Causal state estimation and the Heisenberg uncertainty principle

Junxin Chen,[1,*] Benjamin B. Lane,[1] Su Direkci,[2] Dhruva Ganapathy,[1] Xinghui Yin,[1]
Nergis Mavalvala,[1] Yanbei Chen,[2] and Vivishek Sudhir[1,3]

[1]*LIGO, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA*
[2]*The Division of Physics, Mathematics and Astronomy, California Institute of Technology, California 91125, USA*
[3]*Department of Mechanical Engineering, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA*

(Received 5 May 2023; accepted 11 July 2024; published 5 August 2024)

The observables of a noisy quantum system can be estimated by appropriately filtering the records of their continuous measurement. Such filtering is relevant for state estimation, and if the filter is causal, also relevant for measurement-based feedback control. It is therefore imperative that a pair of conjugate observables estimated causally satisfy the Heisenberg uncertainty principle. In this article, we prove this fact—without assuming Markovian dynamics or Gaussian noises, in the presence or absence of feedback control of the system, and where in the feedback loop (inside or outside) the measurement record is accessed. Indeed, causal estimators using the in-loop measurement record can be as precise as those using the out-of-loop record. These results clarify the role of causal estimators to non-Markovian quantum systems, restore the equanimity of in-loop and out-of-loop measurements in their estimation and control, and simplify future experiments on measurement-based quantum feedback control.

DOI: 10.1103/PhysRevA.110.022207

I. INTRODUCTION

Estimating the state of a system from a noisy measurement record is ubiquitous in engineering and science. For a classical linear system driven by stationary Gaussian noise, the optimal estimator of an observable—in the sense of minimizing the mean-square error—is given by the Wiener filter [1,2]. In essence, the Wiener filter weights the noisy measurement record to get an estimate of the desired observable such that parts of the record with higher signal-to-noise ratio are emphasised. For prediction problems where only the past measurement record is available, the Wiener filter is causal and is given in terms of the spectrum of the measurement record and a model of its cross-spectrum with the desired observable. Additionally, if the dynamics are Markovian, a state-space model of the system and its observation is available, and the Kalman filter uses that description to produce an equivalent, more tractable estimate of the state [3]. State estimation is also a crucial element in optimal control: the separation principle [4,5] asserts that the feedback controller that minimizes a quadratic cost function of a linear system driven by Gaussian noise can be split into a causal state estimator and a linear regulator, with any error in the controlled state set by that of the estimator.

These ideas have been fully transposed to quantum systems: the theory of quantum state estimation [6–11], feedback control [12–16], and the separation principle [17] have been developed. A central distinction between classical and quantum systems is that the latter has to obey Heisenberg's uncertainty principle: a pair of conjugate observables cannot be determined with arbitrary precision simultaneously, even in principle. Consequently, it is believed that causal estimation of conjugate observables cannot be simultaneously more precise than that allowed for the observables themselves. Violation of this expectation would result in an unphysical conditional state (i.e., a state estimated through the measurement record) [10], and, by the separation principle, the possibility of transmuting the unphysical conditional state to an unphysical steady state by feedback control.

The purpose of this paper is to prove, in general and explicitly, that the product of the variance of the errors in the causal estimate of observables is lower bounded by the minimum variance product of corresponding physical observables allowed by the Heisenberg uncertainty principle. In other words, the uncertainties of causal estimation errors respect the uncertainty principle of the corresponding physical observables, and thus the causal conditional state has to be physical. Prior work guarantees this in Markovian and Gaussian settings [6–8,10,18] but only in the case where either a quantum stochastic differential equation for the state is available. Similar to the classical case, in the linear Gaussian Markovian setting, the quantum version of Kalman filter is equivalent to solving the quantum stochastic master equation to get the observable dynamics [10]. However, for non-Markovian or non-Gaussian cases, no mathematical framework is currently available for solving for the observable dynamics directly, and the previous proof of the uncertainty principle is not valid any more. Recent experiments in quantum state estimation and feedback control in systems ranging from atoms [19–22] and solid state qubits [23–26] to mechanical oscillators [27–35] call for a simple and general guarantee that causal state estimation and control of quantum systems will not violate the basic tenet of the uncertainty principle, including

*Contact author: nielschan@gmail.com

non-Markovian cases. In particular, non-Markovian $1/f$ noise limits coherence of solid-state mechanical oscillators [36–40] and superconducting qubits [41–44], both leading candidates for measurement-based feedback control. Moreover, retrodictive (i.e., anti-causal) estimation is known for potential violation of Heisenberg's uncertainty principle [45], which, as a matter of principle, calls for a proof of Heisenberg's uncertainty principle in the causal estimation scenario under general conditions. In the following we address the validity of the uncertainty principle in causal state estimation in the non-Markovian setting generally.

II. STATEMENT OF PROBLEM AND MAIN RESULT

Consider a quantum system, and any pair of observables $\hat{A}(t)$, $\hat{B}(t)$, in the Heisenberg (or interaction) picture, which will be the object of estimation via continuous linear measurement. The uncertainties in them are lower-bounded by the uncertainty principle [46–48]

$$\sigma^2_{A(t)}\sigma^2_{B(t)} \geqslant \left|\tfrac{1}{2}\langle[\hat{A}(t),\hat{B}(t)]\rangle\right|^2 + \left|\tfrac{1}{2}\langle\{\hat{A}(t),\hat{B}(t)\}\rangle\right|^2$$
$$\geqslant \left|\tfrac{1}{2}\langle[\hat{A}(t),\hat{B}(t)]\rangle\right|^2, \qquad (1)$$

where $\sigma^2_O \equiv \langle\hat{O}^2\rangle$ is the variance of the operator $\hat{O}$, $[\cdot,\cdot]$ is the commutator, and $\{\cdot,\cdot\}$ is the anticommutator. Here we assume, without loss of generality, that observables are zero-mean. In passing to the last line, we have also dropped an overall positive term on the right-hand side, which makes the resulting bound weaker, but has the advantage that for bosonic (and linearized fermionic [49–51]) systems the lower bound can be state independent.

