No-broadcasting of magic states

Zijian Zhang, Lingxuan Feng,* and Shunlong Luo
Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China
and School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China

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From both intuitive and physical perspectives, it is generally recognized that within a resource theory framework, free operations cannot broadcast a resource state due to their inability to generate resource from free states. In the stabilizer formalism of fault-tolerant quantum computation, the basic ingredients of the corresponding resource theory consist of stabilizer states as free states and stabilizer operations as free operations. The celebrated Gottesman-Knill theorem shows that quantum advantages over classical computation come from the magic (nonstabilizer) resource, such as magic states or non-Clifford gates. In this work, we prove that broadcasting of any magic state via stabilizer operations is impossible, which is reminiscent of the no-broadcasting theorems for noncommuting states or quantum correlations. We further derive a trade-off relation between the magic resource consumed in the initial system and that gained in the target system. These results characterize magic states in the stabilizer formalism from the broadcasting angle, and may have implications for distributed quantum computation and quantum secret sharing.

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I. INTRODUCTION

In the stabilizer formalism of universal and fault-tolerant quantum computation, magic resources are necessary for elevating stabilizer circuits to quantum computers beyond classical ones [1–5]. Indeed, according to the Gottesman-Knill theorem, quantum circuits that only consist of stabilizer states, Clifford gates, and Pauli measurements (i.e., the so-called stabilizer circuits) can be efficiently simulated classically [5]. To achieve genuine quantum computation, additional resources, such as magic states or non-Clifford operations, are required [4–8]. The concept of magic (nonstabilizer) resources can be conveniently formulated and understood within the broad context of resource theory, which focuses on identifying valuable resources as opposed to free resources. In recent years, various concrete realizations of the general resource framework have been widely studied, such as entanglement [9–12], reference frames [13–15], asymmetry [16–21], coherence [22–26], contextuality [27,28], Wigner negativity [29–31], and so on. An operational characterization of general convex resource theories both within quantum mechanics and in general probabilistic theories was established in Ref. [32]. All these approaches shed considerable lights on foundational aspects of quantum structures and provided powerful tools to explore quantum information processing.

In the resource theory of stabilizer quantum computation, stabilizer states and stabilizer (Clifford) operations are considered free, while magic (nonstabilizer) states and nonstabilizer (non-Clifford) operations are treated as resources [8]. Intuitively, it is expected that magic resources cannot be generated by stabilizer operations from stabilizer states, although they may be generated by nonstabilizer operations from stabilizer states. In this context, a closely related and subtle issue arises: If we start from magic states (rather than stabilizer states), can we generate additional magic resources while keeping the original magic states intact? Put alternatively, can we clone or broadcast magic states?

To address the above issue, we first recall that various no-broadcasting theorems have been extensively investigated ever since the discovery of no-cloning of quantum states by Wootters and Zurek [33] and by Dieks [34]. In 1996, Barnum *et al.* proved a no-broadcasting theorem for noncommuting states [35]. In 2008, Piani *et al.* established a no-broadcasting theorem for quantum correlations [36]. The connections between the above two no-broadcasting theorems were revealed in Refs. [37,38], and a unified picture emerged. Various other no-broadcasting theorems and no-go results have also been established [39–51]. These results, somewhat reminiscent of the Heisenberg uncertainty principle, shed considerable light on the quantum world and played an interesting role in studying quantum information.

From a general viewpoint, it is quite reasonable to expect that, in a framework of resource theory, operations that are considered free cannot broadcast resource states, as they lack the ability to create resources from free states. This belief is indeed supported by the many no-broadcasting theorems established in the literature [33–51]. Of course, for different concrete frameworks of resource theory, the precise mathematical formulations of no-broadcasting are different, and one needs to investigate no-broadcasting in the special context. In this work, we establish the no-broadcasting of magic states via stabilizer operations for the stabilizer formalism of quantum computation, and thus add a no-broadcasting result to the list of no-broadcasting theorems.

*Contact author: fenglingxuan14@mails.ucas.ac.cn

The remainder of the work is organized as follows. In Sec. II, we review the stabilizer formalism of quantum computation and the corresponding resource theory of magic states [8]. We prove some preliminary results concerning stabilizer states, which are of independent interest and will be invoked to establish our main results. In Sec. III, we derive some no-broadcasting results for magic states, and reveal a trade-off relationship for approximate broadcasting of magic states. In Sec. IV, we conclude with a discussion and summary. In the Appendixes, we present detailed mathematical proofs of the main results.

II. PRELIMINARIES

For any natural number d, let $\mathbb{Z}_d = \{0, 1, \ldots, d-1\}$ be the ring of integers modulo d. In this article, we only consider a finite-dimensional quantum system $\mathbb{C}^d$ with d a prime number and a computational basis $\{|j\rangle : j \in \mathbb{Z}_d\}$.

