Non-Markovian refrigeration and heat flow in the quantum switch

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The quantum switch has seen multiple applications in quantum information and thermodynamical tasks. As it is constructed by placing two quantum channels in a coherent superposition of alternating causal orders, a composition known as indefinite causal order, these enhancements are often attributed to this indefinite causality and coherent superposition. However, as recent works have shown that the quantum switch also features non-Markovian effects that can contribute to the enhancement of communication capacities and work extraction, we attempt to show in this work that these non-Markovian effects can also enhance heat extraction tasks. In particular, we compare the quantum switch to the superposition of independent channels where two quantum channels are placed in a superposition, which have no non-Markovian effects, and show that the quantum switch can only outperform the superposition of independent channels in the prethermalization regimes, which also depends on the presence and amount of non-Markovianity. Our work reveals that positive heat extraction is still possible even when the working body is at a higher temperature than the interacting baths, allowing us to construct a refrigeration cycle utilizing this feature.

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I. INTRODUCTION

The quantum switch is a quantum composition of channels where the operations of two quantum channels Φ_A and Φ_B are placed in a controlled superposition of alternating operation orders of $\Phi_B \circ \Phi_A$ and $\Phi_A \circ \Phi_B$, achieving the phenomenon of indefinite causal orders [1] (see Fig. 1). Since its inception, the quantum switch's indefinite causal orders was touted to grant multiple advantages in enhancing certain quantum informational tasks, such as communication capacities [2–6], computational complexity [1,7,8], metrology [9–11], work extraction [12,13], and refrigeration tasks [14]. Without loss of generality, the quantum switch operation can be described by the Kraus operator of

$$K_{ij}^{\text{sw}} = |0\rangle\langle 0| \otimes B_j A_i + |1\rangle\langle 1| \otimes A_i B_j \qquad (1)$$

such that its overall operation is

$$\Phi^{\text{sw}}(\rho^C \otimes \rho^Q) = \sum_{i,j} K_{ij}^{\text{sw}}(\rho^C \otimes \rho^Q) K_{ij}^{\text{sw}\dagger}, \qquad (2)$$

where the channels Φ_A and Φ_B act on the main quantum system Q (denoted by the superscript) with $\Phi_A(\rho^Q) = \sum_i A_i \rho^Q A_i^\dagger$ and $\Phi_B(\rho^Q) = \sum_j B_j \rho^Q B_j^\dagger$, where $\{A_i\}$ and $\{B_j\}$ are their corresponding sets of Kraus operators. Note that the quantum switch operation Φ^{sw} must depend on the choice of Φ_A and Φ_B, i.e., $\Phi^{\text{sw}}(\rho^C \otimes \rho^Q) = \Phi^{\text{sw}}(\Phi_A, \Phi_B)(\rho^C \otimes \rho^Q)$. However, for brevity we will write $\Phi^{\text{sw}}(\Phi_A, \Phi_B)$ as just Φ^{sw}. As evident from the Kraus operator K_{ij}^{sw}, the operational orderings of the two channels are determined or controlled by the control quantum system C such that if it is in a superposition of the two orthogonal bases $|0\rangle^C$ and $|1\rangle^C$, e.g., $\rho^C = |\phi\rangle\langle\phi|^C$, where $|\phi\rangle^C = \sqrt{q}|0\rangle^C + \sqrt{1-q}|1\rangle^C$, we have a superposition of the two causal orders. In Ref. [14] it was shown that if Φ_A and Φ_B are taken to be the fully thermalizing channels of $\mathcal{N}_\beta(\rho) = \sigma_\beta$, where σ_β is the thermal state of

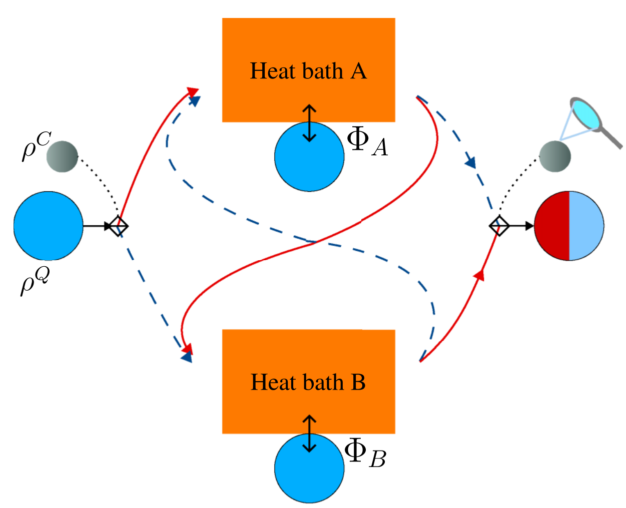

FIG. 1. Quantum switch, where a control qubit ρ^C puts the thermal interaction orders of $\Phi_B \circ \Phi_A$ and $\Phi_A \circ \Phi_B$ in a superposition. In one path of the superposition, the main system ρ^Q interacts with heat bath A and then heat bath B ($\Phi_B \circ \Phi_A$); in the other path it interacts with heat bath B and then heat bath A ($\Phi_A \circ \Phi_B$). At the end of the interaction, the control qubit ρ^C is measured and depending on the measurement basis and outcome, the collapsed main system can be at a higher or lower effective temperature than the heat baths.

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inverse temperature β, we have

$$\Phi^{\rm sw}\big(\rho^C \otimes \rho^Q_{t_0}\big) = [q|0\rangle\langle 0|^C + (1-q)|1\rangle\langle 1|^C] \otimes \sigma_\beta + \sqrt{q(1-q)}[|0\rangle\langle 1|^C \otimes |1\rangle\langle 0|^C] \otimes \sigma_\beta \rho_{t_0} \sigma_\beta, \tag{3}$$

where $\rho^Q_{t_0}$ is the initial state of the main system Q. Physically, this means that the working body of the main system Q thermalizes with two heat baths of the same temperature in an indefinite causal order. If the control qubit of the output state is then measured in the $|\pm\rangle^C$ basis, we obtain the collapsed main system Q of $\rho^Q_\pm = \frac{\sigma_\beta}{2} \pm \sqrt{q(1-q)}\sigma_\beta \rho_{t_0} \sigma_\beta$, depending on the measurement outcome. Despite that both individual channels Φ_A and Φ_B are fully thermalizing such that they transform all states to the thermal state, the second term is dependent on the initial state ρ_{t_0} and allows the collapsed state to deviate from the thermal state σ_β, enabling Ref. [14] to construct a refrigeration cycle utilizing this deviation from the expected equilibrium state. This was also demonstrated experimentally in Refs. [15,16].

Despite the multiple applications that the quantum switch was demonstrated to grant, the underlying resource for its advantage were put into question as other channel compositions with fixed causal orders were shown to replicate or even surpass the quantum switch's enhancements, particularly in the enhancements of communication capacities [2,17,18]. While consensus remains that the quantum switch's indefinite causality does contribute to its advantages [19,20], some recent works have shown that in addition to activating backflow of information for non-Markovian channels that do not violate P (positive) divisibility [21], the quantum switch itself also has intrinsic non-Markovianity and non-Markovian backflow of information [22–24], which can play a role in granting the quantum switch's enhancements to communication capacities [22] and to work extraction [23].

In this work, by taking this non-Markovian perspective, we examine the quantum switch's advantages in refrigeration tasks. Specifically, we show that a superposition of independent channels [18] can achieve the same advantage as the quantum switch in the case of full thermalization [Eq. (3)]. Instead, prethermalization regimes, where the working body does not reach equilibrium with the heat bath, are required for the quantum switch to have an advantage over the superposition of independent channels. Furthermore, we show that these advantages are dependent on the presence and amount of non-Markovianity. We also construct a refrigeration protocol where the working body starts at a hotter temperature than the heat baths and yet heat can still be extracted from the colder heat baths into the working body. This is different from the refrigeration protocol proposed by Ref. [14], where the working body starts at the same temperature as the heat baths. We also note that in our protocol, the requirement of prethermalization regimes for the quantum switch to outperform the superposition of independent channels still holds, as well as its dependence on non-Markovianity.

We will begin by reintroducing the extended quantum switch of Ref. [23] in Sec. II A, which allows the presence and amount of non-Markovianity to be varied in the quantum switch. This is followed by Sec. II B, which demonstrates how the presence of non-Markovianity can contribute to heat flow between a working body and an interacting heat bath. Next, in Sec. III, we examine the conditions where the extended quantum switch can grant an advantage over the superposition of independent channels $\Phi^{\rm indep}$, which covers the full and prethermalization regimes, revealing the possibility of heat extraction even when the working body is at a higher temperature than the interacting heat baths. In Sec. III C we exploit this possibility for heat extraction to propose a refrigeration cycle that is able to perform refrigeration with a hotter working body. Finally, in Sec. IV, we summarize and discuss our results in the context of other work.

