





Semi-device-independent certification of quantum non-Markovianity using sequential random access codes

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The characterization of multi-time correlations in open quantum systems is of fundamental importance. In this work, we investigate multi-time processes using the process matrix formalism and show that the presence of a quantum-memory environment acts as a resource in enhancing the communication capacity in sequential prepare-transform-measure quantum random access codes (QRAC). The correlated environment enables a quantum advantage to multiple parties, even with projective measurements. In particular, we show that Markovian and classical-memory processes, i.e., non-Markovian quantum processes with classical feedback from the environment, do not yield a sequential quantum advantage. In contrast, it is possible to achieve an advantage in the presence of a quantum-memory environment. Therefore, this approach allows a semi-device-independent certification of quantum non-Markovianity. As opposed to entanglement-detection criteria which require knowledge of the complete process, this method allows to certify the presence of a quantum-memory environment from the observed measurement statistics. Moreover, quantum memory ameliorates the unambiguous certifiable region of unsharp instruments in a semi-device-independent manner.

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I. INTRODUCTION

Realistic quantum systems are not isolated and are subject to decoherence due to presence of an environment [1,2]. To develop error mitigation protocols, noise characterization in multi-time processes is of fundamental importance. Current error correction techniques rely on the assumption that the noise across different time steps is uncorrelated, i.e., the process is Markovian [3–6]. However, it turns out that non-Markovianity is the norm rather than an exception, and correlated noise has been identified in the state-of-the-art quantum devices of IBM and Google [7–9].

Characterization of non-Markovianity in a quantum scenario is a nontrivial task, with past approaches lacking a necessary and sufficient condition [10–15]. Although certain works in literature [16,17] argue for a necessary and sufficient condition for Markovian processes, these claims are true only within specific formalism of dynamical maps which capture only two-time correlations. The completely positive (CP) divisibility criteria for dynamical maps do not provide a necessary and sufficient condition [18]. Moreover,

channel description fails to hold in the presence of initial system-environment correlation [19]. Recently, an operational approach to quantum non-Markovianity was developed using process matrices (also known as process tensors or quantum combs for the multi-time causally ordered scenario) [20–23] which surpass the above-mentioned limitations for dynamical maps. The formalism uses the concept of causal-break interventions at each time step to check for conditional dependence of statistics of the future evolution of the system on previous time steps. A Markovian process has a specific structure—the process matrix is a tensor product of channels connecting the labs [20,24]. This characterization was recently used to divide the set of non-Markovian processes into two subsets: classical-memory processes and quantum-memory processes [24]. In the former subclass, the process can be simulated with a classical feedback mechanism, whereas in the latter, such simulation is not possible due to coherent correlation in noise across the time steps. This division of non-Markovian quantum processes is motivated by the fact that for classical correlated noise, it is easier to identify the source and mitigate it compared to coherently correlated noise [25–27]. The study of this division is of both fundamental and practical importance.

Recently, it was shown that entanglement across the relevant bipartition of a process matrix corresponds to the presence of coherently correlated noise [24]. However, this is a sufficient but not necessary criterion for certification of quantum memory [28]. In addition, the certification

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relies on knowing or having a good guess for the process matrix, which is obtained through a resource-expensive process matrix tomography [7,29]. In this work, we provide an alternative criterion for certification of quantum-memory environment that relies only on the joint statistics with minor assumptions on the dimensions of input-output Hilbert spaces and the sharpness of measurements performed at the intermediate time steps. It therefore provides a *semi-device-independent* certification of quantum-memory environment. This form of certification is possible without trusting the preparation and measurement devices in the labs, making it a stronger certification criterion as opposed to one that requires process tomography, where trust in instruments is essential. Device-independent certification techniques have been used widely—for example, in secure quantum key distribution [30,31], randomness certification [32,33], dimension witnessing [34,35], and communication complexity [36,37], which relies on the violation of some form of Bell’s inequality [38]. This enables a certification scheme from nonlocal quantum correlation without considering anything about the internal functioning of the preparation or measurement apparatus. Assuming an upper bound on the Hilbert space dimension, the certification schemes have been extended in a semi-device-independent manner to prepare-measure scenarios [39–48], where the quantum advantage has been shown to violate the preparation-noncontextual [48–50] or Kochen-Specker-noncontextual bound [51].

Here, we use a three-party sequential ($2 \rightarrow 1$) quantum random access code (QRAC) game [43,44,52,53] with rank-one projective (sharp) measurements at the intermediate party, to find a bound on the success probability of the third party which is satisfied by all Markovian and classical-memory processes. Due to the fact that valid process matrices obey linear constraints [54,55] and that the success probability of a QRAC game is a linear function of the process matrix, we cast our problem of finding processes that yield sequential advantage as a semi-definite program (SDP) that can be solved efficiently [56,57]. We show that there exist multi-time processes yielding a sequential advantage and that a higher success probability at the third party is associated with a larger quantum correlation across time steps. Therefore, the technique is a measure of the “quantumness” of the memory in the non-Markovian environment through the statistics of measurement results in a semi-device-independent manner. Moreover, we relax the sharp measurement constraint in the intermediate laboratory and perform a robustness analysis for the certification. In particular, we show that, assuming a lower bound on the unsharpness parameter, it is still possible to certify the presence of a quantum-memory environment. In addition, we find a region to unambiguously certify unsharp instruments at the intermediate laboratory. To our knowledge, this is the first approach that provides a semi-device-independent certification of a class of system-environment interactions, in particular, processes with coherently correlated noise across the time steps.

Our work is organized as follows: In Sec. II, we briefly review the process matrix formalism and the characterization of Markovian and non-Markovian processes (and their subclasses) within this formalism. In Sec. III, we review standard prepare-measure QRAC games in two-party and

sequential scenarios. In Sec. IV, we provide our results on semi-device-independent certification of quantum-memory environment. First, we provide a bound on the success probability in the sequential QRAC for Markovian (Sec. IV A) and classical-memory processes (Sec. IV B) and provide a feasibility problem to search for quantum-memory processes violating the bound (Sec. IV C). Then, we show the existence of processes yielding sequential advantage by using an SDP (Sec. IV D). We discuss the relationship between the amount of quantum memory and the sequential quantum advantage in Sec. IV E. We provide robustness analysis for certifying quantum memory in Sec. IV F. In Sec. IV G we improve the conditions required to unambiguously certify unsharp instruments and provide a tighter bound. In Sec. V we discuss how our results compare with prior works and discuss various extensions.

II. PRELIMINARIES

A. Process matrix formalism

The process matrix formalism is a natural way of studying multi-time processes [3,21,24,58,59]. It captures the most general type of quantum processes that involve the evolution of a system and intervening parties at each time step, which measure and transform the system, to analyze a quantum process. The process matrix gives us a prescription on acquiring the joint probability distribution of results of each party by a generalization of Born’s rule [60].

Each party is represented by a *quantum instrument* $\{\mathcal{M}_{i|x}^A\}$, where $\mathcal{M}_{i|x}^A$ is a completely positive (CP) trace nonincreasing map associated with each outcome i of the party and their sum is a completely positive trace-preserving (CPTP) map for all setting x , $\mathcal{M}_x^A = \sum_i \mathcal{M}_{i|x}^A$. The joint probability distribution is then given by the generalized Born’s rule,

$$P(i, j, k, \dots | x, y, z, \dots) = \text{Tr} [W^T (M_{i|x}^{A_1 A_2} \otimes M_{j|y}^{B_1 B_2} \otimes M_{k|z}^{C_1 C_2} \otimes \dots)], \quad (1)$$

where $M_{i|x}^{A_1 A_2}$ is the Choi-Jamiołkowski (CJ) matrix [61,62] for the map $\mathcal{M}_{i|x}^A$ and $W \in \mathcal{L}(\mathcal{H}^{A_1} \otimes \mathcal{H}^{A_2} \otimes \mathcal{H}^{B_1} \otimes \mathcal{H}^{B_2} \otimes \mathcal{H}^{C_1} \otimes \mathcal{H}^{C_2} \otimes \dots)$ is known as the *process matrix*. Here, $\mathcal{H}^{S(i)}$ represents the input (output) Hilbert space associated with the laboratory S . Note that we have chosen to associate a transpose with the W matrix as compared with some other formulations, e.g., Ref. [24]. The CJ matrix $M_{i|x}^{A_1 A_2} \in \mathcal{L}(\mathcal{H}^{A_1} \otimes \mathcal{H}^{A_2})$ is defined as

$$M_{i|x}^{A_1 A_2} = \mathcal{I} \otimes \mathcal{M}_{i|x}^A(|\mathbb{1}\rangle\langle\mathbb{1}|), \quad (2)$$

where \mathcal{I} is the identity map, $\mathcal{M}_{i|x}^A : \mathcal{L}(\mathcal{H}^{A_1}) \rightarrow \mathcal{L}(\mathcal{H}^{A_2})$, and $|\mathbb{1}\rangle = \sum_{i=1}^{d_{A_1}} |i\rangle|i\rangle \in \mathcal{H}^{A_1} \otimes \mathcal{H}^{A_1}$ is the un-normalized maximally entangled state with orthonormal basis $\{|i\rangle\} \in \mathcal{H}^{A_1}$ and $i = 1, 2, 3, \dots, d_{A_1}$.

