, \Pi^a)$ of ρ^a relative to Π^a provides a lower bound to the total variance of unbiased estimation for the parameters $\boldsymbol{\theta} = (\theta_1, \theta_2, \dots, \theta_{d_a})$ [104],

$$\sum_j \mathrm{Var}(\rho^a_{\boldsymbol{\theta}}, \hat{\theta}^a_j) \geqslant \frac{d_a^2}{F(\rho^a, \Pi^a)}, \tag{34}$$

with $\hat{\boldsymbol{\theta}}^a = (\hat{\theta}^a_1, \hat{\theta}^a_2, \dots, \hat{\theta}^a_{d_a})$ being an unbiased estimator of $\boldsymbol{\theta} = (\theta_1, \theta_2, \dots, \theta_{d_a})$ and $\mathrm{Var}(\rho^a_{\boldsymbol{\theta}}, \hat{\theta}^a_j)$ being the variance of the estimator $\hat{\theta}^a_j$. However, if joint measurements on $\mathcal{H}^a \otimes \mathcal{H}^b$ are allowed, then

$$\sum_j \mathrm{Var}(\rho^{ab}_{\boldsymbol{\theta}}, \hat{\theta}^{ab}_j) \geqslant \frac{d_a^2}{F(\rho^{ab}, \Pi^a \otimes \mathbf{1}^b)}, \tag{35}$$

where the notations are defined analogously. By the fact that $F(\rho^{ab}, \Pi^a \otimes \mathbf{1}^b) \geqslant F(\rho^a, \Pi^a)$, we know that in general using joint measurements instead of local measurements can improve the optimal estimation precision of the parameters $\boldsymbol{\theta}$ [109]. It is easy to verify that block-diagonal product states relative to Π^a satisfy

$$F(\rho^{ab}, \Pi^a \otimes \mathbf{1}^b) = F(\rho^a, \Pi^a), \tag{36}$$

and thus cannot enhance the optimal estimation precision of the parameters. Since both quantum Fisher information and the Wigner-Yanase skew information are special instances of metric-adjusted skew information with similar properties, it is reasonable to expect that if ρ^{ab} satisfies $F(\rho^{ab}, \Pi^a \otimes \mathbf{1}^b) = F(\rho^a, \Pi^a)$, then ρ^{ab} must be a block-diagonal product state relative to Π^a. Actually, the conjecture is true for pure bipartite states $|\psi^{ab}\rangle$ on $\mathcal{H}^a \otimes \mathcal{H}^b$ in that for $\rho^{ab} = |\psi^{ab}\rangle\langle\psi^{ab}|$,

$$F(\rho^{ab}, \Pi^a \otimes \mathbf{1}^b) = F(\rho^a, \Pi^a),$$

if and only if $|\psi^{ab}\rangle$ is a product state, which is a block-diagonal product state relative to Π^a.

If the conjecture is true in general, then the set of block-diagonal product states relative to Π^a can be characterized by the property of possessing no correlations contributing to the enhancement of parameter estimation precision.

V. COMPARISON

To gain some concrete understanding of the three quantifiers for basis-dependent correlations, $D_I(\rho^{ab}, \Pi^a)$,

$D_{C_r}(\rho^{ab}, \Pi^a)$, and $D_{C_{l_1}}(\rho^{ab}, \Pi^a)$, we calculate them for the Werner states, isotropic states, Bell-diagonal states, and a family of classical-quantum states. Throughout this section, $\ln(x)$ represents the natural logarithm of x, and we take $\mathcal{H}^a = \mathcal{H}^b = \mathbb{C}^d$ and choose the computational basis $\{|j\rangle : j = 0, 1, \ldots, d-1\}$ for $\mathcal{H}^a$ as the reference basis, which will be denoted by Π^a as before. For brevity, we omit Π^a in the notations $D_I(\rho^{ab}, \Pi^a)$, $D_{C_r}(\rho^{ab}, \Pi^a)$, and $D_{C_{l_1}}(\rho^{ab}, \Pi^a)$ here.

Example 1. Consider the Werner state on $\mathbb{C}^d \otimes \mathbb{C}^d$ [45,110]

$$\mathbf{w} = \frac{d-x}{d^3-d}\mathbf{1}\otimes\mathbf{1} + \frac{dx-1}{d^3-d}F, \quad x \in [-1, 1],$$

where $\mathbf{1}$ is the identity operator on $\mathbb{C}^d$, and F is the swap operator on $\mathbb{C}^d \otimes \mathbb{C}^d$ defined as $F(|\phi\rangle \otimes |\psi\rangle) = |\psi\rangle \otimes |\phi\rangle,\ \forall\, |\phi\rangle, |\psi\rangle \in \mathbb{C}^d$. It can be directly derived that

$$D_I(\mathbf{w}) = \frac{d-1}{2}\sqrt{\frac{1-x^2}{(d^2-1)}} + \frac{d-x}{2(d+1)},$$

$$D_{C_{l_1}}(\mathbf{w}) = \frac{|dx-1|}{d+1}.$$

In Ref. [111] it was calculated that

$$D_{C_r}(\mathbf{w}) = \ln(d+1) + \frac{1+x}{2}\ln\frac{1+x}{2(d+1)} + \frac{1-x}{2}\ln\frac{1-x}{2(d-1)} - \frac{1+x}{d+1}\ln\frac{1+x}{2} - \frac{d-x}{d+1}\ln\frac{d-x}{2(d-1)}.$$

