

Quantum distinguishability measures: Projectors versus state maximization

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The distinguishability between two quantum states can be defined in terms of their trace distance. The operational meaning of this definition involves a maximization over measurement projectors. Here we introduce an alternative definition of distinguishability which, instead of projectors, is based on maximization over normalized states (density matrices). It is shown that this procedure leads to a distance (between two states) that in contrast to the usual approach based on a 1-norm, is based on an infinite-norm. Properties such as convexity, monotonicity, and invariance under unitary transformations are fulfilled. Equivalent operational implementations based on maximization over classical probabilities and hypothesis-testing scenarios are also established. When considering the action of completely positive transformations, contractivity is only granted for unital maps. This feature allows us to introduce a measure of the quantumness of nonunital maps that can be written in terms of the proposed distinguishability measure and corresponds to the maximal possible deviation from contractivity. Particular examples sustain the main results and conclusions.

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I. INTRODUCTION

Measuring the distinguishability between two quantum states is a central ingredient when evaluating the performance of any quantum information protocol. A solid basis of proposals and results have been developed in recent years [1–5]. Nevertheless, due to its relevance, this issue has been periodically reviewed and still remains as an active area of research [6–14].

A usual and standard definition of distinguishability relies on the following expressions [1–5]. Given two quantum states ρ_A and ρ_B in an arbitrary Hilbert space, their distinguishability is defined as

$$D_{\Pi}(\rho_A, \rho_B) \equiv \max_{\{\Pi\}} |\text{Tr}[\Pi(\rho_A - \rho_B)]|. \quad (1)$$

Here, $\text{Tr}[\cdot]$ is the trace operation. Maximization is performed over arbitrary projectors, $\Pi = \Pi^n$. In general, these projectors may have an arbitrary rank (equal or greater than one). It is well known that the operational definition of $D_{\Pi}(\rho_A, \rho_B)$ is equivalent to the expression [1]

$$D_{\Pi}(\rho_A, \rho_B) = \frac{1}{2} \text{Tr}|\rho_A - \rho_B|. \quad (2)$$

Hence, $D_{\Pi}(\rho_A, \rho_B)$ corresponds to the *trace distance* between the states ρ_A and ρ_B .

Motivated by recent advances in the definition of environment quantumness in open quantum systems [15–24], the main goal of this paper is to introduce an alternative definition of quantum distinguishability, providing, in addition, a full characterization of its properties. While the standard approach (1) relies on maximization over projectors, here we propose to replace projectors with normalized states, that is, density matrices.

We find that this alternative definition can be related to a distance based on an infinite-norm. In contrast, the trace norm is related to a 1-norm [2]. In addition, we prove that some general properties hold in the alternative approach. The proposed distinguishability measure is a metric on the space of density matrices. Furthermore, it is convex on both entries. Monotonicity and invariance under unitary transformations are also fulfilled. Complementarily, we show that equivalent implementations can be defined in terms of maximization over classical probabilities and hypothesis-testing scenarios [5,25]. We also find the conditions under which the state-based and projector-based distinguishability measures are equal.

Added to the intrinsic theoretical and practical interest of the previous results, we find that the alternative definition, in contrast to the standard approach [Eqs. (1) and (2)], allows one to quantify departures from classicality of open quantum dynamics. This quantum-classical border [15–24] is studied by considering the action of completely positive maps. Consistent with the results of Ref. [26], we find that in general, contractivity does not hold here. Hence, the distance between the output states could increase with respect to the distance between the input states. The specific states and maps that lead to maximal violation of contractivity are explicitly stated. These results provide the basis for defining a measure that quantifies the quantumness of dissipative (nonunital) open system dynamics. Furthermore, a close relationship with recently proposed measures of environment quantumness [24] emerges from these analyses.

The manuscript is outlined as follows. In Sec. II, we introduce the distinguishability measure based on maximization over states. Its relationship with an infinite norm is demonstrated. Equivalent operational implementations such as maximization over classical probabilities and hypothesis-

testing scenario are established. Furthermore, we compare the projector and state-based measures, establishing the conditions under which they are equal. In Sec. III, we study its properties when considering the action of completely positive maps. In Sec. IV, we study some specific examples of distance between states and quantum maps. In Sec. V, we provide the conclusions. Extra related results are provided in the Appendices.

II. DISTINGUISHABILITY MEASURE BASED ON MAXIMIZATION OVER STATES

Here, we introduce an alternative definition of distinguishability. In contrast to a maximization over projectors [Eq. (1)], it emerges from a maximization over states. Given two quantum states ρ_A and ρ_B ($\text{Tr}[\rho_A] = \text{Tr}[\rho_B] = 1$), it reads

$$D_\rho(\rho_A, \rho_B) \equiv \max_{\{\rho\}} |\text{Tr}[\rho(\rho_A - \rho_B)]|, \quad (3)$$

where ρ is an arbitrary density matrix, $\text{Tr}[\rho] = 1$. We notice that being a state, ρ has positive eigenvalues in the interval $[0, 1]$. Furthermore, similarly to D_Π , D_ρ is also a dimensionless quantity.

The definition (3) can be read as a maximization over states ρ of the expectation value of the “Hermitian operator” $(\rho_A - \rho_B)$. In Appendix A, we provide a general solution to this problem (arbitrary operator A). Introducing the eigenvalues and eigenvectors associated to $(\rho_A - \rho_B)$,

$$(\rho_A - \rho_B)|i\rangle = \zeta_i|i\rangle, \quad (4)$$

the maximization in Eq. (3) leads to [see Eq. (A5)]

$$D_\rho(\rho_A, \rho_B) = \max_{\{i\}} \{|\zeta_i|\}. \quad (5)$$

Hence, $D_\rho(\rho_A, \rho_B)$ corresponds to the eigenvalue of $(\rho_A - \rho_B)$ with maximal absolute value. In contrast, notice that the projector-based definition [Eq. (2)] can be written as $D_\Pi(\rho_A, \rho_B) = (1/2) \sum_i |\zeta_i|$. On the other hand, the state ρ that solves the maximization in Eq. (3), while in general not unique (see Appendix A), can always be chosen as

$$\rho = |i_{\max}\rangle\langle i_{\max}|, \quad (6)$$

where $|i_{\max}\rangle$ is the eigenstate of $(\rho_A - \rho_B)$ associated to the eigenvalue with maximal absolute value, that is, $\max_{\{i\}} \{|\zeta_i|\}$.

In order to understand the difference between $D_\rho(\rho_A, \rho_B)$ and $D_\Pi(\rho_A, \rho_B)$, we notice that Eq. (5) can be written in the alternative way,

$$D_\rho(\rho_A, \rho_B) = \lim_{\alpha \rightarrow \infty} \sqrt[\alpha]{\text{Tr}|\rho_A - \rho_B|^\alpha}. \quad (7)$$

This expression allows one to read $D_\rho(\rho_A, \rho_B)$ as a distance between states based on a infinite-norm, while $D_\Pi(\rho_A, \rho_B)$ [Eq. (2)] is a distance based on a 1-norm [given an operator A , its α -norm ($\alpha \geq 1$) is given by $|A|_\alpha = \sqrt[\alpha]{\text{Tr}|A|^\alpha}$].

The proposed distinguishability measure is defined by Eq. (3), whose explicit calculation is solved by Eq. (5). In Appendix B, we demonstrate that $D_\rho(\rho_A, \rho_B)$ fulfills some general properties. In particular, it is shown that it defines a metric in the space of states, it is convex in both entries and monotonicity for bipartite systems, and invariance under unitary transformations are also corroborated.

A. Equivalent operational interpretations

Below we study different equivalent operational interpretations of D_ρ .

1. Maximization in terms of probabilities

Let $\{|k\rangle\}$ be the basis where an arbitrary state ρ is diagonal, $\rho = \sum_k p_k |k\rangle\langle k|$. Given two quantum states ρ_A and ρ_B , define $p_A^{(k)} \equiv \langle k|\rho_A|k\rangle$, and $p_B^{(k)} \equiv \langle k|\rho_B|k\rangle$. Then, the distinguishability measure [Eq. (3)] can alternatively be written as

$$D_\rho(\rho_A, \rho_B) = \max_{\{\rho\}} D_c(p_A, p_B), \quad (8)$$

where the maximization is over all possible states $\{\rho\}$. With $p_A \equiv \{p_A^{(k)}\}$ and $p_B \equiv \{p_B^{(k)}\}$, we denote both sets of probabilities. Their distinguishability is

$$D_c(p_A, p_B) \equiv \max_{\{k\}} \{|p_A^{(k)} - p_B^{(k)}|\}. \quad (9)$$

We notice that Eq. (8) implies that $D_\rho(\rho_A, \rho_B)$ is the distinguishability $D_c(p_A, p_B)$ between probabilities maximized over all possible states ρ . A similar result is valid for $D_\Pi(\rho_A, \rho_B)$ [1], but where the probabilities are defined in terms of an arbitrary positive operator value measure [27].