The process matrix W captures the information of everything except the intervening parties in a quantum process and hence makes it a useful tool to study multi-time correlations in non-Markovian quantum processes.

B. Markovian and non-Markovian processes

While there are various characterizations of Markovian and non-Markovian quantum processes in the literature, we take an operational approach discussed in Ref. [24] using the process matrix W , which is the general representation of a multi-time quantum process.

1. Markovian processes

For a classical stochastic process, multi-time correlations for a random variable X_t at time t are given through a joint probability distribution $P(X_t, X_{t-1}, \dots, X_0)$, where X_i is the value of the random variable at that time step. A Markovian process then has the following property:

$$P(X_k|X_{k-1}, X_{k-2}, \dots, X_0) = P(X_k|X_{k-1}) \quad (3)$$

for all time steps. For the causal Markov condition to hold, it is required that the noise from system-environment interactions across the time steps remains uncorrelated. It has been shown that this holds if the process matrix has a factorizable structure as follows [20,22,24]:

$$W_M = \rho^{A^j} \otimes [\mathcal{T}_1]^{A^j_o A^j} \otimes \dots \otimes [\mathcal{T}_{n-1}]^{A^{j-1}_o A^j} \otimes \mathbb{1}^{A^j_o}, \quad (4)$$

where ρ^{A^j} is the state entering the laboratory A^j and $[\mathcal{T}_i]^{A^j_o A^j}$ is the CJ operator corresponding to a completely positive and trace-preserving map $\mathcal{T}_i : A^j_o \rightarrow A^j$. Therefore, the process matrix of a Markovian process is a product of the initial state and channels connecting the labs. It is evident from the structure of the process matrix that the initial state is uncorrelated with the environment and that the dynamics between each laboratory can be simulated using independent environments. Moreover, this structure yields the classical Markov condition in the relevant limit [20,21].

2. Non-Markovian processes

A non-Markovian process is one where the environment exhibits memory effects. In terms of the structure of the process matrices, these processes are the ones that cannot be written in the form of Eq. (4). Depending on the type of feedback, non-Markovian processes have been further classified into classical and quantum-memory processes [24].

Classical-memory processes are processes in which the environment can be simulated using a classical feedback mechanism that correlates the noise. In a general classical-memory process the environment can be thought to measure the system at each time step, say, between t_j and t_{j+1} , and acquire a classical outcome a_j . The evolution of the system after this time can depend on a_j but also on the information stored by the environment up to that point, noted by a classical variable x_j . The most general evolution of this type is represented by a CP map $\mathcal{T}_{a_j|x_j}^j : X_o^j \rightarrow Y_I^{j+1}$ from party X to Y , where the sum $\sum_{a_j} \mathcal{T}_{a_j|x_j}^j$ must be a CPTP map, i.e., each CP map has to be an instrument. In the CJ representation, an instrument satisfies the following:

$$[\mathcal{T}_{a_j|x_j}]^{A^j_o A^{j+1}} \geq 0, \quad \text{Tr}_{A^{j+1}} \sum_{a_j} [\mathcal{T}_{a_j|x_j}]^{A^j_o A^{j+1}} = \mathbb{1}^{A^j_o}. \quad (5)$$

Finally, the classical information stored in the environment is discarded and the process matrix of a classical-memory process is

$$W_{\text{CM}}^{A^1 \dots A^n} = \sum_{\vec{x}\vec{a}} \bigotimes_{j=0}^{n-1} [\mathcal{T}_{a_j|x_j}]^{A^j_o A^{j+1}} P(x_j|\vec{a}_{|j}, \vec{x}_{|j}), \quad (6)$$

where $\vec{a}_{|j} \equiv \{a_0, a_1, \dots, a_{j-1}\}$ and $\vec{x}_{|j} \equiv \{x_0, x_1, \dots, x_{j-1}\}$, and $P(x_j|\vec{a}_{|j}, \vec{x}_{|j})$ are the conditional probability of variable x_j . We also note that $\{[\mathcal{T}_{a_j|x_j}]^{A^j}\}_{a_0} \equiv \{\rho_{a_0|x_0}^{A^j}\}_{a_0}$, where $\{\rho_{a_0|x_0}^{A^j}\}$ are subnormalized states and $P(x_0|\vec{a}_{|0}, \vec{x}_{|0}) = P(x_0)$ is the marginal probability of initial variable x_0 . A process matrix has classical memory if and only if it can be written in this way. For a more detailed discussion on classical-memory processes we refer the reader to Ref. [24].

Quantum-memory processes are the remaining set of process matrices, i.e., $W_{\text{QM}} \in W/W_{\text{CM}}$. An explicit characterization of quantum-memory processes has proved elusive [63]. Owing to the fact that a process matrix can be seen as a multipartite state W , one can characterize a subset of quantum-memory processes by checking for entanglement on the state [24]. This method provides a powerful tool for checking for quantum memory in a non-Markovian process. However, entanglement of the process can either be verified through process tomography or through entanglement witnesses, which requires a good guess of the underlying process.

III. RANDOM ACCESS CODE

Random access codes (RACs) are a special class of communication complexity tasks where a sender has to encode a message in fewer bits than the original message and the receiver has to retrieve any subset of the message with the highest success rate. One can thus think about optimization over different strategies to find the maximum probability to retrieve the required message. The success in RAC games is considered mostly in two types of scenarios, namely, the average success probability and the worst-case success probability. From here on, we only talk about the average success probability whenever we mention success probability. It has been found that the quantum version of the game where the message is encoded in qubits instead of bits yields an advantage in the success probability to retrieve the message [64,65]. In a typical QRAC scenario, Alice prepares a set of quantum states depending on random input messages and sends them to Bob. Bob performs a measurement on received states depending on the random subset of the message to be retrieved. The challenge and interest in QRACs arise from optimizing the choice of quantum states and measurements to achieve specific objectives, such as minimizing the error rate or maximizing the amount of information transferred. There are several settings where one can study QRACs, such as preparation-measurement [64], preparation-transformation-measurement [44], entanglement assisted [66,67], with shared randomness [65], having parity-oblivious constraint [68], etc. Here, we are interested in a preparation-transformation-measurement QRAC game in a $(2 \rightarrow 1)$ setting, i.e., a two-bit message has to be encoded in a qubit.

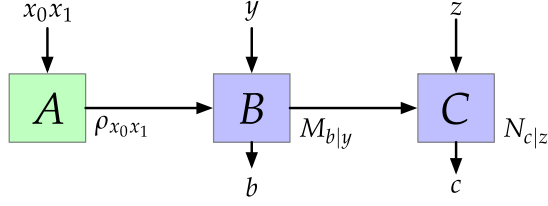


FIG. 1. A sequential ($2 \rightarrow 1$) quantum RAC game. Alice prepares one of four states depending on the bit string she receives $x \in \{00, 01, 10, 11\}$, and Bob and Charlie try to guess the random bit $y \in \{0, 1\}$ and $z \in \{0, 1\}$ in the string, respectively.

A. Two-party QRAC

Consider a two-party ($2 \rightarrow 1$) QRAC protocol. Alice randomly obtains a two-bit input message $x = x_0 x_1$, where $x_i \in \{0, 1\}$, prepares a state $\rho_{x_0 x_1}$ and sends it to Bob. Bob is given a random bit y , and his task is to guess the y th bit of Alice (see Fig. 1). They win the game if Bob's guess is right. In QRAC, Bob applies an instrument, $\{M_{b|y}\}$, on the received system depending on his input $y \in \{0, 1\}$ where $\sum_b M_{b|y} = 1 \forall y$ and b is a binary outcome corresponding to his guess. They win the game when $b = x_y$. Therefore, the average success probability is given by

$$p_{\text{succ}} = \frac{1}{8} \sum_{x,y} p(b = x_y | x, y), \quad (7)$$

where the normalization factor arises from inputs x_0, x_1 , and y being equally probable. Now, if Alice is constrained to send only a classical bit, the optimal success probability is $3/4$. This can be easily seen through the following strategy: Alice always encodes one of her input bits, say, the first one, and sends it to Bob. In this scenario, whenever $y = 0$, Bob can correctly guess Alice's first bit, and whenever $y = 1$, the guess will be random, i.e., $p(b = x_0 | y = 0) = 1$ and $p(b = x_1 | y = 1) = 1/2$, which leads to a success probability of $3/4$. Interestingly, if we allow Alice to encode her input in a qubit, there exist state preparations for Alice and measurements for Bob which can beat the classical optimal success probability.