II. NON-MARKOVIAN HEAT FLOW IN THE QUANTUM SWITCH

A. Extended quantum switch

Here we reintroduce the extended quantum switch of Ref. [23]. It was shown from resource-theoretic arguments that a thermal operation Φ acting on a quantum state ρ^Q at inverse temperature β can be defined as $\Phi(\rho^Q) = {\rm Tr}_B[U(\rho^Q \otimes \sigma^B_\beta)U^\dagger]$, with an energy conserving unitary U acting on ρ^Q and an ancillary quantum system B that is at a thermal state of inverse temperature β, i.e., σ^B_β [25–27]. Therefore, by taking Φ_A and Φ_B as thermal operations, the quantum switch operation of $\Phi^{\rm sw}$ in Eqs. (1) and (2) has the environmental representation [28,29]

$$\Phi^{\rm sw}(\rho^C \otimes \rho^Q) = {\rm Tr}_{A,B}\big[U^{\rm sw}\big(\rho^C \otimes \sigma^A_\beta \otimes \rho^Q \otimes \sigma^B_\beta\big)U^{{\rm sw}\dagger}\big], \tag{4}$$

where subsystems A and B are ancillary environmental subsystems that correspond to the operations of Φ_A and Φ_B, respectively, and are initialized as thermal states. In the qubit case, we refer to these two ancillary systems as the bath qubits.

It was shown in Refs. [22,23] that $U^{\rm sw}$ can be broken into three time steps of

$$\begin{aligned} U^{\rm sw} &= U^{\rm sw}_{t_2\to t_3} I_{t_1\to t_2} U^{\rm sw}_{t_0\to t_1} \\ &= \big[|0\rangle\langle 0|^C \otimes \big(I^A \otimes U^{QB}_B\big) + |1\rangle\langle 1|^C \otimes \big(U^{AQ}_A \otimes I^B\big)\big] \\ &\quad \times [I^C \otimes I^A \otimes I^Q \otimes I^B] \\ &\quad \times \big[|0\rangle\langle 0|^C \otimes \big(U^{AQ}_A \otimes I^B\big) + |1\rangle\langle 1|^C \otimes \big(I^A \otimes U^{QB}_B\big)\big], \end{aligned} \tag{5}$$

where

$$U^{\rm sw}_{t_0\to t_1} = |0\rangle\langle 0|^C \otimes \big(U^{AQ}_A \otimes I^B\big) + |1\rangle\langle 1|^C \otimes \big(I^A \otimes U^{QB}_B\big), \tag{6}$$

$$I_{t_1\to t_2} = I^C \otimes I^A \otimes I^Q \otimes I^B, \tag{7}$$

$$U^{\rm sw}_{t_2\to t_3} = |0\rangle\langle 0|^C \otimes \big(I^A \otimes U^{QB}_B\big) + |1\rangle\langle 1|^C \otimes \big(U^{AQ}_A \otimes I^B\big). \tag{8}$$

Note that the second time step is the identity operation of $I_{t_1\to t_2} = I^C \otimes I^A \otimes I^Q \otimes I^B$ for reasons that will become clear later. Under the completely positive (CP) divisibility definition of non-Markovianity [29,30], whether the quantum switch operation $\Phi^{\rm sw}$ is a Markovian or non-Markovian evolution depends on whether it obeys the divisibility property

$$\Phi_{t\to t''} = \Phi_{t'\to t''}\circ\Phi_{t\to t'}\,\forall\, t < t' < t''. \tag{9}$$

The divisibility property states that a CP map $\Phi_{t\to t''}$ is Markovian if it is divisible into other CP maps for all $t < t' < t''$. In the case of the quantum switch's three-time-step process $U^{\rm sw} = U^{\rm sw}_{t_2\to t_3} I_{t_1\to t_2} U^{\rm sw}_{t_0\to t_1}$, the divisibility property can be written as

$$\Phi^{\rm sw}_{t_0\to t_3} = \Phi^{\rm sw}_{t_2\to t_3}\circ\Phi^{\rm sw}_{t_0\to t_1}, \tag{10}$$

where we have written $\Phi^{\rm sw} = \Phi^{\rm sw}_{t_0\to t_3}$. It was shown in Refs. [22,23] that this divisibility property does not hold for the quantum switch, as the operation in the last time step is not CP in general due to the presence of system-environment (SE) correlations generated in the first interaction time steps [31,32]. This SE correlation is the non-Markovian memory that results from the operation of the quantum switch. Since certain information measures are contractive under CP maps, the non-CP evolution in non-Markovian processes can lead to a revival of these measures. This is referred to as non-Markovian backflow of information [33–35]. Backflow of information in the quantum switch was demonstrated with the revival of a general entanglement monotone in Ref. [22], of a trace distance in Ref. [23], and of general geometric distance measures in Ref. [24].

It is possible to drive the evolution of the quantum switch to a Markovian one by destroying the correlations before the operation of the last time step. This is the reason for the inclusion of the identity $I_{t_1\to t_2}$ in the second time step. By choosing appropriate operations instead of the identity, one can reduce or eliminate the SE correlations to control the amount of non-Markovian memory. For example, in Ref. [22], additional ancillary systems A' and B' were used to interact unitarily with the environmental systems A and B, i.e., $U_{t_1\to t_2} = I^C\otimes U^{A'A}\otimes I^Q\otimes U^{BB'}$, while in Ref. [23], a thermalization process acting on just A and B was used, both of which can reduce the SE correlations between the main systems CQ and the environmental systems A and B, causing the process to be more Markovian. The resulting operation of this extended quantum switch where the SE correlations are controllable is denoted by $\Phi^{\rm ext}$. If the second time step is the identity of $I_{t_1\to t_2}$ as in the quantum switch case, i.e., $\Phi^{\rm ext} = \Phi^{\rm sw}$, no operation is performed to destroy SE correlations and thus we have the fully non-Markovian limit. This means that we expect non-Markovian effects to be most apparent in the quantum switch $\Phi^{\rm sw}$ case. In the other extreme of the fully Markovian limit where all SE correlations are destroyed before the final time step, Refs. [22,23] showed that we obtain the case of a superposition of trajectories [20], denoted by $\Phi^{\rm traj}$ and with Kraus operator

$$K^{\rm traj}_{ijkl} = \alpha_0|0\rangle\langle 0|\otimes B_l A_i + \alpha_1|1\rangle\langle 1|\otimes A_k B_j, \tag{11}$$

where α_0 and α_1 are complex coefficients to ensure $\sum_{ijkl} K^{\rm traj\dagger}_{ijkl} K^{\rm traj}_{ijkl} = I$. The superposition of trajectories $\Phi^{\rm traj}$ is still a superposition of the two operations $\Phi_B\circ\Phi_A$ and $\Phi_A\circ\Phi_B$ like the quantum switch $\Phi^{\rm sw}$, except that the two paths of the superposition are independent of each other. In other words, the second pair of Φ_A and Φ_B operations in $t_2\to t_3$ is independent of the first pair in $t_0\to t_1$, which implies Markovianity. This independence is reflected in the different indices in Eq. (11).

Restricting the discussion to the qubit case, we take the system Hamiltonian of the main system Q and bath qubits A and B to be $\omega|1\rangle\langle 1|$ (the control system is taken to have no energy) such that the total system Hamiltonian $H_0 = H_0^Q + H_0^A + H_0^B$, where

$$H_0^Q = I^C\otimes I^A\otimes\omega|1\rangle\langle 1|^Q\otimes I^B, \tag{12}$$

$$H_0^A = I^C\otimes\omega|1\rangle\langle 1|^A\otimes I^Q\otimes I^B, \tag{13}$$

$$H_0^B = I^C\otimes I^A\otimes I^Q\otimes\omega|1\rangle\langle 1|^B, \tag{14}$$

with ω the energy gap between the excited and ground states. We take the approach of Ref. [23], where the erasure of the SE correlations is performed with a thermalization process $\mathcal{N}_{t_1\to t_2}$ in the second time step with Kraus operators [36]

$$K_1 = \sqrt{1-N}(|0\rangle\langle 0| + \sqrt{1-\gamma}|1\rangle\langle 1|), \tag{15}$$

$$K_2 = \sqrt{\gamma(1-N)}|0\rangle\langle 1|, \tag{16}$$

$$K_3 = \sqrt{N}(\sqrt{1-\gamma}|0\rangle\langle 0| + |1\rangle\langle 1|), \tag{17}$$

$$K_4 = \sqrt{\gamma N}|1\rangle\langle 0| \tag{18}$$

such that

$$\mathcal{N}_{t_1\to t_2}(\rho^{CAQB}) = \sum_{i,j} W_{ij}\rho^{CAQB}W_{ij}^\dagger, \tag{19}$$

where $W_{ij} = I^C\otimes K_i^A\otimes I^Q\otimes K_j^B$ and $N = 1/(1+e^{\beta\omega})$, with β the inverse temperature of the thermalizing bath. Note that the thermal state can be expressed in terms of N with

$$\sigma_\beta = \begin{pmatrix} 1-N & 0 \\ 0 & N\end{pmatrix}, \tag{20}$$

where $N\in[0, 0.5]$, achieving the maximum for infinite temperature and the minimum for zero temperature. This parametrization of the bath temperature with N will aid in our analysis later in this work. For example, we can define N_Q to parametrize the thermal state σ_β^Q of the main quantum system Q at some inverse temperature β that depends on N_Q. On the other hand, note that we will simply use N with no subscript to parametrize the bath temperature of the interacting baths in the extended quantum switch, i.e., the baths of bath qubits A and B. Therefore, the comparison between the temperatures of the main system Q and the heat baths in the extended quantum switch can be written in terms of N_Q and N, e.g., $N_Q < N$ implies $T_Q < T$ or $\beta_Q > \beta$.