Example 2. Consider the isotropic state on $\mathbb{C}^d \otimes \mathbb{C}^d$ [45,110]

$$\tau = \frac{1-y}{d^2-1}\mathbf{1}\otimes\mathbf{1} + \frac{d^2y-1}{d^2-1}|\Psi\rangle\langle\Psi|, \quad y \in [0, 1],$$

where $|\Psi\rangle := 1/\sqrt{d}\sum_{j=0}^{d-1}|jj\rangle$ is the maximally entangled state. It can be directly calculated that

$$D_I(\tau) = \frac{d^2y-2y+1}{d(d+1)} - \frac{2}{d}\sqrt{\frac{(d-1)(1-y)y}{d+1}},$$

$$D_{C_{l_1}}(\tau) = \frac{|d^2y-1|}{d+1}.$$

In Ref. [111] it was calculated that

$$D_{C_r}(\tau) = y\ln y + \frac{1-y}{d+1}\ln\frac{1-y}{d^2-1} - \frac{dy+1}{d+1}\ln\frac{dy+1}{d(d+1)}.$$

Example 3. Consider the Bell-diagonal state on $\mathbb{C}^2 \otimes \mathbb{C}^2$ [52]

$$\rho_B = \lambda_0|\Phi^-\rangle\langle\Phi^-| + \lambda_1|\Psi^-\rangle\langle\Psi^-| + \lambda_2|\Psi^+\rangle\langle\Psi^+| + \lambda_3|\Phi^+\rangle\langle\Phi^+|,$$

where $|\Psi^\pm\rangle := (|00\rangle \pm |11\rangle)/\sqrt{2}$, $|\Phi^\pm\rangle := (|01\rangle \pm |10\rangle)/\sqrt{2}$. Then the three quantifiers of basis-dependent correlations relative to $\Pi^a = \{|0\rangle\langle 0|, |1\rangle\langle 1|\}$ can be calculated as follows:

$$D_I(\rho_B) = 1 - \frac{1}{4}(\sqrt{\lambda_0} + \sqrt{\lambda_1} + \sqrt{\lambda_2} + \sqrt{\lambda_3})^2 - \frac{1}{4}(-\sqrt{\lambda_0} + \sqrt{\lambda_1} + \sqrt{\lambda_2} - \sqrt{\lambda_3})^2,$$

$$\begin{aligned} D_{C_r}(\rho_B) = \frac{1}{4}(&(1-c_1-c_2-c_3)\ln(1-c_1-c_2-c_3) \\ &+ (1-c_1+c_2+c_3)\ln(1-c_1+c_2+c_3) \\ &+ (1+c_1-c_2+c_3)\ln(1+c_1-c_2+c_3) \\ &+ (1+c_1+c_2-c_3)\ln(1+c_1+c_2-c_3)) \\ &- \frac{1-|c_3|}{2}\ln(1-|c_3|) - \frac{1+|c_3|}{2}\ln(1+|c_3|), \end{aligned}$$

$$D_{C_{l_1}}(\rho_B) = |\lambda_2 - \lambda_1| + |\lambda_3 - \lambda_0|,$$

with $c_1 := -\lambda_0 - \lambda_1 + \lambda_2 + \lambda_3$, $c_2 := -\lambda_0 + \lambda_1 - \lambda_2 + \lambda_3$, and $c_3 := -\lambda_0 + \lambda_1 + \lambda_2 - \lambda_3$.

Example 4. Consider a family of classical-quantum states on $\mathbb{C}^2 \otimes \mathbb{C}^2$,

$$\eta = \left(\frac{1}{2} + s + t\right)|+\rangle\langle+| \otimes \rho_+^b + \left(\frac{1}{2} - s - t\right)|-\rangle\langle-| \otimes \rho_-^b,$$

where $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$ and

$$\rho_\pm^b = \frac{1 \pm 4s}{2 \pm 4(s+t)}|0\rangle\langle 0| + \frac{1 \pm 4t}{2 \pm 4(s+t)}|1\rangle\langle 1|,$$

with $-0.25 \leqslant s, t \leqslant 0.25$. η has the following matrix representation in the computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$,

$$\eta = \begin{pmatrix} \frac{1}{4} & 0 & s & 0 \\ 0 & \frac{1}{4} & 0 & t \\ s & 0 & \frac{1}{4} & 0 \\ 0 & t & 0 & \frac{1}{4} \end{pmatrix}.$$

The three quantifiers of basis-dependent correlations relative to $\Pi^a = \{|0\rangle\langle 0|, |1\rangle\langle 1|\}$ can be calculated as follows:

$$D_I(\eta) = \frac{1}{4}(2\sqrt{1-4(s+t)^2} - \sqrt{1-16s^2} - \sqrt{1-16t^2}),$$

$$D_{C_r}(\eta) = \ln 2 + H\left(\left\{\frac{1}{2} \pm (s+t)\right\}\right) - H\left(\left\{\frac{1}{4} \pm s, \frac{1}{4} \pm t\right\}\right),$$

$$D_{C_{l_1}}(\eta) = 2(|s| + |t| - |s+t|),$$

where $H(\{p_i\}) := -\sum_i p_i \ln p_i$ denotes the Shannon entropy of the probability distribution $\{p_i\}$.