In a single-qudit system $\mathbb{C}^d$, the discrete Heisenberg-Weyl group (generalized Pauli group) [52–57]

$$\mathcal{P} = \{\tau^j D_{k,l} : j \in \mathbb{Z}_{2d}, k, l \in \mathbb{Z}_d\}$$

consists of discrete Heisenberg-Weyl operators (displacement operators)

$$D_{kl} = \tau^{kl} X^k Z^l, \quad \tau = -e^{i\pi/d},$$

where

$$X = \sum_{j=0}^{d-1} |j + 1 (\mathrm{mod}\ d)\rangle\langle j|, \quad Z = \sum_{d=0}^{d-1} \omega^j |j\rangle\langle j|,$$
$$\omega = e^{i2\pi/d}.$$

These operators were first introduced by Schwinger in his seminal study of unitary operator bases [58], and are basic ingredients for finite-dimensional quantum mechanics. The normalizer of $\mathcal{P}$ in the full unitary group $\mathcal{U}(d)$ of $\mathbb{C}^d$ is the Clifford group [52]

$$\mathcal{C} = \{V \in \mathcal{U}(d) : V\mathcal{P}V^\dagger = \mathcal{P}\}.$$

One may also define $\mathcal{C}/\mathcal{U}(1)$ as the Clifford group.

For an n-qudit system $(\mathbb{C}^d)^{\otimes n}$, the n-qudit discrete Heisenberg-Weyl group is given by

$$\mathcal{P}_n = \underbrace{\mathcal{P} \otimes \mathcal{P} \otimes \cdots \otimes \mathcal{P}}_{n},$$

and the n-qudit Clifford group is defined by

$$\mathcal{C}_n = \{V \in \mathcal{U}(d^n) : V\mathcal{P}_n V^\dagger = \mathcal{P}_n\}.$$

The stabilizer formalism is based on these discrete Heisenberg-Weyl operators, which are higher-dimensional extensions of the familiar Pauli operators $\sigma_x, \sigma_y, \sigma_z$. The stabilizer states are introduced in fault-tolerant quantum computation [1], and according to the Gottesman-Knill theorem [5], quantum circuits initialized from stabilizer states and involving only Clifford gates and Pauli measurements can be efficiently simulated by classical computers.

In an n-qudit system $(\mathbb{C}^d)^{\otimes n}$, a pure stabilizer state is defined as the common eigenstate (with common eigenvalue 1) of a maximal abelian subgroup $\mathcal{A}$ ($c\mathbf{1} \notin \mathcal{A}$, $c \neq 1$) of the discrete Heisenberg Weyl group $\mathcal{P}_n$ [1,53]. The group $\mathcal{A}$ is called the stabilizer group of the stabilizer state. Thus in the stabilizer formalism, stabilizer states are conveniently and efficiently described by stabilizer (abelian) groups, hence the name. Any pure stabilizer state $|\mathcal{A}\rangle$ associated with its stabilizer group $\mathcal{A}$ can be written in the projection operator form as [57]

$$|\mathcal{A}\rangle\langle\mathcal{A}| = \frac{1}{|\mathcal{A}|}\sum_{D\in\mathcal{A}} D,$$

where $|\mathcal{A}| = d^n$ is the number of elements in $\mathcal{A}$. Mixed stabilizer states are nontrivial probabilistic mixtures of pure stabilizer states. Except for stabilizer states, all other states are called magic states or nonstabilizer states.

More general than the Clifford unitary gates in the Clifford group $\mathcal{C}_n$, a stabilizer (Clifford) operation on $(\mathbb{C}^d)^{\otimes n}$ is a map between quantum states on $(\mathbb{C}^d)^{\otimes n}$ which can be expressed as [59,60]

$$\Lambda(\rho) = \mathrm{tr}_{\mathrm{E}}(U(\rho \otimes \tau^{\mathrm{E}})U^\dagger)$$

for some stabilizer state τ^{E} on an ancillary system (environment) $H^{\mathrm{E}} = (\mathbb{C}^d)^{\otimes m}$ and some Clifford unitary gate $U \in \mathcal{C}_{n+m}$ (the Clifford group of $(\mathbb{C}^d)^{\otimes(n+m)} = (\mathbb{C}^d)^{\otimes n} \otimes (\mathbb{C}^d)^{\otimes m}$). Here ρ is any state on $(\mathbb{C}^d)^{\otimes n}$, and tr_{E} denotes the partial trace over the ancillary system H^{E}.

According to the Gottesman-Knill theorem, to achieve genuine quantum computation, magic resources must be injected into stabilizer circuits. In the resource theory of stabilizer quantum computation [8], stabilizer states and stabilizer (Clifford) operations are considered as free states and free operations, respectively, while magic states and nonstabilizer (non-Clifford) operations are considered as resource states and resource operations, respectively. In a resource theory, states and operations outside the corresponding free sets are recognized as resourceful. To quantify the amount of the resource, various resource measures are proposed. Typically, a well-defined measure $M(\cdot)$ of resource should satisfy the following two properties.

(1) Faithfulness: $M(\rho) \geqslant c$, where $c \geqslant 0$ is a constant and the equality holds if and only if ρ is a free state (stabilizer state).

(2) Monotonicity: For any free operation Λ, we have $M(\Lambda(\rho)) \leqslant M(\rho)$.

