Quantum communication on the bosonic loss-dephasing channel

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Quantum optical systems are typically affected by two types of noise: photon loss and dephasing. Despite extensive research on each noise process individually, a comprehensive understanding of their combined effect is still lacking. A crucial problem lies in determining the values of loss and dephasing for which the resulting loss-dephasing channel is antidegradable, implying the absence of codes capable of correcting its effect or, alternatively, capable of enabling quantum communication. A conjecture [Quantum **6**, 821 (2022)] suggested that the bosonic loss-dephasing channel is not antidegradable if the loss is below 50%. In this paper we refute this conjecture, specifically proving that for any value of the loss, if the dephasing is above a critical value, then the bosonic loss-dephasing channel is antidegradable. While our result identifies a large parameter region where quantum communication is not possible, we also prove that if two-way classical communication is available, then quantum communication—and thus quantum key distribution—is always achievable, even for high values of loss and dephasing.

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I. INTRODUCTION

Quantum optical platforms are key elements of quantum technologies, contributing significantly to both quantum communication and quantum computation [1–9]. Since the potential benefits of quantum technologies are hindered by the presence of decoherence [10], the investigation of decoherence sources affecting bosonic systems and the development of bosonic quantum error-correcting codes have been extensively analyzed in recent years [11–22]. The primary noise processes in bosonic systems that act as dominant sources of decoherence are *photon loss* and *bosonic dephasing* [23–25], which have been both extensively analyzed [13,26–28]. Loss affects the system by causing it to dissipate some of its energy, whereas dephasing works to transform coherent superpositions into probabilistic mixtures. Although loss and dephasing sources can simultaneously affect bosonic systems [15,29], such as in superconducting systems [30,31], the existing literature provides only partial results about their combined effect [32]. On a technical level, understanding the combined effect of loss and dephasing is challenging due to the conflicting behaviors they exhibit: the action of loss takes a simple form when written in the coherent state basis but is complicated to analyze in the Fock basis [2], whereas dephasing demonstrates the opposite pattern, making the analysis of their combined effect quite intricate.

Consider an optical link (e.g., an optical fiber or a free-space link) or a quantum memory affected by both loss and dephasing, where the link is used for quantum communication and the memory for quantum computation. A crucial challenge is to determine the conditions under which there exist protocols capable of enabling reliable quantum communication across the optical link or capable of mitigating the combined noise affecting the quantum memory. This problem is closely related to the antidegradability condition in quantum Shannon theory: if a noise channel is *antidegradable* [33,34], there are no quantum communication protocols for reliable information transmission or quantum error-correcting codes capable of overcoming it. Consequently, it is crucial to understand whether the combined effect of loss and dephasing results in an antidegradable channel. This has been a puzzling problem, to the point that in [32] it was conjectured that the combined loss-dephasing noise does not result in an antidegradable channel if the loss is below 50%.

In this paper we refute the above conjecture; specifically, we prove that for any value of the photon loss there exists a *critical value* of the dephasing above which the resulting *bosonic loss-dephasing channel* is antidegradable. Our discovery thus identifies a large region of the loss-dephasing parameter space where correcting the noise and achieving reliable quantum communication is impossible. On the more positive side, however, we also prove that if the sender and the receiver are assisted by two-way classical communication, then reliable quantum communication—and thus quantum key distribution—is always possible, even in scenarios

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characterized by arbitrarily high levels of loss and dephasing. The analytical results reported here characterize the transmission of quantum information in the presence of both loss and dephasing noise.

The structure of the paper is as follows. In Sec. II we briefly review some preliminary notions necessary for stating our results. In Sec. III we present our results on the antidegradability of the bosonic loss-dephasing channel, refuting a conjecture put forth in [32]. In Sec. IV we analyze quantum communication across the bosonic loss-dephasing channel assisted by two-way classical communication. In Sec. V we discuss the degradability of the loss-dephasing channel. In the Appendixes, we provide detailed derivations and additional results concerning the bosonic loss-dephasing channel.