If Alice encodes her message in a quantum state ρ_x , the success probability is given by

$$p_{\text{succ}} = \frac{1}{8} \sum_{x,y} \text{Tr}[\rho_x M_{x_y|y}]. \quad (8)$$

If we assume that Alice uses the following state assignments for the bit string:

$$\rho_{x_0 x_1} = \frac{1}{2} \left\{ \mathbb{1} + \frac{1}{\sqrt{2}} [(-1)^{x_0} \sigma_z + (-1)^{x_1} \sigma_x] \right\}, \quad (9)$$

and that Bob performs the following projective rank-one measurements:

$$M_{b|y} = \frac{1}{2} \left(\mathbb{1} + \frac{(-1)^b}{2} \{ [1 - (-1)^y] \sigma_x + [1 + (-1)^y] \sigma_z \} \right), \quad (10)$$

the optimal success probability is $p_{\text{succ}} = (2 + \sqrt{2})/4 > 3/4$. This optimal success probability in prepare-measure QRAC self-tests that the prepared states form a square on a great

circle of the Bloch sphere and the measurement operators are projectors on the diagonals [42].

B. Sequential QRAC

In a sequential QRAC game, after performing the desired instrument on the state, Bob relays the postmeasurement state to another party, say, Charlie, who has to decode another independent subset of the message (see Fig. 1). Alice prepares the state $\rho_{x_0 x_1}$ and sends it to Bob, who, depending on a random bit y , applies the instrument $\mathcal{M}_y \equiv \{M_{b|y}\}$. The postmeasurement state $\tilde{\rho}_{x_0 x_1}$ is

$$\tilde{\rho}_{x_0 x_1} = \frac{1}{2} [\mathcal{M}_0(\rho_{x_0 x_1}) + \mathcal{M}_1(\rho_{x_0 x_1})], \quad (11)$$

which is relayed to Charlie, where $\mathcal{M}_y(\rho) = \sum_b M_{b|y} \rho M_{b|y}^\dagger$ and $y = 0, 1$. The factor $1/2$ above is due to y being random (i.e., instrument \mathcal{M}_y is applied with equal probability). Now, depending on a random bit z , Charlie performs an instrument $\mathcal{N}_z \equiv \{N_{c|z}\}$. We are interested in the case where both Bob and Charlie beat the classical optimal success probability (which is $\frac{3}{4}$ irrespective of the number of independent sequential observers). In sequential QRAC, there is a trade-off between the extracted information and measurement-induced disturbance in the postmeasurement state at the laboratory B. If Bob performs the projective measurement as in Eq. (10) on Alice's preparation given in Eq. (9), and the resulting postmeasurement states are sent to C, in the noiseless scenario the optimal success probability for Charlie is $(4 + \sqrt{2})/8$.

In the process matrix formalism, the statistics obtained from three parties A, B, and C are given by

$$p(a, b, c | x, y, z) = \text{Tr} [W^T (M_{a|x}^{A_1 A_2} \otimes M_{b|y}^{B_1 B_2} \otimes M_{c|z}^{C_1 C_2})], \quad (12)$$

where W is the process matrix, and terms of the form $M^{P_1 P_2}$ are the CJ matrices of the operations performed by party P , with P_1 and P_2 representing the input and output Hilbert spaces, respectively. We note that the choice of transpose on W is purely aesthetic. In this work, we have considered the conventional definition of a CJ operator as opposed to one with an overall transpose, like most of the literature. Therefore, this choice is to ensure a simple form of the process matrix as a tensor product of a state and channels without any transpose, as in Eq. (4). Party A performs a deterministic state preparation conditional on the two-bit setting she receives, $x = x_0 x_1$. The CJ matrix for the operation in her laboratory is

$$M_x^{A_1 A_2} = \mathbb{1}^{A_1} \otimes \rho_x^{A_2}, \quad (13)$$

which is equivalent to discarding the state on A_1 and preparing a state deterministically on A_2 , which is sent to B. With this specialisation, Eq. (12) becomes

$$p(b, c | x, y, z) = \text{Tr} [W^T (\mathbb{1}^{A_1} \otimes \rho_x^{A_2} \otimes M_{b|y}^{B_1 B_2} \otimes M_{c|z}^{C_1 C_2})]. \quad (14)$$

Party B performs a measurement and sends the postmeasurement state to C. If the measurement operator corresponding to

the setting y and outcome b is $E_{b|y}$, the CJ matrix of B is

$$M_{b|y}^{B_1 B_2} = |E_{b|y}\rangle\langle E_{b|y}|, \quad (15)$$

where $|E_{b|y}\rangle = \mathbb{1} \otimes E_{b|y}|\mathbb{1}\rangle$ is the vectorized form of operator $E_{b|y}$, where $|\mathbb{1}\rangle$ is the unnormalized maximally entangled state [see Eq. (2)]. Finally, the party C performs a measurement and discards the state. If the Positive Operator Valued Measure (POVM) element corresponding to the setting z and outcome c is given by $E_{c|z}$, then the CJ matrix for party C is given by

$$M_{c|z}^{C_1 C_2} = (E_{c|z})^T \otimes \frac{\mathbb{1}}{2}. \quad (16)$$

With these simplifications, we can define a reduced W_r matrix, where

$$p(b, c|x, y, z) = \text{Tr} \left\{ W_r^T \left[\rho_x^{A_2} \otimes |E_{b|y}\rangle\langle E_{b|y}|^{B_1 B_2} \otimes (E_{c|z})^T \right] \right\}, \quad (17)$$

and

$$W_r = \frac{1}{2} \text{Tr}_{A_1 C_2} [W]. \quad (18)$$

We note that for parties B and C the operations become CPTP when they are summed over the measurement outcomes. To obtain the success probability in the QRAC games played by B and C , we find the reduced statistics from the joint probability distribution $p(b, c, x, y, z)$,

$$p_{\text{succ}}^B = \sum_{xyz} p(b = x_y, c, x, y, z) = \frac{1}{8} \sum_{x,y} p(b = x_y | x, y),$$

$$p_{\text{succ}}^C = \sum_{xyz} p(b, c = x_z, x, y, z) = \frac{1}{16} \sum_{x,y,z} p(c = x_z | x, y, z). \quad (19)$$

IV. RESULTS: SEMI-DEVICE-INDEPENDENT CERTIFICATION OF QUANTUM MEMORY

A. Sequential QRAC with Markovian processes

1. Noiseless Markovian case

In a Markovian process, either there is no noise or the noise across time steps is uncorrelated. In the process matrix formalism, such a process is written as a tensor product of channels connecting the labs [Eq. (4)]. Here, in a sequential QRAC game with projective measurements for B , we provide a bound on the success probability of C and show that the bound is saturated for a noiseless Markovian process.

A three-party noiseless Markovian process, i.e., with identity channels connecting the parties, is written as

$$W = \rho^{A_1} \otimes [\mathcal{I}]^{A_2 B_1} \otimes [\mathcal{I}]^{B_2 C_1} \otimes \mathbb{1}^{C_2}, \quad (20)$$

where $[\mathcal{I}]^{X_2 Y_1}$ is the CJ matrix of the identity channel connecting the output Hilbert space of party X to the input Hilbert space of party Y , and ρ^{A_1} is the input state of A (the choice of which does not affect the joint statistics we are interested in).

The linearity of the success probability in the QRAC game implies that the upper bound will be for pure state preparations of $\rho_{x_0 x_1}$. In this case, we show that if we perform a rank-one projective measurement corresponding to all the settings and outcomes of B and send the postmeasurement state to C , the success probability of C is upper bounded by $3/4$.

The proof is as follows. Consider a noiseless Markovian process as in Eq. (20) with maximally mixed state as the input state on A_1 . Consider A preparing an arbitrary state $\rho_{x_0 x_1}$ corresponding to the random bit $x_0 x_1$. The corresponding operation in the laboratory A will be $M_{x_0 x_1}^{A_1 A_2} = \mathbb{1} \otimes \rho_{x_0 x_1}$. In the laboratory B , a rank-one projective measurement is performed corresponding to each setting y , i.e., the operations in the laboratory B are of the form $M_{b|y}^{B_1 B_2} = P_{b|y} \otimes P_{b|y}$, where $P_{b|y}$ is a rank-one projector for all settings and outcomes b and y . Finally, C performs a measurement and discards the state, which corresponds to the operation $M_{c|z}^{C_1 C_2} = E_{c|z} \otimes \mathbb{1}/2$. With these operations and the process matrix in Eq. (20), we obtain the following probabilities:

$$p(b, c|x_0 x_1, y, z) = \text{Tr} \left[W^T (M_{x_0 x_1}^{A_1 A_2} \otimes M_{b|y}^{B_1 B_2} \otimes M_{c|z}^{C_1 C_2}) \right] \\ = \text{Tr} \left[E_{c|z} (P_{b|y} \rho_{x_0 x_1} P_{b|y}) \right]. \quad (21)$$