We depict the graphs of $D_I(\mathbf{w})$, $D_{C_r}(\mathbf{w})$ and $D_{C_{l_1}}(\mathbf{w})$ versus the parameter $x \in [-1, 1]$ for $d = 2$ in Fig. 3(a), the graphs of $D_I(\tau)$, $D_{C_r}(\tau)$, and $D_{C_{l_1}}(\tau)$ versus the parameter $y \in [0, 1]$ for $d = 2$ in Fig. 3(b), the graphs of $D_I(\rho_B)$, $D_{C_r}(\rho_B)$, and $D_{C_{l_1}}(\rho_B)$ versus the parameter $c_3 \in [-1, 1]$ when $c_1 = c_2 = (1-c_3)/3$ in Fig. 3(c), and the graphs of $D_I(\eta)$, $D_{C_r}(\eta)$, and $D_{C_{l_1}}(\eta)$ versus the parameter $s \in [-0.25, 0.25]$ when $t = 0.25$ in Fig. 3(d). From these graphs we can see that the behaviors of $D_I(\rho^{ab}, \Pi^a)$, $D_{C_r}(\rho^{ab}, \Pi^a)$, and $D_{C_{l_1}}(\rho^{ab}, \Pi^a)$ with the varying parameters are similar in the four examples.

At last, we emphasize that as shown in Fig. 3(d), when $t = 0.25$, $0 \leqslant s < 0.25$,

$$D_I(\eta) > 0, \quad D_{C_r}(\eta) > 0, \quad D_{C_{l_1}}(\eta) = 0,$$

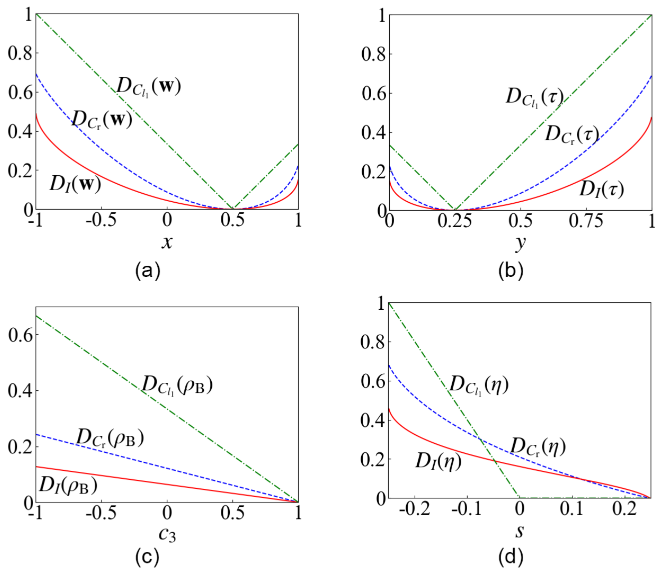

FIG. 3. Graphs of $D_I(\rho^{ab})$ (red solid line), $D_{C_r}(\rho^{ab})$ (blue dashed line), and $D_{C_{l_1}}(\rho^{ab})$ (green dashed-dotted line) versus the parameter $x \in [-1, 1]$ for the Werner states $\mathbf{w}$ when $d = 2$ (a), the parameter $y \in [0, 1]$ for the isotropic states τ when $d = 2$ (b), the parameter $c_3 \in [-1, 1]$ when $c_1 = c_2 = (1 - c_3)/3$ for the Bell-diagonal states ρ_B (c), and the parameter $s \in [-0.25, 0.25]$ when $t = 0.25$ for the classical-quantum states η (d).

which implies that in this case η is not a block-diagonal product state relative to Π^a, i.e., $\eta \notin S_{\Pi^a}$, while $D_{C_{l_1}}(\rho^{ab}, \Pi^a) = 0$. Hence, the set of block-diagonal product states relative to Π^a is strictly contained in the set $\{\rho^{ab} : D_{C_{l_1}}(\rho^{ab}, \Pi^a) = 0\}$.

VI. DISCUSSION

Given a local reference basis for a bipartite quantum system, it is natural to consider the coherence of both the global state and the local state relative to this basis. It is reasonable to expect that the coherence of the global state should be at least as much as that of the local state due to the correlations. We regard the difference in coherence between the global state and the local state as the basis-dependent correlations relative to the local basis. These basis-dependent correlations are what allow the coherence resource in the global state to exceed that in the local state. From an information-theoretic standpoint, a natural question is how to characterize and quantify these basis-dependent correlations.