For our purpose, we will need three well-defined measures of magic resource [8,31,61], the robustness of magic resource, the relative entropy of magic resource, and the regularized relative entropy of magic resource (reviewed in Sec. III and the Appendix C).

The following results, apart from their own independent interest, will be used in the proof of our main results.

Lemma 1. Let $\mathcal{A} \subseteq \mathcal{P}$ be any maximal abelian subgroup of the discrete Heisenberg-Weyl group $\mathcal{P}$ in a d-dimensional system $\mathbb{C}^d$ such that $c\mathbf{1} \notin \mathcal{A}$ for any $c \neq 1$, then all common normalized eigenstates of $\mathcal{A}$ are stabilizer states and constitute an orthonormal basis of $\mathbb{C}^d$.

For the proof, see the Appendix A.

From Lemma 1, we obtain the following sufficient condition for determining whether a state ρ is a stabilizer state or not. This condition depends on its support $\{D_{k,l} : \mathrm{tr}\rho D_{k,l} \neq 0\}$ with respect to the discrete Heisenberg-Weyl group.

Lemma 2. Let ρ be a (pure or mixed) quantum state in a d-dimensional system $\mathbb{C}^d$, and let $\mathcal{A} \subseteq \mathcal{P}$ be the subgroup of $\mathcal{P}$ generated by $\{D_{k,l} : \mathrm{tr}\rho D_{k,l} \neq 0\}$. If $\mathcal{A}$ is an abelian group, then ρ is a stabilizer state.

For the proof, see the Appendix B.

We remark that Lemmas 1 and 2 are actually also true for all finite dimensional systems, not only for prime-dimensional ones.

III. NO-BROADCASTING OF MAGIC STATES

Since no-cloning, on the one hand, is much easier to establish than no-broadcasting, and on the other hand, illustrates some essential points, we first treat no-cloning, even though no-broadcasting includes no-cloning as a special instance.

No-cloning reveals a radical difference between classical and quantum information. In the standard form, it shows that for a family of distinct nonorthogonal and unknown states $\{\rho_j : j = 1, 2, \ldots\}$ on H^a, there exists no operation Φ which copies all the input states in the sense that

$$\Phi(\rho_j \otimes \tau) = \rho_j \otimes \rho_j, \quad j = 1, 2, \ldots,$$

where τ is any fixed state on $H^b = H^a$ [62]. In particular, there is no operation which can clone all quantum states [33]. By relaxing the condition to allow correlations, the family of states $\{\rho_j : j = 1, 2, \ldots\}$ is said to be broadcast by an operation Φ if

$$\mathrm{tr}_a\Phi(\rho_j \otimes \tau) = \rho_j = \mathrm{tr}_b\Phi(\rho_j \otimes \tau), \quad j = 1, 2, \ldots.$$

The celebrated result of Barnum *et al.* shows that a family of quantum states can be broadcast if and only if it is a commuting family [35].

In the framework of quantum resource theories, a natural belief is that resources cannot be generated by free operations from free resources. In the consideration of cloning and broadcasting in resource theories, it is natural to ask whether resources can be cloned or broadcast via *free operations* (rather than general operations as in the traditional approach to no-cloning). In the resource theory of stabilizer quantum computation, a stabilizer state can be cloned by a *stabilizer operation* simply by the preparation operation (which is a stabilizer operation) of the stabilizer state. The question is whether a magic state can be cloned or broadcast by a stabilizer operation. We first treat no-cloning of magic states since this case is easier to prove than no-broadcasting of magic states.

Definition 1 (Cloning of magic states). A magic state ρ^a on a system H^a is said to be cloned by the stabilizer operation Λ on the system $H^a \otimes H^b$ if there exists a stabilizer state τ^b on $H^b = H^a$ such that

$$\Lambda(\rho^a \otimes \tau^b) = \rho^a \otimes \rho^a.$$

With the above definition, we obtain the following no-cloning result for magic states.

Proposition 1. In any prime-dimensional system $\mathbb{C}^d$, a magic state cannot be cloned by any stabilizer operation.

For the proof, see the Appendix C.

The product structure of the output state $\rho^a \otimes \rho^a$ possesses no correlations and simplifies considerably the proof of no-cloning of magic states. If we allow correlations between the two reduced states of the output state, then we come to the scenario of broadcasting.

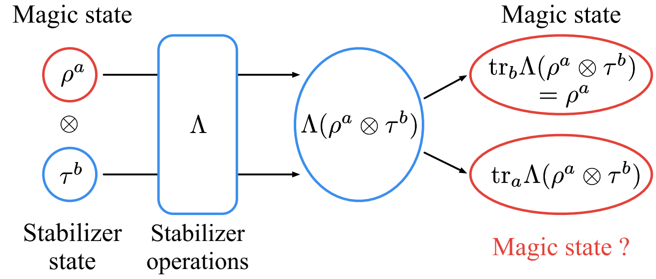

FIG. 1. Magic states broadcasting process.

