Multimode rotation-symmetric bosonic codes from homological rotor codes

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We develop quantum information processing primitives for the planar rotor, the state space of a particle on a circle. The n-rotor Clifford group, $\mathrm{U}(1)^{n(n+1)/2} \rtimes \mathrm{GL}_n(\mathbb{Z})$, is represented by continuous U(1) gates generated by polynomials quadratic in angular momenta, as well as discrete $\mathrm{GL}_n(\mathbb{Z})$ gates generated by momentum sign-flip and sum gates. Our understanding of this group allows us to establish connections between homological rotor error-correcting codes [Vuillot, Ciani, and Terhal, Commun. Math. Phys. **405**, 53 (2024)] and oscillator quantum codes, including Gottesman-Kitaev-Preskill codes and rotation-symmetric bosonic codes. Inspired by homological rotor codes, we provide a systematic construction of multimode rotation-symmetric bosonic codes by making a parallel between oscillator Fock states and rotor states with fixed non-negative angular momentum. This family of homological number-phase codes protects against dephasing and changes in occupation number. Encoding and decoding circuits for these codes can be derived from the corresponding rotor Clifford operations. As a result of independent interest, we show how to nondestructively measure the oscillator phase using conditional occupation-number addition and postselection. We also outline several rotor and oscillator varieties of the Gottesman-Kitaev-Preskill-stabilizer codes [Phys. Rev. Lett. **125**, 080503 (2020).].

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I. INTRODUCTION

Quantum information processing can be done in various quantum platforms, which can be described in the abstract by one of only a few types of state spaces. For example, the qudit state space describes any few-level physical system, irrespective of the physical nature of the levels. Similarly, a continuous-variable harmonic oscillator state space models vibrations in ions and materials, as well as photons confined to cavities or traveling through various media.

Conventional quantum state spaces come with a set of primitives—canonical states and operations—that are both physically motivated and essential for information processing schemes. For the case of a qudit, canonical operations come from the Pauli group as well as its normalizer, the Clifford group; canonical states are eigenstates of the Pauli operators [1,2]. For the case of the oscillator, the operations are the oscillator displacements and more general quadratic Gaussian (also known as Bogoliubov) operations, while the canonical states are states of fixed position or momentum [3,4].

While few-level systems and harmonic oscillators have been well studied for over 100 years, a third "angular" state space—the planar or U(1) rotor [5–17]—has lagged in its development. A planar rotor describes the state space of a quantum system confined to a circle. Such systems have been overlooked in the past due to a lack of controllable quantum platforms amenable to a rotor description. However, rotor systems are gaining traction due to recent improvements in the control of superconducting circuits [18,19], ultracold molecules [20,21], ion traps [22,23], free electrons [24,25], and orbital angular-momentum systems [13,26–29]. The recent exciting experimental progress warrants a deeper quantitative investigation into the rotor's information processing primitives. We perform this investigation in this work, to obtain a firmer understanding of rotor states and operations. We also obtain several immediate applications for quantum error correction of both oscillator and rotor platforms.

We enumerate canonical primitives of the planar rotor and, in particular, determine the group formed by its canonical unitary operations. While the types of primitive rotor operations have long been known [30–33], the specific group formed by them is, to our knowledge, not yet established. We do so by treating the rotor—whose configuration space is described by an angle—as a subspace of the harmonic oscillator that is defined by periodically identifying oscillator positions. This embedding of the rotor into the oscillator allows us to view rotor primitives as oscillator primitives that preserve the embedded rotor subspace. Our embedded rotor treatment also provides insight into quantum error correction, yielding a rotor version of a recent class of oscillator error-correcting codes [34] and revealing several connections between oscillator and rotor codes.

In parallel, we focus on the oscillator itself, developing its "polar" coordinates in terms of occupation number and phase degrees of freedom [5,35,36] that are analogues of the angular momentum and phase degrees of freedom on planar rotor. We show that most of the properties of the planar rotor can be

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TABLE I. Canonical groups for oscillator, rotor, and qubit systems. The displacement group shifts the canonical variables of each system, while the symplectic group preserves the displacement group. The passive symplectic group of the oscillator is the symmetry group of the complex sphere formed by n-mode coherent states of the same energy, while its rotor and qubit counterparts are intersections of this group with the respective rotor and qubit symplectic groups.

	Oscillators [4,42]	Planar rotors (this work)	Qubits [1]
Displacement group	$H_3(\mathbb{R})^n$ (Heisenberg-Weyl)	$\mathcal{P}_n^{\text{rot}}$ (rotor Pauli [113])	$\mathcal{P}_n$ (Pauli)
Symplectic group	$\text{Sp}_{2n}(\mathbb{R})$	$U(1)^{n(n+1)/2} \rtimes \text{GL}_n(\mathbb{Z})$ (rotor Clifford)	$\text{Sp}_{2n}(\mathbb{Z}_2)$ (Clifford)
Passive symplectic group	$U(n) = \text{Sp}_{2n}(\mathbb{R}) \cap \text{SO}(2n)$	$\mathbb{Z}_2 \wr S_n$	S_n

transferred into oscillator Fock space, for example the symplectic representation of Clifford operations. Although some of the unitary Clifford operations on the rotor side become channels on the oscillator side, they are still valid and useful quantum operations.

Our development of number-phase primitives allows us to map several classes of error-correcting codes, including the recently developed homological rotor codes [37], into the codes defined in oscillator Fock basis. These new oscillator codes are polar analogues of lattice codes [34,38,39] and are compatible with oscillator noise channels where photon loss is present but random-rotation (i.e., dephasing) noise is dominant. As such, we anticipate that these codes will be relevant to trapped-ion systems [23,40].

A. Summary of results

Positions of a planar rotor are labeled by an angle, which makes for a compact configuration space. The dual basis of momentum states is labeled by the integers $\mathbb{Z}$, making for a discrete and infinite set of labels called the circle group $\mathbb{T} \cong U(1)$. A planar rotor can thus be thought of as being "in between" the qudit and oscillator, encapsulating both the compactness of the former and the infinite-dimensional nature of the latter.

Rotor Clifford group. The n-qubit Clifford group consists of all unitary operations that preserve Pauli-matrix commutation relations, and we exclude the Pauli group from this definition for simplicity [1]. Similarly, the analogous n-oscillator group of Gaussian transformations consists of all operations that preserve the commutation relations between position and momentum [4,41,42]. In both cases, the commutation-preservation can be tied to preservation of a particular symplectic form, and the two groups correspond to the symplectic groups $\text{Sp}_{2n}(\mathbb{Z}_2)$ and $\text{Sp}_{2n}(\mathbb{R})$, respectively.

The rotor Clifford group is the group of unitary operations that preserve commutation relations between rotor position and momentum shift Pauli-type operators. It has been studied before in the context of efficient simulation [31], and its generators have been detailed before [30,31,33,43]. However, the *structure* of this group has, to our knowledge, not yet been identified. This is, in part, because a rotor's angular positions and momenta are labeled by *different* types of numbers—angles vs integers—which complicates analogous symplectic formulations.

By periodically extending planar rotor wave functions such that they form a subset of oscillator wave functions, we observe that the rotor Clifford group can be thought of as a subgroup of the oscillator symplectic group, $\text{Sp}_{2n}(\mathbb{R})$. Projecting this group into the state subspace of the rotor, we find the n-rotor Clifford group to be a semi-direct product of two groups—the Lie group $U(1)^{n(n+1)/2}$ and the discrete group $\text{GL}_n(\mathbb{Z})$ of unimodular integer-valued invertible matrices,

$$\text{Rotor Clifford group} = U(1)^{n(n+1)/2} \rtimes \text{GL}_n(\mathbb{Z}). \quad (1)$$

The Lie group corresponds to gates generated by products of two rotor momenta, while the discrete group is generated by conditional momentum shifts and momentum sign flips. We summarize our results in Table I.

Classifying homological rotor codes. Equipped with better understanding of the rotor Clifford group and the embedded rotor construction, we investigate the structure of the homological rotor codes [37]—a recent extension of stabilizer codes [44] to rotors.

We classify homological rotor codes using the Smith normal form—the standard tool for homology-group calculation. We show that the Smith normal form of a code cannot be changed to that of another code by any rotor Clifford operation that preserves the code's Calderbank-Steane-Shor (CSS) structure. The different Smith normal forms thus label different equivalence classes of homological rotor codes under such operations. If Clifford operations are the only available operations for an encoding, this implies that such operations have to act on a resource state within the same Smith class. These resource states are tightly related to oscillator Gottesman-Kitaev-Preskill (GKP) states [38].

Homological number-phase codes. Returning to the harmonic oscillator, we study multimode extensions of the number-phase codes [45–47]—polar analogues of bosonic lattice codes that protect against occupation-number loss or gain and dephasing, which correspond to distortions in the oscillator's number and phase degrees of freedom, respectively. The error-correction scheme and performance of random rotation-symmetric codes are recently studied in Ref. [48,49].

Bosonic rotation codes are defined for a single oscillator, and multimode extensions have not been substantially studied [50]. We show how to map the entire class of homological rotor codes into the oscillator, yielding a new class of polar-like codes protecting against loss and dephasing noise.

We also show that the rotor Clifford-group encodings of homological rotor codes can be performed by analogous operations in the number-phase picture, but with some caveats. Mapping rotor Clifford-group transformation into the number-phase interpretation of the oscillator yields a Clifford *semigroup* consisting of some nonunitary transformations. For

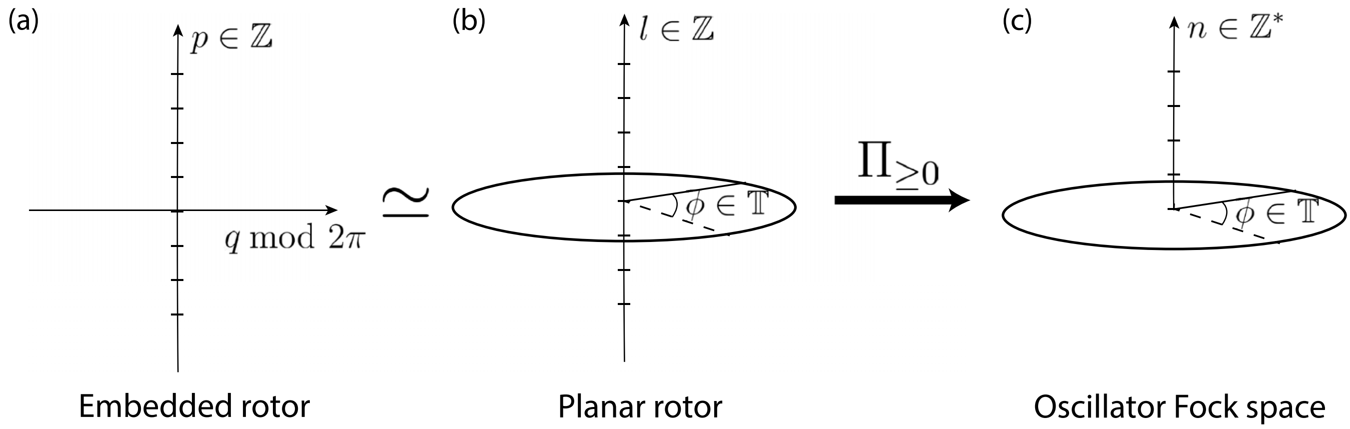

FIG. 1. (b) The planar rotor describes the motion of a particle on a circle $\mathbb{T}$, with said particle admitting only integer-valued angular momenta. This phase space can be embedded into that of the harmonic oscillator by periodically identifying oscillator positions q, yielding (a) the embedded rotor. Alternatively, interpreting oscillator Fock states as the positive-momentum states of a rotor yields (c) the number-phase interpretation of the oscillator. Relations between these constructions allow us to identify primitives necessary for information processing as well as develop various error-correcting codes for all three state spaces.

example, while quadratic momentum gates are mapped to a Kerr interaction, conditional momentum shifts are mapped to conditional photon injection—a nonunitary operation.

Nevertheless, all Clifford semigroup operations are valid quantum channels, and the original rotor Clifford algebra is mostly left intact after the number-phase mapping. As a result, a Clifford-based encoding of homological codes is still possible.

On the other hand, extraction of error syndromes for number-phase codes becomes more difficult. Nevertheless, we show it is possible to measure the effect of an oscillator rotation *nondestructively* with the help of postselection. This provides a probabilistic error recovery alternative to Knill telecorrection for bosonic rotation codes [51] and may be relevant to metrological protocols for determining the oscillator phase in a nondesctructive fashion.

Other new codes and relations. The relation between planar rotors, their embeddings into the oscillator, and the number-phase interpretation of the oscillator (see Fig. 1). allows us to treat various seemingly unrelated error-correcting codes in the same fashion.

We outline how GKP-stabilizer codes [34] can be mapped into the number-phase degrees of freedom of the oscillator, yielding another class of codes protecting a (possibly infinite) logical state space against loss and dephasing noise.

The embedded rotor construction enables us to investigate the underlying connections between rotor and oscillator codes. In particular, for an embedded rotor, a single homological rotor code with torsion is equivalent to an oscillator GKP code encoding a qudit. Moreover, rotor GKP codes [33,38,52] can be included in the same framework as a concatenation of homological rotor codes and modular-qudit GKP codes [38], Sec. II] (see Example 1). While homological number-phase codes are multimode generalizations of the original number-phase codes [45], they can also be viewed as a rotation-symmetric generalization of multimode GKP codes. These relations are illustrated in Fig. 2.

B. Outline of the manuscript

In Sec. II, we introduce the generalized Pauli operators for planar rotors. We describe a method to embed a logical planar rotor into a single-mode harmonic oscillator, as well as the number-phase interpretation of the oscillator.

Next, we identify its group structure in Sec. III A and further investigate the generators of rotor Clifford group and their symplectic representation in Sec. III B. The rotor Clifford group forms the encoding unitaries of rotor codes and motivates us to investigate the classification of rotor codes.

We then revisit the formalism of homological rotor codes and investigate the physical implications of torsion and Smith normal form by calculating the codes' homology group in Sec. IV A. In Sec. IV B, we show that the codes with different torsion parts from different equivalent classes which cannot be related by CSS Clifford transformations. In Sec. IV C, we show that rotor GKP codes are concatenations between homological rotor codes and modular-qudit codes.

In Sec. V, starting from an Example 3 that the codewords of number-phase codes are rotor GKP codes after projecting on the non-negative angular-momentum subspace and analogizing the rotor angular-momentum and oscillator Fock basis, we propose the homological number-phase codes—a multimode generalization of rotation-symmetric bosonic codes called as number-phase codes which is inspired by the Clifford-deformed homological rotor codes. In Proposition 1, we provide a procedure of mapping homological rotor codes to homological number-phase codes. In Sec. V A, we demonstrate how to use the Clifford semigroup of number-phase operations to encode in homological number-phase codes.

In Sec. VI, we show that GKP-stabilizer codes can be generalized to U(1) rotor systems as well as oscillator Fock space and compare their differences.

In the Appendixes, we collect several miscellaneous results related to quantum applications of rotors. In Appendix A, we show that a unitary squeezing automorphism cannot exist for rotors, in contrast to oscillator systems. In Appendix B,

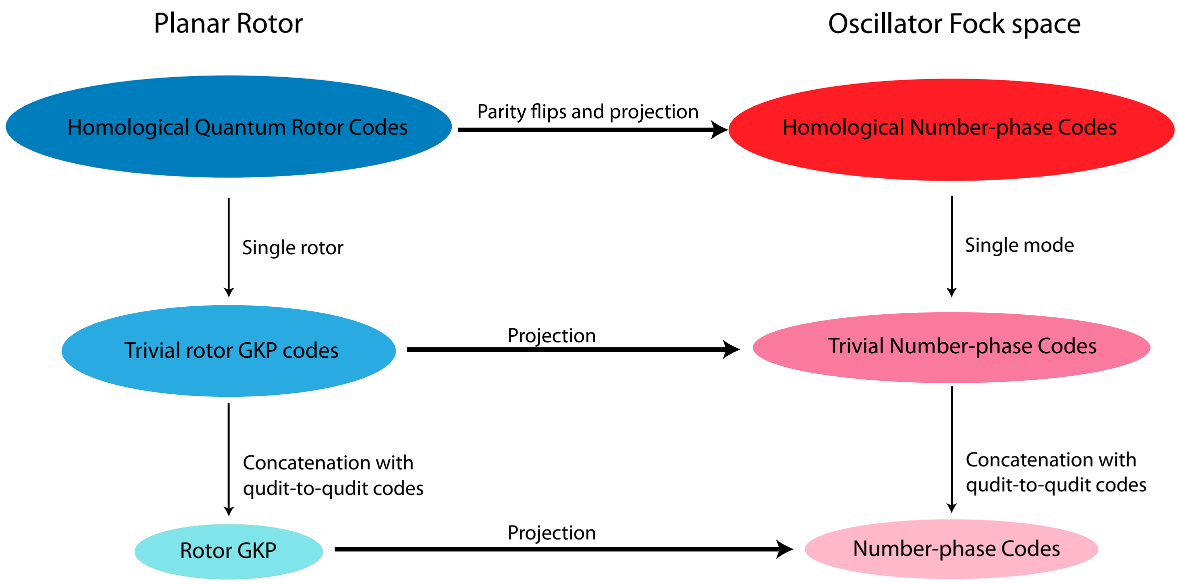

FIG. 2. Relations between homological rotor codes, trivial rotor GKP codes (oscillator GKP codes from the point of view of the embedded rotor), rotor GKP codes, and homological number-phase codes.

we give the subgroup of rotor Clifford group that is also a subgroup of the group of Gaussian transformations that preserve the total occupation number. In Appendix C, we study "Gaussian" states of rotors and their behavior under Clifford group and passive subgroup, including nullifier states (Appendix C 1) and coherent states (Appendix C 2). We also discuss how the Clifford group transforms the Josephson-junction Hamiltonian in Appendix C 3. In Appendix D, we calculate the Wigner function for the rotor GKP codewords and show they indeed have negativity. In Appendix E, we calculate the error-correction conditions for the normalized rotor GKP codes and discuss their relations with Jacobi ϑ functions. In Appendix F we discuss an analogy of Gaussian encoding no-go theorem [53] for rotors.