The premise of the uncertainty principle is the positivity of the quantum state $\hat{\rho}$: clearly, $\mathrm{Tr}[\hat{M}(t)^\dagger\hat{M}(t)\hat{\rho}] \geqslant 0$ for any operator $\hat{M}(t)$; in particular, also for $\hat{M} = \hat{A} + \lambda\hat{B}$ for any constant λ. Further, the inequality must still hold for the minimum value of its left-hand side as a function of λ; this gives Eq. (1). So as long as the quantum state is guaranteed to be positive, the uncertainty relation holds. The problem is that in non-Markovian settings—i.e., where the system may be driven by non-Markovian noises, its measurements may be contaminated by non-Markovian noises, and/or feedback may be non-Markovian—positivity of the conditional state is difficult to extract from models of its evolution.

We therefore analyze the problem in the interaction picture for the observables of interest, assuming only linearity of measurement and feedback. In particular, we will show that for causal linear measurement, the estimation errors $\Delta\hat{A}$, $\Delta\hat{B}$ (to be defined below) of the observables $\hat{A}$, $\hat{B}$ satisfy

$$\sigma^2_{\Delta A(t)}\sigma^2_{\Delta B(t)} \geqslant \left|\tfrac{1}{2}\langle[\hat{A}(t),\hat{B}(t)]\rangle\right|^2, \qquad (2)$$

i.e., causal estimates of conjugate observables are no more precise than the observables themselves.

III. PROOF FOR CASE WITHOUT FEEDBACK

We now consider a setup for quantum state estimation where the system is monitored continuously, the result of which is described by the measurement record $\hat{Y}$ [Fig. 1(a)]. The continuous monitoring condition (self-nondemolition principle [6]) is that [52]

$$[\hat{Y}(t),\hat{Y}(t')] = 0 \quad \text{for all } t, t'. \qquad (3)$$

The observables are estimated by filters $W_{A,B}$ acting linearly on the record [2]:

$$\hat{A}_e(t) = (W_A * \hat{Y})(t), \quad \hat{B}_e(t) = (W_B * \hat{Y})(t), \qquad (4)$$

where $*$ stands for convolution. When $W_{A,B}$ are causal, i.e.,

$$W_{A,B}(t \leqslant 0) = 0, \qquad (5)$$

they *only* act on the past record.

Equation (3) implies that the estimators commute with each other, and so they do not need to obey the uncertainty principle. That is, the uncertainty in the conditional state (i.e., the state estimated based on the measurement record) is not contained in the estimators, but in the estimation errors, described by the operators [10]

$$\Delta\hat{A}(t) \equiv \hat{A}(t) - \hat{A}_e(t) = \hat{A}(t) - (W_A * \hat{Y})(t) \qquad (6)$$

$$\Delta\hat{B}(t) \equiv \hat{B}(t) - \hat{B}_e(t) = \hat{B}(t) - (W_B * \hat{Y})(t), \qquad (7)$$

which characterize how closely the estimators follow the physical operators. To see this, note that in the Heisenberg picture, the positivity of the *initial* state suffices to guarantee that

$$\sigma^2_{\Delta A(t)}\sigma^2_{\Delta B(t)} \geqslant \left|\tfrac{1}{2}\langle[\Delta\hat{A}(t),\Delta\hat{B}(t)]\rangle\right|^2. \qquad (8)$$

That is, the estimation errors cannot be determined simultaneously with arbitrary precision.

We will now show that causality of the estimation filters $W_{A,B}$ [Eq. (5)] and of the measurement implies that $\langle[\Delta\hat{A}(t),\Delta\hat{B}(t)]\rangle = \langle[\hat{A}(t),\hat{B}(t)]\rangle$, so that the lower bound in Eq. (8) becomes the lower bound in the uncertainty principle for the observables $\hat{A}$, $\hat{B}$.

Using Eqs. (6) and (8) together with Eq. (3), we have

$$\langle[\Delta\hat{A}(t),\Delta\hat{B}(t)]\rangle = \langle[\hat{A}(t),\hat{B}(t)]\rangle - \left\langle[\hat{A}(t),\hat{B}_e(t)]\right\rangle$$
$$- \langle[\hat{A}_e(t),\hat{B}(t)]\rangle. \qquad (9)$$

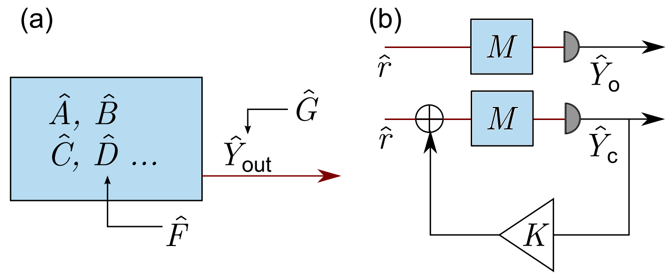

FIG. 1. (a) Schematic diagram for a general open-loop measurement. $\hat{F}$ is a general force coupling to the system through system parameter $\hat{D}$. The effect of this interaction is encoded in the outgoing field $\hat{Y}_{\rm out}$ together with other system information. A general force $\hat{G}$ coupling through $\hat{Y}_{\rm out}$ does not influence the system due to causality. (b) Schematic diagram for an open-loop system (upper panel) and a measurement-based feedback controlled system (lower panel). M and K stand for the system and controller transfer functions respectively. $\hat{r}$, $\hat{Y}_{\rm o}$, and $\hat{Y}_{\rm c}$ are the set point, the open-loop measurement record, and the in-loop measurement record, respectively.