Demonstration. Below we demonstrate the validity of the operational representation defined by Eqs. (8) and (9). By using the explicit expressions of $p_A^{(k)}$ and $p_B^{(k)}$, it is possible to rewrite $D_c(p_A, p_B)$ as

$$D_c(p_A, p_B) = \max_{\{k\}} \{|\langle k|(\rho_A - \rho_B)|k\rangle|\}. \quad (10)$$

From Eq. (4), we write

$$(\rho_A - \rho_B) = \sum_i \zeta_i |i\rangle\langle i|, \quad (11)$$

where $\{|i\rangle\}$ is the basis where $(\rho_A - \rho_B)$ is a diagonal matrix. Hence, the previous expression becomes

$$\begin{aligned} D_c(p_A, p_B) &= \max_{\{k\}} \left\{ \left| \sum_i \zeta_i |\langle k|i\rangle|^2 \right| \right\} \\ &\leq \max_{\{k\}} \left\{ \sum_i |\zeta_i| |\langle k|i\rangle|^2 \right\} \\ &\leq \left(\max_{\{i\}} \{|\zeta_i|\} \right) \max_{\{k\}} \sum_i |\langle k|i\rangle|^2 \\ &= \max_{\{i\}} \{|\zeta_i|\} = D_\rho(\rho_A, \rho_B), \end{aligned}$$

which demonstrates Eq. (8). In fact, the equality is achieved when the basis $\{|k\rangle\}$, where the state ρ is diagonal, is the same basis $\{|i\rangle\}$, where $(\rho_A - \rho_B)$ is a diagonal operator.

2. Hypothesis-testing scenario

Here we demonstrate that under an appropriate constraint, D_ρ plays the same role as D_Π in a “hypothesis-testing scenario” [5, 25]. Let Alice prepare two quantum states ρ_1 and ρ_0 , each one with probability 1/2. Bob can perform a binary “positive operator value measure” with elements $\Lambda = \{\Lambda_1, \Lambda_0\}$ to distinguish the two states. Central for the following arguments, here Λ_1 is restricted to be a 1-rank projector, while Λ_0

is its complement, $\Lambda_1 + \Lambda_0 = I$. For example,

$$\Lambda_1 = |\psi_1\rangle\langle\psi_1|, \quad \Lambda_0 = I - |\psi_1\rangle\langle\psi_1| = \sum_{i \neq 1} |\psi_i\rangle\langle\psi_i|, \quad (12)$$

where $\{|\psi_i\rangle\}$ is a complete basis.

When the outcome 1 or 0 is obtained, Bob guesses the state ρ_1 or ρ_0 , respectively. Thus, the probability $p_{\text{succ}}(\Lambda)$ for this hypothesis-testing scenario is

$$p_{\text{succ}}(\Lambda) = \text{Tr}[\Lambda_1 \rho_1] \frac{1}{2} + \text{Tr}[\Lambda_0 \rho_0] \frac{1}{2} \quad (13a)$$

$$= \frac{1}{2} \{1 + \text{Tr}[\Lambda_1(\rho_1 - \rho_0)]\}, \quad (13b)$$

where we have used that $\Lambda_1 + \Lambda_0 = I$. Now, we assume that Bob can freely choose the projectors $\{\Lambda_1, \Lambda_0\}$ such that $p_{\text{succ}}(\Lambda)$ is maximized. The success probability with respect to all measurements, under the constraint (12), can then be defined as

$$p_{\text{succ}}(\Lambda) = \frac{1}{2} \max \left\{ \begin{array}{l} 1 + \max_{\{\Lambda_1\}} \text{Tr}[\Lambda_1(\rho_1 - \rho_0)] \\ 1 - \min_{\{\Lambda_1\}} \text{Tr}[\Lambda_1(\rho_1 - \rho_0)] \end{array} \right\}. \quad (14)$$

This expression can be rewritten as

$$p_{\text{succ}}(\Lambda) = \frac{1}{2} \left\{ 1 + \max_{\{\Lambda_1\}} |\text{Tr}[\Lambda_1(\rho_1 - \rho_0)]| \right\}. \quad (15)$$

Using that Λ_1 is a one-dimensional projector [Eq. (12)] and given that the states ρ that maximize D_ρ can always be chosen as pure states [Eq. (6)], it follows that

$$\max_{\{\Lambda_1\}} |\text{Tr}[\Lambda_1(\rho_1 - \rho_0)]| = D_\rho(\rho_1, \rho_0), \quad (16)$$

which implies that

$$p_{\text{succ}}(\Lambda) = \frac{1}{2} [1 + D_\rho(\rho_1, \rho_0)]. \quad (17)$$

Consequently, the proposed distinguishability measure D_ρ is related to the maximum success probability in distinguishing two quantum states in a quantum hypothesis-testing experiment. We notice that when the rank of Λ_1 can be greater than one, the success probability $p_{\text{succ}}(\Lambda)$, instead of $D_\rho(\rho_1, \rho_0)$, is defined in terms of $D_\Pi(\rho_1, \rho_0)$ [5].

B. Comparison between metrics

From the previous analysis, one can conclude that $D_\Pi(\rho_A, \rho_B)$ and $D_\rho(\rho_A, \rho_B)$ [Eqs. (1) and (3), respectively] are intrinsically different distinguishability measures. Here, we establish when they are equal and how they differ in general.

Both distinguishability measures always coincide when the Hilbert space dimension $\dim(\mathcal{H})$ is equal to two and three,

$$D_\rho(\rho_A, \rho_B) = D_\Pi(\rho_A, \rho_B), \quad \dim(\mathcal{H}) = 2, 3. \quad (18)$$

Furthermore, when $\dim(\mathcal{H}) \geq 4$, the inequalities

$$D_\rho(\rho_A, \rho_B) \leq D_\Pi(\rho_A, \rho_B) \leq \mathcal{N} D_\rho(\rho_A, \rho_B) \quad (19)$$

are fulfilled, where the constant \mathcal{N} is

$$\mathcal{N} = \text{Int}[\dim(\mathcal{H})/2]. \quad (20)$$

$\text{Int}[a]$ denotes the integer part of real number a .

The conditions under which the equalities in Eq. (19) are satisfied [higher-dimensional spaces, $\dim(\mathcal{H}) \geq 4$] are

also well defined. $D_\rho(\rho_A, \rho_B) = D_\Pi(\rho_A, \rho_B)$ when the eigenvalue of $(\rho_A - \rho_B)$ with maximal absolute value is not degenerate. Equivalently, this occurs when $(\rho_A - \rho_B)$ has a unique positive (or negative) eigenvalue. On the other hand, $D_\Pi(\rho_A, \rho_B) = \mathcal{N} D_\rho(\rho_A, \rho_B)$ when the eigenvalue of $(\rho_A - \rho_B)$ with maximal absolute value has degeneracy \mathcal{N} .

Demonstration. Below we demonstrate the validity of Eqs. (18) and (19). By using Eq. (11), $(\rho_A - \rho_B) = \sum_i \zeta_i |i\rangle\langle i|$, the projector-based measure [Eq. (2)], $D_\Pi(\rho_A, \rho_B) = (1/2)\text{Tr}|\rho_A - \rho_B|$, can be written in terms of the eigenvalues $\{\zeta_i\}$ of $(\rho_A - \rho_B)$ as

$$D_\Pi(\rho_A, \rho_B) = \frac{1}{2} \sum_i |\zeta_i| = \frac{1}{2} \left(\sum_{i=1}^{n_+} \zeta_i^{(+)} + \sum_{j=1}^{n_-} |\zeta_j^{(-)}| \right). \quad (21)$$

In the second equality, we split the addition in positive and negative eigenvalues, $\{\zeta_i^{(+)}\}$ and $\{\zeta_j^{(-)}\}$, respectively. Furthermore, n_+ and n_- count their quantity, respectively, $n_+ + n_- = \dim(\mathcal{H})$ [28]. Given that $\text{Tr}[(\rho_A - \rho_B)] = 0$, it is fulfilled that $\sum_{i=1}^{n_+} \zeta_i^{(+)} = \sum_{j=1}^{n_-} |\zeta_j^{(-)}|$. Hence, straightforwardly, it follows that

$$D_\Pi(\rho_A, \rho_B) = \sum_{i=1}^{n_+} \zeta_i^{(+)} = \sum_{j=1}^{n_-} |\zeta_j^{(-)}|. \quad (22)$$

On the other hand, Eq. (5) tells us that $D_\rho(\rho_A, \rho_B) = \max_{\{i\}} |\zeta_i|$. Consequently, when the number n_+ or n_- of positive and negative eigenvalues is equal to one, both measures coincide, that is,

$$n_+ = 1 \quad \text{or} \quad n_- = 1 \Leftrightarrow D_\rho = D_\Pi. \quad (23)$$

In fact, in this situation, the unique positive (or negative) eigenvalue, due to the equality $\sum_{i=1}^{n_+} \zeta_i^{(+)} = \sum_{j=1}^{n_-} |\zeta_j^{(-)}|$, is also the eigenvalue with maximal absolute value, which, in turn, is not degenerate. This condition can be rephrased as follows: when $(\rho_A - \rho_B)$ has a unique positive (or negative) eigenvalue, then $D_\rho(\rho_A, \rho_B) = D_\Pi(\rho_A, \rho_B)$.

The previous condition [Eq. (23)] is always fulfilled when $\dim(\mathcal{H}) = 2$, where $n_+ = n_- = 1$. The same occurs when $\dim(\mathcal{H}) = 3$ because it can only occur that $n_+ = 2$, $n_- = 1$ or, complementarily, $n_+ = 1$, $n_- = 2$. The same occurs if there is a null eigenvalue, which implies $n_+ = 1$, $n_- = 1$. Consequently, Eq. (18) is established.