This set of Kraus operators of $\mathcal{N}_{t_1\to t_2}$ is simply the generalized amplitude damping channel (GADC), also known as the one-qubit thermal operation [37,38]. The GADC describes an interaction with a thermal environment or heat bath at some finite temperature, also known as the T_1 or energy relaxation, where the Lindblad operators of the master equation are $\sqrt{J_T n}\sigma_-$ and $\sqrt{J_T(n+1)}\sigma_+$, where $\sigma_\pm$ are the raising and lowering operators, J_T is the coupling strength, and $n = e^{\beta\omega} - 1$ is the mean number of excitations in the bath with inverse temperature β and energy gap ω [36,39].

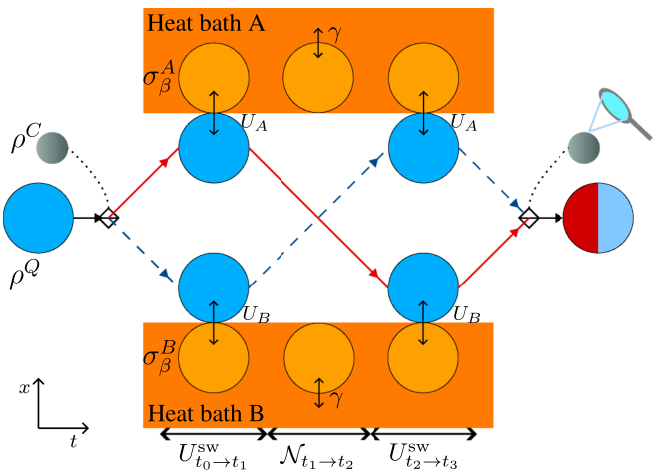

FIG. 2. Extended quantum switch Φ^{ext} unwrapped into a space-time diagram in the environmental representation, which can be broken into three time steps. In addition to the first and last time steps which are present in the standard quantum switch, the extended quantum switch includes a intermediate operation $\mathcal{N}_{t_1\to t_2}$, which allows the bath qubits to thermalize with the heat baths that they are in. Depending on the strength of this thermalization (the parameter γ), the system-environment correlations in the bath qubits can be reduced or destroyed, allowing the presence and amount of non-Markovianity to be controlled.

The thermalization strength is controlled by the parameter $\gamma \in [0, 1]$, where $\gamma = 0$ implies no thermalization, $\gamma = 1$ implies full thermalization, and $0 < \gamma < 1$ implies prethermalization. Physically, full thermalization refers to the case where the interacting system is allowed to equilibrate to the equilibrium state of the heat bath, i.e., the thermal state, while prethermalization refers to the case where the equilibrium state is not reached. How close the system is to the equilibrium or thermal state is determined by how close γ is to 1. Hence, the fully non-Markovian case is achieved with $\gamma = 0$, where the SE correlations are fully retained as the bath qubits are unchanged, while the fully Markovian case is achieved with $\gamma = 1$, where the bath qubits return to the thermal state, which destroys the SE correlations. Therefore, the extended quantum switch operation is

$$\Phi^{ext}(\rho^C \otimes \rho^Q) = \mathrm{Tr}_{A,B}\{U^{sw}_{t_2\to t_3}\mathcal{N}_{t_1\to t_2}[U^{sw}_{t_0\to t_1} \times (\rho^C \otimes \sigma^A_\beta \otimes \rho^Q \otimes \sigma^B_\beta)U^{sw\dagger}_{t_0\to t_1}]U^{sw\dagger}_{t_2\to t_3}\}. \quad (21)$$

Note that while the thermalization operation $\mathcal{N}_{t_1\to t_2}$ is nonunitary, it can always be written as a unitary operation by replacing it with $U_{t_1\to t_2} = I^C \otimes U^{A'A} \otimes I^Q \otimes U^{BB'}$ and adding the ancillary systems A' and B'. This means that Eq. (21) still admits an environmental representation and thus is CP [28,29]. This extended quantum switch operation Φ^{ext} is shown in Fig. 2. The output state at each time step of the extended quantum switch operation can then be obtained as

$$\rho^{CQ}_{t_1} = \mathrm{Tr}_{A,B}\big[U^{sw}_{t_0\to t_1}\big(\rho^C \otimes \sigma^A_\beta \otimes \rho^Q \otimes \sigma^B_\beta\big)U^{sw\dagger}_{t_0\to t_1}\big], \quad (22)$$

$$\rho^{CQ}_{t_2} = \mathrm{Tr}_{A,B}\big[\mathcal{N}_{t_1\to t_2}\big(U^{sw}_{t_0\to t_1}\big(\rho^C \otimes \sigma^A_\beta \otimes \rho^Q \otimes \sigma^B_\beta\big)U^{sw\dagger}_{t_0\to t_1}\big)\big], \quad (23)$$

$$\rho^{CQ}_{t_3} = \Phi^{ext}(\rho^C \otimes \rho^Q). \quad (24)$$

Note that it was shown in Ref. [23] that Eq. (21) indeed reduces to the Kraus representations of the quantum switch Φ^{sw} [Eq. (1)] and the superposition of trajectories Φ^{traj} [Eq. (11)] in the fully non-Markovian ($\gamma = 0$) and the fully Markovian limits ($\gamma = 1$), respectively. As for the operations of $U^{sw}_{t_0\to t_1}$ and $U^{sw}_{t_2\to t_3}$, recall from Eqs. (6) and (8) that they each implement a controlled superposition between an interaction with heat bath A and an interaction with heat bath B. These interactions with the heat baths A and B, defined with U^{AQ}_A and U^{BQ}_B, respectively, are thermal and thus can also be described with the GADC operation, but in its unitary definition of

$$U^{AQ}_A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & x_A & -ie^{i\varphi}\sqrt{1-x_A^2} & 0 \\ 0 & -ie^{-i\varphi}\sqrt{1-x_A^2} & x_A & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (25)$$

$$U^{QB}_B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & x_B & -ie^{-i\varphi}\sqrt{1-x_B^2} & 0 \\ 0 & -ie^{i\varphi}\sqrt{1-x_B^2} & x_B & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (26)$$

where φ is an arbitrary phase typically taken to be $3\pi/2$ [37,38]. Note that U^{AQ}_A can be generated from an energy-conserving interaction Hamiltonian of $H^{AQ}_{int} = J_A(e^{i\varphi}\sigma^A_-\sigma^Q_+ + e^{-i\varphi}\sigma^A_+\sigma^Q_-)$ (and vice versa for U^{QB}_B), where J_A (or J_B for U^{QB}_B) is the coupling strength and $\sigma_\pm$ are the raising and lowering operators such that the parameters $x_A = \cos J_A t$ and $x_B = \cos J_B t$, with $-1 \leqslant x_A, x_B \leqslant 1$ controlling the strength of the thermalizations with bath qubits A and B, respectively. Therefore, each heat bath can be either at full thermalization ($x_A = 0$ or $x_B = 0$) or at prethermalization ($x_A \neq 0$ or $x_B \neq 0$), independently of each other.

Finally, at the end of the three-time-step operation $\Phi^{ext}(\rho^C \otimes \rho^Q)$, the control qubit is measured in the $|\pm\rangle^C$ basis, obtaining the collapsed system Q of $\rho^Q_{\pm,t_3} = \mathrm{Tr}_C[(|\pm\rangle\langle\pm|^C \otimes I^Q)\rho^{CQ}_{t_3}]/p_{\pm,t_3}$, where $p_{\pm,t_3} = \mathrm{Tr}[(|\pm\rangle\langle\pm|^C \otimes I^Q)\rho^{CQ}_{t_3}]$ are the probabilities of measurement. Note that the collapsed state at other time steps, i.e., $\rho^Q_{\pm,t_1}$ and $\rho^Q_{\pm,t_2}$, can also be found from Eq. (21) by applying the operations up to $U^{sw}_{t_0\to t_1}$ or $\mathcal{N}_{t_1\to t_2}$ for t_1 and t_2, respectively. As mentioned in the Introduction, in the case of the quantum switch Φ^{sw} of $\gamma = 0$, it is possible to perform refrigeration tasks depending on the measurement outcome. If the measurement outcome is $|-\rangle^C$, heat is extracted by the collapsed system Q, allowing refrigeration to be performed. On the other hand, if the measurement outcome is $|+\rangle^C$, heat is passed from the collapsed system Q to the heat baths and no refrigeration can be done [14].