Now, the success probability of C becomes

$$p_{\text{succ}}^C = \frac{1}{8} \sum_{x_0 x_1, z} \text{Tr} [\tilde{\rho}_{x_0 x_1} E_{c|z}], \quad (22)$$

where $\tilde{\rho}_{x_0 x_1} = \sum_{b,y} P_{b|y} \rho_{x_0 x_1} P_{b|y} / 2 \equiv C_B(\rho_{x_0 x_1})$. Since $C_B(\cdot)$ and trace are both linear operations, by using the spectral decomposition of the incoming states $\rho_{x_0 x_1}$, it is straightforward to see that the success probability will be upper bounded by the cases when B receives pure states. Therefore, we can restrict $\rho_{x_0 x_1}$ to be pure states to search for a bound on success probability at C . Using a similar argument, for the optimal success probability, POVM elements of C must be rank-one projectors. Therefore, we only need to consider the scenario where B receives pure states, performs rank-one projective measurements, and sends the postmeasurement state to C , who then performs rank-one projective measurements in his laboratory. We have

$$\rho_{x_0 x_1} = |\psi_{x_0 x_1}^A\rangle\langle\psi_{x_0 x_1}^A|, \\ P_{b|y} = |\psi_{b|y}^B\rangle\langle\psi_{b|y}^B|, \\ E_{c|z} = |\psi_{c|z}^C\rangle\langle\psi_{c|z}^C|. \quad (23)$$

The action of B results in the following postmeasurement states:

$$\tilde{\rho}_{x_0 x_1} = \frac{1}{2} \sum_{b,y} p_{b|y}^{x_0 x_1} |\psi_{b|y}^B\rangle\langle\psi_{b|y}^B|, \quad (24)$$

where $p_{b|y}^{x_0 x_1} = \text{Tr}[\rho_{x_0 x_1} P_{b|y}]$ and forms a probability distribution for the settings x_0, x_1 , and y . Using Eq. (24) in Eq. (22), and after some simplification, we obtain

$$p_{\text{succ}}^C = \frac{1}{16} \left[8 + \sum_{b,y} F_{b|y}^{(0)} |\langle\psi_{b|y}^B|\psi_{0|0}^C\rangle|^2 + F_{b|y}^{(1)} |\langle\psi_{b|y}^B|\psi_{0|1}^C\rangle|^2 \right], \quad (25)$$

where

$$F_{b|y}^{(0)} = p_{b|y}^{00} + p_{b|y}^{01} - p_{b|y}^{10} - p_{b|y}^{11} \\ F_{b|y}^{(1)} = p_{b|y}^{00} - p_{b|y}^{01} + p_{b|y}^{10} - p_{b|y}^{11}. \quad (26)$$

Using the normalization condition $\sum_b |\langle \psi_{b|y}^B | \psi_{0|0}^C \rangle|^2 = 1$, we obtain the following:

$$\begin{aligned} & \sum_{b,y} F_{b|y}^{(0)} |\langle \psi_{b|y}^B | \psi_{0|0}^C \rangle|^2 + F_{b|y}^{(1)} |\langle \psi_{b|y}^B | \psi_{0|1}^C \rangle|^2 \\ & \leq \max\{F_{0|0}^{(0)}, F_{1|0}^{(0)}\} + \max\{F_{0|0}^{(1)}, F_{1|0}^{(0)}\} \\ & \quad + \max\{F_{0|1}^{(0)}, F_{1|1}^{(0)}\} + \max\{F_{0|1}^{(1)}, F_{1|1}^{(0)}\}. \end{aligned} \quad (27)$$

Now, consider the terms of following form:

$$\begin{aligned} F_{b|y}^{(0)} + F_{b|y}^{(1)} &= (p_{b|y}^{00} + p_{b|y}^{00}) + (p_{b|y}^{01} - p_{b|y}^{01}) \\ & \quad + (p_{b|y}^{10} - p_{b|y}^{10}) - (p_{b|y}^{11} + p_{b|y}^{11}), \end{aligned} \quad (28)$$

which has a maximum value of 2. Therefore, we obtain the following bound on the success probability at C :

$$p_{\text{succ}}^C \leq \frac{3}{4}. \quad (29)$$

To summarize, we have shown that for arbitrary preparation of A , if B performs any rank-one projective measurement, C cannot have a quantum advantage in the noiseless scenario. Next, we show that the same is true for *all Markovian processes*.

2. General Markovian case

Consider the state received by C as $\tilde{\rho}_{x_0, x_1}$, and the operation performed by C corresponding to the setting z and outcome c as $E_{c|z}$. Then, the optimal success probability can be calculated by the following SDP:

$$\begin{aligned} \max p_{\text{succ}}^C &= \frac{1}{8} \sum_{x,z} p(c = x_z | x, z) \\ \text{s.t. } \sum_i E_{i|z} &= \mathbb{1}, \quad z = 0, 1 \\ E_{i|z} &\geq 0 \quad \forall i, z, \end{aligned} \quad (30)$$

where $p(c|x, z) = \text{Tr}(\tilde{\rho}_{x_0, x_1} E_{c|z})$. We can reduce the number of constraints in the above SDP by redefining the POVMs as

$$E_{0|z} = \frac{\mathbb{1} + M_z}{2}, \quad E_{1|z} = \frac{\mathbb{1} - M_z}{2}, \quad (31)$$

which along with the positivity constraints $E_{i|z} \geq 0$ can be combined into the following constraints:

$$-\mathbb{1} \leq M_z \leq \mathbb{1} \quad z = 0, 1. \quad (32)$$

Further, writing the objective function in terms of variables M_z now provides the following SDP:

$$\max p_{\text{succ}}^C \text{ s.t. } -\mathbb{1} \leq M_z \leq \mathbb{1} \quad z = 0, 1, \quad (33)$$

where the objective function explicitly is

$$\begin{aligned} p_{\text{succ}}^C &= \frac{1}{2} + \frac{1}{16} \text{Tr}[(\tilde{\rho}_{00} + \tilde{\rho}_{01} - \tilde{\rho}_{11} - \tilde{\rho}_{10})M_0] \\ & \quad + \frac{1}{16} \text{Tr}[(\tilde{\rho}_{00} + \tilde{\rho}_{10} - \tilde{\rho}_{11} - \tilde{\rho}_{01})M_1]. \end{aligned} \quad (34)$$

Noticing that the trace norm (also known as l_1 norm) of an operator A , defined as $\|A\| = \text{Tr}[\sqrt{A^\dagger A}]$, has the following SDP representation [57]:

$$\begin{aligned} \|A\|_1 &= \max \text{Tr}(AX) \\ \text{s.t. } -\mathbb{1} &\leq A \leq \mathbb{1}, \end{aligned} \quad (35)$$

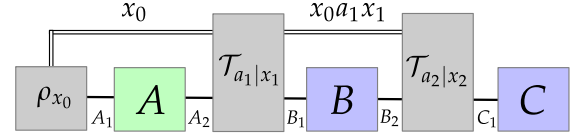


FIG. 2. A general classical-memory process explicitly showing the dependence on the classical variables. The double lines represent a channel for classical information.

we obtain the optimal success probability at C in terms of l_1 norm as

$$\begin{aligned} p_{\text{succ}}^C &= \frac{1}{2} + \frac{1}{8} \left\| \frac{\tilde{\rho}_{00} + \tilde{\rho}_{01}}{2} - \frac{\tilde{\rho}_{11} + \tilde{\rho}_{10}}{2} \right\|_1 \\ & \quad + \frac{1}{8} \left\| \frac{\tilde{\rho}_{00} + \tilde{\rho}_{10}}{2} - \frac{\tilde{\rho}_{11} + \tilde{\rho}_{01}}{2} \right\|_1, \end{aligned} \quad (36)$$

where $\|\cdot\|_1$ is the l_1 norm. Therefore, p_{succ}^C can be interpreted in terms of distance between states. It is well known that under a quantum channel (CPTP map), the l_1 norm distance is nonincreasing, i.e., for a channel $\mathcal{C}(\cdot)$ and states ρ_1 and ρ_2 , we always have $\|\mathcal{C}(\rho_1) - \mathcal{C}(\rho_2)\| \leq \|\rho_1 - \rho_2\|$. Therefore, in the presence of a nontrivial channel from B to C , both the distance terms in (36) can only decrease, leading to a reduced success probability in C . Analogously, a nontrivial channel from A to B , i.e., introducing decoherence in the prepared state before it reaches party B , does not improve the bound on the success probability of C . This follows directly from our previous arguments where we showed that for arbitrary preparation at A , as long as B performs sharp measurement, $p_{\text{succ}}^C \leq 3/4$.

Therefore, we observe that for any process of the form

$$W = \rho^{A_1} \otimes [\mathcal{T}_1]^{A_2 B_1} \otimes [\mathcal{T}_2]^{B_2 C_1} \otimes \mathbb{1}^{C_2}, \quad (37)$$

where ρ is a state and $[\mathcal{T}_i]$ are CJ matrices of the channels, i.e., for all Markovian processes, the optimal success probability for party C is upper bounded by the optimal classical success probability, i.e., $p_{\text{succ}}^C \leq 3/4$.