For Hilbert spaces with $\dim(\mathcal{H}) \geq 4$, the equality of $D_\rho(\rho_A, \rho_B)$ and $D_\Pi(\rho_A, \rho_B)$ is not valid in general, but accidentally it occurs when $n_+ = 1$ or $n_- = 1$. On the other hand, from Eq. (22), it follows that $D_\Pi(\rho_A, \rho_B) \leq n_s \max_{\{i\}} |\zeta_i| = n_s D_\rho(\rho_A, \rho_B)$, where n_s is the number of positive or negative eigenvalues and $s = \pm 1$ gives the sign of the eigenvalue with maximal absolute value. Given that $\sum_{i=1}^{n_+} \zeta_i^{(+)} = \sum_{j=1}^{n_-} |\zeta_j^{(-)}|$, the maximal possible value of n_s is $\mathcal{N} = \text{Int}[\dim(\mathcal{H})/2]$ [29]. These results lead to the upper constraint in Eq. (19). It is achieved when the eigenvalue with maximal absolute value has degeneracy \mathcal{N} . Thus, the conditions under which the equalities in Eq. (19) are fulfilled are established.

III. CONTRACTIVITY UNDER QUANTUM OPERATIONS

Here, we characterize the behavior of D_ρ under quantum operations. Since it is based on an infinite norm [see Eq. (7)],

from Ref. [26], we can anticipate that contractivity is not fulfilled here. The alternative analysis developed below allows us to establish the states and maps that lead to maximal violation of contractivity, which further leads to the formulation of a quantumness measure for nonunital maps.

First, we notice that given a trace-preserving completely positive map, $\rho \rightarrow \mathcal{E}(\rho)$, the projector-based measure [Eqs. (1) and (2)] always satisfies *contractivity* [1],

$$D_{\Pi}(\mathcal{E}[\rho_A], \mathcal{E}[\rho_B]) \leq D_{\Pi}(\rho_A, \rho_B). \quad (24)$$

Hence, the distance between two states can never increase under the action of the map \mathcal{E} . For the state-based measure [Eqs. (3) and (5)], when the Hilbert space dimension is $\dim(\mathcal{H}) = 2$ and $\dim(\mathcal{H}) = 3$, we find that

$$D_{\rho}(\mathcal{E}[\rho_A], \mathcal{E}[\rho_B]) \leq D_{\rho}(\rho_A, \rho_B). \quad (25)$$

This result follows straightforwardly because, with this dimensionality, $D_{\rho}(\rho_A, \rho_B) = D_{\Pi}(\rho_A, \rho_B)$ [Eq. (18)]. In addition, contractivity [Eq. (25)] is always satisfied if \mathcal{E} is a *unital* map, that is, when $\mathcal{E}[I] = I$ (I is the identity operator). For *nonunital* maps, $\mathcal{E}[I] \neq I$, and for higher-dimensional spaces [$\dim(\mathcal{H}) \geq 4$], it is possible to obtain

$$D_{\rho}(\mathcal{E}[\rho_A], \mathcal{E}[\rho_B]) \leq \mathcal{C} D_{\rho}(\rho_A, \rho_B). \quad (26)$$

The constant \mathcal{C} is bounded as

$$1 < \mathcal{C} \leq \dim(\mathcal{H}), \quad (27)$$

implying that standard contractivity is not fulfilled in general ($\mathcal{C} \neq 1$). Furthermore, \mathcal{C} can be written as

$$\mathcal{C} = \max_{\{v\}} \text{Tr}[V_{\mathcal{E}} \rho] = \max_{\{k\}} \{v_k\}. \quad (28)$$

The positive definite operator $V_{\mathcal{E}}$ reads

$$V_{\mathcal{E}} \equiv \mathcal{E}[I] = \sum_{\alpha} V_{\alpha} V_{\alpha}^{\dagger} = \sum_k v_k |k\rangle \langle k|, \quad (29)$$

where $\{v_k\}$ and $\{|k\rangle\}$ are the corresponding eigenvalues and eigenbasis. Thus, \mathcal{C} is the largest eigenvalue of the operator $V_{\mathcal{E}}$. On the other hand, the set of operators $\{V_{\alpha}\}$ defines the Kraus representation [1] of the map \mathcal{E} and its dual $\mathcal{E}^{\#}$, the latter being defined by the relation $\text{Tr}[A \mathcal{E}[\rho]] = \text{Tr}[\rho \mathcal{E}^{\#}[A]]$. Explicitly,

$$\mathcal{E}[\rho] = \sum_{\alpha} V_{\alpha} \rho V_{\alpha}^{\dagger}, \quad \mathcal{E}^{\#}[\rho] = \sum_{\alpha} V_{\alpha}^{\dagger} \rho V_{\alpha}. \quad (30)$$

Notice that trace preservation implies $\sum_{\alpha} V_{\alpha}^{\dagger} V_{\alpha} = I$.

Demonstration. First, we notice that $V_{\mathcal{E}} = \mathcal{E}[I] = \dim(\mathcal{H}) \mathcal{E}[I/\dim(\mathcal{H})]$. Consequently, $\text{Tr}[V_{\mathcal{E}}] = \dim(\mathcal{H}) = \sum_k v_k$, which, for nonunital maps, supports the inequality (27). Furthermore, considering unital maps, $V_{\mathcal{E}} \rightarrow I$ (which implies $v_k = 1 \forall k$) leading to $\mathcal{C} \rightarrow 1$.

Based on the definition (3), we write

$$\begin{aligned} D_{\rho}(\mathcal{E}[\rho_A], \mathcal{E}[\rho_B]) &= \max_{\{\rho\}} |\text{Tr}[\rho(\mathcal{E}[\rho_A] - \mathcal{E}[\rho_B])]| \\ &= \max_{\{\rho\}} |\text{Tr}[\rho \mathcal{E}[\rho_A - \rho_B]]| \\ &= \max_{\{\rho\}} |\text{Tr}[\mathcal{E}^{\#}[\rho](\rho_A - \rho_B)]| \\ &= \max_{\{\rho\}} (\text{Tr}[\mathcal{E}^{\#}[\rho]] |\text{Tr}[\rho \mathcal{E}(\rho_A - \rho_B)]|), \end{aligned} \quad (31)$$

where we have used that $\text{Tr}[\mathcal{E}^{\#}[\rho]] > 0$ and defined the state

$$\rho_{\mathcal{E}} = \rho_{\mathcal{E}}[\rho] \equiv \frac{\mathcal{E}^{\#}[\rho]}{\text{Tr}[\mathcal{E}^{\#}[\rho]]}. \quad (32)$$

Given that $\rho_{\mathcal{E}}$ is a positive definite operator with unit trace, the second factor in the last line of Eq. (31) fulfills $|\text{Tr}[\rho_{\mathcal{E}}(\rho_A - \rho_B)]| \leq D_{\rho}(\rho_A, \rho_B)$, leading to the inequality

$$D_{\rho}(\mathcal{E}[\rho_A], \mathcal{E}[\rho_B]) \leq (\max_{\{\rho\}} \text{Tr}[\mathcal{E}^{\#}[\rho]]) D_{\rho}(\rho_A, \rho_B). \quad (33)$$

From here, we recover Eq. (26) with

$$\mathcal{C} = \max_{\{\rho\}} \text{Tr}[\mathcal{E}^{\#}[\rho]]. \quad (34)$$

Using the Kraus representation [Eq. (30)], the cyclic property of the trace operation, and the maximization defined in Appendix A, this last expression recovers Eq. (28).

A. Maximal departure from contractivity

Given a nonunital map \mathcal{E} , the inequality (26) implies that there may (or may not) exist states ρ_A and ρ_B such that the usual contractivity is violated. Taking into account that the states that maximize the definition of D_{ρ} [Eq. (3)] can always be chosen as pure states [see Eqs. (6) and (16)], we expect that contractivity is not fulfilled for states (ρ_A and ρ_B) whose *purity* is increased by the map. Nevertheless, this relation is not valid in general (over the complete set of possible input states). On the other hand, here we analyze the conditions under which maximal departure could be achieved, $D_{\rho}(\mathcal{E}[\rho_A], \mathcal{E}[\rho_B]) = \mathcal{C} D_{\rho}(\rho_A, \rho_B)$.

Taking into account the last line of Eq. (31), the equality in Eq. (26) is fulfilled when the state ρ^{\max} that maximizes the definition of $D_{\rho}(\rho_A, \rho_B)$ can be written as $\rho^{\max} = \rho_{\mathcal{E}}[\rho_v]$ where ρ_v is the state that maximizes Eq. (34). The state ρ_v can always be chosen as the projector (or mixed state) associated to the space spanned by the eigenstate of $V_{\mathcal{E}}$ with maximal eigenvalue [Eq. (28)] (Appendix A). Due to the action of $\mathcal{E}^{\#}$ [Eq. (32)], $\rho_{\mathcal{E}}$ is, *in general*, a mixed state. Consequently, maximal departure can be reached under the following conditions: (i) The eigenvalue of $(\rho_A - \rho_B)$ with maximal absolute value must be degenerate such that ρ^{\max} can be chosen as an arbitrary statistical superposition (mixed state) of the corresponding eigenvectors (Appendix A). (ii) The equality $\rho^{\max} = \rho_{\mathcal{E}}[\rho_v]$ must be fulfilled.

B. Witnessing maximal violation of contractivity

For an arbitrary map \mathcal{E} , the previous conditions could not be fulfilled. In such a case, maximal violation of contractivity is not observed. In contrast, here we demonstrate that by adding a passive ancillary system, maximal departure from contractivity is always achieved.