II. THE PLANAR ROTOR AND FRIENDS

In this section, we review the setup of the planar, or U(1), quantum rotor and its various connections to the bosonic mode (see Fig. 3) [31–33,54].

A. Planar rotor

The state space of a rotor is the same as that of a particle on a circle, arising naturally from a quantum body rotating in two dimensions. The state space admits bases of fixed particle position and fixed particle momentum. We associate the former with the "Pauli X-basis" of the rotor, denoting basis elements by a phase $\theta \in [0, 2\pi)$, Conversely, the dual "Pauli Z-basis" is characterized by irreducible representations of U(1), which are labeled by the integers $\mathbb{Z}$. The two bases are

$$\begin{aligned} X\text{-basis}: &\ |\theta\rangle, \quad \theta \in \mathbb{T}, \\ Z\text{-basis}: &\ |l\rangle, \quad l \in \mathbb{Z}, \end{aligned} \tag{2}$$

where $|\theta\rangle$ and $|l\rangle$ are called phase states and angular-momentum states, respectively. The former can be expressed in terms of the latter via the Fourier series,

$$|\theta\rangle = \frac{1}{\sqrt{2\pi}} \sum_{\ell\in\mathbb{Z}} e^{i\theta\ell} |\ell\rangle, \tag{3}$$

and vice versa.

The fundamental operators on a single rotor are the generalized Pauli operators. The Pauli X operator is parameterized by an integer $m \in \mathbb{Z}$, and the Pauli Z operator is parameterized by a phase factor $\phi \in \mathbb{T}$. Their actions on the angular position and momentum states are as follows:

$$\begin{aligned} X(m)|\theta\rangle &= e^{im\theta}|\theta\rangle, \quad X(m)|l\rangle = |l+m\rangle, \quad m \in \mathbb{Z}, \\ Z(\phi)|\theta\rangle &= |\theta-\phi\rangle, \quad Z(\phi)|l\rangle = e^{i\phi l}|l\rangle, \quad \phi \in \mathrm{U}(1). \end{aligned} \tag{4}$$

These are natural generalizations of qudit Pauli operators, defined on the space of a particle on the group $\mathbb{Z}_q$, or harmonic oscillator displacement operators defined for $\mathbb{R}$.

We would like to present the Pauli operators in terms of the fundamental degrees of freedom phases and angular momentum. Though the angular-momentum operator $\hat{l}$ is

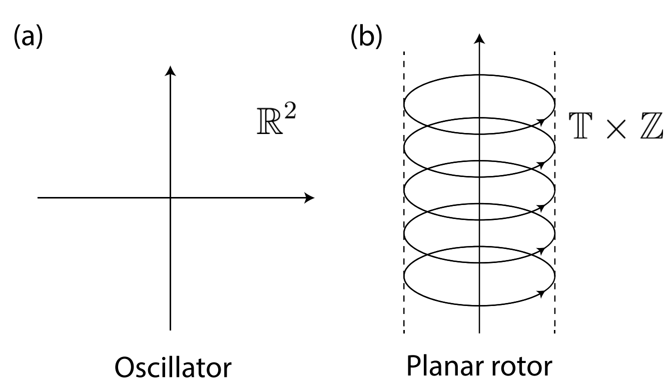

FIG. 3. Comparison between the phase space of (a) oscillator and (b) planar rotor systems. The dashed line denotes that the rotor angular momenta are confined to the integers.

well defined, we cannot define phase operator $\hat{\theta}$ individually because of the ambiguity introduced by the 2π periodicity. Nevertheless, we can still express Pauli operators as exponentials of either operator

$$X(m) = e^{im\hat{\theta}}, \quad Z(\phi) = e^{i\phi\hat{l}}. \tag{5}$$

The commutation relation between Pauli X and Z for rotor is

$$X(m)Z(\phi) = e^{-im\phi}Z(\phi)X(m), \tag{6}$$

where the phase factor arises from the rotor commutation relation $[\hat{l}, e^{i\hat{\theta}}] = e^{i\hat{\theta}}$. Tensor products of these Pauli operators generate the n-rotor Pauli group, $\mathcal{P}_n^{\text{rot}}$.

B. Embedded rotor

A planar rotor can be embedded into a harmonic oscillator by periodically identifying the oscillator's positions [see Figs. 1(a) and 1(b)]. It was pointed out in Ref. [37] that this embedding can be done by restricting to the +1 eigenspace of one of the stabilizers of the GKP code. This stabilizer, $\hat{S}_q = \exp(i2\pi\hat{p})$, written in terms of the oscillator momentum $\hat{p}$, shifts the oscillator position q by 2π. The stabilizer constraint,

$$\hat{S}_q|\psi\rangle = \int dq\psi(q)|q - 2\pi\rangle = \int dq\psi(q)|q\rangle = |\psi\rangle, \tag{7}$$

restricts the possible oscillator wave functions to those satisfying $\psi(q + 2\pi) = \psi(q)$. In other words, imposing the above stabilizer constraint is equivalent to imposing 2π periodicity in the q representation.[1]

The stabilizer $\hat{S}_q$ shifts each oscillator position state by 2π, meaning that its eigenstates are superpositions of a position state with all states related to it by a shift of a multiple of 2π. Such states are labeled by an angle and correspond to the position states of the embedded rotor,

$$|\theta_{\text{emb}}\rangle = \sum_{m\in\mathbb{Z}} |q = 2\pi m + \theta\rangle. \tag{8}$$

Since they are eigenstates of a GKP stabilizer, these are of the same comb-like form as the logical codewords of the GKP code.

The codeword $|\theta_{\text{emb}}\rangle$ can be written in terms of momentum eigenstates via Fourier transformation

$$|\theta_{\text{emb}}\rangle \propto \sum_{\ell\in\mathbb{Z}} \int dp e^{ip(2\pi\ell+\theta)}|p\rangle. \tag{9}$$

The sum of phases over $\ell \in \mathbb{Z}$ is nonvanishing only when the momentum $p \in \mathbb{Z}$. Therefore, the discretized bosonic momentum $\hat{p}$ is identified with the angular momentum of the embedded rotor.

[1]In terms of the oscillator subsystem decomposition from Ref. [55], the above performs the decomposition $\mathbb{T} \otimes 2\pi\mathbb{Z} = \mathbb{R}$, where $\mathbb{T}$ is the embedded rotor space, and the factor $2\pi\mathbb{Z}$ is called the gauge space.

Pauli operators for the embedded rotor,

$$\begin{aligned} X(m) &= e^{im\hat{q}}, \quad m \in \mathbb{Z}, \\ Z(\phi) &= e^{i\phi\hat{p}}, \quad \phi \in \mathbb{R}, \end{aligned} \tag{10}$$

where $\hat{q}$ is the oscillator position operator, form the subset of oscillator displacements that preserve the embedded rotor space. As a logical operator, $X(m)$ must commute with stabilizer $\hat{S}_q$, so m can only take integer values. The oscillator commutation relation $[\hat{q}, \hat{p}] = i$ yields that of the embedded rotor,

$$e^{im\hat{q}}e^{i\phi\hat{p}} = e^{-im\phi}e^{i\phi\hat{p}}e^{im\hat{q}}. \tag{11}$$

The embedded rotor formalism helps understand various features of the planar rotor from the harmonic oscillator perspective.

C. Oscillator Fock space

A different and, in this case, a sparse way to relate the U(1) rotor to an oscillator is to associate rotor momentum states with oscillator Fock bases, $|n\rangle$, $n \in \mathbb{N}$. Since $\mathbb{N} \subset \mathbb{Z}$, the Fock basis of harmonic oscillator shares many of the properties of the non-negative angular-momentum subspace of planar rotor. We can then associate rotor position states from Eq. (3) with the Pegg-Barnett oscillator phase states [5,35,36],

$$|\theta\rangle_{\text{F}} = \sum_{n\in\mathbb{N}} e^{-in\theta}|n\rangle, \quad \theta \in \mathbb{T}. \tag{12}$$

These states play important roles in quantum error correction [46,51] and quantum metrology [56,57], and are relevant in the construction of continuous-variable designs [57]. Indeed, both $|\theta\rangle_{\text{F}}, \forall\, \theta \in \mathbb{T}$ and $|n\rangle, \forall\, n \in \mathbb{N}$ form a complete (but, in the former case, nonorthonormal) basis.

To formally connect planar rotor with non-negative angular momentum to oscillator Fock space, we introduce projection operator to restrict planar rotor to non-negative angular-momentum subspace

$$\Pi_{\geqslant m} = \sum_{l\geqslant m} |l\rangle\langle l|. \tag{13}$$

For $m = 0$, this projection removes all negative momentum states. We denote all projected operators and states in the oscillator Fock space by the subscript "F", i.e., $O_{\text{F}} = \Pi_{\geqslant 0}O\Pi_{\geqslant 0}$ for any operator O. We emphasize projection operator $\Pi_{\geqslant 0}$ is a mathematical treatment that bridges oscillator Fock space and non-negative angular-momentum subspace of planar rotor, instead of a physical operation that needs to be implemented in the laboratory.

The Pauli operators for oscillator Fock space are

$$\begin{aligned} X(m)_{\text{F}} &= \begin{cases} \sum_{n=0}^{\infty} |n+m\rangle\langle n| & m \geqslant 0 \\ \sum_{n=0}^{\infty} |n\rangle\langle n+|m|| & m < 0, \end{cases} \\ Z(\phi)_{\text{F}} &= e^{i\phi\hat{n}}, \end{aligned} \tag{14}$$

where $\hat{n} = \sum_{l=0}^{\infty} l|l\rangle\langle l|$ should be interpreted as the photon number operator.

In the Fock space interpretation, $X(m)_{\rm F}$ performs m-photon injection (subtraction) for positive (negative) m, and both correspond to powers of the Kogut-Susskind phase operator and its adjoint [35,58,59]. The X-type Pauli operator is proposed and realized in cavity systems [60–62]. The $Z(\phi)_{\rm F}$ operator is a single-mode rotation by ϕ, generated by the harmonic oscillator Hamiltonian.

For $m \geqslant 0$, we have

$$X(m)_{\rm F}^{\dagger}X(m)_{\rm F} = \mathbb{I}, \quad X(m)_{\rm F}X(m)_{\rm F}^{\dagger} = \Pi_{\geqslant m} \neq \mathbb{I}. \tag{15}$$

Therefore, $X(m)_{\rm F}$ is not a unitary. This is because one can inject and then subtract the same number of photons on arbitrary states. But subtracting and then injecting m photons can only be applied to states having at least m photons to begin with.

The connection between oscillator Fock space and planar rotor with non-negative angular momentum turns out to be quite fruitful, as much of the algebraic structure of rotors survives under this projection [54], e.g., the Pauli commutation relation,

$$X(m)_{\rm F}Z(\phi)_{\rm F} = e^{-im\phi}Z(\phi)_{\rm F}X(m)_{\rm F}. \tag{16}$$

In Sec. V, we show the mathematical similarity between planar rotor and harmonic oscillator enables us to systematically construct a new class of quantum codes for several oscillators. This generalizes rotation-symmetric bosonic codes [45] to multiple modes.

III. ROTOR CLIFFORD GROUP

The rotor Clifford group is a collection of operations mapping a tensor product of rotor Pauli operators to another tensor product of rotor Pauli operators while preserving the commutation relations among them. To be concrete, given the n-qubit Pauli group $\mathcal{P}_n$, the corresponding n-qubit Clifford group is

$$C(\mathcal{P}_n) = N_{{\rm U}(2^n)}(\mathcal{P}_n)/\mathcal{P}_n, \tag{17}$$

the normalizer of the Pauli group $\mathcal{P}_n$ in the unitary group ${\rm U}(2^n)$, up to multiplication by elements of the Pauli group, as well as any phases.

Operations in the Clifford group are usually regarded as gates that can be easily implemented for quantum computation. In the theory of quantum error correction, the Clifford group plays a fundamental role in designing and characterizing quantum error-correcting codes [63–67], as well as the equivalence and deformation of codes [68–71]. Investigations of the Clifford group also yield efficient classical algorithms for simulating Clifford circuits, guaranteed by the Gottesman-Knill theorem [1] and its generalization to arbitrary Abelian groups [30,31]. On the practical side, the underlying mathematical structure of circuit quantization and its connections to symplectic transformations have also recently been studied [72–76]. Therefore, to facilitate us with the discussion of the encoding and decoding processes of the homological rotor code in Sec. IV, and the code deformation leading to construction of homological number-phase code in Sec. V, we have to have an overall command of the rotor Clifford-group structure.

A nice property of Clifford groups is that their elements can be expressed as symplectic transformations acting on particular vectors—$2n$-dimensional vectors $(X|Z) \in \mathbb{Z}_d^{2n}$ for an n-qudit system, and $2n$-dimensional real vectors $(\vec{q}|\vec{p}) \in \mathbb{R}^{2n}$ for an n-mode system. Since the domain of both quadrature pairs are the same in both cases, it is easy to show that the qudit and oscillator Clifford groups are ${\rm Sp}_{2n}(\mathbb{Z}_d)$ [77] and ${\rm Sp}_{2n}(\mathbb{R})$ [3,4], respectively. Since rotor systems are *hybrid*—behaving like continuous-variable systems in the phase basis and discrete-variable systems in the momentum basis—the rotor Clifford group is not as easy to read off.

In this section, we identify the n-rotor Clifford group to be the semi-direct product group ${\rm U}(1)^{n(n+1)/2} \rtimes {\rm GL}_n(\mathbb{Z})$. We then present the generators of the Clifford group for n rotors.

A. Clifford-group structure

Despite the hybrid nature of rotor systems, a Pauli $Z(\vec{\boldsymbol{\phi}})X(\vec{\boldsymbol{m}})$ can still be represented by a vector $\vec{\boldsymbol{v}}$ [[31,78], Lemma 2.8],

$$\vec{\boldsymbol{v}} = \left(\vec{\boldsymbol{m}}_{\boldsymbol{v}}^T|\vec{\boldsymbol{\phi}}_{\boldsymbol{v}}^T\right)^T, \tag{18}$$

where $\vec{\boldsymbol{m}}_{\boldsymbol{v}}$ is a n-dimensional integer-valued column vector, and $\vec{\boldsymbol{\phi}}_{\boldsymbol{v}}$ is a n-dimensional $\mathbb{T}$-valued column vector. The commutation relation between two Pauli strings $\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{v}}$ can then be represented as

$$\begin{aligned} &Z(\vec{\boldsymbol{\phi}}_{\boldsymbol{u}})X(\vec{\boldsymbol{m}}_{\boldsymbol{u}})Z(\vec{\boldsymbol{\phi}}_{\boldsymbol{v}})X(\vec{\boldsymbol{m}}_{\boldsymbol{v}}) \\ &\quad = e^{-i\vec{\boldsymbol{u}}^T\Lambda\vec{\boldsymbol{v}}}Z(\vec{\boldsymbol{\phi}}_{\boldsymbol{v}})X(\vec{\boldsymbol{m}}_{\boldsymbol{v}})Z(\vec{\boldsymbol{\phi}}_{\boldsymbol{u}})X(\vec{\boldsymbol{m}}_{\boldsymbol{u}}), \end{aligned} \tag{19}$$

where $e^{-i\vec{\boldsymbol{u}}^T\Lambda\vec{\boldsymbol{v}}}$ is the phase factor captured by the symplectic inner product between $\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{v}}$,

$$\vec{\boldsymbol{u}}^T\Lambda\vec{\boldsymbol{v}} = \vec{\boldsymbol{m}}_{\boldsymbol{u}}^T\vec{\boldsymbol{\phi}}_{\boldsymbol{v}} - \vec{\boldsymbol{\phi}}_{\boldsymbol{u}}^T\vec{\boldsymbol{m}}_{\boldsymbol{v}}, \quad \Lambda = \begin{pmatrix} 0 & \mathbb{I}_{n\times n} \\ -\mathbb{I}_{n\times n} & 0 \end{pmatrix}. \tag{20}$$

Each Clifford circuit U can be represented by a symplectic matrix Q_U that transforms a Pauli string $\vec{v}$ as

$$U\vec{\boldsymbol{v}}U^{\dagger} = Q_U\vec{\boldsymbol{v}} = \left(\vec{\boldsymbol{m}}_{Q_U\boldsymbol{v}}^T|\vec{\boldsymbol{\phi}}_{Q_U\boldsymbol{v}}^T\right)^T, \tag{21}$$

for some quadrature transformation Q_U. Since U is a Clifford circuit, $\vec{\boldsymbol{m}}_{Q_U\boldsymbol{v}}$ should be a n-dimensional integer-valued column vector, and $\vec{\boldsymbol{\phi}}_{Q_U\boldsymbol{v}}$ should be a n-dimensional $\mathbb{T}$-valued column vector. Any quadrature transformation Q also has to satisfy

$$Q^T\Lambda Q = \Lambda \text{ (symplectic condition)} \tag{22}$$

because it preserves the rotor commutation relations. This implies that the rotor Clifford group is a particular subgroup of the oscillator Clifford group ${\rm Sp}_{2n}(\mathbb{R})$ that preserves the angle-integer form of the Pauli vectors $\vec{\boldsymbol{v}}$.