Using the relative entropy of coherence, Yadin *et al.* have proven that only block-diagonal product states (relative to a local basis) have vanishing basis-dependent correlations (relative to this basis) [75]. In this work, we have characterized bipartite states with vanishing basis-dependent correlations using the coherence measure based on skew information and the l_1 norm of coherence. To be precise, in terms of the coherence measure based on skew information, we have proven that the set of block-diagonal product states relative to a given local basis can also be characterized by the property of possessing vanishing basis-dependent correlations, while in terms of the l_1 norm of coherence, to be a block-diagonal product state is a sufficient but not necessary condition for a bipartite state to have vanishing basis-dependent correlations.

Block-diagonal product states relative to a given local basis is an interpolation between product states and classical-quantum states. They have three information-theoretic characterizations: (1) They have vanishing basis-dependent discord. (2) They have vanishing basis-dependent correlations via the relative entropy of coherence. (3) They have vanishing basis-dependent correlations via the coherence measure based on skew information. Furthermore, they have an operational interpretation in an interference model as described in Sec. IV.

We emphasize that block-diagonal product states depend on the choice of local bases, distinguishing them from other bipartite states such as product states, separable states, and classical-quantum states, all of which are basis-independent. It is desirable to explore the connections of block-diagonal product states with reference frames [95], pointer states [46], and the no-local-broadcasting theorem [98,112–114]. In addition, a natural question arises and deserves investigation: Can states outside the set of block-diagonal states be customized for specific tasks as resources that cannot be accomplished using block-diagonal product states?

Finally, similar to the block-diagonal product states that emerge from studying basis-dependent correlations relative to a local basis (or, equivalently, a local von Neumann measurement), much richer structures might be revealed when analyzing correlations relative to a local Lüders measurement or even a local quantum channel. This is worthy of further investigation.

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APPENDIX A: PROOF OF PROPOSITION 1

For sufficiency, suppose an ensemble $\{(\lambda_k, \rho_k) : k = 1, \dots, n\}$ satisfies that

$$\rho_k = \bigoplus_{l=1}^{m} p_{k,l}\sigma_l, \quad k = 1, 2, \dots, n,$$

where $\{p_{k,l} : l = 1, 2, \dots, m\}$ is a probability distribution for each k, $\sigma_l \in S(P_l\mathcal{H})$ and $\Pi_L = \{P_l = \sum_{j\in I_l} \Pi_j : l = 1, 2, \dots, m\}$ is a coarse graining of Π. From the property of skew information [96], we have

$$\begin{aligned} I\left(\sum_k \lambda_k\rho_k, \Pi\right) &= I\left(\bigoplus_l \left(\sum_k \lambda_k p_{k,l}\right)\sigma_l, \Pi\right) \\ &= \sum_l \left(\sum_k \lambda_k p_{k,l}\right) \sum_{j\in I_l} I(\sigma_l, \Pi_j) \\ &= \sum_k \lambda_k \sum_l p_{k,l} \sum_{j\in I_l} I(\sigma_l, \Pi_j) \end{aligned}$$

$$= \sum_k \lambda_k I\Big(\bigoplus_l p_{k,l}\sigma_l, \Pi\Big)$$
$$= \sum_k \lambda_k I(\rho_k, \Pi),$$

from which the sufficiency follows.

For necessity, suppose an ensemble $\{(\lambda_k, \rho_k) : k = 1, 2, \ldots, n\}$ satisfies Eq. (25). Let $\rho := \sum_{k=1}^n \lambda_k \rho_k$. For each j, by the convexity of $I(\rho, \Pi_j)$ in ρ [115], we have that

$$I(\rho, \Pi_j) \leqslant \lambda_1 I(\rho_1, \Pi_j) + (1-\lambda_1) I(\rho_2', \Pi_j) \leqslant \sum_k \lambda_k I(\rho_k, \Pi_j), \tag{A1}$$

where $\rho_2' = \sum_{k=2}^n \lambda_k \rho_k/(1-\lambda_1)$. From Eqs. (25) and (A1), we have

$$I(\rho, \Pi_j) = \lambda_1 I(\rho_1, \Pi_j) + (1-\lambda_1) I(\rho_2', \Pi_j), \tag{A2}$$

which together with $I(\rho, \Pi_j) = \langle j|\rho|j\rangle - \langle j|\sqrt{\rho}|j\rangle^2$ implies that

$$\langle j|\sqrt{\rho}|j\rangle^2 = \langle j|\sqrt{\lambda_1\rho_1}|j\rangle^2 + \langle j|\sqrt{(1-\lambda_1)\rho_2'}|j\rangle^2,$$

or, equivalently,

$$\langle jj|\sqrt{\rho}\otimes\sqrt{\rho}|jj\rangle$$
$$= \langle jj|(\lambda_1\sqrt{\rho_1}\otimes\sqrt{\rho_1} + (1-\lambda_1)\sqrt{\rho_2'}\otimes\sqrt{\rho_2'})|jj\rangle.$$