B. Sequential QRAC with classical-memory processes

A general classical-memory process for three parties has the following form [starting with Eq. (6)]:

$$\begin{aligned} W_{\text{CM}} &= \sum_{\vec{x}\vec{a}} p(x_0)p(x_1|x_0)p(x_2|a_1 x_0 x_1) \rho_{x_0}^{A_1} \\ & \quad \otimes [\mathcal{T}_{a_1|x_1}]^{A_2 B_1} \otimes [\mathcal{T}_{a_2|x_2}]^{B_2 C_1} \otimes \mathbb{1}^{C_2}, \end{aligned} \quad (38)$$

where $\vec{a} = (a_1, a_2)$, a_j is a classical outcome of the environment measuring the system between times, $\rho_{x_0}^{A_1}$ is a normalized state, and $[\mathcal{T}_{a_1|x_1}]^{A_2 B_1}$ is a CJ matrix of a CP trace nonincreasing map $\mathcal{T}_{a_1|x_1}$ (see Fig. 2). The relevant statistics depend only on the reduced process from Eq. (18), which leads to the simplified form

$$W_{\text{CM}} = \sum_{j,i} p(i) [\mathcal{T}_{j|i}]^{A_2 B_1} \otimes [\mathcal{C}_{ij}]^{B_2 C_1}, \quad (39)$$

where $\mathcal{T}_{j|i}$ is a CP map and when summed over j becomes a CPTP map, and \mathcal{C}_{ij} is a CPTP map $\forall i, j$. Now, operations

in the labs A , B , and C as in (13), (15), and (16) yield the following probabilities:

$$p(b, c|x, y, z) = \sum_{i,j} p(i) \text{Tr}(E_{c|z} C_{ij} \{ \mathcal{M}_{b|y} [\mathcal{T}_{j|i}(\rho_x)] \}), \quad (40)$$

where we have used the inverse map $\text{Tr}_I((\rho)^T \otimes \mathbb{1}^O[\mathcal{M}]^{I^O}) = \mathcal{M}(\rho)$, and $\mathcal{M}_{b|y}(\cdot) = P_{b|y}(\cdot)P_{b|y}$, such that $P_{b|y}$ is a rank-one projective operator. Note that $\mathcal{T}_{j|i}(\rho_x) = p(j|i)\rho_x^{ij}$, where $p(j|i) = \text{Tr}[\mathcal{T}_{j|i}(\rho_x)]$ and ρ_x^{ij} is a normalized state. Now, to find the bound on success probability of C , we need $p(c|x, z)$, which is obtained to be

$$\begin{aligned} p(c|x, z) &= \sum_{i,j} p(ij) \text{Tr} \{ E_{c|z} C_{ij} [\mathcal{M}^P(\rho_x^{ij})] \} \\ &= \sum_{ij} p(ij) p_{ij}^P(c|x, z), \end{aligned} \quad (41)$$

where the superscript on \mathcal{M}^P represents projective measurements performed in the laboratory B , and $p_{ij}^P(c|x, z)$ represents the reduced statistics obtained when preparation ρ_x^{ij} reaches laboratory B and statistics at C given projective measurements at B , i.e., for each i, j , it represents a statistics obtained from Markovian process. Now, p_{succ}^C is a linear function of statistics, therefore, the resulting success probability for a general classical-memory process will be a convex sum of success probability obtained through some Markovian processes with projective measurement at B , i.e.,

$$p_{\text{succ}}^C(W_{\text{CM}}) = \sum_{ij} p(ij) f_L[p_{ij}^P(c|x, z)] \leq 3/4, \quad (42)$$

where we used the result from the previous section that each term $f_L[p_{ij}^P(c|x, z)]$ is upper bounded by $3/4$, where f_L represents a linear function on $p(c|x, z)$ yielding p_{succ}^C .

Therefore, we observe that a classical-memory process does not improve the bound on the success probability of C , given projective measurements at B : there is no sequential advantage, even with unconstrained success probability at B . Hence, a violation of this bound would semi-device-independently certify *quantum memory*. We show that this is possible in the following sections.

C. Feasibility problem for sequential advantage and quantum memory

We now have a feasibility problem at hand—whether there exist processes yielding sequential quantum advantage. It is a search over all possible environments and system-environment interactions. If we approach this problem traditionally, the possibility of an arbitrary dimensional environment and arbitrary interactions in different time steps makes this feasibility problem highly nonlinear and difficult to solve, even numerically. For the scenario depicted in Fig. 3(a), the statistics at B are

$$p(b|x, y) = \text{Tr}[U_1(\rho_x \otimes \sigma)U_1^\dagger(P_{b|y} \otimes \mathbb{1})], \quad (43)$$

where σ is the initial environment state, U_1 is a joint system-environment unitary, and $P_{b|y}$ is the generalized measurement operator of B , which we assume to be rank-one projectors. The reduced statistics for C in terms of the system-environment

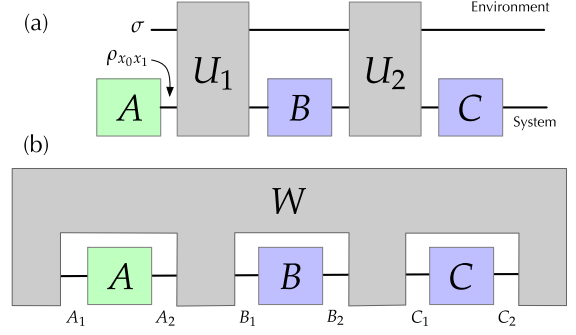


FIG. 3. (a) An environment-system model for the sequential QRAC game. (b) The process matrix as a “comb” surrounding the labs.

unitaries are

$$p(c|x, z) = \text{Tr}[(N_{c|z} \otimes \mathbb{1})(U_2 \tilde{\rho}_x U_2^\dagger)], \quad (44)$$

with

$$\tilde{\rho}_x = \frac{1}{2} \sum_{b,y} (P_{b|y} \otimes \mathbb{1}) U_1(\rho_x \otimes \sigma) U_1^\dagger (P_{b|y}^\dagger \otimes \mathbb{1}), \quad (45)$$

where U_2 is the joint system-environment unitary acting between B and C , and $N_{c|z}$ is the POVM corresponding to setting z and outcome c of party C . Hence, the feasibility problem is a search over all states σ and joint unitaries U_1 and U_2 in $SU(d_S \times d_E)$ such that both $p_{\text{succ}}^B \geq 3/4$ and $p_{\text{succ}}^C \geq 3/4$. The optimization problem is

$$\begin{aligned} \max \quad & p_{\text{succ}}^C \\ \text{s.t.} \quad & p_{\text{succ}}^B > 0.75 \\ & U_1, U_2 \in SU(d_S \times d_E) \\ & \sigma \in \mathcal{L}(H^E), \quad \text{Tr}(\sigma) = 1 \\ & N_{c|z} \geq 0, \quad \sum_c N_{c|z} = \mathbb{1}, \quad \forall z \\ & P_{b|y} \in \text{Herm}(\mathcal{H}^B), \quad \sum_b P_{b|y} = \mathbb{1}, \quad \forall y \\ & P_{b|y}^2 = P_{b|y} \quad \forall b, y, \end{aligned} \quad (46)$$

where $d_{S(E)}$ is the dimension of the system (environment). Note that even though the system dimension is fixed, the environment can have arbitrary dimensions. As is evident, the optimization problem in the above form is a highly nonlinear problem and difficult to solve. In contrast, process matrices provide a way to approach the problem in terms of linear constraints, hence allowing for an SDP formulation that can be solved efficiently, as we discuss next.

D. Process matrix approach to feasibility problem

In this section, we reformulate the feasibility problem with process matrices as an SDP that can be solved efficiently [69]. As a process matrix implicitly includes all system-environment interactions, a search for possible environments and interactions is equivalent to the search over the set of valid processes [Fig. 3(b)]. In addition, a process matrix is a linear

operator on the tensor product of input and output Hilbert spaces, therefore, the search is over a finite-dimensional space; 64 dimensional in our case, which reduces to 16 after discarding A_1 and C_2 . A valid process $W \in A_1 \otimes A_2 \otimes B_1 \otimes B_2 \otimes C_1 \otimes C_2$ satisfies the following constraints:

$$\begin{aligned} W &\geq 0, \\ \text{Tr}(W) &= d_{A_2} d_{B_2} d_{C_2}, \\ L_V(W) &= W, \end{aligned} \quad (47)$$

where

$$L_V(W) = \left(I - \prod_{X \in \{A, B, C\}} (I - X_2 + X_{12}) + A_{12} B_{12} C_{12} \right) W, \quad (48)$$

where $(X)W = \text{Tr}_X(W) \otimes \mathbb{1}^X / d_X$. To restrict the search in a subspace with a specific causal order, namely, $A \rightarrow B \rightarrow C$, we have the additional constraint

$$L_{ABC}(W) = W, \quad (49)$$

$$\begin{aligned} L_{ABC}(W) &= [I - (I - A_2) B_1 B_2 C_1] \\ &\quad \times [I - (I - B_2) C_1] C_2] W. \end{aligned} \quad (50)$$