The key in determining the remaining average of equal-time commutators is an understanding of the *unequal-time* commutators between the system observable and the measurement record. In the context of state estimation and measurement-based quantum control, weak measurement applies to most of the current experiments [19,20,23–32,34,35], thus we treat the effect of measurement on the system as a perturbation, and apply linear response theory [52–55]. The key result of the theory is Kubo's formula, which gives the linear response of the average of a system observable $\langle\hat{C}(t)\rangle$ to an external perturbation $\hat{F}(t)$, whose effect is described by the Hamiltonian $\hat{H}(t) = \hat{H}_0 - \hat{D}\hat{F}(t)$, where $\hat{H}_0$ is the free Hamiltonian and $\hat{D}$ is a system observable. Kubo's formula states

$$\langle\hat{C}(t)\rangle = \langle\hat{C}^{(0)}(t)\rangle + \frac{i}{\hbar}\int_{-\infty}^{t} \chi_{CD}(t,t')\langle\hat{F}(t')\rangle \mathrm{d}t', \qquad (10)$$

where $\hat{C}^{(0)}(t)$ is the observable evolving under $\hat{H}_0$ and

$$\chi_{CD}(t,t') = \langle[\hat{C}^{(0)}(t), \hat{D}^{(0)}(t')]\rangle \qquad (11)$$

is the linear response susceptibility, evaluated on the state of the system evolving under $\hat{H}_0$ (i.e., Kubo's formula holds in the interaction picture).

Consider now the schematic diagram of a general measurement as shown in Fig. 1(a), where $\hat{Y}_{\rm out} = \hat{Y}$ is the *outgoing* measurement record and $\hat{G}$ is the generalized force coupling to the system through $\hat{Y}_{\rm out}$ by the interaction Hamiltonian $-\hat{Y}_{\rm out}\hat{G}$. Focusing on $\hat{A}$ and using Eq. (10), we have

$$\langle\hat{A}(t)\rangle = \langle\hat{A}^{(0)}(t)\rangle + \frac{i}{\hbar}\int_{-\infty}^{t} \chi_{AY}(t,t')\langle\hat{G}(t')\rangle \mathrm{d}t', \qquad (12)$$

with $\chi_{AY}(t,t') = \langle[\hat{A}^{(0)}(t), \hat{Y}^{(0)}_{\rm out}(t')]\rangle$. Now we bring in the crucial ingredient of causality of the measurement: for an open-loop measurement, i.e., where the measurement record is not used for feedback, the measurement record $\hat{Y}$ cannot influence the system dynamics at a later time, implying that

$$\chi_{AY}(t,t') = \langle[\hat{A}^{(0)}(t), \hat{Y}^{(0)}(t')]\rangle = 0 \quad \text{for} \quad t \geqslant t'. \qquad (13)$$

Since the susceptibility is zero for $t \geqslant t'$, the force $\hat{G}$ does not influence the system for such times. Thus,

$$\chi_{AY}(t,t') = \langle[\hat{A}(t), \hat{Y}(t')]\rangle = 0 \quad \text{for} \quad t \geqslant t', \qquad (14)$$

where we have dropped the superscripts "(0)."

Let us now consider $\langle[\hat{A}(t), \hat{B}_e(t)]\rangle$. Without loss of generality, we evaluate it in the interacting picture, where the arbitrary state sandwiching the commutator evolves under $\hat{H}_0$. Using Eq. (4),

$$\langle[\hat{A}(t), \hat{B}_e(t)]\rangle = \int_{-\infty}^{+\infty} W_B(\tau)\langle[\hat{A}(t), \hat{Y}(t-\tau)]\rangle \mathrm{d}\tau. \qquad (15)$$

By causality of the estimation filter, $W_B(\tau) = 0$ for $\tau < 0$. For $\tau \geqslant 0$, Eq. (14) implies that $\langle[\hat{A}(t), \hat{Y}(t-\tau)]\rangle = \chi_{AY}(t, t-\tau) = 0$. Thus, causality—of the measurement interaction and of the estimation filter—implies that $\langle[\hat{A}(t), \hat{B}_e(t)]\rangle = 0$. Similarly, it can be shown that $\langle[\hat{A}_e(t), \hat{B}(t)]\rangle = 0$.

In summary, for causal estimation using open-loop measurements,

$$\langle[\Delta\hat{A}(t), \Delta\hat{B}(t)]\rangle = \langle[\hat{A}(t), \hat{B}(t)]\rangle, \qquad (16)$$

i.e., the estimation errors respect the uncertainty principle of the corresponding physical observables.

IV. PROOF FOR CASE WITH FEEDBACK

We now consider the case where the system is feedback controlled, and using the in-loop measurement record $\hat{Y}_{\rm c}$ for state estimation [see the lower panel of Fig. 1(b)]. In this case the in-loop record does affect the system at later times after the measurement interaction. However, the open-loop record $\hat{Y}_{\rm o}$ does not. In order to use this fact, note that the in-loop and open-loop measurement records are related to each other as

$$\hat{Y}_{\rm c}(t) = (K_{\rm c} * \hat{Y}_{\rm o})(t), \qquad (17)$$

where $K_{\rm c}(t)$ is the inverse Fourier transform of $(1 - M[\omega]K[\omega])^{-1}$ [M and K are the transfer functions of the system and the controller, respectively, as in Fig. 1(b)]. Importantly, if M, K are causal, then so is $K_{\rm c}$.