Let

$$F := \sqrt{\rho}\otimes\sqrt{\rho} - \Big(\lambda_1\sqrt{\rho_1}\otimes\sqrt{\rho_1} + (1-\lambda_1)\sqrt{\rho_2'}\otimes\sqrt{\rho_2'}\Big).$$

By Ando's concavity theorem [116], we know that $F \geqslant 0$. Therefore, for any j,

$$\langle jj|F|jj\rangle = 0,$$

which implies that $\sqrt{F}|jj\rangle = 0$ and thus $F|jj\rangle = 0$, i.e.,

$$\sqrt{\rho}|j\rangle\otimes\sqrt{\rho}|j\rangle = \lambda_1\sqrt{\rho_1}|j\rangle\otimes\sqrt{\rho_1}|j\rangle + (1-\lambda_1)\sqrt{\rho_2'}|j\rangle\otimes\sqrt{\rho_2'}|j\rangle.$$

Since the symmetry rank of the left hand side is 1, by Lemma 5.1 of Ref. [117], we know that for some $b_{1,j}$,

$$\sqrt{\rho_1}|j\rangle = b_{1,j}\sqrt{\rho}|j\rangle,$$

and $b_{1,j} \geqslant 0$ follows from the non-negativity of $\sqrt{\rho}$ and $\sqrt{\rho_1}$. Similarly, for $k > 1$, there exists some $b_{k,j} \geqslant 0$ such that

$$\sqrt{\rho_k}|j\rangle = b_{k,j}\sqrt{\rho}|j\rangle.$$

Let

$$\rho = \bigoplus_{l=1}^m p_l\sigma_l$$

be the minimal direct-sum decomposition of ρ with respect to Π with $p_l = \mathrm{tr}P_l\rho$, $\sigma_l = P_l\rho P_l/p_l$, $\Pi_{\mathrm{L}} = \{P_l = \sum_{j\in I_l}\Pi_j : l = 1, 2, \ldots, m\}$ the corresponding coarse graining of Π. Here a direct-sum decomposition of ρ with respect to Π means that there exists a coarse graining $\Pi_{\mathrm{L}} = \{P_l = \sum_{j\in I_l}\Pi_j : l = 1, 2, \ldots, m\}$ of Π such that

$$\rho = \sum_l P_l\rho P_l.$$

Furthermore, if the coarse graining Π_{L} is the most refined, we call it the minimal direct-sum decomposition of ρ with respect to Π. Then, for $i \in I_l$, $j \in I_{l'}$, $l \neq l'$,

$$\langle i|\sqrt{\rho_k}|j\rangle = b_{k,j}\langle i|\sqrt{\rho}|j\rangle = 0.$$

Thus, for each k, $\sqrt{\rho_k}$ can be represented as

$$\sqrt{\rho_k} = \sum_{l=1}^m P_l\sqrt{\rho_k}P_l.$$

Next, we prove that $P_l\sqrt{\rho_k}P_l$ is equivalent to $\sqrt{\sigma_l}$ up to a constant. Consider the nontrivial case that I_l contains at least two elements. Let $i, j \in I_l$, $i \neq j$. If $\langle i|\sqrt{\rho}|j\rangle \neq 0$,

$$b_{k,j}\langle i|\sqrt{\rho}|j\rangle = \langle i|\sqrt{\rho_k}|j\rangle = \overline{\langle j|\sqrt{\rho_k}|i\rangle} = \overline{b_{k,i}\langle j|\sqrt{\rho}|i\rangle}$$

implies $b_{k,i} = b_{k,j}$. If $\langle i|\sqrt{\rho}|j\rangle = 0$, since $\rho = \bigoplus_l p_l\sigma_l$ is the minimal direct-sum decomposition of ρ with respect to Π, there exist $i_1, \ldots, i_s \in I_l$ such that

$$\langle i|\sqrt{\rho}|i_1\rangle\langle i_1|\sqrt{\rho}|i_2\rangle\cdots\langle i_s|\sqrt{\rho}|j\rangle \neq 0.$$

So each item in the left hand side of the above equation is not 0, which implies that

$$b_{k,i} = b_{k,i_1} = \cdots = b_{k,i_s} = b_{k,j}.$$

Therefore, for any $i, j \in I_l$, $b_{k,i} = b_{k,j}$. Then, letting $c_{k,l} = b_{k,i}$ for $i \in I_l$, we have

$$\sqrt{\rho_k} = \bigoplus_{l=1}^m c_{k,l}\sqrt{\sigma_l},$$

which implies that

$$\rho_k = \bigoplus_{l=1}^m p_{k,l}\sigma_l, \quad k = 1, \ldots, n,$$

where $\{p_{k,l} = (c_{k,l})^2 : l = 1, 2, \ldots, m\}$ constitutes a probabilistic distribution. This completes the proof of Proposition 1.

We remark that from the proof of necessity, it follows that σ_l in Eq. (26) of Proposition 2 can be determined by the minimal direct-sum decomposition of the average state $\rho = \sum_k \lambda_k\rho_k$ with respect to Π.