Most generally, a rotor quadrature transformation can be written as

$$Q = \begin{pmatrix} Q_{XX} & 0 \\ Q_{XZ} & Q_{ZZ} \end{pmatrix}. \tag{23}$$

The upper-left block is a general linear transformation over the integers, $Q_{XX} \in {\rm GL}_n(\mathbb{Z})$; all such transformations have determinant ± 1 (i.e., are *unimodular*). The upper-right block must be zero because a sum of a continuous variable and a discrete variable is not discrete, thereby violating the discreteness of

$\vec{m}$. The lower-left block does not have to be zero since there are no discreteness constraints on the angles $\vec{\phi}$.

Imposing the symplectic condition (22) yields the following constraints on the block matrices:

$$Q_{ZZ}^T Q_{XX} = Q_{XX}^T Q_{ZZ} = \mathbb{I}_{n\times n},\\ Q_{XX}^T Q_{XZ} - (Q_{XX}^T Q_{XZ})^T \equiv 0 \bmod 2\pi. \tag{24}$$

The first equality of Eq. (24) requires that $Q_{ZZ} = (Q_{XX}^T)^{-1}$, indicating that the bottom-right block Q_{ZZ} also has to be unimodular. The second equality requires $Q_{XX}^T Q_{XZ}$ to be symmetric. These requirements are derived directly from the definition of the rotor Clifford group. They constrain the most general form of the rotor Clifford-group element to be

$$Q = \begin{pmatrix} Q_{XX} & 0 \\ (Q_{XX}^T)^{-1}C & (Q_{XX}^T)^{-1} \end{pmatrix} = \begin{pmatrix} Q_{XX} & 0 \\ 0 & (Q_{XX}^T)^{-1} \end{pmatrix}\begin{pmatrix} \mathbb{I}_{n\times n} & 0 \\ C & \mathbb{I}_{n\times n} \end{pmatrix}, \tag{25}$$

in which $Q_{XX} \in \mathrm{GL}_n(\mathbb{Z})$ and C is a $\mathbb{T}$-valued symmetric matrix whose addition generates $\mathrm{U}(1)^{n(n+1)/2}$.

We prove that the Rotor Clifford group has the following structure:

Theorem III.1. Rotor Clifford group is $\mathrm{U}(1)^{n(n+1)/2} \rtimes GL_n(\mathbb{Z})$.

Proof. There are two obvious subgroups of symplectic transformations,

$$H = \left\{ \begin{pmatrix} A & 0 \\ 0 & (A^T)^{-1} \end{pmatrix} \middle| , A \in \mathrm{GL}_n(\mathbb{Z}) \right\},\\ N = \left\{ \begin{pmatrix} \mathbb{I}_{n\times n} & 0 \\ C & \mathbb{I}_{n\times n} \end{pmatrix} \middle| , C \text{ is } n\times n\ \mathbb{T}\text{-valued symmetric matrix} \right\}. \tag{26}$$

The *diagonal* or *CSS* subgroup H form a reducible representation of $\mathrm{GL}_n(\mathbb{Z})$. The *block off-diagonal* subgroup N is the addition group of $n \times n$ symmetric matrices over $\mathbb{T}$, representing the Lie group $\mathrm{U}(1)^{n(n+1)/2}$.

Any element in the form Eq. (25) is in the rotor Clifford group. It can be generated by the two subgroups H and N. For all $g \in G$, there exists $\mathcal{A} \in H$ and $\mathcal{C} \in N$ such that $g = \mathcal{A}\mathcal{C}$. On the other hand, any element generated by the subgroup H, N can also be written as the form of Eq. (25), thus an element of G. Without loss of generality, it can be written as $g = \mathcal{A}_1\mathcal{C}_1 \cdots \mathcal{A}_n\mathcal{C}_n$ for some integer n, with $\mathcal{A}_1, \ldots, \mathcal{A}_n \in H$ and $\mathcal{C}_1, \ldots, \mathcal{C}_n \in G$. This can be proved by induction. The $k = 1$ case is obvious. Suppose for $k = n$, $g = \mathcal{A}_1\mathcal{C}_1 \cdots \mathcal{A}_n\mathcal{C}_n = \mathcal{A}^{(n)}\mathcal{C}^{(n)}$. Then $g' = \mathcal{A}_1\mathcal{C}_1 \cdots \mathcal{A}_{n+1}\mathcal{C}_{n+1} = \mathcal{A}^{(n)}\mathcal{C}^{(n)}\mathcal{A}_{n+1}\mathcal{C}_{n+1} = \mathcal{A}^{(n+1)}\mathcal{C}^{(n+1)}$, where $\mathcal{A}^{(n+1)} = \mathcal{A}^{(n)}\mathcal{A}_{n+1}$ and $\mathcal{C}^{(n+1)} = \mathcal{A}_{n+1}^{-1}\mathcal{C}^{(n)}\mathcal{A}_{n+1}\mathcal{C}_{n+1}$. $\mathcal{C}^{(n+1)} \in N$ can be shown by direct calculations

$$\begin{aligned} &\mathcal{A}_{n+1}^{-1}\mathcal{C}^{(n)}\mathcal{A}_{n+1} \\ &= \begin{pmatrix} A_{n+1}^{-1} & 0 \\ 0 & (A_{n+1}^T) \end{pmatrix}\begin{pmatrix} \mathbb{I}_{n\times n} & 0 \\ C^{(n)} & \mathbb{I}_{n\times n} \end{pmatrix}\begin{pmatrix} A_{n+1} & 0 \\ 0 & (A_{n+1}^T)^{-1} \end{pmatrix} \\ &= \begin{pmatrix} \mathbb{I}_{n\times n} & 0 \\ [(A_{n+1}^T C^{(n)} A_{n+1}] \bmod 2\pi & \mathbb{I}_{n\times n} \end{pmatrix} \\ &= \begin{pmatrix} \mathbb{I}_{n\times n} & 0 \\ C' & \mathbb{I}_{n\times n} \end{pmatrix} \in N. \end{aligned} \tag{27}$$

In the last line of Eq. (27), since $A_{n+1}^T C^{(n)} A_{n+1}$ is a symmetric matrix, $C' \equiv [(A_{n+1}^T C^{(n)} A_{n+1}] \bmod 2\pi$ is also a $\mathbb{T}$-valued symmetric matrix. The above derivation uses the identity

$$[A(C \bmod 2\pi)] \bmod 2\pi = (AC) \bmod 2\pi, \tag{28}$$

where A is a unimodular matrix and C is a $\mathbb{T}$-valued matrix.

Now we proceed to prove that the set generated by H and N is indeed a group and N is the normal subgroup.

(1) *Associativity*. The associativity of group G follows from the associativity of matrix multiplication.

(2) *Identity*. The identity element of group G is

$$e = \begin{pmatrix} \mathbb{I}_{n\times n} & 0 \\ 0 & \mathbb{I}_{n\times n} \end{pmatrix}, \tag{29}$$

which is the identity matrix.

(3) *Inverse*. The existence of the inverse of $g \in G$ follows from the existence of the group inverse of the element of $\mathcal{A} \in H$ and $\mathcal{C} \in N$, as well as the matrix inverse. $g^{-1} = \mathcal{C}^{-1}\mathcal{A}^{-1} = \mathcal{A}^{-1}(\mathcal{A}\mathcal{C}\mathcal{A}^{-1}) \equiv \tilde{\mathcal{A}}\tilde{\mathcal{C}} \in G$.

(4) *Normality*. For all $Q \in G$ and for all $\mathcal{C}_h \in N$, we can write $Q = \mathcal{A}_Q\mathcal{C}_Q$. So $Q\mathcal{C}_hQ^{-1} = \mathcal{A}_Q\mathcal{C}_Q\mathcal{C}_h\mathcal{C}_Q^{-1}\mathcal{A}_Q^{-1} \in N$, following Eq. (27). ■

B. Clifford-group generators

We now identify the generators of the rotor Clifford group [79] and show how they transform rotor Pauli operators. All planar rotor Clifford gates can be expressed as exponentials of quadratic combinations of the phase and angular-momentum operators, except for the parity-flip operation.

(1) The CNOT gate is defined as

$$\mathrm{CNOT}_{1\to 2} = \int_{\mathrm{U}(1)} Z(\phi) \otimes |\phi\rangle\langle\phi| d\phi = \sum_{l\in\mathbb{Z}} |l\rangle\langle l| \otimes X(l), \tag{30a}$$

and it acts on Pauli operators as

$$\begin{aligned} \mathrm{CNOT}_{1\to 2}(X(1)\otimes\mathbb{I})\mathrm{CNOT}_{1\to 2}^\dagger &= X(1)\otimes X(1), \\ \mathrm{CNOT}_{1\to 2}(\mathbb{I}\otimes X(1))\mathrm{CNOT}_{1\to 2}^\dagger &= \mathbb{I}\otimes X(1), \\ \mathrm{CNOT}_{1\to 2}(\mathbb{I}\otimes Z(\phi))\mathrm{CNOT}_{1\to 2}^\dagger &= Z(-\phi)\otimes Z(\phi), \\ \mathrm{CNOT}_{1\to 2}(Z(\phi)\otimes\mathbb{I})\mathrm{CNOT}_{1\to 2}^\dagger &= Z(\phi)\otimes\mathbb{I}. \end{aligned} \tag{30b}$$

TABLE II. The matrix description of rotor Clifford gates. The swap gate can be realized by $\text{SWAP}_{ij} = \text{P}_j\text{CNOT}^\dagger_{ji}\text{CNOT}^\dagger_{ij}\text{CNOT}_{ji}$.

Gate	Matrix representation of rotor Clifford gates: Left multiplication	Right multiplication
SWAP_{ij}	Swap the ith row and the jth row of Q_{XX}, and swap the ith row and the jth row of $(Q_{XZ}\|Q_{ZZ})$.	Swap the ith column and the jth column of Q_{XX}, and swap the ith column and the jth column of Q_{XZ} and Q_{ZZ}.
CNOT_{ij}	Add the ith row of Q_{XX} to the jth row of Q_{XX}, and subtract the jth row of Q_{ZZ} from the ith row of Q_{ZZ}	Add the jth column of Q_{XX} to the ith column of Q_{XX}, and subtract the ith column of Q_{ZZ} from the jth column of Q_{ZZ}
P_i	Multiply the ith rows of Q_{XX}, Q_{XZ}, Q_{ZZ} by -1	Multiply the ith rows of Q_{XX}, Q_{XZ}, Q_{ZZ} by -1
$\text{QUAD}_{\varphi,i}$	Add φ times the ith row of Q_{XX} to the ith row of Q_{XZ}	Add φ times the ith column of $(Q_{XX}^T)^{-1}$ to the ith row of Q_{XZ}
$\text{CPHS}_{\varphi,ij}$	Add φ times the ith row of Q_{XX} to the jth row of Q_{XZ}, and add φ times the jth row of Q_{XX} to the ith row of Q_{XZ}	Add φ times the ith column of $(Q_{XX}^T)^{-1}$ to the jth column of Q_{XZ}, and add φ times the jth column of $(Q_{XX}^T)^{-1}$ to the ith column of Q_{XZ}

(2) The QUAD gate, $\text{QUAD}_\varphi = e^{i\varphi\hat{l}(\hat{l}+1)/2}$, acts on Pauli X operators as

$$\text{QUAD}_\varphi X(1)\text{QUAD}_\varphi^\dagger = Z(\varphi)X(1), \tag{30c}$$

commuting with all Pauli Z operators.

(3) The CPHS gate, $\text{CPHS}_\varphi = e^{i\varphi\hat{l}\otimes\hat{l}}$, commutes with Z operators and acts on X operators as

$$\begin{aligned}\text{CPHS}_\varphi(X(1)\otimes\mathbb{I})\text{CPHS}_\varphi^\dagger &= X(1)\otimes Z(\varphi),\\ \text{CPHS}_\varphi(\mathbb{I}\otimes X(1))\text{CPHS}_\varphi^\dagger &= Z(\varphi)\otimes X(1).\end{aligned} \tag{30d}$$

(4) The parity flip, $\text{P} = \sum_\ell |-\ell\rangle\langle\ell| = \text{P}^\dagger$, acts as

$$\begin{aligned}\text{P}X(m)\text{P} &= X(-m),\\ \text{P}Z(\phi)\text{P} &= Z(-\phi),\end{aligned} \tag{30e}$$

flipping the sign of both position and momentum.

All of the above operators can be generated by evolving under quadratic interactions, with the notable exception of the parity flip. Nevertheless, the flip is a well-defined Clifford operation that can be realized in concrete systems (see Sec. C 3) and that plays a critical role in the study of the relation between the rotor and homological number-phase codes (see Sec. V). It is straightforward to identify how these generators conjugate a Clifford operator defined by the symplectic transformation Q; we tabulate their actions in Table II. The generators of the symplectic group from Eq. (30) indeed generate any rotor Clifford-group element in the form of Eq. (25). Specifically, the subgroup H and N are generated by[2]

$$H = \langle\text{CNOT}, \text{P}\rangle, \quad N = \langle\text{QUAD}, \text{CPHS}\rangle. \tag{31}$$

Therefore, the generators in Eq. (30) form a complete set that generates the whole rotor Clifford group.

IV. HOMOLOGICAL ROTOR CODES

We demonstrate the role of rotor Clifford group in designing and characterizing quantum error-correcting codes, as well as the equivalence and deformation of codes by investigating the homological rotor codes [37]. The relationships we discover in this section between rotor codes and bosonic codes via the embedding rotor formalism will inspire us to design multimode bosonic rotational-symmetric code in Sec. V.

We first review the homological rotor code. Homological rotor codes are stabilizer codes of CSS type, meaning that the codespace is the $+1$-eigenvalue eigenspace of a group of mutually commuting rotor Pauli strings which are either of pure X or pure Z types. The stabilizer generators of an n-rotor code are described by matrices H_X and H_Z of integer-valued entries with size $r_X \times n$ and $r_Z \times n$, respectively. The stabilizer group is

$$\mathcal{S} = \{X(\vec{\boldsymbol{m}}^T H_X)Z(\vec{\boldsymbol{\phi}}^T H_Z)\,\forall\,\vec{\boldsymbol{m}}\in\mathbb{Z}^{r_X}, \forall\,\vec{\boldsymbol{\phi}}\in\mathbb{T}^{r_Z}\}. \tag{32}$$

Mutual commutation imposes the condition

$$\begin{aligned}Z(\vec{\boldsymbol{\phi}}^T H_Z)X(\vec{\boldsymbol{m}}^T H_X) &= \exp(i\vec{\boldsymbol{\phi}}^T H_Z H_X^T\vec{\boldsymbol{m}})X\left(\vec{\boldsymbol{m}}^T H_X\right)Z\left(\vec{\boldsymbol{\phi}}^T H_Z\right)\\ &= X(\vec{\boldsymbol{m}}^T H_X)Z(\vec{\boldsymbol{\phi}}^T H_Z) \Rightarrow H_Z H_X^T = 0.\end{aligned} \tag{33}$$

The matrices H_X and H_Z can also be viewed as maps $\partial = H_X : \mathbb{Z}^{r_X} \to \mathbb{Z}^n$ and $\sigma = H_Z^T : \mathbb{Z}^n \to \mathbb{Z}^{r_Z}$. The above condition implies that the composition of the corresponding maps is zero, $\sigma\circ\partial = H_X H_Z^T = 0$. This enables us to define a chain complex over the integers, with ∂ and σ as its boundary maps. Code properties can be equivalently stated in terms of properties of the chain complex. The logical space is given by the complex's first homology group,

$$\text{H}_1(\mathbb{Z}) = \ker\left(H_Z^T\right)/\text{im}(H_X), \tag{34}$$

which is formed by cosets of elements of the image of H_X $\text{im}(H_X)$ in the kernel of H_Z^T $\ker(H_Z^T)$.

Reference [37] showed that the above homology group is a product of integer factors $\mathbb{Z}$—each denoting a logical rotor—and discrete factors $\mathbb{Z}_d$—each denoting a d-dimensional logical qudit. The latter pieces yield finite-dimensional codespaces and come from what is known as *torsion*. This effect is not present in analogously defined oscillator and Galois-qudit CSS codes.