As a final note, if we were to stop the operation at t_1, i.e., $\Phi^{sw}_{t_0\to t_1}$, we would obtain the superposition of independent channels Φ^{indep} that has the Kraus operator

$$K_{ij}^{\text{indep}} = \nu_0|0\rangle\langle 0| \otimes A_i + \nu_1|1\rangle\langle 1| \otimes B_j, \tag{27}$$

which is simply a superposition of the two channels Φ_A and Φ_B, where ν_0 and ν_1 are complex coefficients to ensure $\sum_{ij} K_{ij}^{\text{indep}\dagger} K_{ij}^{\text{indep}} = I$. The superposition of independent channels Φ^{indep} is often compared to the quantum switch, serving as a counterexample where the advantages of the quantum switch can be achieved even with fixed causal order in some cases [2,18]. It was noted in Refs. [22,23] that since non-Markovian effects in the quantum switch arise in the last time step of $t_2 \to t_3$, which is absent in Φ^{indep}, the question of when one can grant more advantage than the other is a matter of whether advantages from non-Markovian contributions are possible in the quantum switch.

B. Non-Markovian heat flow

Since the quantum switch is a non-Markovian process, we would like to see how non-Markovian effects can contribute to the increase of heat flow. For a quantum system Q interacting with a heat bath system B of inverse temperature β at time t, its nonequilibrium free energy is [40]

$$F(\rho_t^Q) = E(\rho_t^Q) - \frac{1}{\beta} S(\rho_t^Q), \tag{28}$$

where $E(\rho) = \mathrm{Tr}(H_0\rho)$ is the energy of ρ with system Hamiltonian H_0 and $S(\rho) = -\mathrm{Tr}(\rho \ln \rho)$ is the von Neumann entropy. Since the equilibrium state is the thermal state of the heat bath σ_β^Q, the difference in free energy between the current state and the equilibrium state is $F(\rho_t^Q) - F(\sigma_\beta^Q) = \frac{1}{\beta} S(\rho_t^Q||\sigma_\beta^Q)$, where $S(\rho||\sigma) = \mathrm{Tr}(\rho \ln \rho) - \mathrm{Tr}(\rho \ln \sigma)$ is the quantum relative entropy. Therefore, the energy difference of system Q in a time step $t \to t'$ is

$$\begin{aligned} \Delta E(\rho^Q) &= \Delta F(\rho^Q) + \frac{1}{\beta}\Delta S(\rho^Q), \\ \beta \Delta E(\rho^Q) &= \Delta S\big(\rho^Q||\sigma_\beta^Q\big) + \Delta S(\rho^Q), \\ \beta Q_{t\to t'}^Q &= \Delta S\big(\rho^Q||\sigma_\beta^Q\big) + \Delta S(\rho^Q), \end{aligned} \tag{29}$$

where $\Delta E(\rho^Q) = Q_{t\to t'}^Q$ is the heat flow into system Q in the time step $t \to t'$.

Since quantum relative entropy is contractive under CP maps [28,41], we expect $\Delta S(\rho^Q||\sigma_\beta^Q) \leqslant 0$ for thermal operations, which are CP. However, in a non-Markovian process, there are time intervals where the CP-divisibility property can be violated, e.g., the interval $\tau \to t'$ in the entire $t \to \tau \to t'$ process, which means that the monotonic decrease of $\Delta S(\rho^Q||\sigma_\beta^Q) \leqslant 0$ does not need to hold in those intervals, allowing the possibility for backflow of information, e.g., $\Delta_{\tau\to t'} S(\rho^Q||\sigma_\beta^Q) > 0$, resulting in irreversibility mitigation [42,43] or negative-entropy production rates [44,45]. It should be noted, however, that we still have $\Delta_{t\to t'} S(\rho^Q||\sigma_\beta^Q) \leqslant 0$ for the entire process of $t \to t'$ as thermal operations are CP. Furthermore, we can express $\Delta S(\rho^Q||\sigma_\beta^Q)$ as

$$\begin{aligned} \Delta S\big(\rho^Q||\sigma_\beta^Q\big) &= \beta Q_{t\to t'}^Q - \Delta S(\rho^Q) \\ &= -\beta Q_{t\to t'}^B + \Delta S(\rho^B) - \Delta I(\rho^Q;\rho^B) \\ &\quad - \Delta S(\rho^{QB}) \\ &= -\Delta S\big(\rho^B||\sigma_\beta^B\big) - \Delta I\big(\rho^Q;\rho^B\big) - \Delta S(\rho^{QB}), \end{aligned} \tag{30}$$

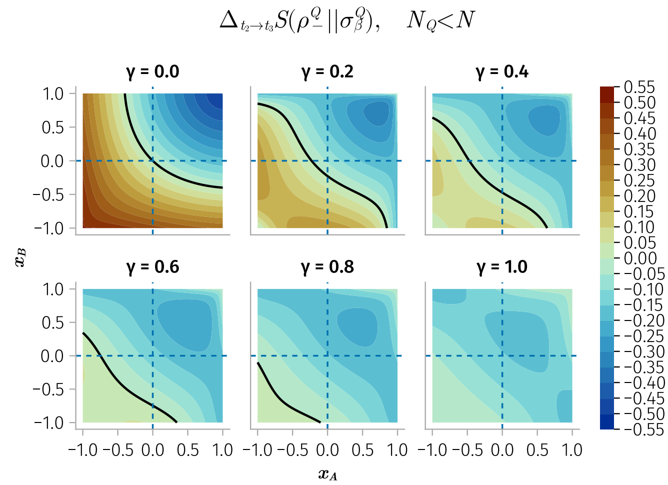

FIG. 3. Violation of the contractive property of quantum relative entropy is most significant in the $|-\rangle\langle -|^C$ measurement outcome for $N = 0.223$ and $N_Q = 0.159$. This violation is indicative of non-Markovian backflow of information and is only present in the prethermalization regimes ($x_A, x_B \neq 0$). The backflow is greatest for the fully non-Markovian case of $\gamma = 0$ and decreases as non-Markovianity decreases, i.e., when γ increases.

where $I(\rho^Q;\rho^B) = S(\rho^Q) + S(\rho^B) - S(\rho^{QB})$ is the quantum mutual information and the SE correlation, which should make clear that the presence of initial SE correlations, as well as a bath qubit that is perturbed away from the thermal state, can grant positive contributions to $\Delta S(\rho^Q||\sigma_\beta^Q)$. In the extended quantum switch, these conditions can happen in $t_2 \to t_3$, i.e., $t = t_2$ and $t' = t_3$. The equilibration process of $\mathcal{N}_{t_1\to t_2}$ (depending on the parameter γ) can prevent this by thermalizing the bath qubits back to the thermal state, which destroys the SE correlations in the process, i.e., $S(\rho_{t_2}^B||\sigma_\beta^B) = 0$ and $I(\rho_{t_2}^Q;\rho_{t_2}^B) = 0$. This leads to positive $\Delta S(\rho^B||\sigma_\beta^B) = S(\rho_{t_3}^B||\sigma_\beta^B)$ and positive $\Delta I(\rho^Q;\rho^B) = I(\rho_{t_3}^Q;\rho_{t_3}^B)$ in the final time step $t_2 \to t_3$, granting a negative contribution to $\Delta S(\rho^Q||\sigma_\beta^Q)$, eliminating the non-Markovian backflow of information.

In the case of the extended quantum switch, we are concerned with the collapsed main system Q after a measurement on the control qubit, and so we have

$$\beta Q_{\pm,t_2\to t_3}^Q = \Delta_{t_2\to t_3} S\big(\rho_\pm^Q||\sigma_\beta^Q\big) + \Delta_{t_2\to t_3} S(\rho_\pm^Q), \tag{31}$$

where the subscript $\pm$ refers to the measurement outcomes $|\pm\rangle^C$. Plotting $\Delta_{t_2\to t_3} S(\rho_-^Q||\sigma_\beta^Q)$ in Fig. 3 reveals the presence of the non-Markovian backflow of information.