APPENDIX B: PROOF OF PROPOSITION 2

For sufficiency, for a local von Neumann measurement $\Pi^a = \{\Pi_j^a = |j\rangle\langle j| : j = 1, 2, \ldots, d_a\}$, let

$$\rho^{ab} = \bigoplus_{l=1}^m p_l\rho_l^a\otimes\rho_l^b$$

be a block-diagonal product state with respect to Π^a. Then

$$\begin{aligned}
I(\rho^{ab}, \Pi^a \otimes \mathbf{1}^b) - I(\rho^a, \Pi^a) \\
&= \sum_j \mathrm{tr}\left(\bigoplus_l \sqrt{p_l \rho_l^a}\Pi_j^a\right)^2 \\
&\quad - \sum_j \mathrm{tr}\left(\bigoplus_l \sqrt{p_l \rho_l^a}\Pi_j^a \otimes \sqrt{\rho_l^b}\right)^2 \\
&= 0,
\end{aligned}$$

which implies the sufficiency.

For necessity, suppose a bipartite state ρ^{ab} satisfies

$$I(\rho^{ab}, \Pi^a \otimes \mathbf{1}^b) - I(\rho^a, \Pi^a) = 0.$$

Following the method in Ref. [57], we expand $\sqrt{\rho^{ab}}$ as

$$\sqrt{\rho^{ab}} = \sum_{k=1}^{d_b^2} A_k \otimes Y_k,$$

where A_k are observables on $\mathcal{H}^a$ and $\{Y_k : k = 1, 2, \ldots, d_b^2\}$ constitutes an orthonormal basis for the Hilbert space of all adjoint operators on d_b-dimensional Hilbert space $\mathcal{H}^b$ with the Hilbert-Schmidt inner product $\langle X|Y\rangle = \mathrm{tr}(X^\dagger Y)$ for operators X and Y. Then

$$\rho^{ab} = \sum_{k,k'} A_k A_{k'} \otimes Y_k Y_{k'}, \quad \rho^a = \mathrm{tr}_b \rho^{ab} = \sum_k A_k^2.$$

Now let X be any observable on $\mathcal{H}^a$, then

$$\begin{aligned}
I(\rho^{ab}, X \otimes \mathbf{1}^b) &= -\frac{1}{2}\mathrm{tr}\left[\sum_k A_k \otimes Y_k, X \otimes \mathbf{1}^b\right]^2 \\
&= -\frac{1}{2}\sum_k \mathrm{tr}[A_k, X]^2 \\
&= \mathrm{tr}\rho^a X^2 - \sum_k \mathrm{tr}(A_k X A_k X)
\end{aligned}$$

and

$$I(\rho^a, X) = \mathrm{tr}\rho^a X^2 - \mathrm{tr}\left(\sqrt{\sum_k A_k^2} X \sqrt{\sum_k A_k^2} X\right).$$

Hence for any j, $I(\rho^{ab}, \Pi_j^a \otimes \mathbf{1}^b) = I(\rho^a, \Pi_j^a)$ implies that

$$\mathrm{tr}\left(\sqrt{\sum_k A_k^2 \Pi_j^a}\sqrt{\sum_k A_k^2 \Pi_j^a}\right) = \sum_k \mathrm{tr}\big(A_k \Pi_j^a A_k \Pi_j^a\big). \tag{B1}$$

Note that

$$\begin{aligned}
\mathrm{tr}\left(\sqrt{\sum_k A_k^2 \Pi_j^a}\sqrt{\sum_k A_k^2 \Pi_j^a}\right) &\geqslant \sum_k \mathrm{tr}\big(|A_k|\Pi_j^a|A_k|\Pi_j^a\big) \\
&\geqslant \sum_k \mathrm{tr}\big(A_k \Pi_j^a A_k \Pi_j^a\big).
\end{aligned} \tag{B2}$$

The first inequality follows from Lieb's concavity theorem [115], and the second one can be directly obtained from Jensen's inequality [101].

From Eqs. (B1) and (B2), we get that

$$\mathrm{tr}\left(\sqrt{\sum_k A_k^2 \Pi_j^a}\sqrt{\sum_k A_k^2 \Pi_j^a}\right) = \sum_k \mathrm{tr}\big(|A_k|\Pi_j^a|A_k|\Pi_j^a\big).$$

Let $\lambda_k := \mathrm{tr}A_k^2$, then $\rho_k = A_k^2/\lambda_k$ for $\lambda_k \neq 0$, and $\rho^a = \sum_k \lambda_k \rho_k$. By Proposition 1, we know that for each k,

$$A_k^2 = \bigoplus_l \lambda_k p_{k,l}\sigma_l, \tag{B3}$$

where $\{p_{k,l}\}$ is a probability distribution and σ_l are determined by the minimal direct-sum decomposition of ρ^a with respect to Π^a denoted by

$$\rho^a = \bigoplus_l p_l \sigma_l. \tag{B4}$$

On the other hand, from Eqs. (B1) and (B2), we have for each k,

$$\mathrm{tr}\big(|A_k|\Pi_j^a|A_k|\Pi_j^a\big) = \mathrm{tr}\big(A_k \Pi_j^a A_k \Pi_j^a\big).$$