The impact of torsion is present already in the case of a single rotor, where one obtains a finite-dimensional codespace that is related to both rotor and oscillator GKP codes.

[2]Although it is known that $\text{GL}_n(\mathbb{Z})$ can be generated by as few as two generators for any order of $n \geqslant 4$, we use CNOT and P because they are local gates that are physically preferred.

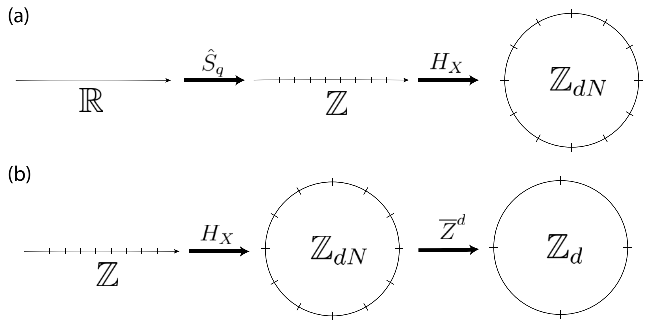

FIG. 4. (a) An oscillator GKP code that encodes a $\mathbb{Z}_{dN}$ qudit is a homological embedded rotor code with $H_X = dN$ and torsion part $\mathbb{Z}_{dN}$. (b) A rotor GKP code can be viewed as a concatenation of a homological rotor code with $H_X = dN$ and a modular-qudit GKP code. We discuss both of these relations in Example 1 and Sec. IV C.

Example 1. For a single rotor homological code, there is only one stabilizer $X(dN) = e^{idN\hat{\theta}}$ represented by a one-by-one matrix $H_X = dN$. The other matrix is just $H_Z = 0$. The codespace is finite dimensional, with the dN codewords given by

$$|j\rangle = \sum_{m\in\mathbb{Z}} |l = mdN + j\rangle \;\forall\, j \in \mathbb{Z}_{dN}. \tag{35}$$

We can relate the above homological rotor code to the ordinary oscillator GKP codes [see Fig. 4(a)]. Embedding this rotor in an oscillator by thinking of the rotor momentum states $|l\rangle$ as oscillator momentum states, we see that the codewords correspond to those of the GKP codes. And the stabilizer for the embedded rotor together with the H_X stabilizer exactly correspond to the pair of the stabilizers of the GKP codes.

Alternatively, if we restrict the exponent of Z-type stabilizers from $\phi \in \mathbb{T}$ to some subgroup $\phi \in \mathbb{Z}_N$, we can observe that the Z-type Pauli $\hat{S}_Z = Z(\frac{2\pi}{N})$ does commute with $X(dN)$. Treating this as an additional stabilizer yields the rotor GKP codes, and adding this stabilizer can be thought of as concatenating the above homological rotor code with modular-qudit GKP codes [see Fig. 4(b)].

We discuss both of the above relations in Sec. IV C.

A. Torsion and Smith normal form

To build up intuition behind torsion, we calculate the homology group from a geometric point of view, by studying the kernels and images in the angular-momentum Hilbert space of rotors which form a lattice $\mathbb{Z}^n$.

To calculate the homology, we first identify the sub-lattice $\ker(H_Z^T)$. H_Z does not contribute to the torsion part. This is because their stabilizer set is a continuum of stabilizers $Z(\vec{\phi}H_Z)$, indexed by the real-valued vectors $\vec{\phi}$. As long as H_Z are integer valued, their rescaled versions yield the same stabilizer group. Therefore, we can effectively rescale each row of H_Z to obtain a vector whose components have no nontrivial common divisor.

We then calculate $\mathrm{im}(H_X)$. Because the stabilizers are $X(\vec{m}H_X)\;\forall\,\vec{m} \in \mathbb{Z}^{r_X}$, as the coefficients are discrete, we are not allowed to rescale row vectors in H_X arbitrarily. These vectors generate a lattice whose spacing is determined by their length. If a row vector $\vec{v}_x$ is k times the unit vector in this direction, then the image of $\vec{v}_x$ under coefficient $\mathbb{Z}$ will skip the grid points in between the starting and ending points of $\vec{v}_x$. When we take the quotient of $\mathrm{im}(H_X)$, the direction of $\vec{v}_x$ becomes a circle with k grid points, giving a k-dimensional qudit and contributing a $\mathbb{Z}_k$ factor to the torsion part.

The systematic method of identifying the elements in $\mathrm{im}(H_X)$ that do not have unit length in one direction but are not multiples of other vectors is to calculate the Smith normal form. For an $r_X \times n$ integer-valued matrix H_X, its *Smith normal form* [80,81] D is given by

$$UH_XV = D,$$

where D is in the diagonal form, meaning $D_{ij} = d_i\delta_{ij}$, $1 \leqslant i \leqslant r_X$. Here d_i are positive integers and satisfy the condition that each d_i is a divisor of d_{i+1}, for $1 \leqslant i < r_X$. U is an $r_X \times r_X$ unimodular matrix and V is an $n \times n$ unimodular matrix. The diagonal entries of D, $\{d_1, \dots, d_{r_X}\}$, are unique and they are called the invariant factors.

The Smith normal form is closely related to the homology group defined by H_Z and H_X. To be concrete, let the diagonal element of D be $d_1, \dots, d_{r_X}$, then the homology group defined by H_Z^T and H_X with $H_XH_Z^T = 0$ can be obtained as

$$\ker\left(H_Z^T\right)/\mathrm{im}(H_X) = \left(\bigoplus_{i=1}^{r_X} \mathbb{Z}_{d_i}\right) \oplus \mathbb{Z}^{n-r_X-r_Z}. \tag{36}$$

The $\bigoplus_{i=1}^{r_X} \mathbb{Z}_{d_i}$ is the torsion part and the $\mathbb{Z}^{n-r_X-r_Z}$ is the free part. If $d_i = 1$, then $\mathbb{Z}_{d_i} = 1$, which is trivial.

Understanding rotor Clifford group helps us identify the physical meanings of matrices U and V. The meaning of U is performing linear combinations of the generators of the X stabilizer group. It has no physical consequences to the logical space of the code. The physical meaning of V is actually the *decoding circuit* of the code. To see this, recall that a Clifford transformation that preserves the CSS structure of the code should not mix the l and θ quadratures [see discussions around Eq. (C15)], it should be only an element of the CSS Clifford subgroup H from Eq. (26). The H_X and H_Z matrices collect the phase and angular-momentum operators so they should change as Eq. (C15) under the CSS Clifford subgroup H,

$$H_X' = H_XA \text{ and } H_Z' = H_Z(A^{-1})^T. \tag{37}$$

Indeed, the CSS condition is preserved as $H_X'H_Z'^T = 0$. So V represents a special decoding Clifford transformation that decouples the entangled stabilizers in H_X to individual X stabilizers on each rotor.

B. Code initialization and equivalence classes

Homological rotor encodings can be performed analogously to those of Gaussian or analog stabilizer codes [82,83]. The oscillator encodings consist of a Gaussian transformation applied to an initial n-mode state, with k of the modes storing logical information, and $n-k$ modes initialized in the zero-position state—the canonical nullifier state. Rotor encodings can be defined analogously using rotor initial states and the rotor Clifford-group circuit V, defined in the previous subsection. A key difference is that obtaining codewords with particular torsion cannot be done by Clifford-group operations and instead requires the initial state on the $n-k$ rotors to be of a particular form.

Applying elements from the CSS Clifford-subgroup H from Eq. (26) to a given homological rotor code does not change the torsion structure of the logical subspace. This can be seen from the invariance of the Smith normal form under unimodular transformation. Consider an X-parity check matrix after transformation $H_X' = BH_XA$ where A is an unimodular matrix representing the Clifford transformation, and B is a change of basis of the stabilizer generators. Its Smith normal form D' is given by

$$D' = U'H_X'V' = (UB^{-1})BH_XA(A^{-1}V) = D. \quad (38)$$

So the Smith normal form D is invariant under the action of unimodular matrices A and B.

This fact implies that quantum rotor codes with distinct Smith normal forms belong to different classes of rotor codes, and cannot be deformed from one class to another by only applying rotor Clifford gates and changing the basis of the stabilizer group, which is called Clifford deformation. The equivalence relation between phase space lattices is also studied in the context of harmonic oscillator GKP states [84–86], [87], Thm. 2]. Quantum systems with the same Hilbert space and the same number of fundamental degrees of freedom are not necessarily in the same equivalence class. For example, for a 16-dimensional Hilbert space with two quantum degrees of freedom, $\mathbb{Z}_2 \bigoplus \mathbb{Z}_8$ and $\mathbb{Z}_4 \bigoplus \mathbb{Z}_4$ are not equivalent via unimodular transformations as they have different Smith normal forms.

Since Clifford deformation does not change the invariant torsion factors $\{d_1, \ldots, d_{r_X}\}$, the initial state of a homological rotor encoding with a given Smith normal form needs to be a resource state with that form. In the embedding formalism of rotors, these resource states become exactly GKP states, like the code shown in Example 1.

Another observation is that the specific form of the Smith normal form (namely, that each d_i is a divisor of d_{i+1}) indicates that not any tensor product of qudits with arbitrary dimensions is allowed in the torsion part. Different tensor products may yield the same Smith normal form. Generally, two qudits with dimensions $a = cq$ and $b = cp$ are equivalent to two qudits with dimensions c and cpq, respectively, in which $\gcd(a, b) = c$ and p, q are coprime. We provide a concrete example of merging two logical qudits into a combined qudit.

Example 2. Given an X-type parity check matrix

$$H_X = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

corresponding to a composite logical system formed by a $\mathbb{Z}_2$ qubit and a $\mathbb{Z}_3$ qudit, it is equivalent to a $\mathbb{Z}_6$ qudit via a sequence of unimodular transformation,

$$H_X = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \to \begin{pmatrix} 2 & 0 \\ -2 & 3 \end{pmatrix} \to \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & -3 \\ 1 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 1 & 6 \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix} = D. \quad (39)$$

A pictorial description of these transformations is shown in Fig. 5.

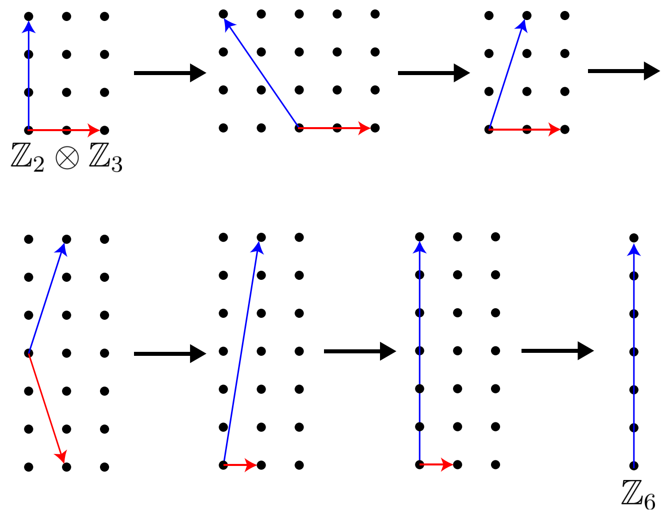

FIG. 5. A sequence of unimodular transformation operations by which the isomorphism $\mathbb{Z}_2 \otimes \mathbb{Z}_3 \cong \mathbb{Z}_6$ is realized.

C. Rotor GKP codes are concatenations of homological rotor codes and modular-qudit GKP codes

Analogously to the GKP codes in harmonic oscillators in Eq. (35), GKP codes in rotors can be defined by definite a discrete lattice in the rotor phase space, $\mathbb{T} \times \mathbb{Z}$. The rotor GKP code has two stabilizers $\hat{S}_X = X(dN)$, $\hat{S}_Z = Z(\frac{2\pi}{N})$. The codewords are

$$|Z_L = \omega^j\rangle_L = \sum_{m\in\mathbb{Z}} |l = jN + mdN\rangle, \quad (40)$$

where $\omega = e^{i2\pi/d}$. Given the codewords, we have logical Pauli operations

$$X_L = X(N) = \hat{S}_X^{1/d}, \quad Z_L = Z\left(\frac{2\pi}{dN}\right) = \hat{S}_Z^{1/d}. \quad (41)$$

Given the similarity between oscillator and rotor GKP codes, we observe a connection between rotor GKP codes and homological rotor codes. We can view the two stabilizers of the rotor GKP code as a two-step concatenation. The two steps are shown in Fig. 4(b). Starting from a single rotor, we first impose the stabilizer $H_X = dN$. This is the code discussed in Example 1 with codewords and logical operators

$$\begin{aligned} |\overline{Z} = e^{i\frac{2\pi j}{dN}}\rangle &= \sum_{m\in\mathbb{Z}} |l = j + mdN\rangle, \\ \overline{X} = X(1) = e^{i\hat{\theta}}, \quad \overline{Z} &= Z\left(\frac{2\pi}{dN}\right) = e^{i\frac{2\pi}{dN}\hat{l}}. \end{aligned} \quad (42)$$

Then we further impose $\overline{Z}^d = Z(\frac{2\pi}{N})$ as the stabilizer for the second step, thereby concatenating the homological code with a modular-qudit GKP code [38], Sec. II] that encodes a logical d-dimensional qudit into a dN-dimensional physical qudit. The effect of the stabilizer $\overline{Z}^d$ is to fix the angular momentum to be a multiple of N. Hence, we obtain a d-dimensional logical qudit Hilbert space, whose codewords are in Eq. (40) and logical operators are in Eq. (41).

In Appendix D, we calculate the Wigner function for GKP states in rotor space and find the function does have negativity which corresponds to the "magic" for continuous-variable systems. Furthermore, in Appendix E, we calculate the error-correction condition for regularized rotor GKP states and find it is related to Jacobi ϑ functions.

V. HOMOLOGICAL NUMBER-PHASE CODES

In this section, we turn back to the oscillator Fock space discussed in Sec. II and construct multimode oscillator codes correcting photon number changes and dephasing errors.

Quantum error-correcting codes only protect certain classes of errors. For example, the oscillator GKP codes are designed to correct (linear) quadrature displacements, but they underperform against (radial) dephasing errors, whose performance was analyzed in Ref. [51]. To protect information from these errors, we need to study bosonic codes with rotational symmetry in phase space which can naturally correct both photon number changes and dephasing noise. Although the landscape of single-mode rotation-symmetric bosonic codes is well investigated theoretically [45,46,88] and experimentally [89,90], constructions of multimode rotation-symmetric bosonic codes without concatenation remain unexplored. In this section, we provide a systematic way to construct multimode rotation-symmetric bosonic codes from homological rotor codes, which is an algebraic construction without directly concatenating single-mode rotation-symmetric bosonic codes with discrete-variable codes. We should emphasize that although these codes are inspired from rotor codes, they are constructed directly on oscillator modes, instead of constraining rotors in the non-negative angular-momentum subspace.

We draw inspiration from the fact that the codewords of number-phase oscillator code can be obtained from the rotor GKP code by projecting the former into the non-negative rotor Hilbert subspace. Extending this intuition, we show that *any* homological rotor code can be similarly mapped into oscillator number-phase code without changes of code parameters, after flipping the signs of some of the rotor momenta. This yields multimode rotation-symmetric bosonic codes which we call *homological number-phase codes* for oscillators, compatible with noise channels where photon loss is present but random-rotation (i.e., dephasing) noise is dominant.

We start from the correspondence between rotor GKP code and bosonic number-phase code as the most straightforward example.

Example 3. The number-phase code is a rotation-symmetric bosonic code protecting against photon loss and dephasing errors [45] (see also Ref. [91]). Its codewords are

$$|\bar{0}\rangle_{\mathrm{F}} = \sum_{k\in\mathbb{N}} |2kN\rangle, \quad |\bar{1}\rangle_{\mathrm{F}} = \sum_{k\in\mathbb{N}} |(2k+1)N\rangle. \tag{43}$$

This code can be obtained by projecting the rotor GKP code from Eq. (40) onto the non-negative rotor Hilbert subspace, and identifying said subspace with the Fock space of the oscillator.

The original rotor GKP code stabilizers, $Z(2k\pi/N)$ and $X(2N)$, become $Z(2k\pi/N)_{\mathrm{F}}$ and $X(2N)_{\mathrm{F}}^{\dagger}$ after projection, respectively. The new Z-type operator is still a stabilizer of the number-phase code, but $X(2N)_{\mathrm{F}}$ is no longer a stabilizer. Nevertheless, the two X- and Z-type operators form a semigroup, and the codestates are $+1$-eigenvalue *right* eigenstates of all semigroup elements [45,54].

Encouraged by the above single-rotor example, we can try to take an arbitrary homological rotor code and project its codewords onto the non-negative momentum subspace of each rotor. However, directly this can sometimes result in trivial codewords, as in Example 4.

Nevertheless, there is a simple remedy. The idea is to flip the orientation of certain rotors so that each codeword has nontrivial support on the non-negative momentum subspace of each rotor. In the example below, we convert the four-rotor current-mirror code [37] into its corresponding four-mode bosonic number-phase code.