III. ADVANTAGE OF NON-MARKOVIANITY

It should be noted that despite that the non-Markovian effects manifest only from $t_2 \to t_3$, the quantification of its advantage is not simply the quantification of $Q_{\pm,t_2\to t_3}^Q$. Instead, we have to compare the performance of the extended quantum switch $\Phi^{\rm ext}$ ($t_0 \to t_3$) and the superposition of independent channels $\Phi^{\rm indep}$ ($t_0 \to t_1$). This is because, while the heat extraction may appear to be equivalent, i.e., $Q^{Q}_{\pm,t_2\to t_3} = Q^{Q}_{\pm,t_0\to t_3} - Q^{Q}_{\pm,t_0\to t_1}$, $\Phi^{\rm indep}$ and $\Phi^{\rm ext}$ actually have different probabilities of measurement for $|\pm\rangle^C$. Therefore, the advantage that the extended quantum switch $\Phi^{\rm ext}$ (which holds the possibility of non-Markovian backflow of information) has over the superposition of independent channels $\Phi^{\rm indep}$ is

$$\Delta\langle Q\rangle^{Q}_{\pm} = p_{\pm,t_3} Q^{Q}_{\pm,t_0\to t_3} - p_{\pm,t_1} Q^{Q}_{\pm,t_0\to t_1}, \tag{32}$$

which is the difference between the amount of heat extracted from $\Phi^{\rm ext}$ and $\Phi^{\rm indep}$, weighted by their respective probability of measurements, and is not equal to $Q^{Q}_{\pm,t_2\to t_3}$. Note that while having $\Delta\langle Q\rangle^{Q}_{\pm} > 0$ implies that the extended quantum switch $\Phi^{\rm ext}$ has an advantage over $\Phi^{\rm indep}$, it does not imply positive heat extraction. Hence, in addition to the conditions for $\Delta\langle Q\rangle^{Q}_{\pm} > 0$, we are also interested in the conditions where positive heat extraction is possible.

A. Full thermalization

Let us consider the case of full thermalization first where $x_A = x_B = 0$, which was the case considered in Ref. [14]. Specifically, Ref. [14] demonstrated that the quantum switch $\Phi^{\rm sw}$ can grant $p_{-,t_3} Q^{Q}_{-,t_0\to t_3} \geqslant 0$ even for $N_Q = N$, i.e., positive heat extraction even when the working body is at the same temperature as the baths. Here we will see that this feature is not unique to the quantum switch. Taking the control qubit to be $\rho_0^C = |\phi\rangle\langle\phi|^C$, where $|\phi\rangle = \sqrt{q}|0\rangle^C + \sqrt{1-q}|1\rangle^C$, and recalling that we have

$$p_{\pm,t} = {\rm Tr}\big[(|\pm\rangle\langle\pm|^C \otimes I^Q)\rho_t^{CQ}\big], \tag{33}$$

$$Q^{Q}_{\pm,t\to t'} = E(\rho^{Q}_{\pm,t'}) - E(\rho^{Q}_{\pm,t}) \tag{34}$$

$$= {\rm Tr}\big(H_0^Q \rho^{Q}_{\pm,t'}\big) - {\rm Tr}\big(H_0^Q \rho^{Q}_{\pm,t}\big), \tag{35}$$

where

$$\rho^{Q}_{\pm,t} = \frac{1}{p_{\pm,t}} {\rm Tr}_C\big[(|\pm\rangle\langle\pm|^C \otimes I^Q)\rho_t^{CQ}\big], \tag{36}$$

we consider the case of general N_Q and compute explicitly the heat flow for the collapsed systems

$$p_{\pm,t_1} Q^{Q}_{\pm,t_0\to t_1} = \omega\sqrt{q(1-q)}\left(\frac{N-N_Q}{2\sqrt{q(1-q)}} \mp N_Q(1-N_Q)(1-2N)\right) \tag{37}$$

and

$$p_{\pm,t_3} Q^{Q}_{\pm,t_0\to t_3} = \omega\sqrt{q(1-q)} \times \left(\frac{N-N_Q}{2\sqrt{q(1-q)}} \mp N_Q(1-N_Q)(1-2N) \mp \nu_\gamma\right), \tag{38}$$

where their difference is an additional term

$$\nu_\gamma = -2\gamma N[N_Q(1-3N) + \sqrt{1-\gamma}N(1-N-2N_Q)] + 2(\gamma + \gamma\sqrt{1-\gamma})N_Q N[N_Q(1-3N) - 2N^2(1-N_Q)], \tag{39}$$

which depends on the non-Markovian parameter γ. Their difference is thus $\Delta\langle Q\rangle^{Q}_{\pm} = \mp\omega\sqrt{q(1-q)}\nu_\gamma$. Equation (39) shows that for all possible values of γ, N_Q, and N, we have $\nu_\gamma \leqslant 0$ with equality only in the quantum switch $\Phi^{\rm sw}$ case of $\gamma = 0$. This means that for the fully non-Markovian case of the quantum switch $\Phi^{\rm sw}$, we have $\Delta\langle Q\rangle^{Q}_{\pm} = 0$. In other words, the superposition of independent channels $\Phi^{\rm indep}$ with its fixed causal order is sufficient to achieve the same advantage as the quantum switch.

Still, $\Delta\langle Q\rangle^{Q}_{\pm}$ can be positive for the case of $\gamma \neq 0$, specifically for the $|+\rangle^C$ measurement outcome. Let us consider the two measurement outcomes separately and find the conditions for positive heat extraction.

1. $|-\rangle\langle-|$ *measurement outcome*

For the case of the $|-\rangle\langle-|$ measurement outcome, since ν_γ is always negative, decreasing the amount of non-Markovianity, or $\gamma > 0$, always results in a negative contribution to the heat flow into main system Q, and so the extended quantum switch $\Phi^{\rm ext}$ does not grant an advantage over $\Phi^{\rm indep}$ and any displacement from the fully non-Markovian case of $\gamma = 0$ will lead to a worse performance than $\Phi^{\rm indep}$.

We can find the condition for positive heat flow for $\Phi^{\rm indep}$ by setting $p_{-,t_1} Q^{Q}_{-,t_0\to t_1} > 0$ to obtain

$$N - N_Q > -2\sqrt{q(1-q)}N_Q(1-N_Q)(1-2N). \tag{40}$$

Note that the right-hand side is always negative, which means that we can get positive heat flow into the main system Q whenever $N_Q < N$, i.e., the left-hand side is positive. This is not surprising since $N_Q < N$ means that the main system Q is in a thermal state of a lower-temperature system interacting with a higher-temperature system, and so heat flow into the main system Q is expected.

Interestingly, the inequality still holds for some cases of $N_Q \geqslant N$. This means that the main system Q can still extract heat from the heat baths even if it is in a thermal state that is of hotter temperature than the heat baths. We can find the boundary for the case of $q = 0.5$ by setting the left-hand and right-hand sides of Eq. (40) to be equal, which gives us the

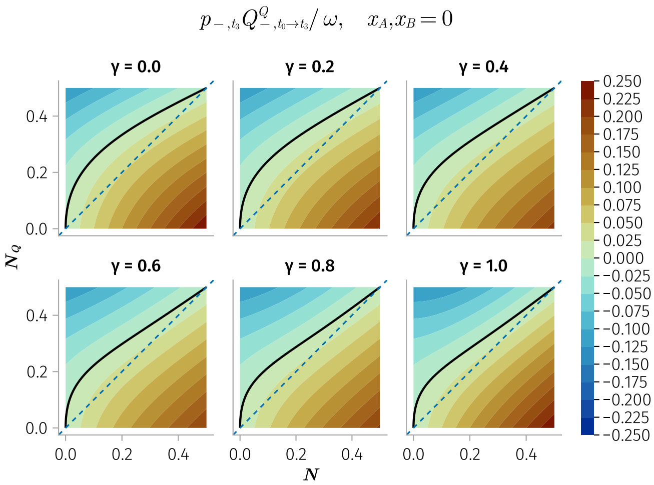

FIG. 4. Heat flow into the collapsed main system Q in the $|-\rangle\langle-|^C$ measurement outcome for the range of all possible N and N_Q for the full thermalization case of $x_A, x_B = 0$. We see that there can be positive heat extraction (bottom right areas bounded by the black line) for some values of $N_Q > N$, which decreases as γ increases.