Note that these linear constraints on W allow for the feasibility problem to be written as an SDP. To search for processes yielding sequential quantum advantage, we first fix the operations of the parties, so the objective function and constraints are linear with the only variable W ,

$$\begin{aligned} M_x^{A_1 A_2} &= \mathbb{1}^{A_1} \otimes \rho_x^{A_2} \\ M_{b|y}^{B_1 B_2} &= P_{b|y}^{B_1} \otimes P_{b|y}^{B_2} \\ M_{c|z}^{C_1 C_2} &= E_{c|z}^{C_1} \otimes \mathbb{1}^{C_2} / 2, \end{aligned} \quad (51)$$

with $M_x^{A_1 A_2}$, $\sum_b M_{b|y}^{B_1 B_2}$, and $\sum_c M_{c|z}^{C_1 C_2}$ are CJ matrices corresponding to CPTP maps for all settings x , y , and z . Here, we have considered preparation as in Eq. (9) and the projective measurement in the labs B and C as in Eq. (10). The objective function is the success probability of C , as our feasibility problem is concerned with obtaining successive quantum advantages. In addition, we can set a minimal bound p_0 on the success probability of B through a linear constraint. The optimization problem as an SDP is

$$\begin{aligned} \max \quad & p_{\text{succ}}^C(W) \\ \text{s.t.} \quad & W \geq 0 \\ & \text{Tr}(W) = 8 \\ & L_V(W) = W \\ & L_{ABC}(W) = W \\ & p_{\text{succ}}^B \geq p_0. \end{aligned} \quad (52)$$

The existence of a process with sequential quantum advantage is guaranteed when the optimal value is greater than $3/4$, for $p_0 \in [3/4, (2 + \sqrt{2})/4]$. Moreover, since the operations by B are projective measurements, the feasible process must have *quantum-memory environment*. We used both MATLAB and JULIA to run the above SDP and found that the feasibility problem is feasible, i.e., there exist processes that yield sequential

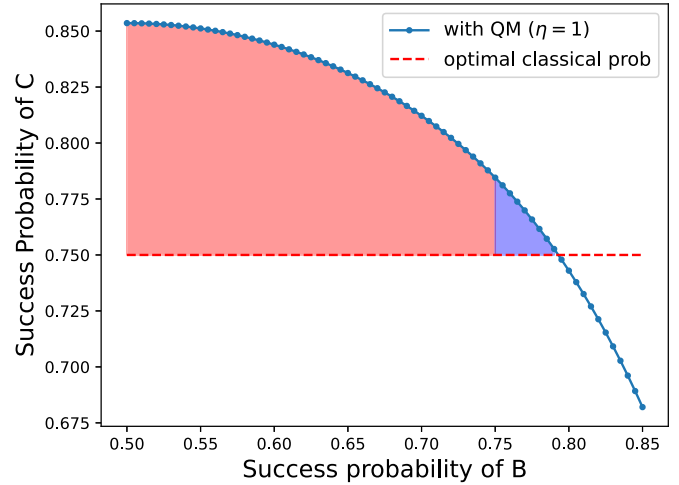


FIG. 4. Optimal success probability of C for various success probabilities of B with sharp measurement ($\eta = 1$) at B . The shaded regions are where one can certify quantum memory (QM). The blue-shaded region represents processes yielding sequential advantage, and in this region, QM can be robustly certified.

quantum advantage. Note that when implementing the SDP and working with the reduced process matrix W_r , it is only necessary to enforce the condition $L_{BC}W = W$, where $L_{BC} = I - C_1 + B_2 C_1$, as by construction W_r has C_2 last and A_1 first. Furthermore, it is important to note that we are optimizing $p_{\text{succ}}^C(W)$ over all valid process matrices W , which includes all possible local deterministic transformations. This means the process matrix automatically includes a unitary rotation after state preparation as well as before the measurement. Hence, all we need to ensure in the beginning is that the measurements are rank-one projectors, as well as maximally incompatible [as in Eq. (51)], and regardless of the initial choice, the search over process matrix will check for various orientations. The same argument holds for the state preparation; all we need to keep track of is that the four prepared states form a square on a great circle of the Bloch sphere, and the search over processes will cover the optimal case. We have checked this numerically and the choice of initial preparation and measurements does not affect the optimal value as long as the above-mentioned conditions hold.

In Fig. 4, we show the optimal value of p_{succ}^C for various lower bounds p_0 of p_{succ}^B with projective measurement at the intermediate party. To compare, for the analogous noiseless, Markovian or classical-memory scenarios, the success probability at C is upper bounded by $3/4$. Hence, $p_{\text{succ}}^C > 3/4$ allows us to validate the presence of a coherently correlated environment. If the instrument at B is completely trusted, namely, if B deterministically performs any arbitrary projective measurement at each round of the game, a quantum advantage at C would certify quantum memory. The pink region in the plot is where $p_{\text{succ}}^B \in [1/2, 3/4]$ and p_{succ}^C gives quantum advantage. However, complete trust in the instrument makes the certification device dependent and unreliable. In particular, there is no robustness bound on the trustability of the instrument in this region, as this joint probability can be simulated in different ways (and consequently loses its practicality). For example, B performs some suboptimal

classical strategy without disturbing the system at all, and C achieves quantum advantage without the presence of any kind of environment. Alternatively, B can perform projective measurements in some rounds, but not all, and C can achieve overall quantum advantage, which can also result in such joint probabilities without the need for an environment. Therefore, to semi-device-independently certify quantum memory with *minimum trust* on the instrument, successive quantum advantages for both parties are necessary.

Both parties getting quantum advantage enforces B to perform some nonclassical strategy, which leaves open only two possibilities that can exhibit sequential success—the unsharpness of the instrument or quantum memory in the environment. This permits a practically feasible scenario where, assuming a minimum trust on the instrument (lower bound on the unsharpness) of B , we can robustly certify the process having quantum memory.

In Fig. 4, the blue-shaded region corresponds to a sequential quantum advantage with a sharp measurement at B and thus certifies quantum memory with only assumptions that Alice encoded the message in qubits, and Bob performed any arbitrary projective measurement. Having a robustness bound on the assumptions about the type of measurement at B makes the certification testable in experiments, as we show in Sec. IV F. Therefore, from the joint measurement statistics, we are able to certify the presence of a quantum memory in the environment. The p_{succ}^C is monotonically decreasing with p_{succ}^B , exhibiting the trade-off relation between the two. As the success probability of B increases, there is a decrease in the information loss to the environment in the first time step (since the operation in the laboratory B is fixed). This leads to a smaller temporal correlation across time steps and consequently lower contribution of the feedback to make up for the decoherence introduced by the projective measurements at B , finally leading to a lower optimal success probability at C .

E. Entanglement in processes and sequential advantage

We have shown that with projective measurement at B , a process with quantum memory is the only resource allowing sequential quantum advantage. Importantly, classical-memory processes, where the correlation across time steps is classical, fails to yield sequential advantage, and thus coherent correlations (nonclassical) across time steps is necessary. We find that correlations across time steps increase monotonically with the optimal quantum advantage at C .

In Fig. 5, we plot a measure of correlations across the time steps with various success probability of C for fixed success probability of B . As a measure of correlations we use entanglement of the bipartite state ρ^{XY} , where $X = A_2B_1$ and $Y = B_2C_1$, as quantified by *negativity*, which is an entanglement monotone, and is quantified by the sum of magnitude of negative eigenvalues of the partially transposed state $(\rho^{XY})^{Tx}$ [70]. To obtain the process corresponding to the success probability p_{succ}^C for a given p_{succ}^B , we used the SDP in Eq. (52) with an additional linear constraint $p_{\text{succ}}^C \leq a$, where a is the success probability required for C . The SDP provides a feasible process for the given $(p_{\text{succ}}^B, p_{\text{succ}}^C)$, in which we investigate the bipartite correlation as described above. We observe that the amount of entanglement manifests in higher optimal

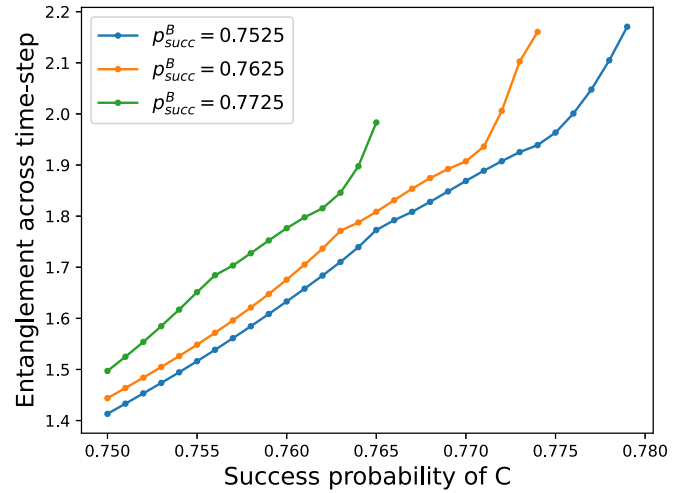


FIG. 5. An increase in the quantum advantage at C corresponds to higher coherent correlation across the time steps. Here, entanglement is quantified across bipartition $A_2B_1|B_2C_1$.

sequential success probability. We emphasize that the entanglement detection in process requires process tomography, which assumes full trust in the measurement, in addition to requiring informationally complete measurements. However, in our approach, the violation of the bound on success probability of C certifies the presence of quantum memory without trusting the preparation and measurements in the labs. Therefore, it is a stronger form of quantum-memory certification.