Definition 2 (Broadcasting of magic states). A magic state ρ^a on a system H^a is said to be broadcast by the stabilizer operation Λ on the system $H^a \otimes H^b$ if there exists a stabilizer state τ^b on $H^b = H^a$ such that

$$\mathrm{tr}_a\Lambda(\rho^a \otimes \tau^b) = \rho^a = \mathrm{tr}_b\Lambda(\rho^a \otimes \tau^b).$$

If we had a superadditive (supermultiplicative) measure of magic resource which was in the meantime faithful and monotonic, then we could easily establish no-broadcasting of magic states. Indeed, if $f(\cdot)$ is such a measure of magic resource and the magic state ρ^a can be broadcast (or, in particular, can be cloned) by the stabilizer operation Λ, then for any stabilizer state ρ^b,

$$\begin{aligned} f(\rho^a) &\geqslant f(\Lambda(\rho^a \otimes \rho^b)) \quad \text{(monotonicity)} \\ &\geqslant f(\mathrm{tr}_b\Lambda(\rho^a \otimes \rho^b)) \\ &\quad + f(\mathrm{tr}_a\Lambda(\rho^a \otimes \rho^b)) \ \ \text{(superadditivity)} \\ &= f(\rho^a) + f(\rho^a) \\ &= 2f(\rho^a), \end{aligned}$$

which is a contradiction since, by the faithfulness of $f(\cdot)$, we have $f(\rho^a) > 0$ for any magic state ρ^a. A similar result can be derived for any well-defined supermultiplicative magic resource measure. However, to the best of our knowledge, we do not have such a measure of magic resource at present. Some counterexamples for the robustness of magic resource, the relative entropy of magic resource, and furthermore any magic resource measure based on quantum divergence can be found in Refs. [8,31,63]. This makes the proofs of no-cloning and no-broadcasting of magic states rather nontrivial. In fact, we will actually prove no-broadcasting of the magic resource, which is a generalization of no-broadcasting of magic states in a stronger sense. For this purpose, we introduce the following notion.

Definition 2′ (Generalized broadcasting of magic states). A magic state ρ^a on a system H^a is said to be broadcast in a generalized sense by the stabilizer operation Λ on the system $H^a \otimes H^b$ if there exists a stabilizer state τ^b on $H^b = H^a$ such that

$$\mathrm{tr}_b\Lambda(\rho^a \otimes \tau^b) = \rho^a,$$

and moreover the reduced state $\mathrm{tr}_a\Lambda(\rho^a \otimes \tau^b)$ on the system H^b is also a magic state (not necessarily equal to ρ^a).

Illustrated in Fig. 1, the above definition focuses on the broadcasting of the resource instead of the state itself, which is about whether we are able to broadcast a certain resource by composite free operations without disturbing the initial resourceful state. Obviously, if the original broadcasting of magic states is possible, then the generalized broadcasting of magic states can be established in the same way. However, the following proposition shows the broadcasting of magic states cannot be accomplished.

Proposition 2. In any prime-dimensional quantum system, a magic state cannot be broadcast in a generalized sense by any stabilizer operation.

For the proof, see the Appendix D.

The above result demonstrates that if the goal is to maintain the initial magic state on H^a, then no magic state can be generated on H^b through stabilizer operations. In other words, the advantages of quantum computation over classical computation cannot be shared via stabilizer operations without the consumption of resources in the original quantum system, i.e., quantum advance cannot be broadcast via stabilizer operations.

By relaxing the requirement that the original magic state ρ^a should be kept intact, we come to approximate broadcasting of magic states and establish a trade-off relationship between the magic resource consumed in system H^a and that generated in system H^b. To make this more precise, we recall an important measure of magic resource.

The robustness of magic resource (ROM) of a quantum state ρ on $(\mathbb{C}^d)^{\otimes n}$ is defined as [31,61]

$$R(\rho) = \min_{\{x_j\}} \left\{ \sum_j |x_j| : \rho = \sum_j x_j |\psi_j\rangle\langle\psi_j|, |\psi_j\rangle \in \mathcal{S} \right\},$$

where $\mathcal{S}$ is the set of all pure stabilizer states on $(\mathbb{C}^d)^{\otimes n}$. The ROM has the following properties:

(1) $R(\rho) \geqslant 1$, the equality holds if and only if ρ is a stabilizer state;

(2) For any stabilizer operation Λ, $R(\Lambda(\rho)) \leqslant R(\rho)$;

(3) $R(\rho^a \otimes \rho^b) \leqslant R(\rho^a)R(\rho^b)$, i.e., the ROM is submultiplicative for product states;

(4) $R(\sum_j p_j\rho_j) \leqslant \sum_j p_j R(\rho_j)$ where $\{p_j\}$ is a probability distribution and ρ_j are quantum states, i.e., the ROM is convex.

Proposition 3. In a prime-dimensional system H^a, let ρ^a be a magic state and Λ a stabilizer operation on $H^a \otimes H^b$ with $H^b = H^a$, then

$$R(\mathrm{tr}_b\Lambda(\rho^a \otimes \tau^b)) + R(\mathrm{tr}_a\Lambda(\rho^a \otimes \tau^b)) \leqslant R(\rho^a) + R(\tau^b).$$

Here $R(\cdot)$ is the robustness of magic resource and τ^b is a stabilizer state on H^b. Notice that actually $R(\tau^b) = 1$ for any stabilizer state τ^b.