II. PRELIMINARIES

The *quantum capacity* $Q(\mathcal{N})$ of a quantum channel $\mathcal{N}$ quantifies the efficiency in transmitting qubits reliably across $\mathcal{N}$ [33,34]. The condition $Q(\mathcal{N})=0$ implies that there exist neither reliable quantum communication protocols across $\mathcal{N}$ nor codes capable of correcting the errors induced by $\mathcal{N}$. Accordingly, if $\mathcal{N}$ is *antidegradable* its quantum capacity vanishes [33]. This underscores the significance of determining whether a channel is antidegradable, as the noise associated with such a channel cannot be corrected. By definition, a channel $\mathcal{N}$ is antidegradable if there exists a channel $\mathcal{A}$—called the *antidegrading map*—such that $\mathcal{A}\circ\mathcal{N}^{c}=\mathcal{N}$, where $\mathcal{N}^{c}$ denotes a complementary channel of $\mathcal{N}$ [33]. Conversely, a channel $\mathcal{N}$ is degradable if there exists a channel $\mathcal{D}$ such that $\mathcal{D}\circ\mathcal{N}=\mathcal{N}^{c}$. Degradable channels are theoretically important because their quantum capacity can be calculated as the single-letter coherent information of the channel [33–35].

The phenomenon of photon loss is mathematically modeled by the *pure-loss channel* $\mathcal{E}_\lambda$ [2,4], a single-mode continuous-variable channel that acts on the input state ρ by mixing it with an environmental vacuum state in a beam splitter of transmissivity $\lambda\in[0,1]$:

$$\mathcal{E}_\lambda(\rho):=\mathrm{Tr}_E[U_\lambda(\rho_S\otimes|0\rangle\langle 0|_E)U_\lambda^\dagger],\tag{1}$$

where $U_\lambda:=\exp[\arccos(\sqrt{\lambda})(\hat{a}^\dagger\hat{e}-\hat{a}\,\hat{e}^\dagger)]$ is the beam splitter unitary, $\hat{a}$ and $\hat{e}$ are the annihilation operators of the input system S and of the environment E, and Tr_E is the partial trace w.r.t. E. When a single photon is fed into $\mathcal{E}_\lambda$, it is transmitted to the output with probability λ, while it is lost to the environment with probability $1-\lambda$. More generally, if n photons are fed into the channel, the output is given by the binomial probability mixture $\mathcal{E}_\lambda(|n\rangle\langle n|)=\sum_{\ell=0}^{n}\binom{n}{l}(1-\lambda)^\ell\lambda^{n-\ell}|n-\ell\rangle\langle n-\ell|$, where $|n\rangle$ denotes the Fock state with n photons [2]. When $\lambda=1$ the pure-loss channel is noiseless, while when $\lambda=0$ it is completely noisy—it maps any state into the vacuum. It is known that the pure-loss channel is antidegradable for $\lambda\in[0,\frac{1}{2}]$ and degradable for $\lambda\in[\frac{1}{2},1]$ [36–39].

The phenomenon of bosonic dephasing is mathematically described by the *bosonic dephasing channel* $\mathcal{D}_\gamma$ [28,32,40], which maps the state $\rho=\sum_{m,n=0}^{\infty}\rho_{mn}|m\rangle\langle n|$, written in the Fock basis, to

$$\mathcal{D}_\gamma(\rho):=\sum_{m,n=0}^{\infty}\rho_{mn}e^{-\frac{\gamma}{2}(m-n)^2}|m\rangle\langle n|,\tag{2}$$

resulting in a reduction in magnitude of the off-diagonal elements. When $\gamma=0$, the bosonic dephasing channel is noiseless. In contrast, when $\gamma\to\infty$, it completely annihilates all off-diagonal components of the input density matrix, reducing it to an incoherent probabilistic mixture of Fock states. Moreover, the bosonic dephasing channel is never antidegradable and it is always degradable [40].

Consider an optical system undergoing simultaneous loss and dephasing over a finite time interval. At each instant, the system is susceptible to both an infinitesimal pure-loss channel and an infinitesimal bosonic dephasing channel. Hence, the overall channel, which describes the simultaneous effect of loss and dephasing, results in a suitable composition of numerous concatenations between infinitesimal pure-loss and bosonic dephasing channels. However, given that (i) the pure-loss channel and the bosonic dephasing channel commute, $\mathcal{E}_\lambda\circ\mathcal{D}_\gamma=\mathcal{D}_\gamma\circ\mathcal{E}_\lambda$; (ii) the composition of pure-loss channels is a pure-loss channel, $\mathcal{E}_{\lambda_1}\circ\mathcal{E}_{\lambda_2}=\mathcal{E}_{\lambda_1\lambda_2}$; and (iii) the composition of bosonic dephasing channels is a bosonic dephasing channel, $\mathcal{D}_{\gamma_1}\circ\mathcal{D}_{\gamma_2}=\mathcal{D}_{\gamma_1+\gamma_2}$; it follows that the combined effect of loss and dephasing can be modeled by the composition

$$\mathcal{N}_{\lambda,\gamma}:=\mathcal{E}_\lambda\circ\mathcal{D}_\gamma,\tag{3}$$

which we will refer to as the *bosonic loss-dephasing channel*.