Imagine now estimation based on the in-loop measurement record $\hat{Y}_{\rm c}$: $\hat{A}_{\rm ec}(t) = (W^{\rm c}_A * \hat{Y}_{\rm c})(t)$ and $\hat{B}_{\rm ec}(t) = (W^{\rm c}_B * \hat{Y}_{\rm c})(t)$, with $W^{\rm c}_A$ and $W^{\rm c}_B$ the estimation filters in the in-loop case. Following the same line of reasoning as in the open-loop case, the question of whether the estimated observables respect the Heisenberg uncertainty principle of the original observables boils down to whether $\langle[\hat{A}(t), \hat{Y}_c(t')]\rangle = 0$ for $t \geqslant t'$. Clearly,

$$\langle[\hat{A}(t), \hat{Y}_c(t')]\rangle = \int_{-\infty}^{\infty} K_c(t'-\tau)\langle[\hat{A}(t), \hat{Y}_{\rm o}(\tau)]\rangle \mathrm{d}\tau.$$

We consider the integrand in two complementary intervals, $t' < \tau$ and $t' \geqslant \tau$. In the former region, $K_{\rm c}(t'-\tau) = 0$ since $K_{\rm c}$ is causal. In the latter region, since $t \geqslant t' \geqslant \tau$, we have that $\langle[\hat{A}(t), \hat{Y}_{\rm o}(\tau)]\rangle = 0$ using causality of open-loop dynamics. In sum, the integral is zero in both intervals due to causality. Thus, $\langle[\hat{A}(t), \hat{Y}_c(t')]\rangle$ vanishes as long as $t \geqslant t'$. A similar argument holds for $\langle[\hat{B}(t), \hat{Y}_c(t')]\rangle$. Thus, $\langle[\hat{A}(t), \hat{Y}_c(t')]\rangle = 0 = \langle[\hat{B}(t), \hat{Y}_c(t')]\rangle$ for $t \geqslant t'$. In summary, the in-loop estimation errors $\Delta\hat{A}_c = \hat{A} - \hat{A}_{\rm ec}$, $\Delta\hat{B}_c = \hat{B} - \hat{B}_{\rm ec}$ satisfy

$$\langle[\Delta\hat{A}_{\rm c}(t), \Delta\hat{B}_{\rm c}(t)]\rangle = \langle[\hat{A}(t), \hat{B}(t)]\rangle, \qquad (18)$$

just as in the open-loop case. Thus, the in-loop causal estimation errors respect the uncertainty principle of the corresponding physical observables just as in the open-loop case.

V. EFFECT OF FEEDBACK ON ESTIMATION ERROR

The in-loop measurement record $\hat{Y}_{\rm c}$ is often deemed untrustworthy for state estimation. Historically, this view originated from the apparent violation of the Heisenberg uncertainty principle by the in-loop field in linear measurement-based feedback control of optical fields [56]. This behavior, called "noise squashing" [57], arises because fields inside a feedback loop are not freely propagating and so need not satisfy the canonical commutation relations [58,59].

In fact, if a model of the feedback loop is available, then state estimation using the in-loop measurement record is as accurate as an estimate based on the out-of-loop record. We can always write the system observable in the presence of feedback $\hat{A}_{\rm c}$ as the sum of the open-loop one $\hat{A}_{\rm o}$ plus a feedback term:

$$\hat{A}_{\rm c}(t) = \hat{A}_{\rm o}(t) + \hat{A}_{\rm fb}(t) = \hat{A}_{\rm o}(t) + (K * \hat{Y}_{\rm c})(t), \qquad (19)$$

where K is a causal feedback control filter. Now consider two filters $W_A^i[\omega]$ ($i = \mathrm{c, o}$), one estimating the observable $\hat{A}$ from the in-loop record and the other from the out-of-loop record. The respective errors are $\Delta\hat{A}_i(t) = \hat{A}_i(t) - (W_A^i * \hat{Y}_i)(t)$. It is straightforward to show that if the in-loop filter is chosen to be

$$W_A^{\mathrm{c}}[\omega] = \frac{W_A^{\mathrm{o}}[\omega]}{K_{\mathrm{c}}[\omega]} + K[\omega]$$
$$= \left(1 - W_A^{\mathrm{o}}[\omega]M[\omega]\right)K[\omega] + W_A^{\mathrm{o}}[\omega],$$

then $\Delta\hat{A}_{\mathrm{c}} = \Delta\hat{A}_{\mathrm{o}}$. Note that if the feedback loop and W_A^{o} are stable and causal, so is W_A^{c}. Therefore, by proper filter design, state estimation can be performed using the in-loop measurement record without loss of fidelity.

VI. EXAMPLE OF A STRUCTURALLY DAMPED MECHANICAL OSCILLATOR

A canonical example of non-Markovian behavior of a subject of contemporary interest to quantum state estimation is a structurally damped mechanical oscillator [33,36–40]. Such a system is described by the linear response of its displacement $\hat{x}[\omega] = \chi[\omega]\hat{F}[\omega]$, to the force $\hat{F}$, by the susceptibility [60]

$$\chi[\omega] = \left[m\left(-\omega^2 + \omega_0^2 + i\omega_0^2\phi[\omega]\right)\right]^{-1}, \tag{20}$$

where m is the mass, ω_0 is the resonance frequency, and ϕ is the loss angle. The thermal displacement spectrum of the oscillator, given by the fluctuation-dissipation theorem [61,62], is

$$S_{xx}^{\mathrm{th}}[\omega] = 2\hbar\left(n_{\mathrm{th}}[\omega] + \frac{1}{2}\right)\mathrm{Im}\,\chi[\omega]$$
$$= \frac{2\hbar\left(n_{\mathrm{th}}[\omega] + \frac{1}{2}\right)}{m\left[\left(\omega^2 - \omega_0^2\right) + (\omega_0\phi[\omega])^2\right]}, \tag{21}$$

with $n_{\mathrm{th}}[\omega] = 1/(\exp[\hbar\omega/k_{\mathrm{B}}T] - 1)$ being the thermal occupancy of the bath, where k_{B} is the Boltzmann constant and T the bath temperature.