For simplicity, the ancilla is taken as a two-level system with associated basis of states $\{|\pm\rangle\}$. The map is extended to the “system-ancilla” Hilbert space as

$$\tilde{\mathcal{E}} = \mathcal{E} \otimes I_a, \quad (35)$$

where I_a is the identity operator for the ancilla system. Furthermore, we consider the states

$$\rho_A = \frac{I}{\dim(\mathcal{H})} \otimes |+\rangle\langle +|, \quad \rho_B = \frac{I}{\dim(\mathcal{H})} \otimes |-\rangle\langle -|. \quad (36)$$

Therefore, it is simple to obtain

$$\tilde{\mathcal{E}}[\rho_A] - \tilde{\mathcal{E}}[\rho_B] = \frac{1}{\dim(\mathcal{H})} V_{\mathcal{E}} \otimes (|+\rangle\langle +| - |-\rangle\langle -|), \quad (37)$$

where we have used that $\mathcal{E}[I] = V_{\mathcal{E}}$. From the previous two expressions, it follows that $D_{\rho}(\rho_A, \rho_B) = 1/\dim(\mathcal{H})$ and $D_{\rho}(\tilde{\mathcal{E}}[\rho_A], \tilde{\mathcal{E}}[\rho_B]) = \mathcal{C}/\dim(\mathcal{H})$, where \mathcal{C} is defined by Eq. (28). Consequently,

$$D_{\rho}(\tilde{\mathcal{E}}[\rho_A], \tilde{\mathcal{E}}[\rho_B]) = \mathcal{C} D_{\rho}(\rho_A, \rho_B). \quad (38)$$

Consistently, the previously defined conditions (i) and (ii) are satisfied. Furthermore, in agreement with the qualitative argument based on the purity of the states, here $\text{Tr}[(\tilde{\mathcal{E}}[\rho_A])^2] + \text{Tr}[(\tilde{\mathcal{E}}[\rho_B])^2] > \text{Tr}[\rho_A^2] + \text{Tr}[\rho_B^2]$. On the other hand, for the same states [Eq. (36)], we have $D_{\Pi}(\rho_A, \rho_B) = 1$ and $D_{\Pi}(\tilde{\mathcal{E}}[\rho_A], \tilde{\mathcal{E}}[\rho_B]) = \text{Tr}[V_{\mathcal{E}}]/\dim(\mathcal{H}) = 1$.

It is important to notice that the states ρ_A and ρ_B that lead to the previous result are not unique. In fact, under the replacements $\rho_A \rightarrow (1-w)\rho_{sa} + w\rho_A$ and $\rho_B \rightarrow (1-w)\rho_{sa} + w\rho_B$, where $0 < w \leq 1$ and ρ_{sa} is an arbitrary system-ancilla state, one again arrives at the equality (38).

C. Quantumness of nonunital maps

The classicality, or complementarily the quantumness, of a given open system evolution can be tackled from different perspectives [15–24]. Consistent with Refs. [15,24], here a map $\rho \rightarrow \mathcal{E}(\rho)$ with the structure

$$\mathcal{E}(\rho) = \sum_c p_c U_c \rho U_c^{\dagger}, \quad (39)$$

where U_c is a unitary transformation and whose weight is p_c , is read as a classical one. In fact, this structure can always be implemented without involving any quantum feature of the environment. Notice that all maps that admit this classical interpretation are also unital (the inverse implication in general is not true; see, for example, [30]). Consequently, in contrast with D_{Π} , the lack of contractivity of D_{ρ} witnesses the nonclassicality of nonunital maps. This property allows us to introduce a degree of map quantumness \mathcal{M}_Q , which gives one the main supports of the present approach. Given that the constant \mathcal{C} measures the maximal departure from contractivity, \mathcal{M}_Q is defined as

$$\mathcal{M}_Q \equiv \mathcal{C} - 1 = \max_{\{\rho\}} |\text{Tr}[\mathcal{E}^{\#}[\rho]] - 1|, \quad (40)$$

where the equality is based on Eq. (34). Furthermore, it is bounded as $0 \leq \mathcal{M}_Q \leq \dim(\mathcal{H}) - 1$.

Using the relation between a map and its dual, Eq. (40) can equivalently be rewritten as

$$\frac{\mathcal{M}_Q}{\dim(\mathcal{H})} = D_{\rho}(\mathcal{E}[\rho_I], \rho_I), \quad (41)$$

where $\rho_I \equiv I/\dim(\mathcal{H})$ is the maximal mixed state. This equality explicitly shows the role of D_{ρ} in the present definition. Furthermore, it allows one to understand the scheme that

permits its determination [Eq. (38)]. In fact, the states (36) involve the (system) maximally mixed state. They lead to maximal departure from contractivity but, in addition, they lead to $\mathcal{M}_Q = 0$ [Eq. (41)] when the map is unital.

Even when D_{ρ} is contractive when $\dim(\mathcal{H}) = 2$ and $\dim(\mathcal{H}) = 3$, \mathcal{M}_Q can be determined in these cases because the extra ancilla leads to a higher-dimensional space (see Sec. IV C). With this dimensionality, the constant \mathcal{C} must be read from the general expression (28). On the other hand, we remark that \mathcal{M}_Q also applies to time-dependent open system dynamics after identifying the map \mathcal{E} with the propagator of the system density matrix. Equation (41) also recovers the degree of environment quantumness introduced in Ref. [24] when studying continuous-in-time evolutions characterized by a unique stationary state [see analysis below Eq. (48)].

IV. EXAMPLES

Here we characterize the proposed distinguishability measure for some particular quantum states. In addition, its behavior under different completely positive maps is studied in detail.

A. Particular cases

(a) When both states are pure, $\rho_A = |\psi_A\rangle\langle\psi_A|$, $\rho_B = |\psi_B\rangle\langle\psi_B|$, from Eq. (3) we get,

$$D_{\rho}(\rho_A, \rho_B) = \max_{\{\rho\}} |\langle\psi_A|\rho|\psi_A\rangle - \langle\psi_B|\rho|\psi_B\rangle|. \quad (42)$$

This expression can be solved after calculating the eigenvalues ζ defined by $(\rho_A - \rho_B)|\psi\rangle = \zeta|\psi\rangle$, where $|\psi\rangle = a|\psi_A\rangle + b|\psi_B\rangle$. We get $\zeta = \pm\sqrt{1 - |\langle\psi_A|\psi_B\rangle|^2}$. The rest of the eigenvalues, $\zeta = 0$, correspond to eigenvectors that are perpendicular to the plane spanned by $|\psi_A\rangle$ and $|\psi_B\rangle$. Thus, from Eq. (5), it follows that

$$D_{\rho}(\rho_A, \rho_B) = \sqrt{1 - |\langle\psi_A|\psi_B\rangle|^2}. \quad (43)$$

Given that $D_{\Pi}(\rho_A, \rho_B) = \sqrt{1 - |\langle\psi_A|\psi_B\rangle|^2}$ [1], $D_{\rho}(\rho_A, \rho_B) = D_{\Pi}(\rho_A, \rho_B)$. In fact, $(\rho_A - \rho_B)$ has a unique positive (negative) eigenvalue [see Eq. (23)].

For orthogonal states, Eq. (43) leads to

$$\langle\psi_A|\psi_B\rangle = 0, \Rightarrow D_{\rho}(\rho_A, \rho_B) = 1. \quad (44)$$

Nevertheless, the inverse implication is not valid, that is, $D_{\rho}(\rho_A, \rho_B) = 1$ does not imply that ρ_A and ρ_B are pure states. Take, for example, $\rho_A = |\psi_A\rangle\langle\psi_A|$ and $\rho_B = \sum_k w_k |\psi_B^k\rangle\langle\psi_B^k|$, where the positive weights are normalized, $\sum_k w_k = 1$, and $\langle\psi_B^k|\psi_A\rangle = 0 \forall k$.

In general, it is simple to realize that $D_{\rho}(\rho_A, \rho_B) = 1$ if and only if ρ_A and ρ_B have support on orthogonal subspaces and ρ_A or ρ_B is a pure state. Instead, $D_{\Pi}(\rho_A, \rho_B) = 1$, whenever ρ_A and ρ_B have support on orthogonal subspaces.

(b) Here we consider two qubit states,

$$\rho_A = (1/2)(I + \alpha \cdot \sigma), \quad \rho_B = (1/2)(I + \beta \cdot \sigma), \quad (45)$$

where α and β are the Bloch vectors and σ is the vector of Pauli matrices. Then, $\rho_A - \rho_B = (1/2)(\alpha - \beta) \cdot \sigma = (1/2)|\alpha - \beta|(\mathbf{n} \cdot \sigma)$, where $\mathbf{n} = (\alpha - \beta)/|\alpha - \beta|$. Given

that the eigenvalues of $(\mathbf{n} \cdot \sigma)$ are ± 1 , it follows that

$$D_\rho(\rho_A, \rho_B) = \frac{1}{2}|\alpha - \beta|. \quad (46)$$

In an alternative way, this result explicitly confirms that when $\dim(\mathcal{H}) = 2$, both measures coincide: in fact, $D_\rho(\rho_A, \rho_B) = D_\Pi(\rho_A, \rho_B) = (1/2)|\alpha - \beta|$ [1].