For the proof, see the Appendix E.

Proposition 3 shows that magic resource obtained on H^b is bounded above by the consumed magic resource on H^a. Moreover, if we even simply want to preserve the amount of magic resource on H^a, i.e., $R(\mathrm{tr}_b\Lambda(\rho^a \otimes \tau^b)) = R(\rho^a)$, then no magic resource can be obtained on H^b through stabilizer operations.

Considering Proposition 3, a natural question is whether it would still hold if τ^b is also a magic state. This is an important issue worth further investigations. Our present method cannot establish this result affirmably since when τ^b is also a magic state, we cannot derive that one of $\mathrm{tr}_b\Lambda(\rho^a \otimes \tau^b)$ and $\mathrm{tr}_a\Lambda(\rho^a \otimes \tau^b)$ must be a stabilizer state. Notice that, in a qubit system, Proposition 3 can be established for any quantum states ρ^a and τ^b. It shows that free operations cannot amplify the magic resource, different from the case of asymmetry [64], although states with either magic or asymmetry resource cannot be broadcast [47,48].

IV. DISCUSSION

In this work, we established some results concerning no-broadcasting of magic states in a resource theory of stabilizer formalism. For this purpose, we provided a criterion for stabilizer states via groups generated by the supports of characteristic functions of the states, which may be of independent interests. Furthermore, we derived a trade-off relationship which sets an upper bound to the amount of magic resource generated by the process of approximate broadcasting of magic states.

We focused only on systems with prime dimensions. It is desirable to consider broadcasting of magic states for the case of general dimensions. However, for non-prime-dimensional systems, some difficulties and subtleties arise. On the one hand, in non-prime-dimensional systems, there is no clearly defined simple resource framework for magic resource as that in prime-dimensional systems. On the other hand, in non-prime dimensions, a maximal abelian subgroup of the discrete Heisenberg-Weyl group may have two generators, rendering our current proof invalid. To generalize our no-broadcasting of magic states to any finite-dimensional systems, further knowledge about the group structure of stabilizer formalism and the condition of equality for the monotonicity of magic resource measures are indispensable.

Finally, we make some remarks on two open issues. First, if there existed a superadditive bona fide measure of magic resource, then no-broadcasting would follow immediately and this issue would be trivial. We conjecture that no such measures exist, although we have not provided a proof of this speculation. Second, notice that the correlated catalysis, a special type of catalysis which allows correlations between the catalyst and the target state, is defined as follows [65]: A state τ^b on H^b is said to be correlated catalyzed to σ^b on H^b if there exist a free operation $\Lambda \in \mathcal{F}$ and a state ρ^a on H^a such that

$$\mathrm{tr}_b\Lambda(\rho^a \otimes \tau^b) = \rho^a, \quad \mathrm{tr}_a\Lambda(\rho^a \otimes \tau^b) = \sigma^b.$$

Obviously, when τ^b is a stabilizer state and σ^b is an arbitrary magic state, this reduces to our scenario of broadcasting of magic states. Here, a natural question arises: If we remove the requirement that τ^b is a stabilizer state, can τ^b be correlated catalyzed to a state σ^b containing more magic resource than τ^b (with the magic state ρ^a as a catalyst)? In other words, does Proposition 3 still hold when τ^b is a magic state? Some discussions on catalysis of magic states can be found in Ref. [66], and the catalysis of entanglement manipulation for mixed states is investigated in Ref. [67]. It is desirable to further investigate similar issues in the context of stabilizer formalism.

APPENDIX A: PROOF OF LEMMA 1

Since d is a prime number, any maximal abelian subgroup $\mathcal{A} \subseteq \mathcal{P}$ of the Heisenberg-Weyl group on $\mathbb{C}^d$ is isomorphic to $\mathbb{Z}_d$, and thus can be expressed as

$$\mathcal{A} = \{\mathbf{1}, D, D^2, \dots, D^{d-1}\},$$

with a generator operator D. Notice that $c\mathbf{1}$ do not belong to $\mathcal{A}$ for $c \neq 1$ by assumption.

Since the unitary operator D has order d, and $\mathbf{1}, D, D^2, \dots, D^{d-1}$ are linearly independent, the minimal polynomial of D is then $x^d - 1$. Consequently, D has d eigenstates $|\psi_j\rangle$ with eigenvalues ω^j and constitute an orthonormal base of $\mathbb{C}^d$: $D|\psi_j\rangle = \omega^j|\psi_j\rangle,\ j \in \mathbb{Z}_d, \omega = e^{i2\pi/d}$. These eigenstates are common eigenstates of all elements of $\mathcal{A}:\ D^k|\psi_j\rangle = \omega^{jk}|\psi_j\rangle,\ j, k \in \mathbb{Z}_d$. It can be checked that

$$|\psi_j\rangle\langle\psi_j| = \sum_{k=0}^{d-1} \omega^{-jk} D^k, \quad j \in \mathbb{Z}_d,$$

which imply that $|\psi_j\rangle$ are stabilizer states.