Example 4. The four-rotor current-mirror code is defined by parity check matrices

$$H_X = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ -1 & -1 & 1 & 1 \end{pmatrix}, \quad H_Z = (1 \quad 1 \quad 1 \quad 1). \tag{44}$$

This code encodes a qubit, with logical codewords

$$\begin{aligned} |\bar{0}\rangle &= \sum_{l_1,l_2,l_3\in\mathbb{Z}} |l_1, 2l_2 - l_1, l_3, -2l_2 - l_3\rangle, \\ |\bar{1}\rangle &= \sum_{l_1,l_2,l_3\in\mathbb{Z}} |l_1, 2l_2 + 1 - l_1, l_3, -2l_2 - 1 - l_3\rangle, \end{aligned} \tag{45}$$

and logical operators $\overline{X} = X_2(1)^{\dagger} X_4(1), \overline{Z} = Z_3(\pi) Z_4(\pi)$. Applying the number-phase projection prematurely by cutting off all negative momenta, we see that the only survived codeword is the trivial state $|0, 0, 0, 0\rangle$ where the logical qubit subspace is eliminated. However, if we properly apply sign flips to H_X such that it becomes non-negative, then the codewords of deformed H_X can still store logical information.

For such an example, the required transformation can be done by flipping the momenta of the second and third rotors. The H_X matrix is then

$$H_X^+ = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 2 & 0 & 2 \end{pmatrix}, \quad H_Z^+ = (1 \quad -1 \quad -1 \quad 1), \tag{46}$$

and the code words become

$$\begin{aligned} |\bar{0}\rangle_+ &= \sum_{l_1,l_2,l_3\in\mathbb{Z}} |l_1, 2l_2 + l_1, l_3, 2l_2 + l_3\rangle, \\ |\bar{1}\rangle_+ &= \sum_{l_1,l_2,l_3\in\mathbb{Z}} |l_1, 2l_2 + 1 + l_1, l_3, 2l_2 + 1 + l_3\rangle. \end{aligned} \tag{47}$$

Applying the projector $\Pi_{\geqslant 0}^{\otimes 4}$ to the codewords yields

$$\begin{aligned} |\bar{0}\rangle_{\mathrm{F}} &= \Pi_{\geqslant 0}^{\otimes 4} |\bar{0}\rangle_+ = \sum_{l_1,l_2,l_3\in\mathbb{N}} |l_1, 2l_2 + l_1, l_3, 2l_2 + l_3\rangle, \\ |\bar{1}\rangle_{\mathrm{F}} &= \Pi_{\geqslant 0}^{\otimes 4} |\bar{1}\rangle_+ = \sum_{l_1,l_2,l_3\in\mathbb{N}} |l_1, 2l_2 + 1 + l_1, l_3, 2l_2 + 1 + l_3\rangle, \end{aligned} \tag{48}$$

which are the codewords of four-mode bosonic number-phase code with logical operators

$$\overline{X} = X_2(1)_{\mathrm{F}}^{\dagger} X_4(1)_{\mathrm{F}}^{\dagger}, \quad \overline{Z} = Z_3(-\pi) Z_4(\pi). \tag{49}$$

We called this code as current-mirror number-phase code. The stabilizer semigroup of Eq. (48) is generated by

$$\{X_1(1)_{\mathrm{F}}^{\dagger} X_2(1)_{\mathrm{F}}^{\dagger}, X_3(1)_{\mathrm{F}}^{\dagger} X_4(1)_{\mathrm{F}}^{\dagger}, X_2(2)_{\mathrm{F}}^{\dagger} X_4(2)_{\mathrm{F}}^{\dagger}, \\ Z_1(\phi)_{\mathrm{F}} Z_2(\phi)_{\mathrm{F}}^{\dagger} Z_3(\phi)_{\mathrm{F}}^{\dagger} Z_4(\phi)_{\mathrm{F}}\}. \tag{50}$$

In Example. 4, we explicitly construct the four-mode number-phase code by flipping and projecting the four-rotor current-mirror code. The next question would be whether the above technique is applicable to general homological rotor codes such that we can always construct multimode rotation-symmetric bosonic codes by flipping and projecting homological rotor codes. The following proposition proves that a sequence of flipping operations that gives non-negative H_X can always be found.

Proposition 1. For a given homological n-rotor code with matrices H_X and H_Z, there exists an unimodular and diagonal matrix S, such that all logical codewords of the code defined by $H_X' = H_X S$ and $H_Z' = H_Z (S^T)^{-1}$ have support on the orthant where each rotor has non-negative momentum.

Proof. The code subspace is a sublattice that passes through the origin in $\mathbb{Z}^n$ lattice grids. So it must have support on some certain orthants. Because code words are defined by the quotient under $\mathrm{Im}(H_X)$, each code word is translation invariant in the direction of the row vectors of H_X. This means if the ker H_Z^T sub-lattice passes through an orthant, this orthant should overlap with the support of each code word.

There always exists a series of flipping operations to transform a given orthant to the orthant in which all the integers are positive, denoted as $(+,+,+,\dots)$. Specifically, an orthant labeled by $(+,-,+,-,-,\dots)$ is mapped to $(+,+,+,\dots)$ via the $\mathbb{Z}_2^n$ element $I \otimes \mathrm{P} \otimes I \otimes \mathrm{P} \otimes \mathrm{P} \cdots$, with the isomorphism $+ \to I$, $- \to \mathrm{P}$. ■

The procedure to map a homological rotor code to its corresponding homological number-phase code is as follows:

(1) Given a n-rotor homological rotor code with H_X, we flip the parities of part of rotors to make $H_X + = H_X S$ so that all its row vectors are in the all positive orthant. This transformation will affect H_Z as well, such that $H_Z^+ = H_Z S$. Here, S is a diagonal matrix composed of parity flips P whose diagonal elements are ±1. The effect of S is to convert all CNOT† gates included in the encoding circuit to CNOT gates.[3] Hence, in practice, we just need to replace every CNOT† by CNOT and eliminate pre-existing parity flips P in the encoding circuit.

(2) Project onto non-negative angular-momentum subspace by applying $\Pi_{\geqslant 0}^{\otimes n}$ on each rotor to obtain a homological number-phase code that is stabilized by a stabilizer semigroup given by H_X^+ and H_Z^+. The resulting homological number-phase code is stabilized by the stabilizer semigroup

$$\mathcal{S}^+ = \{X(\vec{\boldsymbol{m}}^T H_X^+)_{\mathrm{F}}^{\dagger} Z(\vec{\boldsymbol{\phi}}^T H_Z^+)_{\mathrm{F}} \,\forall\, \vec{\boldsymbol{m}} \in \mathbb{N}^{r_X}, \forall\, \vec{\boldsymbol{\phi}} \in \mathbb{T}^{r_Z}\}. \tag{51}$$

[3]It can be shown that $\mathrm{P}_1 \mathrm{CNOT}_{1\to 2}^{\dagger} \mathrm{P}_1 = \mathrm{P}_2 \mathrm{CNOT}_{1\to 2}^{\dagger} \mathrm{P}_2 = \mathrm{CNOT}_{1\to 2}$.

Similar to the single-mode case, we can assign a distance against rotation errors to the resulting homological number-phase code. Phase states are not quite orthogonal (see Ref. [54] for the case of a single mode), so homological number-phase codes are not exactly error correcting. Nevertheless, the sign flipping is done by the single-rotor Clifford operation P from Eq. (26), so the entanglement structure of the code, and thus its intended degree of protection, both carry over. Since configuration-space distances $|(\theta - \phi) \bmod 2\pi|$ are invariant under sign flipping, homological number-phase codes can be described by the distances of their planar-rotor counterparts, but protect against rotation errors in an approximate sense.

A. Number-phase Clifford encodings

Similar to our treatment in Sec. II C, we can construct the Clifford operations in oscillator Fock basis by projecting the rotor Clifford group onto the non-negative angular-momentum subspace. The comparison of the generators for both groups are shown in Table III. Since encoding circuits of stabilizer codes are composed of Clifford operations, investigating the Clifford operations for oscillator Fock space helps us construct and understand the encoding circuit of homological number-phase code. In this subsection, we utilize this Clifford semigroup for oscillator Fock space to encode and decode information into homological number-phase codes.

After applying the mapping between rotor and oscillator Fock space, we see that the rotor Clifford-group operations $\mathrm{QUAD}_{\mathrm{F},\varphi}$ and $\mathrm{CPHS}_{\mathrm{F},\varphi}$ are unitary operators, but $\mathrm{CNOT}_{\mathrm{F}}$ is not. Indeed, we can show that

$$\mathrm{CNOT}_{\mathrm{F}}^{\dagger}\mathrm{CNOT}_{\mathrm{F}} = \mathbb{I}, \\ \mathrm{CNOT}_{\mathrm{F}}\mathrm{CNOT}_{\mathrm{F}}^{\dagger} = \sum_{n=0}^{\infty} |n\rangle\langle n| \otimes \Pi_{\geqslant n}. \tag{52}$$

Nevertheless, $\mathrm{CNOT}_{\mathrm{F}}(\cdot)\mathrm{CNOT}_{\mathrm{F}}^{\dagger}$ remains a valid quantum channel that can be utilized for encoding.

Analogous to Eqs. (30b), (30c), and (30d), we can determine how the Clifford gates for oscillator Fock space transform the occupation number and phase quadratures:

$$\mathrm{CNOT}_{1\to 2,\mathrm{F}}(X(1)_{\mathrm{F}} \otimes \mathbb{I})\mathrm{CNOT}_{1\to 2,\mathrm{F}}^{\dagger} = X(1)_{\mathrm{F}} \otimes X(1)_{\mathrm{F}}, \\ \mathrm{CNOT}_{1\to 2,\mathrm{F}}(X(-1)_{\mathrm{F}} \otimes \mathbb{I})\mathrm{CNOT}_{1\to 2,\mathrm{F}}^{\dagger} = X(-1)_{\mathrm{F}} \otimes X(-1)_{\mathrm{F}}, \\ \mathrm{CNOT}_{1\to 2,\mathrm{F}}(\mathbb{I} \otimes Z(\phi)_{\mathrm{F}})\mathrm{CNOT}_{1\to 2,\mathrm{F}}^{\dagger} = Z(-\phi)_{\mathrm{F}} \otimes Z(\phi)_{\mathrm{F}}, \\ \mathrm{QUAD}_{\mathrm{F},\varphi} X(1)_{\mathrm{F}} \mathrm{QUAD}_{\mathrm{F},\varphi}^{\dagger} = Z(\varphi)_{\mathrm{F}} X(1)_{\mathrm{F}}, \\ \mathrm{CPHS}_{\mathrm{F},\varphi}(X(1)_{\mathrm{F}} \otimes \mathbb{I})\mathrm{CPHS}_{\mathrm{F},\varphi}^{\dagger} = X(1)_{\mathrm{F}} \otimes Z(\varphi)_{\mathrm{F}}, \\ \mathrm{CPHS}_{\mathrm{F},\varphi}(\mathbb{I} \otimes X(1)_{\mathrm{F}})\mathrm{CPHS}_{\mathrm{F},\varphi}^{\dagger} = Z(\varphi)_{\mathrm{F}} \otimes X(1)_{\mathrm{F}}. \tag{53}$$

This shows that, despite the presence of nonunitarity, number-phase Clifford operations can be used to perform conditional operations and extract syndrome information for homological number-phase codes. Notably, the number-phase Clifford operations transforms the number-phase Pauli semigroup in

TABLE III. Planar-rotor Clifford operations as realized by the "number-phase" interpretation of the oscillator from Sec. II; each number-phase operator has the subscript F in the main text. These are obtained by projecting planar-rotor operators into the subspace of non-negative momentum states and interpreting said states as oscillator Fock states $|n\rangle$. Parity flip operations are not present because they flip the sign of the momentum states. Pauli-X rotations are mapped into powers of the nonunitary Pegg-Barnett phase operator [35,58,59], which performs photon subtraction and injection. The CNOT gate is mapped to a nonunitary controlled photon injection operator. The Pauli and symplectic groups become semigroups with identity (i.e., monoids), but both nonunitary operators U remain valid quantum channels because $U^\dagger U = \mathbb{I}$.

Planar rotor	Oscillator Fock space	
Pauli X (5)	Photon injection	$\sum_{n=0}^{\infty} \|m+n\rangle\langle m\|$
Pauli Z (5)	Rotation	$e^{i\varphi\hat{n}}$
CNOT (30b)	Controlled photon injection	$\sum_{n,m=0}^{\infty} \|n\rangle\langle n\| \otimes \|m+n\rangle\langle m\|$
QUAD (30c)	Kerr interaction	$e^{i\varphi\hat{n}(\hat{n}+1)/2}$
CPHS (30d)	Cross-Kerr interaction	$e^{i\varphi\hat{n}\otimes\hat{n}}$

the same way as the relevant part of the rotor Clifford group transforms the rotor Pauli operators. Hence, the symplectic representation we constructed in Sec. III is applicable to calculate the Clifford transformation of Pauli operators for the oscillator Fock basis.

Example 5. We use the same current-mirror number-phase code as in Example 4. The initial state is stabilized by the stabilizers

$$\begin{aligned}\mathcal{S}_0 &= \langle X_1(1)_{\mathrm{F}}^\dagger, X_2(2)_{\mathrm{F}}^\dagger, X_3(1)_{\mathrm{F}}^\dagger, Z_4(\phi)_{\mathrm{F}}\ \forall\,\phi\in\mathbb{T}\rangle \\ &= \langle(-1,0,0,0|\mathbf{0})^T, (0,-2,0,0|\mathbf{0})^T, (0,0,-1,0|\mathbf{0})^T, \\ &\quad (\mathbf{0}|0,0,0,\phi)^T\ \forall\,\phi\in\mathbb{T}\rangle, \end{aligned} \tag{54}$$

in which we use $\mathbf{0}$ to denote (0,0,0,0). The second rotor encodes logical qubit information via a single rotor code with torsion $\mathbb{Z}_2$. We can explicitly write down the initial state as

$$|\psi_0\rangle = \sum_{l_1,l_2,l_3\in\mathbb{N}} |l_1\rangle\otimes(\alpha|2l_2\rangle+\beta|2l_2+1\rangle)\otimes|l_3\rangle\otimes|0\rangle. \tag{55}$$

The logical Pauli operators for the initial state are $X = X_2(1)^\dagger$, $Z = Z_2(\pi)$. They admit a vector representation as

$$X = (0,-1,0,0|\mathbf{0})^T, \quad Z = (\mathbf{0}|0,\pi,0,0)^T. \tag{56}$$

The symplectic representation of the encoding circuit is

$$Q^+ = \begin{pmatrix} A_{\mathrm{enc}}^+ & 0 \\ 0 & (A_{\mathrm{enc}}^{+T})^{-1}\end{pmatrix}, \quad A_{\mathrm{enc}}^+ = \begin{pmatrix} 1&1&0&0\\0&1&0&1\\0&0&1&1\\0&0&0&1\end{pmatrix}, \tag{57}$$

which corresponds to the encoding circuit $U_{\mathrm{enc}}^+ = \mathrm{CNOT}_{1\to2,\mathrm{F}}\mathrm{CNOT}_{3\to4,\mathrm{F}}\mathrm{CNOT}_{2\to4,\mathrm{F}}$. The original encoding circuit (before parity flips) of stabilizers shown in Eq. (44) is $U_{\mathrm{enc}} = \mathrm{CNOT}_{1\to2}^\dagger\mathrm{CNOT}_{3\to4}^\dagger\mathrm{CNOT}_{2\to4}^\dagger = \mathrm{P}_2\mathrm{P}_3U_{\mathrm{enc}}^+\mathrm{P}_2\mathrm{P}_3$.

The stabilizers and logical operators of those codewords can be calculated according to the method in Sec. III. We then transform $\mathcal{S}_0$, X, Z by multiplying them by Q^+, and we obtain

$$\begin{aligned}\mathcal{S} &= \{(-1,-1,0,0|\mathbf{0})^T, (0,-2,0,-2|\mathbf{0})^T, \\ &\quad (0,0,-1,-1|\mathbf{0})^T, (\mathbf{0}|\phi,-\phi,-\phi,\phi)^T\}\ \forall\,\phi\in\mathbb{T}, \\ \overline{X} &= (0,-1,0,-1|\mathbf{0})^T, \quad \overline{Z} = (\mathbf{0}|-\pi,\pi,0,0)^T, \end{aligned} \tag{58}$$

which are the stabilizer semigroup and logical operators of the codewords.

After we apply U_{enc}^+ to $|\psi_0\rangle$, we can see that the codestate is written as

$$|\psi\rangle = U_{\mathrm{enc}}^+|\psi_0\rangle = \alpha|\overline{0}\rangle_{\mathrm{F}} + \beta|\overline{1}\rangle_{\mathrm{F}}, \tag{59}$$

where $|\overline{0}\rangle_{\mathrm{F}}$, $|\overline{1}\rangle_{\mathrm{F}}$ are codewords shown in Eq. (48).