(c) Now we consider that one of the density matrices is the maximally mixed state. Under the replacements $\rho_A \rightarrow \varrho$, where ϱ is an arbitrary density matrix, and $\rho_B \rightarrow \rho_I = I/\dim(\mathcal{H})$, from Eq. (5), we get

$$D_\rho(\varrho, \rho_I) = \max_{\{i\}} \left\{ \left| \lambda_i - \frac{1}{\dim(\mathcal{H})} \right| \right\}, \quad (47)$$

where $\{\lambda_i\}$ are the eigenvalues of ϱ . This expression can be rewritten as

$$D_\rho(\varrho, \rho_I) = \frac{1}{\dim(\mathcal{H})} \max(D_\rho^{\max}, D_\rho^{\min}), \quad (48)$$

where the coefficients are

$$D_\rho^{\max} \equiv \dim(\mathcal{H}) \max\{\lambda_i\} - 1, \quad (49a)$$

$$D_\rho^{\min} \equiv 1 - \dim(\mathcal{H}) \min\{\lambda_i\}. \quad (49b)$$

Here, $\max\{\lambda_i\}$ and $\min\{\lambda_i\}$ are the maximal and minimal eigenvalues of ϱ . These expressions recover the degree of environment quantumness D_Q introduced in Ref. [24]. With the present notation, it can be written as $D_Q = \dim(\mathcal{H})D_\rho(\tilde{\rho}_\infty, \rho_I)$, where $\tilde{\rho}_\infty = \lim_{t \rightarrow \infty} \rho_t$ (disregarding a technical time-inversion operation) is the system stationary state. Under the identification $\tilde{\rho}_\infty \rightarrow \mathcal{E}[\rho_I]$, this last expression for D_Q assumes the structure of Eq. (41).

(d) Take both density matrices as diagonal ones, with

$$\begin{aligned} \rho_A &= (1/10)\text{diag}\{5, 2, 2, 1\}, \\ \rho_B &= (1/4)\text{diag}\{1, 1, 1, 1\}. \end{aligned}$$

Given that ρ_B is the maximally mixed state, ρ_A can be read as an arbitrary quantum state written in its eigenbasis. We notice that $(\rho_A - \rho_B)$ only has one positive eigenvalue. This eigenvalue is not degenerate and coincides with the eigenvalue with maximal absolute value. Thus, both measures [Eqs. (2) and (5)] coincide. In fact,

$$D_\rho(\rho_A, \rho_B) = D_\Pi(\rho_A, \rho_B) = 0.25. \quad (50)$$

(e) Instead, taking

$$\begin{aligned} \rho_A &= (1/10)\text{diag}\{5, 3, 1, 1\}, \\ \rho_B &= (1/4)\text{diag}\{1, 1, 1, 1\}, \end{aligned}$$

it follows that $\rho_A - \rho_B$ has two positive and two negative eigenvalues. In this case, both measures differ [Eqs. (2) and (5)]. We get

$$D_\rho(\rho_A, \rho_B) = 0.25 < D_\Pi(\rho_A, \rho_B) = 0.3. \quad (51)$$

(f) In this example,

$$\begin{aligned} \rho_A &= (1/10)\text{diag}\{4, 4, 1, 1\}, \\ \rho_B &= (1/4)\text{diag}\{1, 1, 1, 1\}. \end{aligned}$$

Hence, $\rho_A - \rho_B$ has two degenerate positive eigenvalues, as well as two degenerate negative eigenvalues. In this case, both measures differ [Eqs. (2) and (5)]. It is fulfilled that

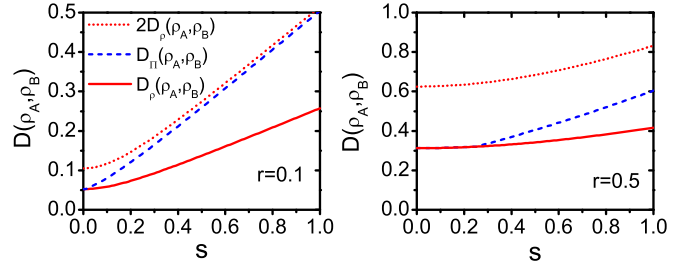


FIG. 1. Distances between the quantum states defined by Eq. (53). The full lines correspond to $D_\rho(\rho_A, \rho_B)$, the dashed lines to $D_\Pi(\rho_A, \rho_B)$, while the dotted lines correspond to $2D_\rho(\rho_A, \rho_B)$. The figures show the dependence with the parameter s associated to ρ_B . The left and right panels correspond to $r = 0.1$ and $r = 0.5$, respectively, where r is the parameter associated to ρ_A .

$0.15 = D_\rho(\rho_A, \rho_B) < D_\Pi(\rho_A, \rho_B) = 0.3$. In addition, the eigenvalue with maximal absolute value has degeneracy equal to two. Consistently with Eq. (19), it is fulfilled that

$$D_\Pi(\rho_A, \rho_B) = 0.3 = 2D_\rho(\rho_A, \rho_B). \quad (52)$$

(g) Here we take the quantum states

$$\rho_A = \frac{1}{2}(I_2 + r\sigma_z) \otimes \frac{1}{2}(I_2 + r\sigma_z), \quad (53a)$$

$$\rho_B = \frac{1}{4}(I_4 + s\sigma_x \otimes \sigma_x), \quad (53b)$$

where the parameters are constrained as $0 \leq r \leq 1$ and $0 \leq s \leq 1$. The dimensionality of the identity matrix I is denoted with its subindex. Furthermore, σ_i are the Pauli matrices. We notice that while ρ_A (a separable state) is diagonal in the natural basis, ρ_B is diagonal in the Bell basis. The four eigenvalues of $(\rho_A - \rho_B)$ are $\{\zeta_i\} = (1/4)\{(\pm s - r^2), (r^2 \pm \sqrt{4r^2 + s^2})\}$. Hence, $D_\rho(\rho_A, \rho_B)$ and $D_\Pi(\rho_A, \rho_B)$ follow from Eqs. (5) and (22), respectively. After some algebra, we find that $D_\rho(\rho_A, \rho_B) = D_\Pi(\rho_A, \rho_B)$ if $s \leq r^2$. In Fig. 1, we plot both distinguishability measures as a function of s for two different values of r . Consistently, the behaviors confirm both the inequalities ($\mathcal{N} = 2$) and equalities defined by Eq. (19).

B. Depolarizing maps

Depolarizing maps (in any Hilbert space dimension) can be defined as

$$\rho \rightarrow \mathcal{E}_w[\rho] = w\rho + (1-w)\frac{I}{\dim(\mathcal{H})}, \quad (54)$$

where $0 \leq w < 1$. Given that this map is unital [1], our previous analysis guarantees that contractivity is fulfilled [Eq. (25)]. In fact, by writing $\rho_A - \rho_B = \sum \xi_i |i\rangle\langle i|$, it follows that $\mathcal{E}[\rho_A] - \mathcal{E}[\rho_B] = w(\rho_A - \rho_B) = w \sum \xi_i |i\rangle\langle i|$. Given that $w|\xi_i| < |\xi_i| \forall i$, using Eq. (5), it follows that

$$D_\rho[\mathcal{E}_w(\rho_A), \mathcal{E}_w(\rho_B)] < D_\rho(\rho_A, \rho_B), \quad (55)$$

where $D_\rho[\mathcal{E}_w(\rho_A), \mathcal{E}_w(\rho_B)] = w \max_{\{i\}} \{|\xi_i|\}$, while $D_\rho(\rho_A, \rho_B) = \max_{\{i\}} \{|\xi_i|\}$.

C. Zero-temperature qubit map

A qubit system coupled to a zero-temperature reservoir can be described by the map $\mathcal{E}[\rho] = V_0 \rho V_0^\dagger + V_1 \rho V_1^\dagger$, with Kraus

operators

$$V_0 = \begin{pmatrix} \sqrt{1-\gamma} & 0 \\ 0 & 1 \end{pmatrix}, \quad V_1 = \begin{pmatrix} 0 & 0 \\ \sqrt{\gamma} & 0 \end{pmatrix}, \quad (56)$$

where $\gamma \in [0, 1]$. The action over an arbitrary state ρ is

$$\rho = \begin{pmatrix} p & c \\ c^* & q \end{pmatrix} \rightarrow \mathcal{E}[\rho] = \begin{pmatrix} (1-\gamma)p & \sqrt{1-\gamma}c \\ \sqrt{1-\gamma}c^* & q + \gamma p \end{pmatrix}, \quad (57)$$

where p and q denote populations, while c denotes coherence. Notice that the parameter γ gives the probability for a transition from the upper to the lower level, $|+\rangle \rightarrow |-\rangle$.

Consistent with the trace-preservation property, it is fulfilled that $V_0^\dagger V_0 + V_1^\dagger V_1 = I$. On the other hand,

$$\mathcal{E}[I] = V_0 V_0^\dagger + V_1 V_1^\dagger = \begin{pmatrix} 1-\gamma & 0 \\ 0 & 1+\gamma \end{pmatrix} \neq I. \quad (58)$$

Thus, the map is not unital (also nonclassical). Nevertheless, given the system dimensionality, $\dim(\mathcal{H}) = 2$, contractivity must be fulfilled [Eq. (25)]. This property is corroborated in Appendix C.

Here we study the two qubits map,

$$\mathcal{E} = \mathcal{E}_a \otimes \mathcal{E}_b, \quad (59)$$

where the maps \mathcal{E}_a and \mathcal{E}_b are defined by the Kraus operators (56) under the replacements $\gamma \rightarrow \gamma_a$ and $\gamma \rightarrow \gamma_b$, respectively.