APPENDIX B: PROOF OF LEMMA 2

Suppose that the subgroup $\mathcal{A}_1 \subseteq \mathcal{P}$ generated by $\{D_{k,l} : \mathrm{tr}\rho D_{k,l} \neq 0\}$ is an abelian group. We can find a subgroup $\mathcal{A}_0$ of $\mathcal{A}_1$ such that $c\mathbf{1} \notin \mathcal{A}_0$ with $c \neq 1$. Then $|\mathcal{A}_0| \leqslant d$. By extending $\mathcal{A}_0$ to a maximal abelian subgroup $\mathcal{A}$ ($c\mathbf{1} \notin \mathcal{A}$ with $c \neq 1$) and in view of Lemma 1, the common eigenstates $|\psi_j\rangle, j \in \mathbb{Z}_d$, of $\mathcal{A}$ are stabilizer states and constitute an orthonormal basis of $\mathbb{C}^d$. Thus all operators in $\mathcal{A}$ are diagonal under the basis $\{|\psi_j\rangle : j \in \mathbb{Z}_d\}$ and therefore ρ is diagonal under the basis $\{|\psi_j\rangle : j \in \mathbb{Z}_d\}$. This implies that ρ is a probabilistic mixture of the d pure stabilizer states. Consequently, ρ is also a stabilizer state (possibly mixed).

APPENDIX C: PROOF OF PROPOSITION 1

First, we recall that the relative entropy of magic resource of a quantum state ρ on $(\mathbb{C}^d)^{\otimes n}$ is defined as [8]

$$R_S(\rho) = \min_{\sigma \in \mathrm{Stab}} S(\rho|\sigma),$$

where Stab is the set of all (pure or mixed) stabilizer states on $(\mathbb{C}^d)^{\otimes n}$ and $S(\rho|\sigma) = \mathrm{tr}(\rho \ln \rho) - \mathrm{tr}(\rho \ln \sigma)$ is the relative entropy between quantum states. The relative entropy of magic resource has the following properties:

(1) $R_S(\rho) \geqslant 0$, the equality holds if and only if ρ is a stabilizer state;

(2) For any stabilizer operation Λ, $R_S(\Lambda(\rho)) \leqslant R_S(\rho)$.

The regularized relative entropy of magic resource of a quantum state ρ on $(\mathbb{C}^d)^{\otimes n}$ is defined as [8]

$$R_S^\infty(\rho) = \lim_{m\to\infty} \frac{1}{m} R_S(\rho^{\otimes m}),$$

and has similar properties as the relative entropy of magic resource.

Now suppose that the magic state ρ^a on H^a can be cloned by the stabilizer operation Λ. By iteratively applying Λ, we can generate k (an arbitrary finite number) copies of ρ^a, denoted as $(\rho^a)^{\otimes k}$. Since ρ^a is a magic state, we have

$$0 < R_S^\infty(\rho^a) = \lim_{m\to\infty} \frac{1}{m} R_S((\rho^a)^{\otimes m}),$$

which implies that

$$\lim_{m\to\infty} R_S((\rho^a)^{\otimes m}) = \infty.$$

Thus, for sufficiently large t,

$$R_S((\rho^a)^{\otimes t}) > R_S(\rho^a).$$

However, since $(\rho^a)^{\otimes t}$ can be produced from ρ^a by stabilizer operations, from the monotonicity of the relative entropy of magic resource, we have

$$R_S((\rho^a)^{\otimes t}) \leqslant R_S(\rho^a),$$

which implies that the magic state ρ^a cannot be cloned.

APPENDIX D: PROOF OF PROPOSITION 2

Without loss of generality, we may ignore the phases of operators. Suppose that the magic state ρ^a on $H^a = \mathbb{C}^d$ can be broadcast by a stabilizer operation Λ, then there exists a stabilizer state τ^b on $H^b = H^a$ such that

$$\mathrm{tr}_b \Lambda(\rho^a \otimes \tau^b) = \rho^a,$$

and moreover $\mathrm{tr}_a \Lambda(\rho^a \otimes \tau^b)$ is a magic state on H^b, where

$$\Lambda(\rho^a \otimes \tau^b) = \mathrm{tr}_{\mathrm{E}} U(\rho^a \otimes \tau^b \otimes \tau^{\mathrm{E}}) U^\dagger \tag{D1}$$

for some stabilizer state τ^{E} on an ancillary system $(\mathbb{C}^d)^{\otimes n}$ and $U \in \mathcal{C}_{n+2}$ is a Clifford unitary operator on $(\mathbb{C}^d)^{\otimes(n+2)}$.

We first establish the desired result when τ^b and τ^{E} are pure states on $\mathbb{C}^d$ and $(\mathbb{C}^d)^{\otimes n}$, respectively. Since τ^b and τ^{E} are stabilizer states, they are stabilized by some maximal abelian subgroups $\mathcal{A}^b \subset \mathcal{P}_1 = \mathcal{P}$ and $\mathcal{A}^{\mathrm{E}} \subset \mathcal{P}_n$, respectively. For any m-qudit quantum state ρ, let

$$S_m(\rho) = \left\{ \bigotimes_{j=1}^m D_{k_j,l_j} :\ \mathrm{tr}\left(\rho \bigotimes_{j=1}^m D_{k_j,l_j}\right) \neq 0,\ k_j, l_j \in \mathbb{Z}_d \right\}$$

be the support of ρ. We consider groups generated by supports of the input state and the output state.