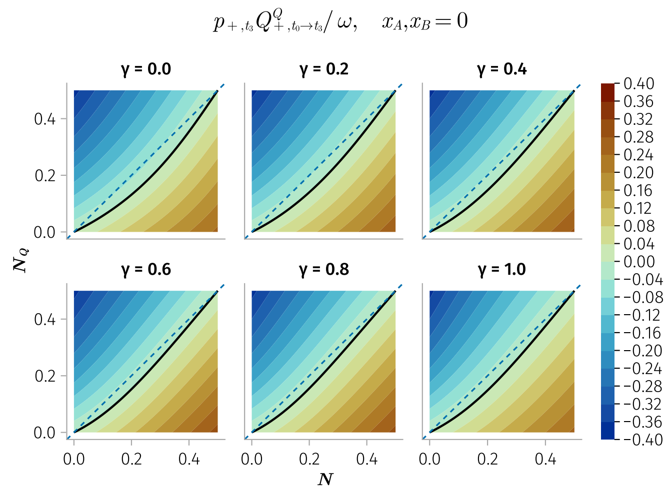

FIG. 5. Heat flow into the collapsed main system Q in the $|+\rangle\langle+|^C$ measurement outcome for the range of all possible N and N_Q for the full thermalization case of $x_A, x_B = 0$. Different from the case of the $|-\rangle\langle-|^C$ outcome, there is no positive heat extraction (bottom right areas bounded by the black line) for $N_Q > N$.

upper bound of N_Q or β_Q that allows positive heat flow:

$$\begin{aligned} N - N_Q + N_Q(1-N_Q)(1-2N) &= 0, \\ \frac{e^{2\beta_Q\omega} - e^{\beta\omega}}{(1+e^{2\beta_Q\omega})(1+e^{\beta\omega})} &= 0, \\ e^{2\beta_Q\omega} &= e^{\beta\omega}, \\ 2\beta_Q &= \beta. \end{aligned} \tag{41}$$

Therefore, positive heat flow into the main system Q is possible if the thermal state of the main system Q has less than twice the temperature of the interacting baths, i.e., $T_Q < 2T$.

Likewise, for Φ^{ext}, we have

$$\begin{aligned} &N - N_Q + 2\sqrt{q(1-q)}\nu_\gamma \\ &\quad > -2\sqrt{q(1-q)}N_Q(1-N_Q)(1-2N), \end{aligned} \tag{42}$$

where despite the presence of the negative contribution of $2\sqrt{q(1-q)}\nu_\gamma$ (since $\nu_\gamma \leqslant 0$), there are still cases of $N_Q \geqslant N$ where the inequality holds. We plot $p_{-,t_3}Q^{Q}_{-,t_0\to t_3}$ in units of ω for different values of N_Q and N in Fig. 4 to demonstrate this. Therefore, this opens the possibility for novel refrigeration tasks where the working body can be at a higher temperature than the baths being cooled. However, we reiterate that the superposition of independent channels Φ^{indep} is sufficient to implement these tasks.

2. $|+\rangle\langle+|$ *measurement outcome*

The case of the $|+\rangle\langle+|$ measurement outcome differs from the $|-\rangle\langle-|$ case simply by the signs of the $N_Q(1-N_Q)(1-2N)$ and ν_γ terms. In this case, $\Delta\langle Q\rangle^Q_+$ can be positive, allowing Φ^{ext} to grant higher heat flow than Φ^{indep}. However, we will see that this advantage is only possible for the trivial case of $N_Q < N$ and thus is not useful for novel refrigeration tasks. This case is not of interest as it is achievable by simpler classical compositions.

Specifically, we have the following two conditions for positive heat flow for Φ^{indep} and Φ^{ext}, respectively:

$$N - N_Q > 2\sqrt{q(1-q)}N_Q(1-N_Q)(1-2N), \tag{43}$$

$$\begin{aligned} &N - N_Q - 2\sqrt{q(1-q)}\nu_\gamma \\ &\quad > 2\sqrt{q(1-q)}N_Q(1-N_Q)(1-2N). \end{aligned} \tag{44}$$

Different from the $|-\rangle\langle-|$ case, the right-hand side is positive, which means that positive heat flow is possible for $N_Q < N$, but not for $N_Q \geqslant N$ in the case of Φ^{indep}. On the other hand, for the case of Φ^{ext}, we have a positive contribution from the $-2\sqrt{q(1-q)}\nu_\gamma$ term, and so it might be possible for the inequality to hold even for $N_Q \geqslant N$. However, plotting $p_{+,t_3}Q^{Q}_{+,t_0\to t_3}$ in units of ω in Fig. 5 reveals that this is still not possible. Therefore, despite the extended quantum switch Φ^{ext} granting an advantage over Φ^{indep} with $\Delta\langle Q\rangle^Q_+ > 0$, the extended quantum switch does not grant any new application for heat extraction that cannot be achieved by other simpler compositions that make use of the temperature difference $N_Q < N$.

B. Prethermalization

Let us now consider the general case of $-1 \leqslant x_A, x_B \leqslant 1$, which includes the prethermalization regimes. For Φ^{indep} we have

$$\begin{aligned} &p_{\pm,t_1}Q^{Q}_{\pm,t_0\to t_1} \\ &\quad = \omega\sqrt{q(1-q)}\bigg(\frac{N-N_Q}{2\sqrt{q(1-q)}}\big[1 - qx_A^2 - (1-q)x_B^2\big] \\ &\qquad \mp N_Q(1-N_Q)(1-2N)(1-x_Ax_B)\bigg), \end{aligned} \tag{45}$$

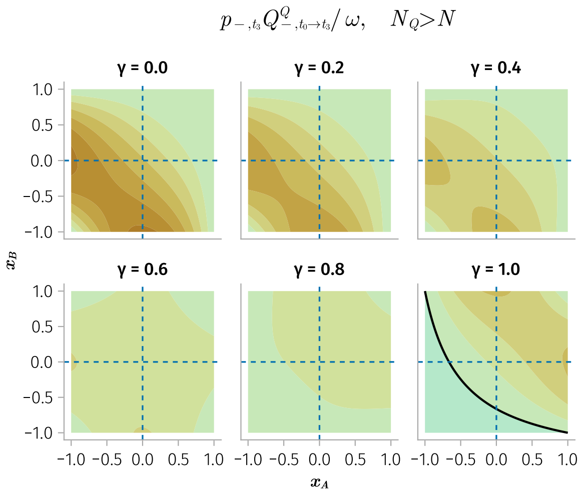

FIG. 6. Positive heat extraction by the main system Q is still possible in the $|-\rangle\langle-|^C$ measurement outcome for the case of $N_Q > N$, which decreases as γ increases. The prethermalization regimes can extract more heat than the full thermalization regime.

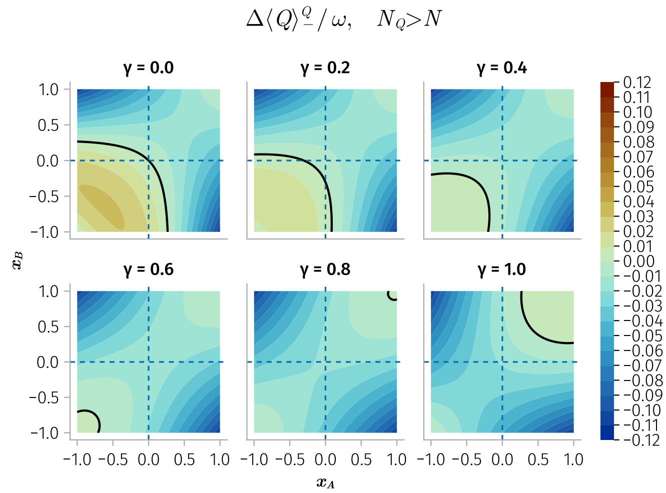

FIG. 7. Advantage in positive heat extraction of the extended quantum switch Φ^{ext} over the superposition of independent channels Φ^{indep}. The advantage is only present in the prethermalization regimes, which decreases as the amount of non-Markovianity decreases (increasing γ).

which is the same as the full thermalization case except for an additional factor that depends on x_A and x_B. Note that we are unable to provide a closed-form solution for the case of the extended quantum switch $p_{\pm,t_3}Q^{Q}_{\pm,t_0\to t_3}$ and will instead provide a numerical analysis of $p_{-,t_3}Q^{Q}_{-,t_0\to t_3}$ and $p_{+,t_3}Q^{Q}_{+,t_0\to t_3}$ separately.

1. $|-\rangle\langle-|$ measurement outcome

In the case of the $|-\rangle\langle-|$ measurement outcome, the condition for $p_{-,t_1}Q^{Q}_{-,t_0\to t_1} > 0$ is

$$N - N_Q > \frac{-(1 - x_A x_B)}{1 - q x_A^2 - (1-q)x_B^2} \times 2\sqrt{q(1-q)}N_Q(1-N_Q)(1-2N). \qquad (46)$$

Note that this is again simply the condition for the case of full thermalization except for an additional term that depends on x_A and x_B. This additional term decreases the right-hand side further; e.g., in the case of $q = 0.5$, the right-hand side has a range of $[-\infty, -1]$, achieving the maximum of -1 for $x_A = x_B = 0$. Hence, prethermalization can allow greater heat extraction and can grant more allowable values of N_Q and N for positive heat extraction. Furthermore, we see that the case of $N_Q > N$ can still grant positive heat flow in the prethermalization regimes.