The structural damping model is unphysical and inconsistent if the loss angle $\phi[\omega]$ is frequency independent. Unphysical, because the $1/\omega$ scaling of the oscillator's displacement spectrum would preclude a finite variance for the displacement. Mathematically inconsistent, because the susceptibility between hermitian operators must have the symmetry $\chi[\omega]^* = \chi[-\omega]$, or equivalently, $\mathrm{Re}\,\chi^{-1}[\omega] = \mathrm{Re}\,\chi^{-1}[-\omega]$ and $\mathrm{Im}\,\chi^{-1}[\omega] = -\mathrm{Im}\,\chi^{-1}[-\omega]$. For a structurally damped oscillator, $\mathrm{Im}\,\chi^{-1}[\omega] = m\omega_0^2\phi[\omega]$ is not antisymmetric in frequency if the loss angle is finite and frequency independent. In order to satisfy the antisymmetry, to leading order, $\mathrm{Im}\,\chi^{-1}[\omega \to 0] \propto \omega$, which precisely cancels the $1/\omega$ pathology in the spectrum. The simplest example of such a loss angle is velocity-proportional damping, which however is inconsistent with observations on high-quality elastic oscillators. The Zener model [63]

$$\phi[\omega] = \phi_0\frac{\omega\tau}{1 + (\omega\tau)^2}, \tag{22}$$

although not frequency independent, is slowly varying around $\omega \approx \tau^{-1}$, consistent with observations, and resolves the pathologies of a truly frequency-independent loss angle. Since the $1/\omega$ low-frequency divergence of the displacement spectrum is mollified, most of the thermal energy of the oscillator is concentrated around resonance; thus, the approximation $\chi[\omega]^{-1} \approx m[2\omega_0(-\omega + \omega_0) + i\Gamma_0\omega_0]$ is viable (here $\Gamma_0 = \omega_0\phi[\omega_0]$).

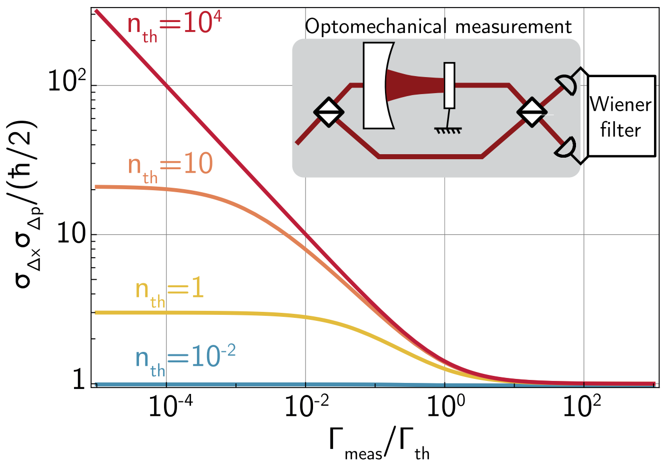

FIG. 2. Product of the uncertainty in the errors of the causal estimates of displacement and momentum of a structurally damped oscillator estimated from an interferometric measurement (inset).

We will now show that a causal Wiener filter applied to an interferometric measurement of a structurally damped oscillator produces estimates of its position $\hat{x}(t)$ and momentum $\hat{p}(t)$, whose estimation errors satisfy

$$\sigma_{\Delta x(t)}^2\sigma_{\Delta p(t)}^2 \geqslant \left|\frac{1}{2}\langle[\Delta\hat{x}(t), \Delta\hat{p}(t)]\rangle\right|^2$$
$$= \left|\frac{1}{2}\langle[\hat{x}(t), \hat{p}(t)]\rangle\right|^2 = \frac{\hbar^2}{4}. \tag{23}$$

We consider that the motion of the Zener-damped oscillator is measured using a cavity interferometer [64], as shown in the insert of Fig. 2. The motion of the oscillator changes the resonance frequency of the cavity, and thus the phase of light leaking out. Homodyne measurement of the light produces a photocurrent that is linearly proportional to the oscillator's displacement, together with detection noise; we denote by $\hat{y}$ these photocurrent fluctuations referred to apparent displacement. Applying a causal Wiener filter to $\hat{y}$ gives an estimate of the physical displacement $\hat{x}$. The physical momentum $\hat{p}$ is then (using the close-to-resonance approximation) $\hat{p}[\omega] = \mathrm{i}m\omega_0\hat{x}[\omega]$. The estimation errors $\Delta\hat{x}$ and $\Delta\hat{p}$ are related similarly. The spectra of the errors in the estimate produced by causal Wiener filtering is [2,65]

$$S_{\Delta x_\ell \Delta x_{\ell'}} = S_{x_\ell x_{\ell'}} - \left[\frac{S_{x_\ell y}}{S_y^-}\right]_+ \left[\frac{S_{x_{\ell'} y}}{S_y^-}\right]_+^*, \tag{24}$$

where $x_\ell \in \{\hat{x}$ or $\hat{p}\}$; $S_{x_\ell x_{\ell'}}$ is the spectrum of the corresponding observables; S_y^- is the anticausal factor of the measured spectrum S_{yy}, such that $S_{yy} = S_y^+ S_y^-$ with $S_y^+ = \{S_y^-\}^*$; and $[.]_+$ takes the causal components of the expression in the bracket. The variance in the estimate Δx_ℓ is the integral of the spectrum $S_{\Delta x_\ell \Delta x_\ell}$.

For measurement using a cavity interferometer, in the bad-cavity regime (i.e.. cavity linewidth $\kappa \gg \omega_0$), with probe laser on-resonance with the cavity and homodyne detection of phase quadrature of the probe laser, direct computation using standard techniques [64] shows that

$$S_{xx}[\omega] = \frac{2x_{\rm zpf}^2(\Gamma_{\rm th}+\Gamma_{\rm meas})}{(\omega-\omega_0)^2+(\Gamma_0/2)^2} = S_{xy}[\omega] \tag{25}$$

$$\left[\frac{S_{xy}}{S_y^-}\right]_+ = i\frac{2x_{\rm zpf}\sqrt{2\Gamma_{\rm meas}}(\Gamma_{\rm th}+\Gamma_{\rm meas})}{(\Gamma_0/2+\Gamma_{\rm W})(\omega-\omega_0+i\Gamma_0/2)}, \tag{26}$$