The main challenge of syndrome measurement of homological number-phase codes using the Clifford semigroup is measuring X-type stabilizers. This is because the conditional photon subtraction, $\mathrm{CNOT}_{np}^\dagger$, is not a completely positive trace-preserving (CPTP) map. However, we can construct the following CPTP map

$$\begin{aligned}\mathcal{D}:\rho &\to \mathrm{CNOT}_{\mathrm{F}}^\dagger\rho\,\mathrm{CNOT}_{\mathrm{F}} + \mathcal{P}_C^\dagger\rho\mathcal{P}_C \\ &= \mathcal{D}_1(\rho)+\mathcal{D}_2(\rho),\end{aligned} \tag{60}$$

where $\mathcal{P}_C = \sum_{n=0}^{\infty}|n\rangle\langle n|\otimes(\sum_{m=0}^{n-1}|m\rangle\langle m|)$ is a projector into the subspace where the photon number on the second mode is smaller than the photon number on the first mode. This CPTP map can decomposed to two non-CPTP maps $\mathcal{D}_1$ and $\mathcal{D}_2$. The $\mathcal{D}_1$ is the map we desire to implement for X-type syndrome extraction while the $\mathcal{D}_2$ is undesired. Hence, the syndrome extraction process $\mathcal{D}$ is a probabilistic process. Nevertheless, the syndrome information can be obtained after postselecting on $\mathcal{D}_1$.

Non-destructive but probabilistic syndrome extraction is likely not the optimal way to readout syndromes for this code. We anticipate that Knill's (destructive) teleportation-based quantum error-correction scheme [92], used to correct rotation-symmetric quantum codes [46,51], can be generalized to this multimode case. We leave the investigation of this scheme to a follow-up work on related codes.

TABLE IV. Comparison between planar rotor GKP-stabilizer, embedded rotor GKP-stabilizer, and number-phase GKP-stabilizer codes outlined in Sec. VI.

	Planar rotor GKP-stabilizer code	Embedded rotor GKP-stabilizer code	Number-phase GKP-stabilizer code
Physical space	Rotors	Oscillators	Oscillators
Logical space	k rotors	k rotors	k oscillators
State on $n-k$ subsystems	Rotor GKP code	Oscillator GKP code	Number-phase code
Encoding circuit	Rotor Clifford	Rotor Clifford	Clifford semigroup
Error	Rotor Pauli (5)	Oscillator displacements	Photon loss and dephasing (14)

VI. GKP-STABILIZER CODES FOR ROTORS

GKP-stabilizer codes [34] have been proposed as a way to encode logical oscillators into physical oscillators and utilize oscillator GKP states to protect against displacement noise. In this section, we consider rotor versions of GKP-stabilizer codes [34], which allow one to protect an infinite-dimensional logical space against displacement noise. The interpretation of the harmonic oscillator number-phase basis allows us to map such codes back into the oscillator, yielding codes protecting against bosonic dephasing errors.

The encoding of each version consists of placing logical information into k subsystems (either rotors or oscillators), initializing the remaining $n-k$ subsystems in a particular resource state, and applying a Clifford circuit. The noise that the code is suitable for depends on the resource state. The ingredients for each code are summarized in Table IV.

(1) *Planar rotor GKP-stabilizer codes.* We encode k logical rotors into n physical rotors, with $n-k$ rotors initialized in rotor GKP states. This provides a way to protect logical rotors, which are compact and infinite-dimensional, from physical rotor Pauli X and Z errors. Although the homological rotor code [37] can also encode logical rotors into physical rotors, its Z-type stabilizers are continuous, as shown in Eq. (32), while the rotor GKP-stabilizer code inherits the discrete stabilizer group of the rotor GKP states on the $n-k$ rotors. This construction can correct the rotor Pauli X and Z error acting on the k logical rotors.

(2) *Embedded rotor GKP-stabilizer codes.* This code can be regarded as a GKP-stabilizer code whose logical space forms rotors instead of oscillators due to the extra embedded-rotor stabilizer constraint placed on the k logical modes. We first encode k logical rotors into k logical oscillator modes via the embedded rotor technique. Then, we encode the k logical oscillator modes into n multiple oscillator modes by initializing $n-k$ modes in oscillator GKP states[4] and applying a Gaussian transformation. We can also regard this code as a concatenation between rotor-to-oscillator code and oscillator GKP-stabilizer code. As with the regular GKP-stabilizer code, this code can correct position and momentum displacements.

(3) *Number-phase GKP-stabilizer codes.* Analogous to the planar rotor GKP-stabilizer codes, we encode k logical oscillators into n physical oscillators[5] by initializing $n-k$ oscillators in single-mode number-phase codestates and applying a Clifford semigroup circuit as encoder. Such construction can be regarded as the polar coordinate generalization of oscillator GKP-stabilizer codes which are formulated in the lattices of Cartesian coordinates (position and momentum). The number-phase GKP-stabilizer codes can protect logical oscillator modes from the photon number changes and dephasing noise.

In the following of this section, we use the GKP-repetition code as an example to demonstrate all three constructions.

Example 6. We would like to encode a single logical rotor into two physical rotors while being able to detect Z-type rotor Pauli errors. We place the logical information, denoted by the function $\psi(k)$ for integer k, into the first rotor, initialize the second rotor in a rotor GKP state, and apply a CNOT gate. This yields

$$\begin{aligned}|\psi\rangle_{\text{code}} &= \text{CNOT}_{2\to 1}\left(\sum_{k\in\mathbb{Z}}\psi(k)|k\rangle\right)\otimes\left(\sum_{\ell\in\mathbb{Z}}|m\ell\rangle\right)\\ &= \sum_{k,\ell\in\mathbb{Z}}\psi(k)|m\ell+k\rangle\otimes|m\ell\rangle,\end{aligned}\tag{61}$$

where all constituent states are rotor momentum states. This code is stabilized by rotor Pauli strings $X(m)\otimes X(m)$ and $\mathbb{I}\otimes Z(\frac{2\pi}{m})$. The logical Pauli operator of the encoded rotor becomes $\overline{X}(l)=X(l)\otimes\mathbb{I}\ \forall\ l\in\mathbb{Z}$ and $\overline{Z}(\phi)=Z(\phi)\otimes Z(-\phi)$ $\forall\ \phi\in\mathbb{T}$.

Using Eq. (30b), rotor Pauli errors propagate as

$$\begin{aligned}\text{CNOT}_{1\to 2}(X(l)\otimes\mathbb{I}) &= (X(l)\otimes X(l))\text{CNOT}_{1\to 2},\\ \text{CNOT}_{1\to 2}(\mathbb{I}\otimes Z(\phi)) &= (Z(-\phi)\otimes Z(\phi))\text{CNOT}_{1\to 2},\end{aligned}\tag{62}$$

enabling us to extract error syndromes as follows.

Suppose we have a codeword corrupted by a dephasing error $Z(\xi_1^Z)\otimes Z(\xi_2^Z)$, the noisy codeword is written as $Z(\xi_1^Z)\otimes Z(\xi_2^Z)|\psi\rangle_{\text{code}}$. Then we initialize an ancillary third mode in the state

$$|+\rangle_{\text{U(1)}}\propto\sum_{n\in\mathbb{Z}}|nm\rangle=\sum_{n=0}^{m-1}\left|\theta=\frac{2\pi n}{m}\right\rangle,\tag{63}$$

[4]They are concatenation between oscillator GKP codes and qudit GKP codes (see Sec. IV C), here we still call them as oscillator GKP codes because of its comb structure.

[5]Note that number-phase basis is another description of the oscillator, hence this construction is for oscillator-to-oscillator codes that correct photon number changing and dephasing errors.

where $|\theta\rangle$ is a rotor position state. Finally, we apply $\text{CNOT}_{3\to1}\text{CNOT}_{3\to2}$. This yields

$$\text{CNOT}_{3\to1}\text{CNOT}_{3\to2}\big(Z(\xi_1^Z)\otimes Z(\xi_2^Z)\big)|\psi\rangle_{\text{code}}\otimes|+\rangle_{\text{U(1)}} \tag{64a}$$

$$=\big(Z(\xi_1^Z)\otimes Z(\xi_2^Z)\otimes Z(-\xi_1^Z-\xi_2^Z)\big)\text{CNOT}_{3\to1}\text{CNOT}_{3\to2}|\psi\rangle_{\text{code}}\otimes|+\rangle_{\text{U(1)}} \tag{64b}$$

$$=\big(Z(\xi_1^Z)\otimes Z(\xi_2^Z)\otimes Z(-\xi_1^Z-\xi_2^Z)\big)\sum_{k,n,f\in\mathbb{Z}}\psi(k)|m(n+f)+k\rangle\otimes|m(n+f)\rangle\otimes|mf\rangle \tag{64c}$$

$$=\big(Z(\xi_1^Z)\otimes Z(\xi_2^Z)\otimes Z(-\xi_1^Z-\xi_2^Z)\big)|\psi\rangle_{\text{code}}\otimes|+\rangle_{\text{U(1)}} \tag{64d}$$

$$=\big(Z(\xi_1^Z)\otimes Z(\xi_2^Z)\otimes Z(-\xi_1^Z-\xi_2^Z)\big)|\psi\rangle_{\text{code}}\otimes\sum_{f=0}^{m-1}\left|\theta=\frac{2\pi f}{m}\right\rangle \tag{64e}$$

$$=\big(Z(\xi_1^Z)\otimes Z(\xi_2^Z)\otimes\mathbb{I}\big)|\psi\rangle_{\text{code}}\otimes\sum_{f=0}^{m-1}\left|\theta=\frac{2\pi f}{m}+\xi_1^Z+\xi_2^Z\right\rangle. \tag{64f}$$

Then, by performing a phase-basis projective measurement on the third ancillary mode, we extract the syndrome $(\xi_1^Z+\xi_2^Z) \bmod 2\pi/m$. The correctable Z error would be in the interval $(\xi_1^Z+\xi_2^Z)\in[-\pi/m,\pi/m)$ where $(\xi_1^Z+\xi_2^Z) \bmod 2\pi/m=\xi_1^Z+\xi_2^Z$. Then we apply the decoding unitary $\text{CNOT}_{2\to1}^\dagger$ to the noisy codeword and obtain

$$\big(Z(\xi_1^Z+\xi_2^Z)\otimes\mathbb{I}\big)\left(\sum_{k\in\mathbb{Z}}\psi(k)|k\rangle\right)\otimes\left(\sum_{l\in\mathbb{Z}}|ml\rangle\right). \tag{65}$$

Then we perform error correction by applying $Z[-(\xi_1^Z+\xi_2^Z)/2]$ such that the logical Z variance is $\text{Var}[(\xi_1^Z+\xi_2^Z)/2]$. In the general case, where the X and Z errors appear simultaneously such as $X(\xi_1^X)Z(\xi_1^Z)\otimes X(\xi_2^X)Z(\xi_2^Z)$, aside from the above analysis, we can use another ancilla to measure the X error acting on the second rotor by measuring stabilizer $\mathbb{I}\otimes Z(\frac{2\pi}{m})$. The error-correction procedure of this code for the general case is to prepare two ancilla rotors in $|+\rangle_{U(1)}$ (one for the Z error and another for the X error). For the X error, we can extract the syndrome $\xi_2^X \bmod m$ by applying $\text{CNOT}_{2\to4}$ and measure the angular momentum on the fourth rotor. The correctable error X error would be in the interval $\xi_2^X\in[-\frac{m}{2},\frac{m}{2})$. Sharing the spirit of oscillator GKP-stabilizer codes [34], if we assume the X- and Z-type errors are identical and independent random variables and their variances are much smaller than π/m and m, respectively, then such construction can reduce the variance of Z-type error acting on the logical rotor to $1/2$ while it does not amplify the X-type error.

We can repeat the projection procedure we discussed in Sec. V to obtain number-phase resource states for this version of GKP-stabilizer codes. For this number-phase version, all ingredients are replaced by their number-phase counterparts. The stabilizer becomes $X(m)_{\text{F}}^\dagger\otimes X(m)_{\text{F}}^\dagger$ and $\mathbb{I}\otimes Z(\frac{2\pi}{m})_{\text{F}}$. The Z-type errors correspond to the dephasing channels in bosonic systems, and X-type errors (momentum kicks) correspond to the photon loss channel in bosonic systems.

VII. DISCUSSION AND OUTLOOK

In Sec. II, we discuss how to faithfully realize a planar rotor, which we call the embedded rotor, inside a harmonic oscillator. This provides a way to investigate quantum systems described by U(1) degrees of freedom with harmonic oscillators with modular variables [93]. We then treat the Fock space of the oscillator as the non-negative angular-momentum subspace of the rotor by using the analog between the rotor Pauli operator and the Pauli semigroup of oscillator Fock basis. This enables us to construct oscillator codes against photon number changing and dephasing errors by adapting homological rotor codes in Sec. V. This yields a multimode generalization of number-phase codes [51] which we call homological number-phase codes. The performance of homological number-phase codes and the number-phase uncertainty relation of these codes are left for future studies. Moreover, our treatment of oscillator Fock space provides another approach to realize rotor algebras in harmonic oscillator systems. Its use in quantum simulation of many-body physics is also an interesting question.

We investigate rotor Clifford operations in Sec. III. We identify its structure and its generators. They are fundamental for analyzing the encoding/decoding of rotor codes, as well as code deformation, which leads us to the development of bosonic number-phase codes. It is a practical direction to study the realization of the whole set of rotor Clifford generators in experimental platforms.

In Sec. IV, we explain the method of calculating the logical space of homological rotor codes via Smith normal form. Homological rotor codes are classified by their torsion parts, and the torsion structure is invariant under CSS rotor Clifford transformations. This classification is tightly related to the classification of GKP code lattices. We find a relation between the single-rotor code with torsion and the oscillator GKP qudit in the embedded rotor formalism. Also, we show that rotor GKP codes can be treated as a concatenation between homological rotor codes and modular-qudit GKP codes.

Drawing an analogy to oscillator Gaussian states, we study the rotor nullifier states (analogues of position and momentum eigenstates), coherent states, Josephson junction Hamiltonians, as well as their transformations under the rotor Clifford group in Appendix C. Due to the structure of the rotor Clifford group, rotor nullifier states and coherent states are not related via rotor Clifford transformations. These problems call for a suitable definition of the Gaussian and non-Gaussian states for rotors, as well as a quantification of non-Gaussianity, or "magic" [94–98], in rotor states. It is worthwhile to continue this direction since studies of magic and non-Gaussianity of quantum states on discrete and continuous-variable quantum systems yield various results on realizing fault-tolerant quantum computing [98–104], estimation of quantum information resources [96,105–107], and characterizing exotic quantum phases of matter [108].

Having quantified some of the basic computational primitives for the planar rotor, we leave open the question of the Clifford hierarchy [109–111]. We believe that, once this hierarchy is determined for oscillator systems, projecting each member of the hierarchy into the oscillator's embedded rotor should be useful in backing out the corresponding operators for the planar rotor.

It would be theoretically important as well as practically useful to develop analogous symplectic primitives for more general quantum systems, such as molecules, rigid bodies and other group-valued qudits. They may open the possibilities of utilizing other physical platforms, as well as providing better candidates for the study of holographic gravity. This would be an interesting direction to pursue in the future.

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APPENDIX A: NO SQUEEZING FOR PLANAR ROTORS

In this Appendix, we recap that (unitary) squeezing is not a Clifford operation for rotors because it is an automorphism of $\mathbb{R}^n$ but not of $\mathbb{T}^n$ or $\mathbb{Z}^n$.

The squeezing of a single-mode harmonic oscillator squeezes one quadrature and dilates its conjugate pair, $\hat{q} \to e^r\hat{q}$ and $\hat{p} \to e^{-r}\hat{p}$ for some real parameter r. It is a symplectic automorphism in $\mathbb{R}$ so is an element of the oscillator Clifford group. Notably, squeezing preserves the spectrum of both the position and momentum operators, which is all of the reals.

For rotor systems, a map multiplying the angular position by a constant c, $\theta \to c\theta$, can only be an automorphism for $c = \pm 1$. Hence, squeezing a rotor is impossible. The squeezing we defined is analogous to the quadrature squeezing of harmonic oscillator: $\hat{q} \to c\hat{q}$ such that $\text{Var}(q) \to c\text{Var}(q)$. In the context of quantum metrology, the spin squeezing is usually defined as $\text{Var}(S_z) \to c\text{Var}(S_z)$ without requiring $\hat{S}_z \to c\hat{S}_z$, the interplays between squeezed rotor states [112] in the context of quantum metrology and rotor Clifford operations will be left for future investigation.

For multiple rotors, the automorphism of $\mathbb{T}^n$ is $\text{GL}_n(\mathbb{Z})$. Because any matrix of $\text{GL}_n(\mathbb{Z})$ has determinant ± 1, there should not exist an overall squeezing in the phase basis. Because a $\text{GL}_n(\mathbb{Z})$ matrix can be diagonalized via unimodular matrices, and unimodular matrices have determinant ± 1, this implies that the resulting diagonal matrix should also have determinant ± 1. Since the entries of it should be integers, the eigenvalues can only be ± 1. This means for multiple rotor modes, there does not exist an automorphism that squeezes one collective mode while stretches another collective mode under the phase basis.