Instead of proposing a set of states ρ_A and ρ_B , we write their difference ($\rho_A - \rho_B$) in its proper eigenbasis as

$$\Delta = \rho_A - \rho_B = \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & y & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & -(x+y+z) \end{pmatrix}. \quad (60)$$

Under appropriate constraints on these parameters [eigenvalues x, y, z and $-(x+y+z)$],

$$|x| \leq 1, \quad |y| \leq 1, \quad |z| \leq 1, \quad |x+y+z| \leq 1, \quad (61a)$$

jointly with

$$|x+y| \leq 1, \quad |x+z| \leq 1, \quad |y+z| \leq 1, \quad (61b)$$

the matrix Δ represents a difference of two arbitrary density matrices (see derivation in Appendix D). From its definition (5), the distance between the input states [$D_\rho(\rho_A, \rho_B) = D_\rho(\Delta)$] is

$$D_\rho(\Delta) = \max\{|x|, |y|, |z|, |x+y+z|\}. \quad (62)$$

In order to solve the action of the map on the difference of states, Δ , we need to explicitly specify the basis where it is diagonal. For simplicity, we take the same basis where the Kraus operators are defined, $\{|++\rangle, |+-\rangle, |-+\rangle, |--\rangle\}$. In this case, the application of the map (59) over Δ leads to a diagonal matrix $\mathcal{E}[\Delta]$, whose four elements are

$$\mathcal{E}[\Delta]_{++} = (1-\gamma_a)(1-\gamma_b)x, \quad (63a)$$

$$\mathcal{E}[\Delta]_{+-} = (1-\gamma_a)(x\gamma_b + y), \quad (63b)$$

$$\mathcal{E}[\Delta]_{-+} = (1-\gamma_b)(x\gamma_a + z), \quad (63c)$$

$$\mathcal{E}[\Delta]_{--} = -(1-\gamma_a\gamma_b)x - (1-\gamma_a)y - (1-\gamma_b)z. \quad (63d)$$

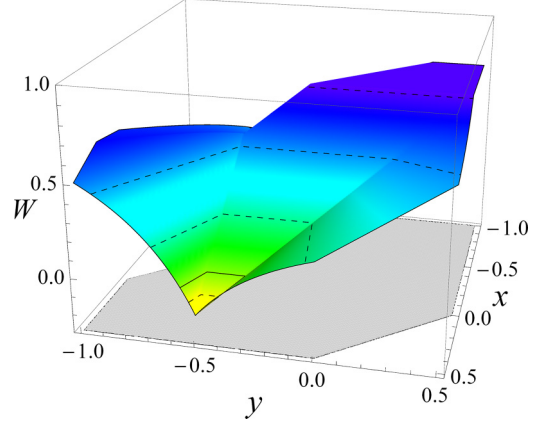


FIG. 2. Witness W [Eq. (65)] for the two qubit map (59) as a function of (x, y) and fixed z [Eq. (60)]. The map parameters are $\gamma_a = 1/2$ and $\gamma_b = 1/4$. The horizontal full line corresponds to the level curve $W = 0$. The gray plane corresponds to the domain of (x, y) given that here, $z = 0.5$.

We notice that here the symmetry under interchange of subsystems, $a \leftrightarrow b$, is consistently fulfilled under the parameter changes $\gamma_a \leftrightarrow \gamma_b$ and $y \leftrightarrow z$. The distance between the output states, from (5), can be written as

$$D_\rho(\mathcal{E}[\Delta]) = \max_{\{s,s'\}} \{|\mathcal{E}[\Delta]_{ss'}|\}, \quad s = \pm 1, \quad s' = \pm 1. \quad (64)$$

Both $D_\rho(\Delta)$ and $D_\rho(\mathcal{E}[\Delta])$ [Eqs. (62) and (64)] depend on (x, y, z) . This dependence labels different possible states ρ_A and ρ_B . In order to quantify the violation of (standard) contractivity [Eq. (26)], we introduce the (dimensionless) witness,

$$W \equiv \frac{D_\rho(\rho_A, \rho_B) - D_\rho(\mathcal{E}[\rho_A], \mathcal{E}[\rho_B])}{D_\rho(\rho_A, \rho_B)(C-1)}. \quad (65)$$

If $W \geq 0$, the usual contractivity is fulfilled. Whenever $W < 0$, the usual contractivity is not fulfilled. When $W = -1$, the maximal violation of contractivity is achieved. In fact, in this case, $D_\rho(\mathcal{E}[\rho_A], \mathcal{E}[\rho_B]) = C D_\rho(\rho_A, \rho_B)$. Furthermore, notice that $W = W(x, y, z)$ where $D_\rho(\rho_A, \rho_B) = D_\rho(\Delta)$ [Eq. (62)] and $D_\rho(\mathcal{E}[\rho_A], \mathcal{E}[\rho_B]) = D_\rho(\mathcal{E}[\Delta])$ [Eq. (64)].

For the bipartite map (59), the constant C , from Eqs. (28) and (58), is

$$C = (1+\gamma_a)(1+\gamma_b) \leq 4. \quad (66)$$

Consistent with our definitions [$\mathcal{M}_Q = C - 1$; see Eq. (40)], classicality is only achieved when $\gamma_a = \gamma_b = 0$, which reduces the map [Eq. (59)] to the identity.

In Figs. 2 and 3, we plot the contractivity witness W as a function of (x, y) and fixed z . Given z , the domain of the (x, y) variables corresponds to the surface defined by $z = \text{const}$ in the three-dimensional body defined by Eq. (61) (see Fig. 4 in Appendix D).

In Fig. 2, the map parameters [Eqs. (57) and (59)] are $\gamma_a = 1/2$ and $\gamma_b = 1/4$. Furthermore, we take $z = 0.5$. Depending on the values of (x, y) , we observe a transition between contractivity ($W > 0$) and its violation ($W < 0$). Furthermore, we observe that the limit of maximal departure from contractivity

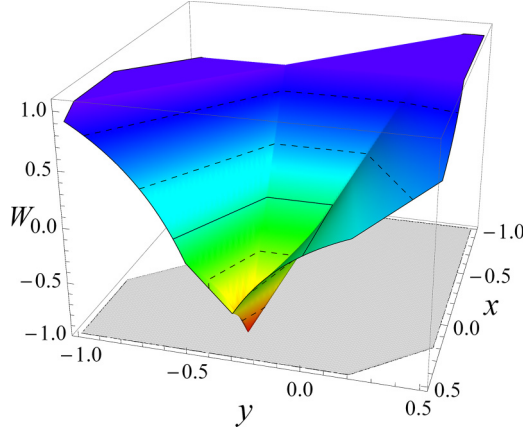


FIG. 3. Witness W [Eq. (65)] for the two qubit map (59) as a function of (x, y) and fixed z [Eq. (60)]. The map parameters are $\gamma_a = 1/2$ and $\gamma_b = 0$. The horizontal full line corresponds to the level curve $W = 0$. The gray plane corresponds to the domain of (x, y) given that here, $z = 0.3$.

is not achieved ($W \neq -1$). We checked that these properties remain the same when considering other possible values of z .

In general, the dependence of W on (x, y, z) defines a complex landscape. It may include regions where W is constant or even develops nonsmooth nonderivable behaviors. These features are inherited from the expressions for the input and output distances, Eqs. (62) and (64), which involve a maximization associated to the definition of D_ρ . Given this feature, in general, it is not easy or even possible to infer (analytically) general properties of W as a function of the underlying parameters, here γ_a and γ_b . Nevertheless, for this example, it is possible to check the following properties.

Assuming that $\gamma_b \leq \gamma_a$, the witness $W = W(x, y, z)$ assumes its minimal value,

$$W_{\min} = 1 - \frac{2\gamma_a}{(\gamma_a + \gamma_b + \gamma_a\gamma_b)}, \quad (67)$$

when $x = z$, $y = -z$ and arbitrary z in its domain. For this choice, its domain is $|z| \leq 1/2$ [see Eq. (61)]. From the expression of W_{\min} , it follows that when

$$\gamma_b < \frac{\gamma_a}{(1 + \gamma_a)}, \quad (68)$$

there exist input states [values $(x, y, z) = (z, -z, z)$, with $z \neq 0$] where contractivity is not fulfilled ($W_{\min} < 0$). The parameters of Fig. 2 are in this regime, where $W_{\min} \cong -0.14$ at $x = 0.5$, $y = -0.5$, $z = 0.5$.

From Eq. (67), it follows that maximal departure ($W_{\min} = -1$) can only be achieved when $\gamma_b = 0$. Hence, the subsystem b can be read as the passive ancillary system associated to the scheme of Sec. III B, which allows one to determine the quantumness of the two-dimensional qubit map [Eq. (57)],

$$\mathcal{C} = (1 + \gamma_a), \quad \mathcal{M}_Q = \gamma_a. \quad (69)$$

In Fig. 3, we check this regime. The map parameters are $\gamma_a = 1/2$, $\gamma_b = 0$. Furthermore, $z = 0.3$. Consistently, when $x = 0.3$, $y = -0.3$, $W = -1$ is achieved. Here, the degeneracy of the value of z for getting $W = -1$ [$(x, y, z) = (z, -z, z)$] can straightforwardly be related to the nonunique-

ness of the states that achieve maximal departure in the proposed scheme (Sec. III B). Explicitly, here the states can be taken as $\rho_A = (1 - 2|z|)\varrho_{ab} + |z|(I_2 \otimes |+\rangle\langle +|)$ and $\rho_B = (1 - 2|z|)\varrho_{ab} + |z|(I_2 \otimes |-\rangle\langle -|)$, where ϱ_{ab} is an arbitrary density matrix for two qubits. Hence, $\rho_A - \rho_B$ (jointly with $\rho_B - \rho_A$) recovers Eq. (60) with $x = -y = z$.

V. SUMMARY AND CONCLUSIONS

We have introduced an alternative distinguishability measure between quantum states. In contrast to the standard definition based on maximization over projectors, the proposed measure relies on a maximization over states [Eq. (3)]. This operation can be explicitly performed [Eq. (5)], which allowed us to demonstrate that the proposed measure is a metric in the space of density matrices based on an operator-infinite-norm. In addition, it was shown that properties such as convexity, monotonicity in bipartite Hilbert spaces, and invariance under unitary transformations are also fulfilled.