where $x_{\rm zpf} = \sqrt{\hbar/2m\omega_0}$ is the zero-point fluctuation in displacement, $\Gamma_{\rm th} = \Gamma_0(n_{\rm th}+1/2)$ is the thermal decoherence rate, and $\Gamma_{\rm meas} = 4g^2/\kappa \equiv n_{\rm meas}\Gamma_0$ is the measurement rate, where $n_{\rm meas} = 4g^2/(\kappa\Gamma_0)$ is the occupation due to quantum back-action of the measurement and g the multiphoton optomechanical coupling rate. $\Gamma_{\rm W}$ is the characteristic estimation bandwidth of the Wiener filter, given by $\Gamma_{\rm W}^2 = 4\Gamma_{\rm th}\Gamma_{\rm meas} + 4\Gamma_{\rm meas}^2 + (\Gamma_0/2)^2$. The spectrum of the estimation error [Eq. (24)] is obtained using Eqs. (25) and (26):

$$S_{\Delta x\Delta x}[\omega] = \frac{8x_{\rm zpf}^2(\Gamma_{\rm meas}+\Gamma_{\rm th})}{\Gamma_0^2+4(\omega-\omega_0)^2} \times\left[1-\frac{16\Gamma_{\rm meas}(\Gamma_{\rm meas}+\Gamma_{\rm th})}{(\Gamma_0+\Gamma_{\rm W})^2}\right]. \tag{27}$$

Integrating it gives the variance

$$\sigma_{\Delta x}^2 = x_{\rm zpf}^2\frac{4(\Gamma_{\rm meas}+\Gamma_{\rm th})}{\Gamma_0+2\Gamma_{\rm W}}. \tag{28}$$

A similar computation for the momentum gives

$$\sigma_{\Delta p}^2 = p_{\rm zpf}^2\frac{4(\Gamma_{\rm meas}+\Gamma_{\rm th})}{\Gamma_0+2\Gamma_{\rm W}}, \tag{29}$$

where $p_{\rm zpf} = (\hbar/2)/x_{\rm zpf}$ is the zero-point fluctuation in the momentum. In both cases, straightforward analysis of the right-hand side shows that the variances are bounded as

$$(2n_{\rm th}+1)x_{\rm zpf}^2 \geqslant \sigma_{\Delta x}^2 \geqslant x_{\rm zpf}^2$$
$$(2n_{\rm th}+1)p_{\rm zpf}^2 \geqslant \sigma_{\Delta p}^2 \geqslant p_{\rm zpf}^2; \tag{30}$$

here the lower bound is attained in the regime of strong measurement (i.e., $\Gamma_{\rm meas} \gg \Gamma_{\rm th}$), which is favorable for estimation, while the upper bound is attained in the opposite regime. It is worth noting that even though quantum back action contributes to the physical motion of the oscillator, as indicated by $\Gamma_{\rm meas}$ in Eq. (25), the estimated motion can be free from it in the strong measurement regime for unity detection efficiency. Clearly the variances of the estimate errors satisfy

$$(2n_{\rm th}+1)^2\frac{\hbar^2}{4} \geqslant \sigma_{\Delta x}^2\sigma_{\Delta p}^2 \geqslant \frac{\hbar^2}{4}, \tag{31}$$

where the lower bound is the claim in Eq. (23). Figure 2 shows how the product $\sigma_{\Delta x}\sigma_{\Delta p}$ is bounded as the measurement rate $\Gamma_{\rm meas}$ is increased.

VII. CONCLUSION

We have proved—with or without feedback, and without invoking any Markovian or Gaussian character of the system dynamics or measurement—that the errors in causal estimation of quantum observables from a linear continuous measurement respect the Heisenberg uncertainty principle for the corresponding physical observables. We utilize linear response theory, widely applicable to current experimental measurement-based quantum control. Furthermore, in the scenario with feedback control, we clarify that despite "noise squashing," the in-loop measurement record can provide as faithful an estimate of an observable as the out-of-loop record. As a matter of practice, this vastly simplifies experiments and extends the reach of quantum state estimation to non-Markovian scenarios [33,38,66]. Importantly, this insight eliminates the compromise in measurement efficiency that is required in having two simultaneous measurements on the same system.

ACKNOWLEDGMENTS

We wish to thank C. Meng and W. Bowen for fruitful discussions. We acknowledge the generous support from the National Science Foundation (NSF). Especially, J.C. and B.B. L. are supported through NSF Grant No. PHY-20122088, D.G. by NSF Grant No. PHY-0823459, and S.D. and Y.C. by NSF Grants No. PHY-2011961, No. PHY-2011968, and No. PHY-1836809. In addition, S.D. and Y.C. acknowledge support by the Simons Foundation (Award No. 568762).

[1] N. Wiener, *Extrapolation, Interpolation, and Smoothing of Stationary Time Series* (MIT Press, Cambridge, 1942).
[2] T. Kailath, *Lectures on Wiener and Kalman Filtering* (Springer, New York, 1981).
[3] R. E. Kalman, A new approach to linear filtering and prediction problems, J. Basic Eng. **82**, 35 (1960).
[4] W. M. Wonham, On the separation theorem of stochastic control, SIAM J. Control **6**, 312 (1968).
[5] A. Lindquist, On feedback control of linear stochastic systems, SIAM J. Control **11**, 323 (1973).
[6] V. P. Belavkin, A new wave equation for a continuous nondemolition measurement, Phys. Lett. A **140**, 355 (1989).
[7] A. Barchielli and V. Belavkin, Measurements continuous in time and a posteriori states in quantum mechanics, J. Phys. A **24**, 1495 (1991).
[8] V. P. Belavkin, Quantum continual measurements and a posteriori collapse on CCR, Commun. Math. Phys. **146**, 611 (1992).
[9] H. M. Wiseman, Quantum trajectories and quantum measurement theory, Quantum Semiclass. Opt. **8**, 205 (1996).

[10] A. C. Doherty, S. M. Tan, A. S. Parkins, and D. F. Walls, State determination in continuous measurement, Phys. Rev. A **60**, 2380 (1999).