As for the case of the extended quantum switch Φ^{ext}, we plot $p_{-,t_3}Q^{Q}_{-,t_0\to t_3}$ in units of ω in Fig. 6 with $N = 0.223$, $N_Q = 0.269$, and $q = 0.5$. Note that this corresponds to the case of $N_Q > N$. We see that Φ^{ext} is still able to grant positive heat flow into the main system Q, despite having $N_Q > N$, and that the maximum possible heat flow occurs in the prethermalization regime ($x_A, x_B \neq 0$) and is greater as γ approaches 0, i.e., for increasing non-Markovianity. Finally, we plot the advantage $\Delta\langle Q\rangle^{Q}_{-}$ in units of ω in Fig. 7. We see that there are thermalization regimes where the extended quantum switch Φ^{ext} can perform better than Φ^{indep}, i.e., $\Delta\langle Q\rangle^{Q}_{-} > 0$, and that they are dependent on the presence and amount of non-Markovianity.

Therefore, we see that not only can Φ^{ext} grant positive heat flow into the main system Q with $N_Q > N$, but it can also grant a better performance over Φ^{indep} in the prethermalization regimes. This advantage is dependent on the non-Markovianity parameter γ, and we see that the maximum is achieved for the fully non-Markovian case of $\gamma = 0$. We can further probe the non-Markovian contribution to positive heat flow by plotting $p_{-,t_3}Q^{Q}_{-,t_2\to t_3}$ in units of ω in Fig. 8, where we see that the enhancement to positive heat flow decreases with increasing γ or Markovianity. Different from the case of $\Delta\langle Q\rangle^{Q}_{-}$, there is no positive enhancement for large

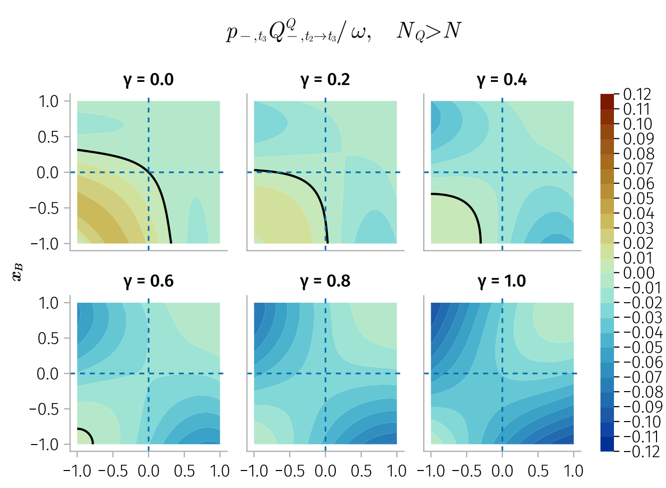

FIG. 8. Non-Markovian contribution to the amount of heat extraction for the extended quantum switch Φ^{ext}, which decreases as non-Markovianity decreases (increasing γ). This makes clear that the advantage of Φ^{ext} seen in Fig. 7 is largely due to the contribution from the non-Markovian time step from $t_2 \to t_3$.

values of γ. This is expected as the non-Markovian backflow of information $\Delta S(\rho_{-}^{Q}||\sigma_{\beta}^{Q})$ should decrease as the system becomes more Markovian. Furthermore, the regimes where non-Markovian backflow of information is possible (Fig. 8) closely coincide with the regimes where the extended quantum switch Φ^{ext} has an advantage over Φ^{indep}, i.e., $\Delta\langle Q\rangle_{-}^{Q} > 0$, for small γ (Fig. 7).

2. $|+\rangle\langle+|$ *measurement outcome*

Let us briefly take a look at the case of the $|+\rangle\langle+|$ measurement outcome, where the difference from the $|-\rangle\langle-|$ case is simply a sign difference for the last term and so the condition for positive heat extraction for $p_{+,t_1}Q_{+,t_0\to t_1}^{Q}$ is

$$N - N_Q > \frac{1 - x_A x_B}{1 - q x_A^2 - (1-q) x_B^2} \times 2\sqrt{q(1-q)} N_Q(1-N_Q)(1-2N), \tag{47}$$

where the right-hand side now has a range $[0, \infty]$. Therefore, similar to the full thermalization case, positive heat extraction for Φ^{indep} is still impossible for $N_Q > N$. For the case of Φ^{ext}, a numerical search over the entire range of values $N, N_Q \in [0, 0.5]$, $x_A, x_B \in [-1, 1]$, and $\gamma \in [0, 1]$, for $q = 0.5$, reveals that it is also impossible to have positive heat extraction for $N_Q > N$.

C. Refrigeration with $N_Q > N$

To recap, our results agree with Ref. [14] that the quantum switch is able to grant positive heat extraction even for $N_Q = N$ in the $|-\rangle\langle-|$ measurement outcome. However, we showed that this advantage is already achievable with the superposition of independent channels Φ^{indep} and that it is only in the prethermalization regimes that the quantum switch has an advantage over it due to the presence of contributions from the non-Markovian backflow of information.

Furthermore, we revealed that in addition to $N_Q = N$, positive heat extraction is still possible with $N_Q > N$. Positive heat extraction in the case of $N_Q = N$ is novel as we expect no net flow of heat, and in the case of $N_Q > N$ we would expect a negative heat flow, i.e., heat flows out of the hotter main system Q into the colder baths of β, yet positive heat flow into the main system Q is possible up to a certain N_Q with respect to N. This might present an opportunity for more novel refrigeration applications where the working body can be at a higher temperature than the system being cooled. In this section we construct such a refrigeration cycle for $N_Q > N$ and compare the cycle's coefficient of performance of the extended quantum switch Φ^{ext} and the superposition of independent channels Φ^{indep}.

We illustrate the refrigeration cycle in Fig. 9. The cycle involves heat baths of two different temperatures: the cold baths of inverse temperature β that the working body or main system Q interacts with in the Φ^{ext} operation and a hot bath of inverse temperature β_Q, with $\beta_Q < \beta$, $T_Q > T$, or $N_Q > N$. The main system Q is the working body of the cycle and starts in a thermal state of the hot bath of inverse temperature β_Q. It undergoes the extended quantum switch Φ^{ext} operation interacting with the cold baths of inverse temperature β. After the interaction, the control qubit is measured. If the

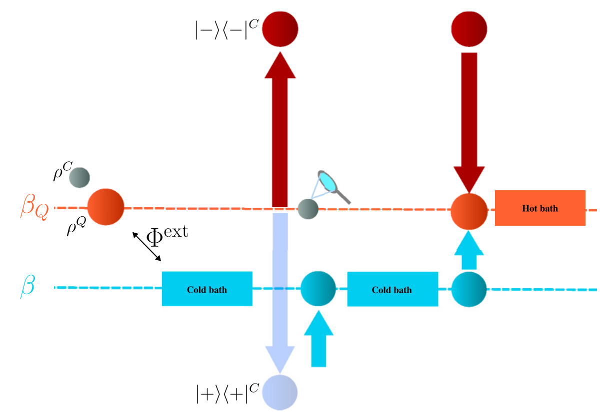

FIG. 9. Refrigeration cycle for the case of a main system or working body Q that starts at inverse temperature β_Q of a hot bath that is hotter than the cold baths in the Φ^{ext} operation, i.e., $\beta_Q < \beta$ or $N_Q > N$. After measurement of the control qubit, depending on the measurement outcome, there can be positive heat flow into the working body Q, despite it being at a higher temperature than the interacting baths. This extracted heat is passed to the β_Q hot bath, resetting the cycle. In the measurement outcome where heat flows from the working body into the cold baths of β, we interact the working body again with the cold bath of β before resetting the working body with the hot bath of β_Q. We can then find regimes where on average heat flows out of the cold bath and into the hot bath.

measurement outcome is $|-\rangle\langle-|$, we reinitialize the collapsed working body Q back to the thermal state of β_Q by thermalizing with the β_Q hot bath. If the measurement outcome is $|+\rangle\langle+|$, we thermalize the collapsed working body Q with the β cold baths before reinitializing it back to the thermal state of β_Q with the hot bath.