On the input side, ignoring the global phase, supports of stabilizer states τ^b and τ^{E} coincide with $\mathcal{A}^b$ and $\mathcal{A}^{\mathrm{E}}$, respectively. Since ρ^a is a magic state, the group generated by the support of ρ^a is $\langle S_1(\rho^a)\rangle = \mathcal{P}$. Thus, we obtain that

$$\begin{aligned}\langle S_{n+2}(\rho^a \otimes \tau^b \otimes \tau^{\mathrm{E}})\rangle &= \langle S_1(\rho^a) \otimes S_1(\tau^b) \otimes S_n(\tau^{\mathrm{E}})\rangle \\ &= \mathcal{P} \otimes \mathcal{A}^b \otimes \mathcal{A}^{\mathrm{E}}.\end{aligned}$$

On the output side, according to the assumption, both $\mathrm{tr}_b \Lambda(\rho^a \otimes \tau^b)$ and $\mathrm{tr}_a \Lambda(\rho^a \otimes \tau^b)$ are magic states on $\mathbb{C}^d$, which imply that

$$\langle S_1(\mathrm{tr}_b \Lambda(\rho^a \otimes \tau^b))\rangle = \langle S_1(\mathrm{tr}_a \Lambda(\rho^a \otimes \tau^b))\rangle = \mathcal{P}.$$

Thus, we obtain that

$$\langle S_2(\Lambda(\rho^a \otimes \tau^b))\rangle = \mathcal{P}_2.$$

However, from Eq. (D1) we know that $S_2(\Lambda(\rho^a\otimes\tau^b))\otimes\mathbf{1}^{\otimes n}$ is a subset of $S_{n+2}(U(\rho^a\otimes\tau^b\otimes\tau^{\mathrm{E}})U^\dagger)$ and thus

$$\begin{aligned}\langle S_2(\Lambda(\rho^a\otimes\tau^b))\rangle\otimes\mathbf{1}^{\otimes n}&=\mathcal{P}_2\otimes\mathbf{1}^{\otimes n}\\&\subseteq\langle S_{n+2}(U(\rho^a\otimes\tau^b\otimes\tau^{\mathrm{E}})U^\dagger)\rangle.\end{aligned}\tag{D2}$$

Notice that the Clifford operator $U\in\mathcal{C}_{n+2}$ induces an automorphism of the group $\mathcal{P}_{n+2}$ by conjugation, and thus we have (up to global phase)

$$\begin{aligned}\langle S_{n+2}(U(\rho^a\otimes\tau^b\otimes\tau^{\mathrm{E}})U^\dagger)\rangle&\cong\langle S_{n+2}(\rho^a\otimes\tau^b\otimes\tau^{\mathrm{E}})\rangle\\&=\mathcal{P}\otimes\mathcal{A}^b\otimes\mathcal{A}^{\mathrm{E}}.\end{aligned}\tag{D3}$$

Now combining Eqs. (D2) and (D3), we obtain that

$$\mathcal{P}_2\otimes\mathbf{1}^{\otimes n}\subseteq\langle S_{n+2}(U(\rho^a\otimes\tau^b\otimes\tau^{\mathrm{E}})U^\dagger)\rangle\cong\mathcal{P}\otimes\mathcal{A}^b\otimes\mathcal{A}^{\mathrm{E}},$$

which contradicts to the fact that $\mathcal{A}^b$ and $\mathcal{A}^{\mathrm{E}}$ are commutative proper subsets of $\mathcal{P}$ and $\mathcal{P}_n$, respectively.

From the discussion above, for pure stabilizer states τ^b and τ^{E}, if $\mathrm{tr}_a\Lambda(\rho^a\otimes\tau^b)$ becomes a magic state via a stabilizer operation Λ, then $\langle S_1(\mathrm{tr}_a\Lambda(\rho^a\otimes\tau^b))\rangle=\mathcal{P}$ and $\langle S_1(\mathrm{tr}_b\Lambda(\rho^a\otimes\tau^b))\rangle$ should remain an abelian group. From Lemma 2, we conclude that $\mathrm{tr}_b\Lambda(\rho^a\otimes\tau^b)$ is a stabilizer state.