F. Robustness analysis: Quantum-memory certification with unsharp measurements at laboratory B

No measurement is perfect in a realistic experimental setup. Therefore, the assumption that Bob's instrument performs arbitrary projective measurements is quite restrictive. Since the optimal success probability for C , with projective measurements at B , is upper bounded by the optimal *classical* success probability (plot in green in Fig. 6), a very small sequential quantum advantage would be insufficient to faithfully certify quantum memory. This is because a sequential advantage could be obtained if B performs *unsharp* measurements, causing less disturbance to the measured state, which leaves more extractable information at C . Hence, we introduce an unsharpness parameter η on the measurement operator for B [44,53],

$$M_{b|y} = \sqrt{\frac{1 + (-1)^b \eta}{2}} P_{0|y} + \sqrt{\frac{1 + (-1)^{b+1} \eta}{2}} P_{1|y}, \quad (53)$$

where $\eta \in [0, 1]$, and $P_{b|y}$ are arbitrary rank-one projective measurements. The measurement operator reduces to the identity (noninteractive) measurement for $\eta = 0$ and to sharp measurements for $\eta = 1$. We can now calculate p_{succ}^C in the case of unsharp measurements.

Consider arbitrary prepared states from laboratory A , $\rho_{x_0 x_1}$ entering the laboratory B . Considering the unsharp measurements as in Eq. (53), the postmeasurement states from

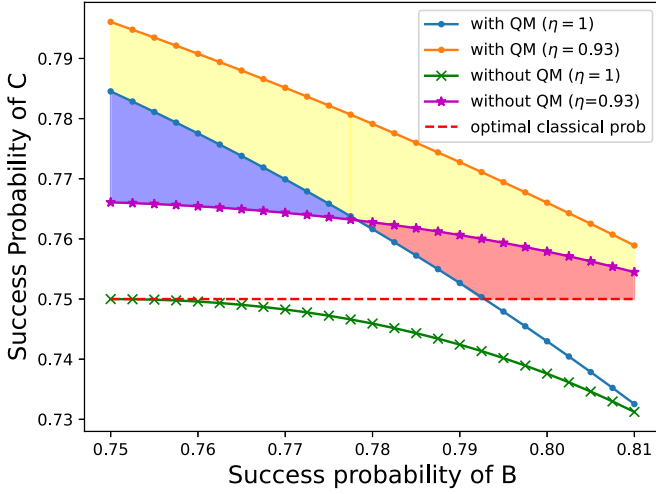


FIG. 6. The blue and green plots represent the upper bound on p_{succ}^C for sharp measurement ($\eta = 1$) with and without environment, respectively. The orange and purple plots represent the upper bound on p_{succ}^C for unsharp measurement (lower bound $\eta = 0.93$) with and without environment, respectively.

laboratory B , i.e., $\tilde{\rho}_{x_0x_1}$ will be

$$\tilde{\rho}_{x_0x_1} = \rho_{x_0x_1}^S + \frac{\sqrt{1-\eta^2}}{2} \mathcal{L}(\rho_{x_0x_1}), \quad (54)$$

where

$$\rho_{x_0x_1}^S = \frac{1}{2} \sum_{b,y} P_{b|y} \rho_{x_0x_1} P_{b|y} \quad (55)$$

is the postmeasured state when sharp measurements $P_{b|y}$ are performed and

$$\begin{aligned} \mathcal{L}(\rho_{x_0x_1}) = & P_{0|0} \rho_{x_0x_1} P_{1|0} + P_{1|0} \rho_{x_0x_1} P_{0|0} \\ & + P_{0|1} \rho_{x_0x_1} P_{1|1} + P_{1|1} \rho_{x_0x_1} P_{0|1} \end{aligned} \quad (56)$$

is the correction term containing the cross terms.

Using these postmeasurement states $\tilde{\rho}_{x_0x_1}$ in Eq. (30) to find the success probability of C , we obtain

$$p_{\text{succ}}^C = p_C^S + \frac{\sqrt{1-\eta^2}}{16} \sum_{x_0, x_1, z} \text{Tr} [E_{x_z|z} \mathcal{L}(\rho_{x_0x_1})], \quad (57)$$

where $E_{c|z}$ are the measurement operators in the laboratory C and p_C^S is the success probability at laboratory C if sharp measurements were applied at laboratory B ($\eta = 1$).

A slight rearrangement of $\mathcal{L}(\rho_{x_0x_1})$ gives

$$\mathcal{L}(\rho_{x_0x_1}) = 2(\rho_{x_0x_1} - \rho_{x_0x_1}^S), \quad (58)$$

which upon substitution in Eq. (57) yields

$$p_{\text{succ}}^C = \sqrt{1-\eta^2} p_C^I + (1 - \sqrt{1-\eta^2}) p_C^S, \quad (59)$$

where p_C^I is the success probability of C if the prepared state directly reached C without any disturbance (or B performed identity measurements). In the above form, success probability is a convex sum of success probabilities at C corresponding to the two extreme choices of B , namely, performing sharp measurements or performing trivial (identity) measurement. The probabilities p_C^I and p_C^S do not maximize for the same

choices of measurements at C , therefore, there will be a non-trivial trade-off which will be dependent on the unsharpness parameter η . It is straightforward to observe that for a given η and measurements at B and C , p_{succ}^C optimizes for the optimal preparations [for example, Eq. (9)]. We numerically optimize Eq. (59) over measurements at B and C to find the optimal p_{succ}^C in the absence of environment. In the presence of quantum memory in the environment, the optimization is done using SDP, starting with a specific choice of measurement operators. Note, however, this is without loss of generality as the search over process implicitly optimizes over various measurements as it includes local unitary transformations.

In Fig. 6 we compare the optimal p_{succ}^C for varying p_{succ}^B with and without the presence of quantum memory as well as with and without unsharp measurements at B . In particular, the green plot represents optimal p_{succ}^C for $\eta = 1$, i.e., sharp measurement at B in a noiseless scenario. If Bob's instrument is unsharp, the upper bound on p_{succ}^C increases, as is evident in the purple plot in Fig. 6, which is for a noiseless scenario with unsharpness $\eta = 0.93$. The blue plot represents the optimal achievable sequential advantage with sharp measurement and quantum memory. As seen from the blue-shaded area in Fig. 6, we can still certify the presence of quantum memory through the violation of the new bound, assuming a lower bound on the unsharpness parameter, i.e., $\eta \geq 0.93$. This gives a robustness bound to experimentally certify quantum memory with partial trust in the instrument at B . However, the sequential advantage in the pink-shaded region cannot be achieved only through the coherent correlation in the environment. Therefore, a sequential advantage in this region certifies an unsharp instrument at B . Finally, the sequential advantage in the yellow-shaded region implies the presence of quantum memory as well as unsharpness at B , where the upper bound has been obtained with $\eta \geq 0.93$ (orange plot). Therefore, we observe that assuming a lower bound on the unsharpness parameter allows certification of either quantum non-Markovianity or unsharp instruments, or both.

Note that whenever the optimal sequential success probability curve for quantum memory with sharp measurement is above the curve for unsharp measurement without environment, one can find a region of joint probabilities that uniquely attributes the sequential advantage to quantum memory. This is possible by assuming a lower bound on the unsharpness parameter. In Fig. 7, we plot optimal p_{succ}^C as a function of the unsharpness parameter for various fixed p_{succ}^B , in the absence of an environment. The horizontal dashed lines correspond to the optimal p_{succ}^C for given p_{succ}^B with a coherently correlated environment and sharp measurement at B . The intersection provides the upper bound on $1 - \eta$, above which optimal sequential advantage from unsharpness takes over the successive advantages from quantum memory. Therefore, the regions in the plots before the intersection provide certification of quantum memory, assuming the intersection point is the minimum value of the unsharpness parameter. It is evident that the lower bound on η allowing certification of quantum memory decreases with an increase in p_{succ}^B , i.e., for a larger success probability of B , the certifiable region for quantum memory reduces.

In Fig. 8 we plot optimal p_{succ}^C as a function of the unsharpness at B 's instrument in a noiseless setting (blue plots) as well

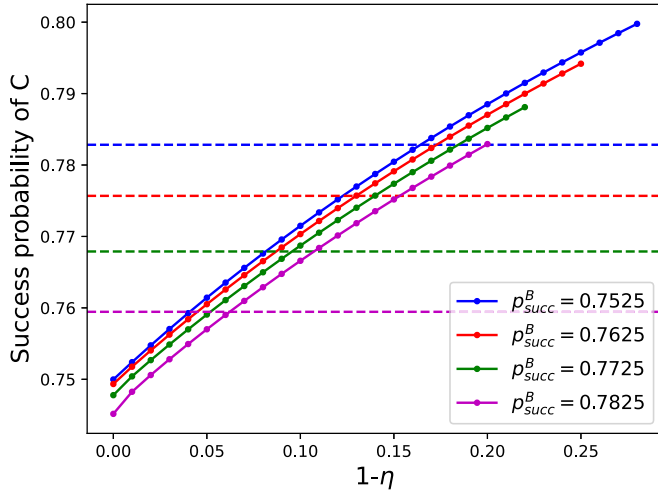


FIG. 7. The upper bound on p_{succ}^C for different unsharpness parameter η given a fixed value of p_{succ}^B for process without quantum memory. The horizontal line corresponds to the optimal feasible process with quantum memory and sharp measurement at B .

as with a coherently correlated environment (green plots) for different fixed values of p_{succ}^B in the feasible region and compare it with the optimal case with quantum memory and sharp measurement (red dashed line in horizontal). Note that the blue plots being below the reds gives us certifiable regions for quantum memory, while the regions after intersection certify unsharpness. In all plots, the optimal achievable sequential success probability is higher for a correlated environment, showing that quantum memory is a resource, even with unsharp measurements. For the minimum η with a feasible solution for fixed p_{succ}^B , the blue and the green plots coincide, implying the optimal process is the noiseless scenario, i.e., where there are identity channels between the different labs.