Since each observable A_k can be decomposed as $A_k = A_{k,+} - A_{k,-}$ with $A_{k,+} > 0$ and $A_{k,-} > 0$, we can directly obtain

$$\langle j|A_{k,+}|j\rangle\langle j|A_{k,-}|j\rangle = 0,$$

which implies that $\langle j|A_{k,+}|j\rangle = 0$ or $\langle j|A_{k,-}|j\rangle = 0$. Let $I := \{1, 2, \ldots, d_a\}$,

$$I_{k,+} := \{j : \langle j|A_{k,+}|j\rangle \neq 0\}, \quad I_{k,+}^c := I \setminus I_{k,+}$$

and

$$P_{k,+} := \sum_{j\in I_{k,+}} \Pi_j^a, \quad P_{k,-} := \sum_{j\in I_{k,+}^c} \Pi_j^a.$$

Then it can be directly verified that

$$A_k^2 = P_{k,+}A_k^2 P_{k,+} + P_{k,-}A_k^2 P_{k,-}. \tag{B5}$$

Substituting Eq. (B3) into Eq. (B5), we have

$$\bigoplus_l \lambda_k p_{k,l}\sigma_l = \bigoplus_l \lambda_k p_{k,l}(P_{k,+}\sigma_l P_{k,+} + P_{k,-}\sigma_l P_{k,-}). \tag{B6}$$

Since $\bigoplus_l \lambda_k p_{k,l}\sigma_l$ is the minimal direct-sum decomposition of ρ^a with respect to Π^a, it holds that for each l such that $\lambda_k p_{k,l} > 0$,

$$\sigma_l = P_{k,+}\sigma_l P_{k,+} \quad \text{or} \quad \sigma_l = P_{k,-}\sigma_l P_{k,-}.$$

Therefore, A_k can be written as

$$A_k = \bigoplus_l a_{k,l}\sqrt{\sigma_l},$$

where $a_{k,l} = \sqrt{\lambda_k p_{k,l}}$ if $\sigma_l = P_{k,+}\sigma_l P_{k,+}$, and $a_{k,l} = -\sqrt{\lambda_k p_{k,l}}$ if $\sigma_l = P_{k,-}\sigma_l P_{k,-}$. Thus,

$$\sqrt{\rho^{ab}} = \sum_k \bigoplus_l a_{k,l}\sqrt{\sigma_l} \otimes Y_k = \bigoplus_l \sqrt{\sigma_l} \otimes \sum_k a_{k,l} Y_k$$

and

$$\rho^{ab} = \bigoplus_l \sigma_l \otimes \left(\sum_k a_{k,l} Y_k\right)^2.$$

Then from Eq. (B4) we have $p_l = \mathrm{tr}(\sum_k a_{k,l}Y_k)^2$. Let $\rho_l^b := (\sum_k a_{k,l}Y_k)^2/p_l$ and $\rho_l^a := \sigma_l$. Then $\rho^{ab} = \bigoplus_l p_l\rho_l^a \otimes \rho_l^b$ is a block-diagonal product state relative to Π^a, and thus the necessity is obtained.

APPENDIX C: PROOF OF PROPOSITION 3

For sufficiency, if ρ^{ab} is a block-diagonal product state relative to both Π^a and Π^b as described by Eq. (29), then

$$\rho^{ab} = \bigoplus_{l=1}^{m} \rho_l^a \otimes \left(\bigoplus_{s=1}^{n} p_{l,s}\rho_s^b\right) = \bigoplus_{s=1}^{n}\left(\bigoplus_{l=1}^{m} p_{l,s}\rho_l^a\right)\otimes \rho_s^b,$$

which implies that ρ^{ab} satisfies Eq. (30) from Proposition 2.

For necessity, suppose ρ^{ab} satisfies Eq. (30), then from Proposition 2, there exist a coarse graining $\Pi_{\mathrm{L}}^a = \{P_l^a = \sum_{j\in I_l^a}\Pi_j^a : l = 1,2,\ldots,m\}$ of $\Pi^a = \{\Pi_j^a = |j\rangle\langle j| : j = 1,2,\ldots,d_a\}$ and a coarse graining $\Pi_{\mathrm{L}}^b = \{P_s^b = \sum_{k\in I_s^b}\Pi_k^b : s = 1,2,\ldots,n\}$ of $\Pi^b = \{\Pi_k^b = |k\rangle\langle k| : k = 1,2,\ldots,d_b\}$ such that ρ^{ab} can be represented as the following two forms:

$$\rho^{ab} = \bigoplus_{l=1}^{m} p_l\rho_l^a \otimes \sigma_l^b = \bigoplus_{s=1}^{n} q_s\gamma_s^a \otimes \rho_s^b, \tag{C1}$$