The angular-momentum basis has group structure $\mathbb{Z}^n$ and its group automorphism group is $\text{GL}_n(\mathbb{Z})$ as well. The same argument shows that squeezing is not an automorphism of this basis either.

Not having unitary squeezing is a generic feature in other systems as well, e.g., modular-qudit systems whose states are valued in $\mathbb{Z}_d$. More generally, if one of the quadratures is valued in a compact group, squeezing should not be able to be realized via a unitary operation, as it cannot be an automorphism of the group.

APPENDIX B: PASSIVE SYMPLECTIC SUBGROUP

The oscillator symplectic group has an important subgroup—the group of *passive* transformations that preserve the total energy (i.e., photon number) of the oscillators [4]. The passive symplectic group for rotor systems can be similarly defined as preserving the total energy of the rotors. For n identical rotors, the total energy should be proportional to the sum of angular momentum squared of each rotor, $\sum_i l_i^2$. This yields the corresponding passive symplectic group of the rotor.

The passive symplectic group of the oscillator preserves the total photon number, $\sum_i \hat{a}_i^\dagger \hat{a}_i = n/2 + \sum_i(\hat{q}_i^2 + \hat{p}_i^2)$. Collecting positions and momenta into a $2n$-dimensional vector $\vec{v}$, we see that passive transformations have to preserve the inner product $\vec{v}^T\vec{v}$. This constraint defines a $2n$-dimensional real sphere in phase space, meaning that any passive transformation has to be an element of the sphere's proper-rotation symmetry group, $\text{SO}(2n)$. Taking the intersection of this group with the symplectic group yields $\text{Sp}_{2n}(\mathbb{R}) \cap \text{SO}(2n) \cong \text{U}(n)$, the n-dimensional unitary group.

The real $2n$-dimensional sphere can equivalently be thought of as a complex n-dimensional sphere, whose corresponding constraint can be formulated using the vector of annihilation operators, $\vec{a} = (\hat{a}_1, \hat{a}_2, \dots, \hat{a}_n)$. Passive transformations form that sphere's symmetry group, $\text{U}(n)$. Since coherent states are eigenstates of annihilation operators, they can be thought of as points on said sphere. Passive transformations rotate these points, preserving the tensor product structure of coherent states on different modes,

$$U|\alpha_1\rangle \otimes \cdots |\alpha_n\rangle = |\tilde{\alpha}_1\rangle \otimes \cdots |\tilde{\alpha}_n\rangle, \tag{B1}$$

where $\tilde{\alpha}_i = \sum_j U_{ij}\alpha_j$.

In the symplectic representation, a general passive symplectic element can be written as

$$S = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}, \quad \text{where} \begin{cases} AB^T - BA^T = 0, \\ AA^T + BB^T = \mathbb{I}_{n\times n}. \end{cases} \tag{B2}$$

In the rotor case, according to the analysis in Sec. III A, A should be in $\mathrm{GL}(n, \mathbb{Z})$ and the upper-right block B must be empty. Hence, rotor passive transformations are written as

$$S = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}, \quad \text{where} \begin{cases} AA^T = \mathbb{I}_{n\times n}, \\ A \in \mathrm{GL}_n(\mathbb{Z}). \end{cases} \tag{B3}$$

In words, to preserve the square of the total angular momentum, $\sum_i l_i^2$, a passive transformation should be an element of $\mathrm{O}(n)$. In the meantime, l_i are integers, so the transformation should take value in $\mathrm{GL}_n(\mathbb{Z})$. These conditions fix the rotor passive transformation group to be the signed symmetric group (also known as the hyperoctahedral group),

$$\mathrm{O}(n) \cap \mathrm{GL}_n(\mathbb{Z}) = \mathbb{Z}_2 \wr S_n, \tag{B4}$$

generated by SWAP and parity flip P (where $\wr$ is the wreath product).

While the oscillator group is a rich and complex amalgamation of SU(2) mode-mixing transformations (beam splitters) and U(1) single-mode rotations (phase shifters) [4], its rotor counterpart consists of only permutations and momentum parity flips. A parity flip is inherited from the π-phase shift, while a SWAP is passed down from the 50-50 beam splitter.

In passing, we note that the lack of beam-splitters and squeezing precludes us from mapping any interesting Gaussian bosonic channels [3] into the rotor state space.

APPENDIX C: ROTOR GAUSSIAN STATES

Gaussian states play a fundamental role in many aspects of quantum physics of harmonic oscillator systems. They are a class of states with non-negative Wigner functions which can be described just by their first and second moments. The evolution of bosonic Gaussian states under symplectic unitaries can be efficiently simulated by classical algorithms via tracking the changes of their first and second moments [41,114]. The definition of rotor Gaussian states are the rotor states with non-negative Wigner functions. In the cases of planar rotor, the symplectic representation of Clifford circuits also gives us a classically efficient simulation algorithm [31].

The oscillator nullifier states, coherent states, squeezed states, as well as thermal states of free Hamiltonians, are all important examples of oscillator Gaussian states. We wish to enumerate the counterparts of nullifier, coherent, and squeezed coherent states of the rotor system with similar properties to oscillator Gaussian states.

In this Appendix, we discuss the analogy of nullifier states and coherent states in rotor systems and compare them to their counterparts in oscillator systems. We also discuss the transformation of Josephson-junction rotor Hamiltonians under the rotor Clifford group.

1. Rotor nullifier states

For oscillator systems, the position and momentum eigenstates are called *nullifier states* [83]. They are δ functions localized at given position or momentum values and can thought of as infinitely squeezed coherent states. Oscillator nullifier states are an important class of oscillator Gaussian states and have been studied in the context of continuous-variable quantum computing and error correction [53,83,115–118].

Single-rotor nullifier states are the angle phase states and the angular-momentum eigenstates. However, only rotor angular-momentum eigenstates, whose Wigner functions are delta functions corresponding to the limit of sharp Gaussian with variance closed to zero, are proper Gaussian states for rotors while the phase states are not Gaussian [27]. One way to understand this is to project oscillator Gaussian states into the embedded rotor subspace and look for those states that remain Gaussian after projection. Only momentum states satisfy this constraint, as rotor position states—an infinite superposition of periodically identified oscillator position states—are no longer Gaussian.

A multirotor nullifier state is defined by a vector $\vec{\boldsymbol{l}} = (l_1, \ldots, l_n)^T$ denoting the momentum of each rotor,

$$|\vec{\boldsymbol{l}}\rangle = \bigotimes_{j=1}^{n} |\hat{l}_j = l_j\rangle. \tag{C1}$$

We show that rotor nullifier states are closed under the Clifford operation.

If we apply a rotor Clifford circuit U on $|\vec{\boldsymbol{l}}\rangle$, the evolution of the eigenoperator is

$$\hat{\mathbf{l}} \rightarrow U\hat{\mathbf{l}}U^\dagger. \tag{C2}$$

This can be represented by multiplying symplectic matrices on the quadrature vector. Importantly, we only need to track the changes of angular-momentum operators since the rotor Clifford group does not mix in (continuous) positions into (integer) momenta. In other words, since the group has a semidirect product structure, we can always write an element as $g = g_H g_N$, where $g_H \in H$, $g_N \in N$, and Eq. (C2) becomes

$$U\hat{\mathbf{l}}U^\dagger = g_H g_N \hat{\mathbf{l}} = g_H \hat{\mathbf{l}} = A_U^{-1}\hat{\mathbf{l}}. \tag{C3}$$

Here we use $g_N\hat{\mathbf{l}} = \hat{\mathbf{l}}$, because QUAD and CPHS gates (block off-diagonal gates) all commute with $\hat{\mathbf{l}}$.

Re-expressing the above in the Schrodinger picture, any nullifier state $|\vec{\boldsymbol{l}}\rangle$—an eigenstate of $\hat{\mathbf{l}}$ with eigenvalues $\vec{\boldsymbol{l}}$—transforms into the state $U|\vec{\boldsymbol{l}}\rangle$—an eigenstate of $A_U^{-1}\hat{\mathbf{l}}$ with eigenvalues $A_U^{-1}\vec{\boldsymbol{l}}$. This new state is still a tensor product of angular-momentum eigenstates. Therefore, rotor nullifier states are closed under the action of rotor Clifford group and this evolution is fully captured by symplectic transformations.

2. Coherent states

In this Appendix, we review rotor coherent states [9] and show that they arise from projecting oscillator coherent states into the embedded rotor subspace. Although, unlike to the oscillator case, it is shown that rotor coherent states do not have non-negative Wigner functions [27], we can discussed some the the properties of rotor coherent states such as the displaced coherent states and the closed orbits of rotor coherent states under the action of passive symplectic subgroup. Per the discussion from Sec. A, squeezing such coherent states cannot be implemented unitarily. However, we show that squeezed rotor states previously introduced in Ref. [112] are equivalent to applying an adjustable regularizer $e^{-\Delta\hat{l}^2/2}$ to a fiducial coherent state. This "regularization" is similar to the manner of realizing finite-energy GKP states [119].

Oscillator coherent states are (right) eigenstates of the oscillator annihilation operator. A class of rotor coherent states [9] can be defined analogously by the equation

$$e^{i\hat{a}}|\xi\rangle = e^{i(\hat{\theta}+i\hat{l})}|\xi\rangle. \tag{C4}$$

Above, $\hat{a}$ can be thought of as an effective lowering operator for the rotor. It has to be in the exponent because $\hat{\theta}$ is not well defined as a standalone operator. Using the commutation relation between $\hat{\theta}$ and $\hat{l}$,

$$e^{i\hat{a}} = e^{i\hat{\theta}}e^{-\hat{l}-\frac{1}{2}} = X(1)e^{-\hat{l}-\frac{1}{2}}. \tag{C5}$$

Expressing rotor coherent states in the momentum basis yields

$$\begin{aligned} |\xi\rangle &= \sum_{l\in\mathbb{Z}} \xi^{-l}e^{-l^2/2}|l\rangle, \\ e^{i\hat{a}}|\xi\rangle &= \xi|\xi\rangle, \quad \xi\in\mathbb{C}-\{0\}, \\ e^{-(\ln\chi)\hat{l}}|\xi\rangle &= |\chi\xi\rangle, \quad \chi>0. \end{aligned} \tag{C6}$$

Note that $|\xi=0\rangle$ is not allowed because its wave function diverges. In this form, we can see that rotor coherent-state coefficients are evaluations of the momentum Gaussian wave function of the oscillator coherent state at integer momenta. Contrary to the oscillator case, the (rotor) Wigner functions of coherent states $|\xi\rangle$ have negative parts [27].

One can define a rotor displacement operator $D(\alpha)$ as

$$\begin{aligned} D(\alpha) &= \exp\left(\frac{\alpha}{2}\hat{a}^\dagger - \frac{\alpha^*}{2}\hat{a}\right) = \exp\left[i(d\hat{\theta}-c\hat{l})\right], \\ &= e^{-icd/2}X(d)Z(-c) \\ \text{where}\quad & \alpha=c+id,\quad c\in\mathbb{T},\quad d\in\mathbb{Z}. \end{aligned} \tag{C7}$$

Because of the discrete nature of the angular-momentum basis, we can only apply discrete displacement along the angular-momentum direction. Displacement operators are an alternative way to express the rotor Pauli group $\mathcal{P}_n^{\text{rot}}$. The difference between the displacement operators of rotors and oscillators is that $\alpha\in\mathbb{C}$ for oscillator systems, whereas $\text{Im}(\alpha)\in\mathbb{Z}$ in rotors.

Conjugating $e^{i\hat{a}}$ by $D(\alpha)$ performs a displacement transformation on $e^{i\hat{a}}$:

$$\begin{aligned} D(\alpha)e^{i\hat{a}}D(\alpha)^\dagger &= X(d)Z(-c)e^{i\hat{a}}Z(-c)^\dagger X(d)^\dagger \\ &= e^{i(\hat{a}-\alpha)}. \end{aligned} \tag{C8}$$

When the displacement operator $D(\alpha)$ acts on a rotor coherent state $|\xi\rangle$, we have

$$D(\alpha)|\xi\rangle = (\xi e^{\alpha/2})^d|e^{i\alpha}\xi\rangle, \tag{C9}$$

giving rise to another coherent state, up to a constant factor. In contrast to oscillator coherent states, here the displaced state has to be renormalized relative to the initial state when d is nonzero.

As discussed in Appendix B, the transformations that leave the direct product structure of rotor coherent states invariant form the group $\mathcal{P}_n^{\text{rot}} \rtimes (\mathbb{Z}_2 \wr S_n)$. The action of the Pauli part $\mathcal{P}_n^{\text{rot}}$ is shown in Eq. (C9). The action of the permutation group S_n swaps the order of the rotors in the direct product sequence. The action of the $\mathbb{Z}_2$ part, generated by parity P, acts on a rotor coherent state as

$$\text{P}|\xi\rangle = |\xi^{-1}\rangle. \tag{C10}$$

Rotor coherent states cannot be used to approximate rotor nullifier states by applying a squeezing operator, because squeezing is not a unitary operation for rotor systems (see Sec. A). However, one can still introduce one more parameter to the rotor coherent state to "simulate" squeezing,

$$|\xi=1\rangle_\Delta \propto E_\Delta|\theta=0\rangle = \sum_{l\in\mathbb{Z}} e^{-\frac{\Delta l^2}{2}}|l\rangle, \tag{C11}$$

where $E_\Delta(\hat{l}) = \exp(-\Delta\hat{l}^2/2)$ is called a regularizer, and $\Delta\in[0,\infty)$ is a regularization parameter.

The above states simulate squeezing by smoothly interpolating between states of fixed position and momentum. If we take $\Delta\to 0$, then the "squeezed" rotor coherent state approaches the phase state as $|\xi=1\rangle_{\Delta\to 0}\to|\theta=0\rangle$. If we take $\Delta\to\infty$, then the squeezed rotor coherent state approaches the angular-momentum state $|\xi=1\rangle_{\Delta\to\infty}\to|l=0\rangle$.

Following the finite-energy approach in Refs. [84,119,120], we define the finite-energy version of Pauli operators via

$$\begin{aligned} X(m)_\Delta &= E_\Delta(\hat{l})X(m)E_\Delta^{-1}(\hat{l}) = X(m)E_\Delta(\hat{l}+m)E_\Delta^{-1}(\hat{l}), \\ Z(\phi)_\Delta &= E_\Delta(\hat{l})Z(\phi)E_\Delta^{-1}(\hat{l}) = Z(\phi). \end{aligned} \tag{C12}$$

Though the finite-energy version of operators are nonunitary, they follow the same Pauli algebra as regular Pauli operators. The Pauli Z component is unaffected by the energy regularizer. We only need to apply regulator in l bases as $\exp(-\Delta\hat{l}^2/2)$ because the phase states are un-normalizable while angular-momentum states are normalized. This regularizer can also be regarded as an unnormalized thermal state density operator whose Hamiltonian is quadratic in angular momentum.

3. Clifford-group orbits of the Josephson junction

In this Appendix, we discuss how the Josephson junction Hamiltonian [121,122] transforms under the action of the rotor Clifford group (*without* performing the cosine expansion and approximating all rotors as oscillators). Rotor Clifford operations perform a unique mixture of oscillator-like Gaussian manipulations and qubit-like Clifford conjugations.

The Josephson junction allows for tunneling of superconducting paired electrons called Cooper pairs between the two islands that the junction connects. The planar rotor angular momentum $\hat{l}$ can be associated with the difference of Cooper pairs $\hat{l} = \hat{n}_R - \hat{n}_L$ across the junction, where $\hat{n}_{L/R}$ is the Cooper-pair number operator of the left and right island. In such a case, the parity flip transformation P corresponds to swapping the definition of the two islands.

The Josephson junction Hamiltonian is

$$H = -2E_J \cos\hat{\theta} - E_C\hat{l}^2 = -E_J(e^{i\hat{\theta}} + e^{-i\hat{\theta}}) - E_C\hat{l}^2, \tag{C13}$$

which includes a quadratic kinematic term $E_C\hat{l}^2$ and a periodic potential $E_J\cos\hat{\theta}$. If we consider this Hamiltonian in the picture of embedded rotor where $\hat{\theta} \to \hat{q}$, $\hat{l} \to \hat{p}$, it is equivalent to a boson with quadratic dispersion relation $\hat{p}^2$ that moves in a one-dimensional periodic potential $\cos\hat{q}$. This can be regarded as a GKP Hamiltonian [38,123,124] with only θ variable periodic while leaving l variable noncompact.