Now consider the general case that

$$\rho^a\otimes\tau^b\otimes\tau^{\mathrm{E}}=\rho^a\otimes\left(\sum_{j,k}p_{jk}|\psi_j^b\rangle\langle\psi_j^b|\otimes|\psi_k^{\mathrm{E}}\rangle\langle\psi_k^{\mathrm{E}}|\right),$$

where p_{jk} satisfy $p_{jk}\geqslant 0$, $\sum_{jk}p_{jk}=1$, both $|\psi_j^b\rangle\in\mathbb{C}^d$ and $|\psi_k^{\mathrm{E}}\rangle\in(\mathbb{C}^d)^{\otimes n}$ are pure stabilizer states. From the linearity of Λ, we have

$$\mathrm{tr}_b\Lambda(\rho^a\otimes\tau^b)=\sum_{j,k}p_{jk}\mathrm{tr}_{b\mathrm{E}}U(\rho_{jk}^{ab\mathrm{E}})U^\dagger=\sum_{j,k}p_{jk}\rho_{jk}^a,$$

$$\mathrm{tr}_a\Lambda(\rho^a\otimes\tau^b)=\sum_{j,k}p_{jk}\mathrm{tr}_{a\mathrm{E}}U(\rho_{jk}^{ab\mathrm{E}})U^\dagger=\sum_{j,k}p_{jk}\rho_{jk}^b,$$

where

$$\begin{aligned}\rho_{jk}^{ab\mathrm{E}}&=\rho^a\otimes|\psi_j^b\rangle\langle\psi_j^b|\otimes|\psi_k^{\mathrm{E}}\rangle\langle\psi_k^{\mathrm{E}}|,\\\rho_{jk}^a&=\mathrm{tr}_{b\mathrm{E}}U(\rho_{jk}^{ab\mathrm{E}})U^\dagger,\\\rho_{jk}^b&=\mathrm{tr}_{a\mathrm{E}}U(\rho_{jk}^{ab\mathrm{E}})U^\dagger.\end{aligned}$$

According to the assumption, there exist indices $j_0,k_0\in\mathbb{N}$ such that $\rho_{j_0k_0}^b$ is a magic state since otherwise, $\mathrm{tr}_a\Lambda(\rho^a\otimes\tau^b)$ would be a stabilizer state. By the discussion about no-broadcasting theorem for the case of τ^b and τ^{E} being pure states, we know if $\rho_{j_0k_0}^b$ is a magic state, then $\rho_{j_0k_0}^a$ must be a stabilizer state. Considering the channels

$$\Lambda_{jk}(\rho^a)=\mathrm{tr}_{b\mathrm{E}}U(\rho^a\otimes|\psi_j^b\rangle\langle\psi_j^b|\otimes|\psi_k^{\mathrm{E}}\rangle\langle\psi_k^{\mathrm{E}}|)U^\dagger=\rho_{jk}^a,$$

which are stabilizer operations for all $j,k=1,2,\dots$. From the monotonicity of the ROM, we have

$$R(\rho_{jk}^a)=R(\Lambda_{jk}(\rho^a))\leqslant R(\rho^a).$$

In addition, in view of the convexity of the ROM, we have

$$R(\mathrm{tr}_b\Lambda(\rho^a\otimes\tau^b))\leqslant\sum_{jk}p_{jk}R(\rho_{jk}^a)\leqslant\sum_{jk}p_{jk}R(\rho^a)=R(\rho^a).$$

Since $\mathrm{tr}_b\Lambda(\rho^a\otimes\tau^b)=\rho^a$, i.e., $R(\mathrm{tr}_b\Lambda(\rho^a\otimes\tau^b))=R(\rho^a)$, which implies that the above two inequalities are saturated, we have

$$R(\rho_{jk}^a)=R(\rho^a)>1,\quad\forall\, j,k=1,2,\dots,$$

which contradict to the fact that there exist indices $j_0,k_0\in\mathbb{N}$ such that $\rho_{j_0k_0}^a$ is a stabilizer state and $R(\rho_{j_0k_0}^a)=1$. Consequently, the magic state ρ^a cannot be broadcast by the stabilizer operation Λ (in the generalized sense).

APPENDIX E: PROOF OF PROPOSITION 3

Following the notation in the proof of Proposition 2, for each pair (j,k), from the monotonicity of the ROM, we have

$$\begin{aligned}R(\rho_{jk}^a)&=R(\mathrm{tr}_{b\mathrm{E}}U(\rho_{jk}^{ab\mathrm{E}})U^\dagger)\leqslant R(\rho_{jk}^{ab\mathrm{E}})\leqslant R(\rho^a),\\R(\rho_{jk}^b)&=R(\mathrm{tr}_{a\mathrm{E}}U(\rho_{jk}^{ab\mathrm{E}})U^\dagger)\leqslant R(\rho_{jk}^{ab\mathrm{E}})\leqslant R(\rho^a).\end{aligned}$$

It follows from the previous proof of Proposition 2 that at least one of ρ_{jk}^a and ρ_{jk}^b must be a stabilizer state, and thus

$$\begin{aligned}R(\rho_{jk}^a)+R(\rho_{jk}^b)&=\max\{R(\rho_{jk}^a),R(\rho_{jk}^b)\}+1\\&\leqslant R(\rho^a)+1.\end{aligned}$$

Consequently,

$$\begin{aligned}&R(\mathrm{tr}_b\Lambda(\rho^a\otimes\tau^b))+R(\mathrm{tr}_a\Lambda(\rho^a\otimes\tau^b))\\&\leqslant\sum_{j,k}p_{jk}(R(\rho_{jk}^a)+R(\rho_{jk}^b))\\&\leqslant R(\rho^a)+1,\end{aligned}$$

where the first inequality comes from the convexity of the ROM.

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