The average heat flow per cycle out of the cold bath (and into the hot bath) is thus

$$\langle Q\rangle_t^H = p_{-,t}Q_{-,t_0\to t}^{Q} - p_{+,t}\left[E\left(\sigma_{\beta_Q}\right) - E(\sigma_\beta)\right], \tag{48}$$

where the superscript H represents heat flow into the hot bath. Note that the entropic or work cost comes about due to Maxwell's demonlike measurement process, where after the measurement of the control qubit, some amount of information about the measurement result is stored in a memory register, which must be erased to reset the thermodynamic cycle [14]. This erasure of information would then require work expenditure in accordance with Landauer's principle [46]. Given that the erasure is performed with a resetting bath of inverse temperature β_R, the work cost of the cycle is thus $\Delta W_t = H(p_{\pm,t})/\beta_R$, or

$$\Delta W_t = -\frac{1}{\beta_R}(p_{-,t}\ln p_{-,t} + p_{+,t}\ln p_{+,t}), \tag{49}$$

where $H(\cdot)$ is the Shannon entropy. We can then calculate the efficiency or coefficient of performance (COP) of the refrigeration cycle as

$$\eta_t = \frac{\max\left\{0, \langle Q\rangle_t^H\right\}}{\Delta W_t}, \tag{50}$$

where we only consider the COP where refrigeration is possible, i.e., $\langle Q\rangle_t^H \geqslant 0$. Therefore, we can find the difference in

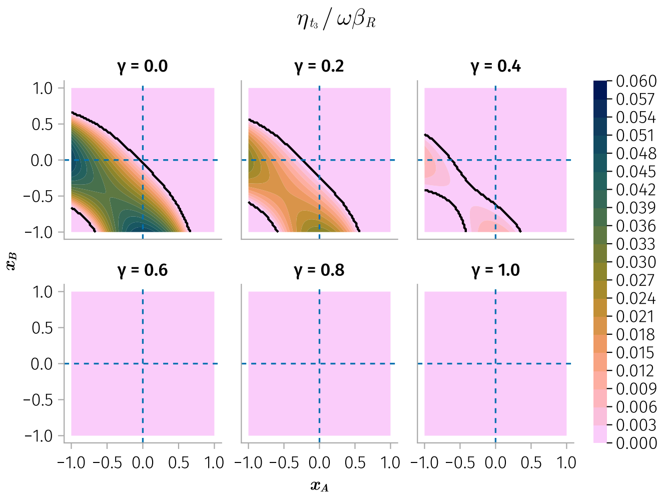

FIG. 10. The COP of the refrigeration cycle of Fig. 9 utilizing the extended quantum switch Φ^{ext} in units of β_R. It is greatest for the fully non-Markovian case of $\gamma = 0$ and decreases with decreasing non-Markovianity (increasing γ). It is also only nonzero in the prethermalization regimes.

COP of Φ^{ext} and Φ^{indep} as

$$\Delta\eta = \eta_{t_3} - \eta_{t_1}. \tag{51}$$

Taking $N = 0.223$, $N_Q = 0.269$, and $q = 0.5$, we plot η_{t_3} in units of $\omega\beta_R$ in Fig. 10, demonstrating that there are thermalization regimes (areas bounded by the black lines) where refrigeration is possible for the extended quantum switch Φ^{ext}. Likewise, we plot $\Delta\eta$ in units of $\omega\beta_R$ in Fig. 11, revealing the regimes where the extended quantum switch Φ^{ext} can perform better than the superposition of independent channels Φ^{indep} (areas bounded by the black lines). There are three important things to note. First, the ability for Φ^{ext} to perform refrigeration and its advantage over Φ^{indep} are maximal for the fully non-Markovian case of $\gamma = 0$, which decreases with decreasing non-Markovianity, i.e., increasing γ. Second, this ability and its advantage are absent in the fully thermalization regime, i.e., $x_A, x_B = 0$. This is despite the fact that heat extraction is possible in the $|-\rangle\langle-|^C$ measurement outcome for full thermalization as seen in Fig. 4, which means that this extractable heat is balanced out by the $|+\rangle\langle+|^C$ measurement outcome, resulting in no average heat extraction. Third, there are regimes where Φ^{indep} still grants a higher COP than the extended quantum switch Φ^{ext}, i.e., $\Delta\eta < 0$. These three results note that while the quantum switch has additional advantage granted by its non-Markovianity, deliberately chosen prethermalization regimes is necessary for this advantage to be significant and for it to outperform the simpler superposition of independent channels. For example, consider a setup where heat bath A is fully thermalizing ($x_A = 0$), while heat bath B can be prethermalizing ($x_B \neq 0$). We can see in Fig. 11 that there are prethermalization regimes for heat bath B where the quantum switch Φ^{sw} can outperforms Φ^{indep} ($x_B < 0$), whereas in regimes where the converse is true ($x_B > 0$), Φ^{indep} can only grant a lesser advantage close to zero.

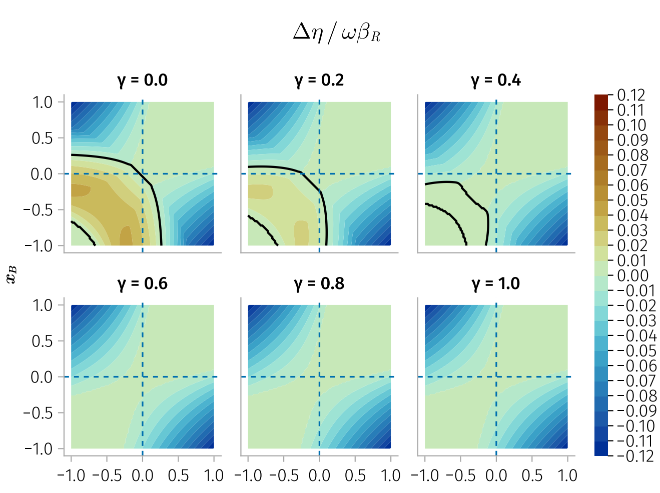

FIG. 11. Advantage in COP of the refrigeration cycle utilizing Φ^{ext} as compared to Φ^{indep}. Again, the advantage is greatest in the fully non-Markovian case of $\gamma = 0$ and decreases with decreasing non-Markovianity (increasing γ). The advantage is also only present in the prethermalization regimes. However, it can be seen that there are other regimes where Φ^{indep} can outperform the extended quantum switch Φ^{ext}.

IV. CONCLUSION

Many works have demonstrated that when the quantum switch operation Φ^{sw} is used to perform full thermalization between a working body and the baths, the working body is able to extract heat from the baths even when they are initialized at the same temperature [14–16]. However, here we showed that the same advantage is achieved by the superposition of independent channels Φ^{indep}, which has fixed causal order. Instead, the quantum switch Φ^{sw} only has an advantage over Φ^{indep} in certain prethermalizing regimes, and by extending the quantum switch into the extended quantum switch Φ^{ext} of Ref. [23] such that its non-Markovianity is variable, we are able to show that this advantage and its regimes decrease as the amount of non-Markovianity decreases.

Furthermore, we computed the conditions for positive heat extraction in terms of the temperatures of the baths and working body, revealing that positive heat extraction is possible even when the working body is at a higher temperature than the baths. This allows us to construct a refrigeration cycle that is able to perform refrigeration with a working body of a higher temperature than the bath being cooled in certain prethermalization regimes. The coefficient of performance of the cycle is greatest for the fully non-Markovian case of the quantum switch Φ^{sw} and decreases as the amount of non-Markovianity decreases. This dependence on the amount of non-Markovianity is also true for the quantum switch's advantage over the superposition of independent channels Φ^{indep}.

Our work is consistent with previous works on the presence and contributions of non-Markovianity to some advantages seen in the quantum switch [22–24]. The ability to perform refrigeration with a working body of a higher temperature might also present an opportunity for novel quantum refrigeration applications. However, it is important to note that the advantage of the quantum switch over Φ^{indep}, as well as the advantage granted by non-Markovian contributions, is only present in certain prethermalization regimes and that there are other regimes where Φ^{indep} can perform much better than the quantum switch. This means that while non-Markovianity can grant additional advantage, its advantage can be easily surpassed with just a coherent superposition. This is in line with recent works that showed that the advantages of the quantum switch can be replicated with just coherent superpositions, e.g., in communication advantages [2,17,18]. Furthermore, our work, which includes the prethermalization regimes, extends the work of Ref. [47], where it was shown that the superposition of N independent channels can outperform a quantum switch with N cyclic causal orders, which considers only the full thermalization regime, although in this work we only have $N = 2$. Therefore, combined with the low COP, practical quantum refrigeration applications utilizing this feature of the quantum switch are circumstantial and depend on the thermalization regimes.

Future works could look into how to better utilize the non-Markovian advantage granted by the quantum switch in quantum refrigeration tasks. For example, Ref. [48] showed that non-Markovian memory can be enhanced by increasing the dimensions of the control qubit. Otherwise, one can also examine if the advantages granted by the quantum switch are always redundant and achievable with just coherent superpositions. Other forms of thermal interactions such as ones with squeezed baths [39] or non-Markovian baths [49] (on top of the non-Markovianity of the quantum switch) might also enhance refrigeration beyond what the quantum switch can offer. The feature of non-Markovianity is also not exclusive to the quantum switch and is applicable to other channel compositions. While our work focused on its presence in the quantum switch, other future works can extend the same framework to other channel compositions that might grant a greater coefficient of performance than the quantum switch. Ultimately, our work further demonstrated the role of non-Markovianity in the quantum switch in granting its various advantages, and one important open question is whether its feature of indefinite causal order, as well as the advantages that it grants, always has a dual characterization in terms of non-Markovianity.

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