G. Unambiguous semi-device-independent certification of unsharp measurement

In the transformation step of standard prepare-transform-measure QRAC, only measurement-induced reduction of postmeasurement states has been considered in the literature [43,44,46,48,52,53]. In a noiseless scenario, any sequential quantum advantage can certify the unsharpness of Bob's instrument. However, more general transformations include system-environment interactions that can aid in sequential quantum advantage, as we have shown. To be specific, if more general transformations are considered, any sequential quantum advantage *cannot unambiguously* certify unsharpness in the instrument, as the advantage can come either from quantum memory or from unsharpness, or both. However, there are joint statistics that cannot come only from quantum memory with sharp measurement, i.e., the instrument at B must perform unsharp measurement. These joint probabilities are represented in the green-shaded region in Fig. 9. The red curve represents the joint optimal success probability for both observers for arbitrary unsharp measurement, while the blue curve represents quantum memory with only sharp measurement. Evidently, any success probability above the blue curve semi-device-independently certifies the presence of an

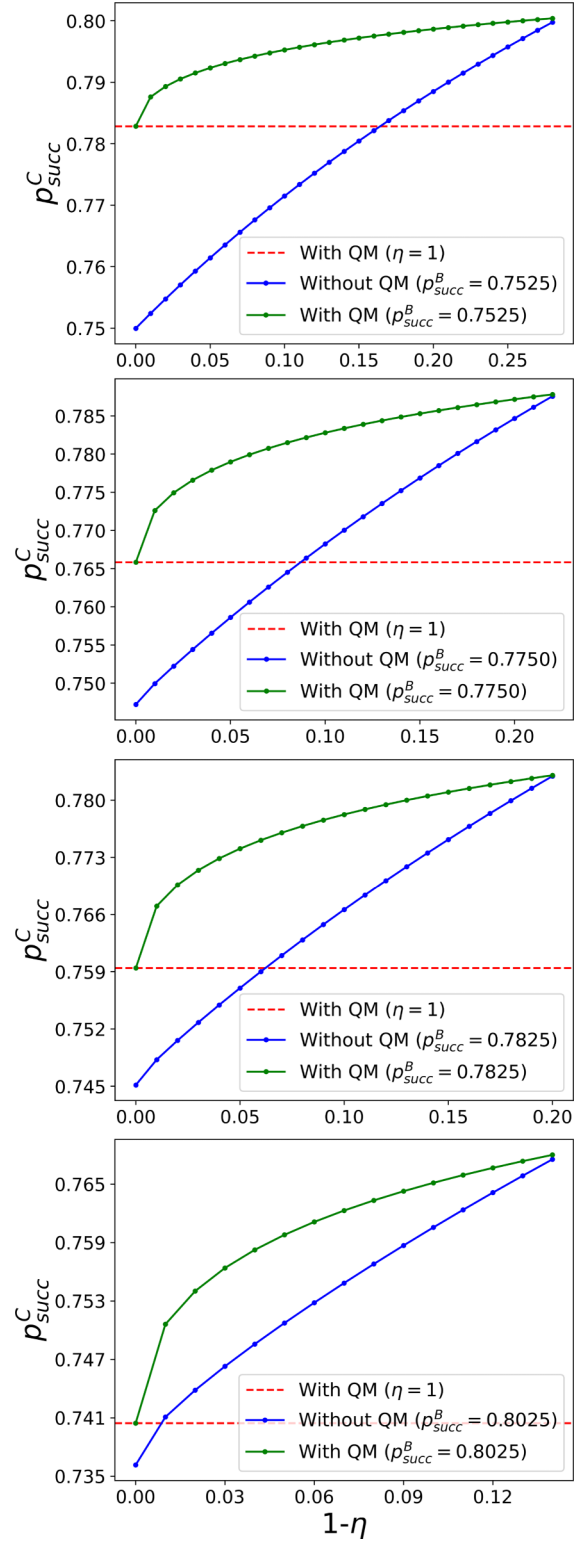


FIG. 8. The optimal success probability of C with (blue curve) and without (green curve) quantum-memory environment for varying unsharpness parameter for fixed values of p_{succ}^B in the respective feasible regions.

unsharp instrument at B in an unambiguous manner with only assumptions that qubits were prepared. Also, we note that if the unsharpness parameter is large, precisely for $\eta \geq 0.97$, the sequential quantum advantage from unsharpness never

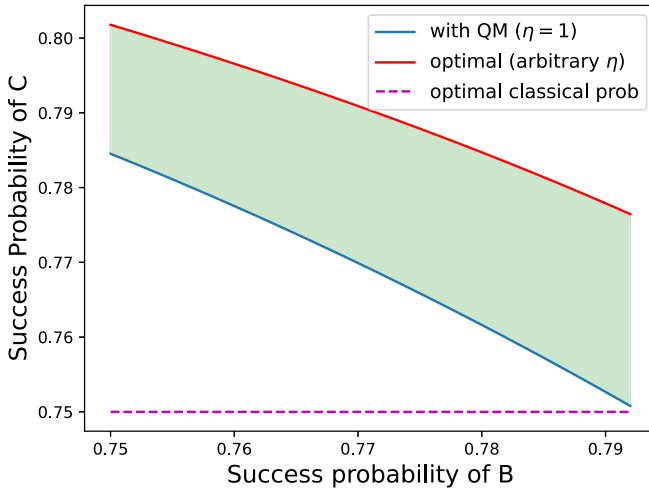


FIG. 9. The optimal joint success probability for B and C for arbitrary unsharp measurement (red curve) as compared to the optimal success probability with quantum memory and sharp measurement at B (blue curve) in the region where sequential wins are feasible. The shaded region is where one can unambiguously certify unsharpness.

goes above the QM curve. Therefore, if the unsharpness parameter $\eta \geq 0.97$, it can never be unambiguously certified in a semi-device-independent manner without assumption on the system-environment interaction.

V. DISCUSSION AND CONCLUSIONS

While QRAC games have been traditionally studied in the noiseless scenario [43,44,71,72], since the presence of environment is usually associated with detrimental decoherence effects on the system, which worsens the success probabilities, here we focus on the regime of strong system-environment interactions. In particular, we show that an environment with quantum memory can act as a resource to provide sequential wins at QRAC games with projective measurement at the intermediate party, something not possible in the noiseless scenario. We show that with assumptions on the dimension of the system and a minimum of trust in the intermediate instrument, we can unambiguously certify a quantum-memory environment. Alternatively, we can view the environment with quantum memory as allowing sequential wins in a QRAC game *despite* having intermediate projective measurements.

In addition, we argue that the *unsharpness* of the instrument can be semi-device-independently certified for certain joint probabilities even in the presence of quantum memory. In contrast to previous works [44,53], we show that the more general setting allowing system-environment interactions leads to tighter bounds on the unambiguous certification of the unsharpness (see Fig. 9).

In future work, it will be interesting to find the classes of system-environment interaction resulting in such resourceful non-Markovian processes. For example, a system-environment interaction model which simulates all the quantum advantage at laboratory C when success at B is unconstrained is given by partial swap operations between the labs. This model exhibits quantum memory but does not lead to *sequential* quantum advantage. Importantly, the set of processes is a convex set, therefore with any finite set of processes yielding successive quantum advantage, one can find an infinite number of resourceful processes through convex linear combinations.

Furthermore, it is worthwhile to ask if the presence of a non-Markovian environment allows for an extension of sequential quantum advantage to more than two independent observers. Interestingly, we find that there is no process allowing sequential quantum advantage at another laboratory D , with rank-one projective measurements at intermediate laboratories B and C . Even allowing unsharp measurements at intermediate laboratories does not permit the optimal success probability at D to be more than $3/4$. Moreover, when we lift the causal restriction $L_{ABCD}(W) = W$, which corresponds to allowing indefinite causal order between laboratories B and C , there is no quantum advantage at D . In the noiseless scenario, it has been suggested in the previous works [44,46] that the sequential win in QRAC games cannot possibly be extended to another third sequential independent observer, based on its connection to the temporal Clauser-Horne-Shimony-Holt (CHSH) game. Interestingly, in a recent article [73] the authors show that even with sharp measurement, nonlocal correlation is extractable for sequential CHSH game for three independent observers. A rigorous theoretical investigation of sequential quantum advantage in a prepare-transform-measure scenario is left for future work. It will also be interesting to explore whether using higher-dimensional input and output Hilbert spaces leads to extending the sequential advantage to more than two parties.

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