Similarly to the usual projector-based definition, different operational interpretations of the proposed distinguishability measure have been established. It can be read as a maximization over states of a distance between probabilities, each one being associated to each input state [Eq. (8)]. The distinguishability measure also defines the probability of success in a hypothesis-testing scenario [Eq. (17)] where a state is guessed in terms of a measurement process consisting of a 1-rank projector and its complement [Eq. (12)].

The projector- and state-based definitions are equal when the Hilbert space dimension is two or three [Eq. (18)]. For higher-dimensional spaces [Eq. (19)], the relationship between both objects depends on the eigenvalues of the difference of states. When the eigenvalue with maximal absolute value is not degenerate, both measures coincide. When this eigenvalue has maximal degeneracy, the state-based definition achieves its minimal value with respect to the projector-based definition.

In contrast to other distances in Hilbert space, we demonstrated that the proposed measure is able to quantify the quantum character of dissipative open system dynamics. This result relies on the contractivity properties of the proposed measure. For unital maps, contractivity is always satisfied, while for nonunital maps, violation of contractivity is expected, meaning that there could be states such that their distance increases after application of the map. It was shown that maximal violation of contractivity is always achieved when expanding the map to an extra ancillary Hilbert space without dynamics [Eqs. (35) and (38)]. The quantumness measure for nonunital maps is defined by the constant associated to this scheme which, in turn, can be written in terms of the proposed distinguishability measure [Eqs. (40) and (41)].

We have studied some particular cases and examples that sustain the main results and conclusions. The proposed measure may find applications in quantum information tasks as well as in the characterization of open quantum system dynamics. In particular, given that dissipative nonclassical (quantum) system-environment interactions lead to nonunital dynamics, the present measure plays a central role when characterizing this quantum-classical border.

ACKNOWLEDGMENTS

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APPENDIX A: MAXIMIZATION OVER STATES OF AN OPERATOR EXPECTATION VALUE

Let A be an arbitrary Hermitian operator, $A = A^\dagger$. Define its maximized expectation value by

$$\langle A \rangle_{\max} \equiv \max_{\{\rho\}} |\text{Tr}[\rho A]|, \quad (\text{A1})$$

where the maximization is performed over positive-definite normalized density matrices, $\text{Tr}[\rho] = 1$. Introducing the eigenbasis $\{|i\rangle\}$ of the operator A ,

$$A|i\rangle = \lambda_i|i\rangle, \quad (\text{A2})$$

where $\{\lambda_i\}$ are the corresponding eigenvalues, it follows that

$$\langle A \rangle_{\max} = \max_{\{\rho\}} \left| \sum_i \lambda_i \langle i|\rho|i\rangle \right| = \max_{\{P_i\}} \left| \sum_i \lambda_i P_i \right|. \quad (\text{A3})$$

Here, $P_i \equiv \langle i|\rho|i\rangle$, $0 \leq P_i \leq 1$. Using the triangular inequality $|a + b| \leq |a| + |b|$, it follows that

$$\left| \sum_i \lambda_i P_i \right| \leq \sum_i |\lambda_i P_i| \leq \max_{\{i\}} (|\lambda_i|) \sum_i P_i, \quad (\text{A4})$$

where $\max_{\{i\}} (|\lambda_i|)$ is the eigenvalue of A with maximal absolute value. Using that $\sum_i P_i = 1$, we obtain

$$\langle A \rangle_{\max} = \max_{\{i\}} (|\lambda_i|). \quad (\text{A5})$$

Hence, $\langle A \rangle_{\max}$ is the eigenvalue of the operator A with maximal absolute value. On the other hand, we notice that $\langle A \rangle_{\max} = 0 \Leftrightarrow A = 0$. Both implications follow straightforwardly from Eqs. (A1) and (A5), respectively.

The state ρ that achieves the maximal value in the definition (A1) can always be chosen as $\rho = |i_{\max}\rangle\langle i_{\max}|$, where $|i_{\max}\rangle$ is the eigenstate associated to $\max_{\{i\}} (|\lambda_i|)$. If this eigenvalue (with a given sign) is degenerate, ρ can be taken as an arbitrary mixed state over the corresponding subspace. On the other hand, if there exists a subspace with null eigenvalues, $\{\lambda_k = 0\}$, the demonstration remains the same because $\sum_i P_i$ can always be normalized to one on the subspace with non-null eigenvalues.

APPENDIX B: GENERAL PROPERTIES OF D_ρ

The distinguishability measure $D_\rho(\rho_A, \rho_B)$ fulfills some general properties whose formulation and demonstration are provided below.

(a) $D_\rho(\rho_A, \rho_B)$ is positive and bounded,

$$0 \leq D_\rho(\rho_A, \rho_B) \leq 1. \quad (\text{B1})$$

This results follows from Eq. (5) after noticing that Eq. (4) implies that $\langle i|(\rho_A - \rho_B)|i\rangle = \langle i|\rho_A|i\rangle - \langle i|\rho_B|i\rangle = \zeta_i$, which is a difference between two populations leading to $-1 \leq \zeta_i \leq 1$.

(b) $D_\rho(\rho_A, \rho_B)$ is null if and only if $\rho_A = \rho_B$,

$$D_\rho(\rho_A, \rho_B) = 0 \Leftrightarrow \rho_A = \rho_B. \quad (\text{B2})$$

Both implications follow from Eqs. (3) and (5).

(c) $D_\rho(\rho_A, \rho_B)$ is a distance or *metric* in the space of density operators, that is, in addition it satisfies

$$D_\rho(\rho_A, \rho_C) \leq D_\rho(\rho_A, \rho_B) + D_\rho(\rho_B, \rho_C), \quad (\text{B3})$$

the triangular inequality.

Demonstration. By its definition [Eq. (3)], there exists a state ρ^{\max} such that

$$\begin{aligned} D_\rho(\rho_A, \rho_C) &= |\text{Tr}[\rho^{\max}(\rho_A - \rho_C)]| \\ &= |\text{Tr}[\rho^{\max}(\rho_A - \rho_B)] + \text{Tr}[\rho^{\max}(\rho_B - \rho_C)]| \\ &\leq |\text{Tr}[\rho^{\max}(\rho_A - \rho_B)]| + |\text{Tr}[\rho^{\max}(\rho_B - \rho_C)]| \\ &\leq D_\rho(\rho_A, \rho_B) + D_\rho(\rho_B, \rho_C), \end{aligned}$$

establishing that $D_\rho(\rho_A, \rho_B)$ is a metric. The inequality in the third line relies on the usual triangular inequality $|a + b| \leq |a| + |b|$.

(d) Given a set of positive normalized weights, $\sum_i p_i = 1$, *convexity* is

$$D_\rho\left(\sum_i p_i \rho_i, \sum_i p_i \sigma_i\right) \leq \sum_i p_i D_\rho(\rho_i, \sigma_i), \quad (\text{B4})$$

where the sets of states $\{\rho_i\}$ and $\{\sigma_i\}$ are arbitrary ones. In the case in which $\sigma_i \rightarrow \sigma$, it follows that

$$D_\rho\left(\sum_i p_i \rho_i, \sigma\right) \leq \sum_i p_i D_\rho(\rho_i, \sigma). \quad (\text{B5})$$

Thus, D_ρ is convex in both entries.

Demonstration. By its definition, there exists a state ρ^{\max} such that

$$\begin{aligned} D_\rho\left(\sum_i p_i \rho_i, \sum_i p_i \sigma_i\right) &= \left| \text{Tr}\left[\rho^{\max} \sum_i p_i (\rho_i - \sigma_i)\right] \right| \\ &= \left| \sum_i p_i \text{Tr}[\rho^{\max}(\rho_i - \sigma_i)] \right| \\ &\leq \sum_i p_i |\text{Tr}[\rho^{\max}(\rho_i - \sigma_i)]| \\ &\leq \sum_i p_i \max_{\{\rho\}} |\text{Tr}[\rho(\rho_i - \sigma_i)]| \\ &= \sum_i p_i D_\rho(\rho_i, \sigma_i), \end{aligned}$$

where the triangular inequality was used in the third line. The demonstration of Eq. (B5) is the same as that before, replacing $\sigma_i \rightarrow \sigma$ and using that $\sum_i p_i = 1$.

(e) In a bipartite Hilbert space with subparts a and b , *monotonicity* is

$$D_\rho(\rho_a, \sigma_a) \leq D_\rho(\rho_{ab}, \sigma_{ab}), \quad (\text{B6})$$

where $\rho_a = \text{Tr}_b[\rho_{ab}]$ and $\sigma_a = \text{Tr}_b[\sigma_{ab}]$.

Demonstration. There exists a state ρ_a^{\max} that leads to maximization,

$$\begin{aligned} D_\rho(\rho_a, \sigma_a) &= |\text{Tr}_a[\rho_a^{\max}(\rho_a - \sigma_a)]| \\ &= |\text{Tr}_{ab}[(\rho_a^{\max} \otimes I_b)(\rho_{ab} - \sigma_{ab})]| \\ &\leq \max_{\{\rho\}} |\text{Tr}_{ab}[\rho(\rho_{ab} - \sigma_{ab})]| \\ &= D_\rho(\rho_{ab}, \sigma_{ab}), \end{aligned}$$

where, consistently, ρ (in the third line) is an arbitrary bipartite state.

(f) *Invariance under unitary rotations,*

$$D_\rho(U\rho_A U^\dagger, U\rho_B U^\dagger) = D_\rho(\rho_A, \rho_B), \quad (\text{B7})$$

where $UU^\dagger = I$. This result straightforwardly follows from the definition (3) after using the cyclic property of the trace and noting that $U^\dagger \rho U$ is also an arbitrary state.