[11] L. Bouten, M. Guta, and H. Maassen, Stochastic Schrödinger equations, J. Phys. A: Math. Gen. **37**, 3189 (2004).

[12] H. M. Wiseman, Quantum theory of continuous feedback, Phys. Rev. A **49**, 2133 (1994).

[13] H. M. Wiseman and A. C. Doherty, Optimal unravellings for feedback control in linear quantum systems, Phys. Rev. Lett. **94**, 070405 (2005).

[14] M. G. Genoni, S. Mancini, and A. Serafini, Optimal feedback control of linear quantum systems in the presence of thermal noise, Phys. Rev. A **87**, 042333 (2013).

[15] H. M. Wiseman and G. J. Milburn, *Quantum Measurement and Control* (Cambridge University Press, Cambridge, 2010).

[16] K. Jacobs, *Quantum Measurement Theory and its Applications* (Cambridge University Press, Cambridge, 2014).

[17] L. Bouten and R. V. Handel, On the separation principle in quantum control, in *Quantum Stochastics and Information* (World Scientific, Singapore, 2008), pp. 206–238.

[18] M. Tsang, J. H. Shapiro, and S. Lloyd, Quantum theory of optical temporal phase and instantaneous frequency. II. Continuous-time limit and state-variable approach to phase-locked loop design, Phys. Rev. A **79**, 053843 (2009).

[19] C. J. Hood, T. W. Lynn, A. C. Doherty, A. S. Parkins, and H. J. Kimble, The atom-cavity microscope: Single atoms bound in orbit by single photons, Science **287**, 1447 (2000).

[20] A. Kubanek, M. Koch, C. Sames, A. Ourjoumtsev, P. W. H. Pinkse, K. Murr, and G. Rempe, Photon-by-photon feedback control of a single-atom trajectory, Nature (London) **462**, 898 (2009).

[21] C. Sayrin, I. Dotsenko, X. Zhou, B. Peaudecerf, T. Rybarczyk, S. Gleyzes, P. Rouchon, M. Mirrahimi, H. Amini, M. Brune, J.-M. Raimond, and S. Haroche, Real-time quantum feedback prepares and stabilizes photon number states, Nature (London) **477**, 73 (2011).

[22] X. Zhou, I. Dotsenko, B. Peaudecerf, T. Rybarczyk, C. Sayrin, S. Gleyzes, J. M. Raimond, M. Brune, and S. Haroche, Field locked to a Fock state by quantum feedback with single photon corrections, Phys. Rev. Lett. **108**, 243602 (2012).

[23] R. Vijay, C. Macklin, D. H. Slichter, S. J. Weber, K. W. Murch, R. Naik, A. N. Korotkov, and I. Siddiqi, Stabilizing Rabi oscillations in a superconducting qubit using quantum feedback, Nature (London) **490**, 77 (2012).

[24] M. Hatridge, S. Shankar, M. Mirrahimi, F. Schackert, K. Geerlings, T. Brecht, K. M. Sliwa, B. Abdo, L. Frunzio, S. M. Girvin, R. J. Schoelkopf, and M. H. Devoret, Quantum back-action of an individual variable-strength measurement, Science **339**, 178 (2013).

[25] K. Murch, S. Weber, C. Macklin, and I. Siddiqi, Observing single quantum trajectories of a superconducting quantum bit, Nature (London) **502**, 211 (2013).

[26] S. Weber, A. Chantasri, J. Dressel, A. N. Jordan, K. Murch, and I. Siddiqi, Mapping the optimal route between two quantum states, Nature (London) **511**, 570 (2014).

[27] D. J. Wilson, V. Sudhir, N. Piro, R. Schilling, A. Ghadimi, and T. J. Kippenberg, Measurement-based control of a mechanical oscillator at its thermal decoherence rate, Nature (London) **524**, 325 (2015).

[28] S. Hacohen-Gourgy, L. S. Martin, E. Flurin, V. V. Ramasesh, K. B. Whaley, and I. Siddiqi, Quantum dynamics of simultaneously measured non-commuting observables, Nature (London) **538**, 491 (2016).

[29] V. Sudhir, D. J. Wilson, R. Schilling, H. Schütz, S. A. Fedorov, A. H. Ghadimi, A. Nunnenkamp, and T. J. Kippenberg, Appearance and disappearance of quantum correlations in measurement-based feedback control of a mechanical oscillator, Phys. Rev. X **7**, 011001 (2017).

[30] M. Rossi, D. Mason, J. Chen, Y. Tsaturyan, and A. Schliesser, Measurement-based quantum control of mechanical motion, Nature (London) **563**, 53 (2018).

[31] M. Rossi, D. Mason, J. Chen, and A. Schliesser, Observing and verifying the quantum trajectory of a mechanical resonator, Phys. Rev. Lett. **123**, 163601 (2019).

[32] Z. K. Minev, S. O. Mundhada, S. Shankar, P. Reinhold, R. Gutiérrez-Jáuregui, R. J. Schoelkopf, M. Mirrahimi, H. J. Carmichael, and M. H. Devoret, To catch and reverse a quantum jump mid-flight, Nature (London) **570**, 200 (2019).

[33] C. Whittle *et al.*, Approaching the motional ground state of a 10-kg object, Science **372**, 1333 (2021).

[34] F. Tebbenjohanns, M. L. Mattana, M. Rossi, M. Frimmer, and L. Novotny, Quantum control of a nanoparticle optically levitated in cryogenic free space, Nature (London) **595**, 378 (2021).

[35] L. Magrini, P. Rosenzweig, C. Bach, A. Deutschmann-Olek, S. G. Hofer, S. Hong, N. Kiesel, A. Kugi, and M. Aspelmeyer, Real-time optimal quantum control of mechanical motion at room temperature, Nature (London) **595**, 373 (2021).

[36] G. I. González and P. R. Saulson, Brownian motion of a torsion pendulum with internal friction, Phys. Lett. A **201**, 12 (1995).