where $\rho_l^a \in S(P_l^a\mathcal{H}^a)$, $\sigma_l^b \in S(\mathcal{H}^b)$, $\gamma_s^a \in S(\mathcal{H}^a)$, $\rho_s^b \in S(P_s^b\mathcal{H}^b)$, and $\{p_l\}$ and $\{q_s\}$ are two probability distributions. Hence,

$$\begin{aligned}\rho^{ab} &= \sum_{l=1}^{m}\sum_{s=1}^{n}\left(P_l^a\otimes P_s^b\right)\rho^{ab}\left(P_l^a\otimes P_s^b\right)\\ &= \sum_{s=1}^{n}\bigoplus_{l=1}^{m} p_l\rho_l^a\otimes P_s^b\sigma_l^bP_s^b\\ &= \sum_{l=1}^{m}\bigoplus_{s=1}^{n} q_sP_l^a\gamma_s^aP_l^a\otimes\rho_s^b,\end{aligned}$$

which implies that for any l, s,

$$p_l\rho_l^a\otimes P_s^b\sigma_l^bP_s^b = q_sP_l^a\gamma_s^aP_l^a\otimes\rho_s^b.$$

Let $p_{l,s} = p_l\mathrm{tr}(P_s^b\sigma_l^b) = q_s\mathrm{tr}(P_l^a\gamma_s^a)$, then

$$\rho^{ab} = \bigoplus_{l=1}^{m}\bigoplus_{s=1}^{n} p_{l,s}\rho_l^a\otimes\rho_s^b,$$

which completes the proof of the necessity.

APPENDIX D: PROOF OF PROPOSITION 4

We first expand ρ^{ab} as

$$\rho^{ab} = \sum_{j,j'}|j\rangle\langle j'|\otimes\langle j|\rho^{ab}|j'\rangle.$$

From the definition of the l_1 norm of coherence $C_{l_1}(\rho,\Pi_L)$, we have

$$\begin{aligned}&C_{l_1}(\rho^{ab},\Pi^a\otimes\mathbf{1}^b)\\ &\quad=\sum_{j\neq j'}||\Pi_j^a\otimes\mathbf{1}^b\rho^{ab}\Pi_{j'}^a\otimes\mathbf{1}^b||_{\mathrm{tr}}\\ &\quad=\sum_{j\neq j'}\mathrm{tr}\sqrt{\Pi_j^a\otimes\mathbf{1}^b\rho^{ab}\Pi_{j'}^a\otimes\mathbf{1}^b\rho^{ab}\Pi_j^a\otimes\mathbf{1}^b}\\ &\quad=\sum_{j\neq j'}\mathrm{tr}\sqrt{\Pi_j^a\otimes\langle j|\rho^{ab}|j'\rangle\langle j'|\rho^{ab}|j\rangle}\\ &\quad=\sum_{j\neq j'}\mathrm{tr}|\langle j|\rho^{ab}|j'\rangle|\end{aligned}$$

and

$$C_{l_1}(\rho^a,\Pi^a) = \sum_{j\neq j'}|\langle j|\rho^a|j'\rangle| = \sum_{j\neq j'}|\mathrm{tr}\langle j|\rho^{ab}|j'\rangle|.$$

Let $A_{jj'} := \langle j|\rho^{ab}|j'\rangle$ for any j, j', then

$$\begin{aligned}D_{C_{l_1}}(\rho^{ab},\Pi^a) &= C_{l_1}(\rho^{ab},\Pi^a\otimes\mathbf{1}^b) - C_{l_1}(\rho^a,\Pi^a)\\ &= \sum_{j\neq j'}(\mathrm{tr}|\langle j|\rho^{ab}|j'\rangle| - |\mathrm{tr}\langle j|\rho^{ab}|j'\rangle|)\\ &= \sum_{j\neq j'}(\mathrm{tr}|A_{jj'}| - |\mathrm{tr}A_{jj'}|)\\ &\geqslant 0.\end{aligned}$$

By the polar decomposition $A_{jj'} = U_{jj'}|A_{jj'}|$ for some unitary operators $U_{jj'}$ and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}|\mathrm{tr}A_{jj'}| = |\mathrm{tr}(U_{jj'}|A_{jj'}|)| &= |\mathrm{tr}(U_{jj'}|A_{jj'}|^{1/2}|A_{jj'}|^{1/2})|\\ &\leqslant \mathrm{tr}|A_{jj'}|,\end{aligned}$$

and the equality holds if and only if for some $\varphi_{jj'}\in[0,2\pi)$,

$$U_{jj'}|A_{jj'}|^{1/2} = e^{i\varphi_{jj'}}|A_{jj'}|^{1/2},$$

which is equivalent to

$$U_{jj'}|A_{jj'}| = e^{i\varphi_{jj'}}|A_{jj'}|.$$

Thus, $D_{C_{l_1}}(\rho^{ab},\Pi^a) = 0$ if and only if for any j and j',

$$\langle j|\rho^{ab}|j'\rangle = e^{i\varphi_{jj'}}|\langle j|\rho^{ab}|j'\rangle|$$

with $\varphi_{jj'}\in[0,2\pi)$. This completes the proof of Proposition 4.

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