APPENDIX C: CONTRACTIVITY OF THE ZERO-TEMPERATURE QUBIT MAP

Given two arbitrary states ρ_A and ρ_B , their difference is denoted as (see, also, Appendix D)

$$\Delta \equiv \rho_A - \rho_B = \begin{pmatrix} \delta p & \delta c \\ \delta c^* & -\delta p \end{pmatrix}. \quad (\text{C1})$$

The eigenvalues of Δ are $\pm\sqrt{\delta p^2 + |\delta c|^2}$. Consequently,

$$D_\rho[\Delta] = \sqrt{\delta p^2 + |\delta c|^2}. \quad (\text{C2})$$

The action of the map on the difference of states Δ , from Eq. (57), is

$$\mathcal{E}[\Delta] = \begin{pmatrix} (1-\gamma)\delta p & \sqrt{1-\gamma}\delta c \\ \sqrt{1-\gamma}\delta c^* & -(1-\gamma)\delta p \end{pmatrix}. \quad (\text{C3})$$

The eigenvalues of $\mathcal{E}[\Delta]$ are $\pm\sqrt{(1-\gamma)^2\delta p^2 + (1-\gamma)|\delta c|^2}$. Consequently,

$$D_\rho[\mathcal{E}[\Delta]] = \sqrt{(1-\gamma)} \sqrt{(1-\gamma)\delta p^2 + |\delta c|^2}. \quad (\text{C4})$$

From Eqs. (C2) and (C4), it follows that

$$D_\rho[\mathcal{E}[\Delta]] \leq D_\rho[\Delta]. \quad (\text{C5})$$

As expected, the usual contractivity [Eq. (25)] is fulfilled for any input state.

APPENDIX D: SPACE ASSOCIATED TO DIFFERENCE OF QUANTUM STATES

Here we establish how to parametrize, in a general way, the difference between two density matrices. Given two states ρ_A and ρ_B , define

$$\Delta \equiv \rho_A - \rho_B. \quad (\text{D1})$$

Hence, instead of ρ_A and ρ_B , the goal is to parametrize Δ in an independent way. Written in terms of the eigensystem $(\rho_A - \rho_B)|i\rangle = \zeta_i|i\rangle$, it follows that

$$\Delta = \sum_i \zeta_i |i\rangle\langle i|. \quad (\text{D2})$$

Thus, Δ can be characterized in terms of an arbitrary basis $\{|i\rangle\}$ and the eigenvalues $\{\zeta_i\}$. Given that $\text{Tr}[\Delta] = 0$, the addition of the eigenvalues must vanish. Furthermore, each eigenvalue must be in the interval $[-1, 1]$, that is,

$$|\zeta_i| \leq 1, \quad \sum_{i=1}^{\dim(H)} \zeta_i = 0, \quad (\text{D3})$$

where $\dim(H)$ is the dimension of the Hilbert space. Added to these conditions, the sum of an arbitrary number of eigenvalues also must be in the interval $[-1, 1]$. This condition can be explicitly written by introducing the vector of eigenvalues $\zeta = (\zeta_1, \zeta_1, \dots, \zeta_n)$ and the vector $\mathbf{b} = (b_1, b_2, \dots, b_n)$, whose components are $b_i = 0$ or $b_i = 1$. Thus, it must be satisfied that for all vectors \mathbf{b} [$\mathbf{b} \neq (1, 1, \dots, 1)$],

$$|\zeta \cdot \mathbf{b}| = \left| \sum_{k=1}^{\dim(H)} \zeta_k b_k \right| \leq 1. \quad (\text{D4})$$

We notice that the condition $|\zeta_i| \leq 1$ is recovered when \mathbf{b} is the canonical basis, $b_k = \delta_{ki}$. On the other hand, the condition $\sum_{i=1}^{\dim(H)} \zeta_i = 0$ can be written as $|\zeta \cdot \mathbf{b}| = 0$, where $\mathbf{b} = (1, 1, \dots, 1)$.

Demonstration. The previous conditions [Eqs. (D3) and (D4)] can be derived as follows. Straightforwardly, the condition $\text{Tr}[\Delta] = 0$ implies that the eigenvalues fulfill $\sum_i \zeta_i = 0$. On the other hand, given Δ , there must exist states ρ and σ such that $\Delta + \sigma = \rho$. Taking matrix elements in the basis $\{|i\rangle\}$ associated to Δ , it follows that $\langle i|\Delta|i\rangle + \langle i|\sigma|i\rangle = \langle i|\rho|i\rangle$. Given that $0 \leq \langle i|\rho|i\rangle \leq 1$ and $0 \leq \langle i|\sigma|i\rangle \leq 1$, using that $\langle i|\Delta|i\rangle = \zeta_i$, it follows that $|\zeta_i| \leq 1$ [Eq. (D3)]. Furthermore, the addition of an arbitrary number of diagonal components must be less than one. For example, $\langle i|\Delta|i\rangle + \langle k|\Delta|k\rangle + \langle i|\sigma|i\rangle + \langle k|\sigma|k\rangle = \langle i|\rho|i\rangle + \langle k|\rho|k\rangle \leq 1$. In general, $\sum_{k=1}^{\dim(H)} b_k \langle k|\Delta|k\rangle + \sum_{k=1}^{\dim(H)} b_k \langle k|\sigma|k\rangle = \sum_{k=1}^{\dim(H)} b_k \langle k|\rho|k\rangle \leq 1$, which leads to Eq. (D4).

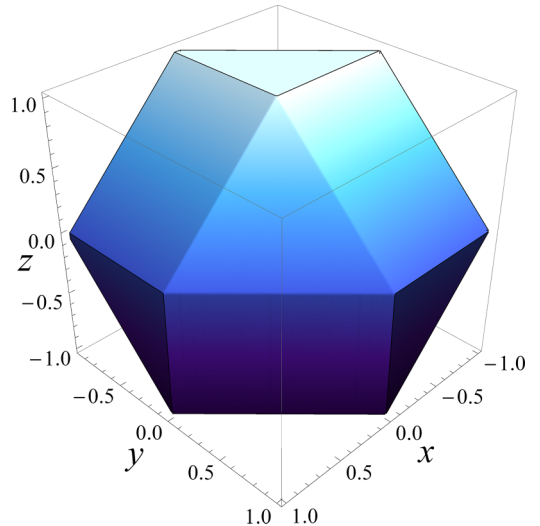


FIG. 4. Domain of the parameters (x, y, z) that set the eigenvalues of the difference of states $\Delta = \rho_A - \rho_B$ defined by Eq. (D5) (four-dimensional Hilbert space). The constraints on (x, y, z) are defined by Eqs. (D6) and (D7).

Four-dimensional case. Below, we characterize the difference of states, Δ , in a four-dimensional space. The basis where it is diagonal remains unspecified. Thus, we write

$$\Delta = \rho_A - \rho_B = \text{diag}\{x, y, z, -(x + y + z)\}. \quad (\text{D5})$$

The condition $\text{Tr}[\Delta] = 0$ is automatically fulfilled. Furthermore, the condition Eq. (D3) on the eigenvalues, here denoted

as x , y , z , and $-(x + y + z)$, is satisfied under the conditions

$$|x| \leq 1, \quad |y| \leq 1, \quad |z| \leq 1, \quad |x + y + z| \leq 1. \quad (\text{D6})$$

In addition, Eq. (D4) leads to the extra constraints,

$$|x + y| \leq 1, \quad |x + z| \leq 1, \quad |y + z| \leq 1. \quad (\text{D7})$$

The inequalities (D6) and (D7), in the space defined by (x, y, z) , define a three-dimensional body with 14 faces. It is plotted in Fig. 4.

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- [27] Let $E = \{E_m\}$ be a set of positive operator value measure, $\sum_m E_m = I$. Define $p_A^{(m)} \equiv \text{Tr}[E_m \rho_A]$ and $p_B^{(m)} \equiv \text{Tr}[E_m \rho_B]$. Then, $D_{\Pi}(\rho_A, \rho_B) = \max_{\{E\}} D_c(p_A, p_B)$, where the (classical-incoherent) probability distance is $D_c(p_A, p_B) \equiv (1/2) \sum_m |p_A^{(m)} - p_B^{(m)}|$.
- [28] If there are n_0 null eigenvalues, the argument remains the same under the replacement $\dim(\mathcal{H}) \rightarrow \dim(\mathcal{H}) - n_0$.
- [29] Assume that $\max_{\{i\}} \{|\zeta_i|\} = \zeta_{\max}^{(+)}$ is positive. It maximal degeneracy occurs when $\zeta_i^{(+)} = \zeta_{\max}^{(+)} \forall i$. Using that $\sum_{i=1}^{n^+} \zeta_i^{(+)} = \sum_{j=1}^{n^-} |\zeta_j^{(-)}|$, it follows that $\zeta_{\max}^{(+)} n^+ = \sum_{j=1}^{n^-} |\zeta_j^{(-)}| \leq \zeta_{\max}^{(+)} n^-$, implying $n^+ \leq n^-$. Given that $n^+ + n^- = \dim(\mathcal{H})$, it is fulfilled that $2n^+ \leq \dim(\mathcal{H})$. Hence, the maximal value of n^+ is $\mathcal{N} = \text{Int}[\dim(\mathcal{H})/2]$. The same reasoning applies when $\max_{\{i\}} \{|\zeta_i|\} = |\zeta_{\max}^{(-)}|$ is negative.
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