[37] M. Kajima, N. Kusumi, S. Moriwaki, and N. Mio, Wide-band measurement of mechanical thermal noise using a laser interferometer, Phys. Lett. A **264**, 251 (1999).

[38] S. Gröblacher, A. Trubarov, N. Prigge, G. Cole, M. Aspelmeyer, and J. Eisert, Observation of non-Markovian micromechanical Brownian motion, Nat. Commun. **6**, 7606 (2015).

[39] A. R. Neben, T. P. Bodiya, C. Wipf, E. Oelker, T. Corbitt, and N. Mavalvala, Structural thermal noise in gram-scale mirror oscillators, New J. Phys. **14**, 115008 (2012).

[40] S. A. Fedorov, V. Sudhir, R. Schilling, H. Schütz, D. J. Wilson, and T. J. Kippenberg, Evidence for structural damping in a high-stress silicon nitride nanobeam and its implications for quantum optomechanics, Phys. Lett. A **382**, 2251 (2018).

[41] F. C. Wellstood, C. Urbina, and J. Clarke, Low-frequency noise in dc superconducting quantum interference devices below 1 K, Appl. Phys. Lett. **50**, 772 (1987).

[42] F. Yoshihara, K. Harrabi, A. O. Niskanen, Y. Nakamura, and J. S. Tsai, Decoherence of flux qubits due to $1/f$ flux noise, Phys. Rev. Lett. **97**, 167001 (2006).

[43] R. C. Bialczak, R. McDermott, M. Ansmann, M. Hofheinz, N. Katz, E. Lucero, M. Neeley, A. D. O'Connell, H. Wang, A. N. Cleland, and J. M. Martinis, $1/f$ flux noise in Josephson phase qubits, Phys. Rev. Lett. **99**, 187006 (2007).

[44] P. Kumar, S. Sendelbach, M. A. Beck, J. W. Freeland, Z. Wang, H. Wang, C. C. Yu, R. Q. Wu, D. P. Pappas, and R. McDermott, Origin and reduction of $1/f$ magnetic flux noise in superconducting devices, Phys. Rev. Appl. **6**, 041001(R) (2016).

[45] M. Khanahmadi and K. Mølmer, Guessing the outcome of separate and joint quantum measurements of noncommuting observables, Phys. Rev. A **104**, 022204 (2021).
[46] E. Schrödinger, *Zum Heisenbergschen Unschärfeprinzip* (Akademie der Wissenschaften, Hamburg, 1930).
[47] H. P. Robertson, An indeterminacy relation for several observables and its classical interpretation, Phys. Rev. **46**, 794 (1934).
[48] L. Maccone and A. K. Pati, Stronger uncertainty relations for all incompatible observables, Phys. Rev. Lett. **113**, 260401 (2014).
[49] A. Klein and E. R. Marshalek, Boson realizations of Lie algebras with applications to nuclear physics, Rev. Mod. Phys. **63**, 375 (1991).
[50] A. Auerbach, *Interacting Electrons and Quantum Magnetism* (Springer, New York, 1994).
[51] H. Yuan, Y. Cao, A. Kamra, R. A. Duine, and P. Yan, Quantum magnonics: When magnon spintronics meets quantum information science, Phys. Rep. **965**, 1 (2022).
[52] V. B. Braginsky and F. Khalili, *Quantum Measurement* (Cambridge University Press, Cambridge, 1992).
[53] R. Kubo, Statistical-mechanical theory of irreversible processes. I. General theory and simple applications to magnetic and conduction problems, J. Phys. Soc. Jpn. **12**, 570 (1957).
[54] A. Buonanno and Y. Chen, Signal recycled laser-interferometer gravitational-wave detectors as optical springs, Phys. Rev. D **65**, 042001 (2002).
[55] A. A. Clerk, Quantum-limited position detection and amplification: A linear response perspective, Phys. Rev. B **70**, 245306 (2004).
[56] Y. Yamamoto, N. Imoto, and S. Machida, Amplitude squeezing in a semiconductor laser using quantum nondemolition measurement and negative feedback, Phys. Rev. A **33**, 3243 (1986).
[57] H. Wiseman, Squashed states of light: Theory and applications to quantum spectroscopy, J. Opt. B: Quantum Semiclass. Opt. **1**, 459 (1999).
[58] J. Shapiro, G. Saplakoglu, S.-T. Ho, P. Kumar, B. Saleh, and M. Teich, Theory of light detection in the presence of feedback, J. Opt. Soc. Am. B **4**, 1604 (1987).
[59] M. S. Taubman, H. Wiseman, D. E. McClelland, and H.-A. Bachor, Intensity feedback effects on quantum-limited noise, J. Opt. Soc. Am. B **12**, 1792 (1995).
[60] P. R. Saulson, Thermal noise in mechanical experiments, Phys. Rev. D **42**, 2437 (1990).
[61] H. Callen and T. Welton, Irreversibility and generalized noise, Phys. Rev. **83**, 34 (1951).
[62] R. Kubo, The fluctuation-dissipation theorem, Rep. Prog. Phys. **29**, 255 (1966).
[63] C. Zener, Internal friction in solids, Proc. Phys. Soc. **52**, 152 (1940).
[64] M. Aspelmeyer, T. J. Kippenberg, and F. Marquardt, Cavity optomechanics, Rev. Mod. Phys. **86**, 1391 (2014).
[65] H. Müller-Ebhardt, H. Rehbein, C. Li, Y. Mino, K. Somiya, R. Schnabel, K. Danzmann, and Y. Chen, Quantum-state preparation and macroscopic entanglement in gravitational-wave detectors, Phys. Rev. A **80**, 043802 (2009).
[66] C. Meng, G. A. Brawley, S. Khademi, E. M. Bridge, J. S. Bennett, and W. P. Bowen, Measurement-based preparation of multimode mechanical states, Sci. Adv. **8**, eabm7585 (2022).