Suppose we have n decoupled Josephson junctions, where the total Hamiltonian is written as

$$\begin{aligned} H_{\text{tot}} &= \sum_{j=1}^{n} \left[-E_J(e^{i\hat{\theta}_j} + e^{-i\hat{\theta}_j}) - E_C\hat{l}_j^2\right] \\ &= \sum_{j=1}^{n} -E_J[X_j(1) + X_j(1)^\dagger] - E_C\vec{\mathbf{l}}^T\hat{\mathbf{l}}. \end{aligned} \tag{C14}$$

To see how the above Hamiltonian transforms under a rotor Clifford transformation, we recall that in Eq. (21) how the symplectic matrix acts on the coefficient vector. Instead of directly transforming the momentum-position coefficients in a Pauli string, we use the Heisenberg picture and act on the position-momentum *operators*. We abuse the notation and suppose $\hat{\theta}$ could be defined without exponentiation and write the collection of them as a vector $\hat{\boldsymbol{\theta}} = (\hat{\theta}_1, \dots, \hat{\theta}_n)^T$. The symplectic matrix acting on the operators can be written as

$$(\hat{\boldsymbol{\theta}}^T|\hat{\mathbf{l}}^T)\begin{pmatrix} A & 0 \\ CA & (A^T)^{-1}\end{pmatrix} = [(A^T\hat{\boldsymbol{\theta}})^T + (A^T C\hat{\mathbf{l}})^T|(A^{-1}\hat{\mathbf{l}})^T]. \tag{C15}$$

The position-operator part of the Hamiltonian becomes mixture of position and momentum Paulis,

$$\sum_{j=1}^{n}(e^{i\hat{\theta}_j} + \text{H.c.}) \to \sum_{j=1}^{n}\left(e^{i\sum_{k=1}^{n}A_{kj}\hat{\theta}_k}e^{i\sum_{k=1}^{n}(A^TC)_{jk}\hat{l}_k} + \text{H.c.}\right). \tag{C16}$$

The Z part of Pauli operators comes in because of the off diagonal matrix $Q_{XZ} = CA$, corresponding to CPHS and QUAD gates. If we only applied an element of the CSS subgroup H from Eq. (26), then the original $X_j(1)$ only transforms into a product of Pauli X operators of rotors.

On the other hand, the angular-momentum part of the Hamiltonian always transforms as

$$\hat{\mathbf{l}}^T\hat{\mathbf{l}} \to \hat{\mathbf{l}}^T(A)^{-1}(A^T)^{-1}\hat{\mathbf{l}}, \tag{C17}$$

remaining a quadratic form after any symplectic transformation.

The difference between the transformation behaviours of the θ and l variables illustrates the difference between periodic bounded variables and unbounded variables. For an unbounded variable l, the quadratic term in the Hamiltonian remains a quadratic term. This is the same as in the oscillator system where a quadratic Hamiltonian remains quadratic after Gaussian transformation.

However, for periodic variable θ, if we start from a trivial Hamiltonian $H = \sum_{j=1}^{n} X_j$ and apply a Clifford circuit, we get a stabilizer Hamiltonian $\hat{H}' = \sum_{j=1}^{n}\hat{S}_j$, where $\hat{S}_j$ are Pauli strings of multiple rotors. This behavior resembles the Clifford transformation of Pauli operators for qudits. This is due to the exponential required to properly express the periodic rotor position.

APPENDIX D: WIGNER FUNCTION OF ROTOR GKP CODES

In the oscillator cases, GKP states are non-Gaussian states that have Wigner negativity and infinite Stellar rank [97,100–102,125,126]. The studies of the phase space structure and non-Gaussianity of bosonic modes yield various discoveries of classical simulability and quantum magic of bosonic systems [97,102,103,126–132].

In this Appendix, we calculate the Wigner function of rotor GKP code as an example, and show its Wigner function has a similar form as the Wigner function of oscillator GKP states, and indeed, the rotor GKP states have Wigner negativity and can be written as a sum of Kronecker and Dirac delta functions. The presence of Kronecker delta functions is because of the discrete-variable nature of angular momentum, while, in contrast, both position and momentum are continuous variables in harmonic oscillators.

The studies of the Wigner function of the planar rotor are initialized in the context of orbital angular-momentum states of light [13,26–29], which is also described by planar rotor. The modular Wigner function is also studied in the context of oscillator GKP codes, considering the modular nature of position and momentum variables [133].

The definition of Wigner function of planar rotor is written as

$$W_\rho(l,\phi) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left\langle \phi - \frac{\phi'}{2}\middle|\rho\middle|\phi + \frac{\phi'}{2}\right\rangle e^{i\phi' l}d\phi', \tag{D1}$$

where the Wigner function has two canonical variable: phase ϕ and angular momentum l. This function is defined on a infinite cylinder $\mathbb{T}\times\mathbb{Z}$ as shown in Fig. 3.

Here we consider $\rho = |0\rangle_L\langle 0|_L$ and calculate its Wigner function

$$\begin{aligned} W_\rho(l,\phi) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{m,m'=0}^{N-1}\left\langle \phi - \frac{\phi'}{2}\middle|\frac{2\pi m}{N}\right\rangle \\ &\quad\times\left\langle\frac{2\pi m'}{N}\middle|\phi + \frac{\phi'}{2}\right\rangle e^{i\phi' l}d\phi' \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{m,m'=0}^{N-1}\delta_{2\pi}\left(\phi - \frac{\phi'}{2} - \frac{2\pi m}{N}\right) \\ &\quad\times\delta_{2\pi}\left(\phi + \frac{\phi'}{2} - \frac{2\pi m}{N}\right)e^{i\phi' l}d\phi'. \end{aligned} \tag{D2}$$

Here the $\delta_{2\pi}(x)$ represents periodic delta function, which is

$$\delta_{2\pi}(x) = \begin{cases} \delta(0) & \text{if } x \bmod 2\pi = 0 \\ 0 & \text{otherwise.} \end{cases} \tag{D3}$$

Hence, we can rewrite Eq. (D2) as

$$\begin{aligned} W_\rho(l,\phi) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{m,m'\in\mathbb{Z}}\delta\left(\phi-\frac{\phi'}{2}-\frac{2\pi m}{N}\right) \\ &\quad\times\delta\left(\phi+\frac{\phi'}{2}-\frac{2\pi m'}{N}\right)e^{i\phi' l}d\phi' \\ &\propto \frac{1}{2\pi}\sum_{m,m'\in\mathbb{Z}}\delta\left(2\phi-\frac{2\pi}{N}(m+m')\right)e^{il(-2\phi+\frac{4\pi m'}{N})} \\ &= \frac{1}{2\pi}\sum_{c,d\in\mathbb{Z}}\delta\left(\phi-\frac{\pi c}{N}\right)(-1)^{cd}\delta_{l,Nd/2}. \end{aligned} \tag{D4}$$

Similarly, the Wigner function of $|1\rangle\langle 1|$ is

$$W_{|1\rangle\langle 1|}(l,\phi) \propto \frac{1}{2\pi}\sum_{c,d\in\mathbb{Z}}\delta\left(\phi-\frac{\pi(c+1)}{N}\right)(-1)^{cd}\delta_{l,Nd/2}. \tag{D5}$$

The Wigner function of rotor GKP states shows strong negativity relative to its oscillator counterparts. The difference between the Wigner functions of oscillator and rotor GKP states is that the former is a sum of products of Dirac delta functions in both position and momentum, while the latter is a sum of products of Dirac delta functions in phase and Kronecker deltas in angular momentum.

Another interesting phenomenon is that, in the rotor case, the angular-momentum eigenstates $|l\rangle$, $l\in\mathbb{Z}$ are the only normalizable states with non-negative Wigner function [27]. This statement matches our understanding of oscillator systems with discrete translational symmetry in position direction $S_q = e^{i2\sqrt{\pi}\hat{p}}$. The oscillator system with spatial periodicity will have quantized angular momentum defined on $\mathbb{Z}$, and it can be regarded as a rotor (mathematically). In this picture, the angular-momentum eigenstates of the rotor correspond to the momentum eigenstates of a periodic oscillator, which are Gaussian states with non-negative Wigner functions.

APPENDIX E: QUANTUM ERROR-CORRECTION CONDITION FOR NORMALIZED ROTOR GKP CODES

For the ideal oscillator GKP states, the codewords are equal-weight superpositions of infinite numbers of Delta functions which are un-normalizable and require unbounded energy to prepare. In practice, we typically impose various regularizers to impose the normalization and finite-energy conditions, and the Gaussian regularizer can be implemented experimentally in trapped-ion, superconducting-circuit, and optical platforms [40,84,134–137]. In this section, we study the quantum error-correction conditions [138,139] for normalized rotor GKP states that are regularized by the Gaussian regularizer $E_\Delta(\hat{l})$.

The un-normalizable codeword of rotor GKP code is

$$|0\rangle_L = \sum_{k\in\mathbb{Z}}|l=kN\rangle,\quad |1\rangle_L = \sum_{k\in\mathbb{Z}}(-1)^k|l=kN\rangle. \tag{E1}$$

To normalize them, we impose a Gaussian envelope $e^{-\Delta\hat{l}^2}$ such that

$$\begin{aligned} |0\rangle_\Delta &= \sum_{k\in\mathbb{Z}}e^{-\Delta(kN)^2}|l=kN\rangle, \\ |1\rangle_\Delta &= \sum_{k\in\mathbb{Z}}e^{-\Delta(kN)^2}(-1)^k|l=kN\rangle. \end{aligned} \tag{E2}$$

The error operator is written as $E_m(\theta) = Z(\theta)X(m)$. Hence, we can calculate the quantum error-correction condition for the normalized rotor GKP states

$$\begin{aligned} &\langle\phi_i|E_{m'}(\theta')^\dagger E_m(\theta)|\phi_j\rangle \\ &\quad= e^{i(\theta-\theta')m}\langle\phi_i|X(m-m')Z(\theta-\theta')|\phi_j\rangle. \end{aligned} \tag{E3}$$

We have

$$\begin{aligned} &X(m-m')Z(\theta-\theta')|0\rangle_\Delta \\ &\quad= \sum_{k\in\mathbb{Z}}e^{-\Delta(kN)^2}e^{i(\theta-\theta')kN}|l=kN+m-m'\rangle, \\ &X(m-m')Z(\theta-\theta')|1\rangle_\Delta \\ &\quad= \sum_{k\in\mathbb{Z}}(-1)^k e^{-\Delta(kN)^2}e^{i(\theta-\theta')kN}|l=kN+m-m'\rangle. \end{aligned} \tag{E4}$$

Then we calculate Eq. (E3) utilizing Jacobi theta functions

$$\begin{aligned} \text{(a) } e^{i(\theta-\theta')m}\langle 0|_\Delta X(m-m')Z(\theta-\theta')|0\rangle_\Delta &= e^{i(\theta-\theta')m}\sum_{k,k'\in\mathbb{Z}}\delta_{k'N,kN+m-m'}e^{-\Delta N^2(k^2+k'^2)}e^{i(\theta-\theta')kN} \\ &= e^{i(\theta-\theta')m}\sum_{k\in\mathbb{Z}}e^{-\Delta N^2[k^2+(k+\frac{m-m'}{N})^2]}e^{i(\theta-\theta')kN}\delta_{m-m'\bmod N,0} \\ &= e^{i(\theta-\theta')\frac{m+m'}{2}}e^{-\Delta\frac{(m-m')^2}{2}}\sum_{k\in\mathbb{Z}}e^{-2\Delta N^2(k+\frac{m'-m}{2N})^2}e^{i(\theta-\theta')N(k+\frac{m'-m}{2N})}\delta_{m-m'\bmod N,0} \\ &= \begin{cases} e^{i(\theta-\theta')\frac{m+m'}{2}}e^{-\Delta\frac{(m-m')^2}{2}}\delta_{m-m'\bmod N,0}\vartheta_2\left(z=\frac{(\theta-\theta')N}{2},q=e^{-2\Delta N^2}\right) & \text{if } \frac{m'-m}{N} \text{ is odd} \\ e^{i(\theta-\theta')\frac{m+m'}{2}}e^{-\Delta\frac{(m-m')^2}{2}}\delta_{m-m'\bmod N,0}\vartheta_3\left(z=\frac{(\theta-\theta')N}{2},q=e^{-2\Delta N^2}\right) & \text{if } \frac{m'-m}{N} \text{ is even.} \end{cases} \end{aligned}$$

$$\text{(b) } e^{i(\theta-\theta')m}\langle 1|_\Delta X(m-m')Z(\theta-\theta')|1\rangle_\Delta = \begin{cases} -e^{i(\theta-\theta')\frac{m+m'}{2}}e^{-\Delta\frac{(m-m')^2}{2}}\delta_{m-m'\bmod N,0}\vartheta_2\left(z=\frac{(\theta-\theta')N}{2},q=e^{-2\Delta N^2}\right) & \text{if } \frac{m'-m}{N} \text{ is odd} \\ e^{i(\theta-\theta')\frac{m+m'}{2}}e^{-\Delta\frac{(m-m')^2}{2}}\delta_{m-m'\bmod N,0}\vartheta_3\left(z=\frac{(\theta-\theta')N}{2},q=e^{-2\Delta N^2}\right) & \text{if } \frac{m'-m}{N} \text{ is even.} \end{cases}$$

$$\text{(c) } e^{i(\theta-\theta')m}\langle 1|_\Delta X(m-m')Z(\theta-\theta')|0\rangle_\Delta = \begin{cases} -e^{i(\theta-\theta')\frac{m+m'}{2}}e^{-\Delta\frac{(m-m')^2}{2}}\delta_{m-m'\,\mathrm{mod}N,0}\vartheta_1\left(z=\frac{(\theta-\theta')N}{2}, q=e^{-2\Delta N^2}\right) & \text{if } \frac{m'-m}{N} \text{ is odd} \\ e^{i(\theta-\theta')\frac{m+m'}{2}}e^{-\Delta\frac{(m-m')^2}{2}}\delta_{m-m'\,\mathrm{mod}N,0}\vartheta_4\left(z=\frac{(\theta-\theta')N}{2}, q=e^{-2\Delta N^2}\right) & \text{if } \frac{m'-m}{N} \text{ is even.} \end{cases} \quad \text{(E5)}$$

APPENDIX F: DISCUSSIONS ON THE NO-GO THEOREM FOR OSCILLATOR

In Ref. [53], authors proved a no-go theorem for Gaussian-stabilizer codes. The statement is for mode-to-mode codes, if the encoding, error correction, and decoding all consist of only Gaussian operations, then these codes cannot correct Gaussian quadrature displacement errors. Suppose the logical quadrature errors follow a Gaussian distribution $\mathcal{N}(0, \sigma_{q/p}^2)$, then $\sigma_{q_L}^2\sigma_{p_L}^2 = \sigma_q^2\sigma_p^2$ after encoding and decoding. This no-go theorem indicates that Gaussian stabilizer codes can only rotate or squeeze Gaussian errors, but will never reduce variance on both quadratures.

In this section, we will briefly review the derivation of Gaussian no-go theorem and its limitation, then we will comment its relevance to homological rotor codes.

We first state the conditions for the no-go theorem to be true:

(1) Encoding unitary is a Gaussian operation (symplectic transformation), and ancilla states are all initialized in infinitely squeezed states (Gaussian states).

(2) The error correction is adding linear combinations of nullifiers onto logical quadratures. For example, the maximum-likelihood error correction is adding $-CG^T(GG^T)^{-1}G$ onto the logical quadrature C.

(3) Quadratures are defined on $\mathbb{R}$.

The derivation utilizes the linearity and orthogonality of symplectic vectors. Although the analog rotor codes also share a symplectic structure, their phase quadrature is a modular quadrature which doesn't have linearity. The lack of linearity in rotor systems provides an obstruction to generalizing the Gaussian no-go theorem for oscillators to rotor systems.

For a $[[n, k, d]]$ Gaussian stabilizer codes, the Gaussian unitary encoders U_{enc} are symplectic transformations

$$U_{\mathrm{enc}}\vec{r}U_{\mathrm{enc}}^\dagger = U_{\mathrm{enc}}(\hat{q}_1, \ldots, \hat{q}_n, \hat{p}_1, \ldots, \hat{p}_n)^T U_{\mathrm{enc}}^\dagger = A\vec{r}. \quad \text{(F1)}$$

We can decompose the symplectic matrix A as [53]

$$A = \begin{pmatrix} Q \\ G \\ P \\ D \end{pmatrix}. \quad \text{(F2)}$$

The syndrome $\vec{z}$ is given by following equation:

$$\vec{z} = G\vec{\xi}, \quad \text{(F3)}$$

where $\vec{\xi}$ is a $2n$-dimensional noise vector. The error-corrected logical quadrature can be written as

$$C' = C - CG^T(GG^T)^{-1}G = C + \Lambda G. \quad \text{(F4)}$$

And the covariance matrix of error-corrected logical quadratures can be diagonalized

$$K(C'C'^T)K^{-1} = \begin{pmatrix} \mathrm{diag}\left(\sigma_{q,j}^2\right) & 0 \\ 0 & \mathrm{diag}\left(\sigma_{p,j}^2\right) \end{pmatrix}. \quad \text{(F5)}$$

This no-go theorem is applicable once Eqs. (F3) and (F4) are linear. However, in rotor case, Eq. (F3) is no longer linear, the rotor syndrome has modular structure,

$$\vec{z}_{\mathrm{rotor}} = R_{2\pi}(G\vec{\xi}), \quad \text{(F6)}$$

where $R_{2\pi}$ is a rounding function that rounds the input to the nearest multiplicity of 2π. The modular structure is nonlinear, hence, the no-go theorem will not hold true in rotor systems. However, if we drop the modularity by assuming the variance of syndrome is much smaller than 2π, the rotors will be reduced to regular oscillators where the Gaussian no-go theorems holds true.

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