III. ANTIDEGRADABILITY

Prior to this work, the only result on the antidegradability of the bosonic loss-dephasing channel was that it is antidegradable if the transmissivity is below $\frac{1}{2}$ [32]. This result trivially follows from the antidegradability of the pure-loss channel for transmissivities below $\frac{1}{2}$, and the fact that the composition of an antidegradable channel with another channel inherits the property of being antidegradable (see Lemma 55 in the Appendixes). Notably, in the regime $\lambda>\frac{1}{2}$, it was an open question to understand whether or not the bosonic loss-dephasing channel $\mathcal{N}_{\lambda,\gamma}$ is antidegradable for some values of the dephasing γ, and in [32] the answer was conjectured to be negative. However, in the forthcoming Theorem 1, we show that the latter conjecture is incorrect, specifically, we prove that for all $\lambda\in[0,1)$, if γ is sufficiently large, then $\mathcal{N}_{\lambda,\gamma}$ becomes antidegradable.

Theorem 1. The bosonic loss-dephasing channel $\mathcal{N}_{\lambda,\gamma}$ is antidegradable if the transmissivity λ and the dephasing γ fall within one of the following regions: (i) $\lambda\in[0,\frac{1}{2}]$ and $\gamma\geqslant 0$ and (ii) $\lambda\in(\frac{1}{2},1)$ and γ such that $\theta(e^{-\gamma/2},\sqrt{\frac{\lambda}{1-\lambda}})\leqslant\frac{3}{2}$, where $\theta(x,y):=\sum_{n=0}^{\infty}x^{n^2}y^n$. A weaker but simpler sufficient condition that implies antidegradability is given by $\lambda\leqslant\max(\frac{1}{2},\frac{1}{1+9e^{-\gamma}})$.

Here we present a sketch of the proof, with a detailed version of the proof provided in Theorem 30 in the Appendixes.

Proof sketch. Any finite dimensional channel $\mathcal{N}$ is antidegradable if and only if its Choi state is *two-extendible* [41], meaning that there exists a tripartite state $\rho_{AB_1B_2}$ such that the reduced states on AB_1 and AB_2 both coincide with the Choi state:

$$\mathrm{Tr}_{B_2}\left[\rho_{AB_1B_2}\right] = C_{AB_1}(\mathcal{N}),$$
$$\mathrm{Tr}_{B_1}\left[\rho_{AB_1B_2}\right] = C_{AB_2}(\mathcal{N}), \tag{4}$$

where the Choi state is defined as $C_{AB}(\mathcal{N}) := \mathrm{id}_A \otimes \mathcal{N}_{A'\to B}(\Phi_{AA'})$, with $\Phi_{AA'}$ being the maximally entangled state. Such a characterization extends to infinite dimension by considering the generalized Choi state $C^{(r)}_{AB}(\mathcal{N}) := \mathrm{id}_A \otimes \mathcal{N}_{A'\to B}(\Psi^{(r)}_{AA'})$ [42], obtained by replacing $\Phi_{AA'}$ by the two-mode squeezed vacuum state $\Psi^{(r)}_{AA'}$ with squeezing parameter $r > 0$ [2]. The crux of our proof is to find a two-extension of $C^{(r)}_{AB}(\mathcal{N}_{\lambda,\gamma})$ in the region identified by condition (ii). We do this in two steps.

First, after scrutinizing the matrix $C^{(r)}_{AB}(\mathcal{N}_{\lambda,\gamma})$ written in the Fock basis, we construct a tripartite state $\tau_{AB_1B_2}$ such that the reduced states on AB_1 and AB_2 have the same diagonal as $C^{(r)}_{AB}(\mathcal{N}_{\lambda,\gamma})$, and the same pattern of vanishing off-diagonal entries. The construction of this tripartite state involves applying several channels—namely, beam splitter unitaries, squeezing unitary, partial trace, and a three-mode controlled-add-add isometry—to a four-mode vacuum state.

The second step consists in transforming $\tau_{AB_1B_2}$ into a two-extension of $C^{(r)}_{AB}(\mathcal{N}_{\lambda,\gamma})$ by tweaking its off-diagonal entries. This is done by using the toolbox of *Hadamard maps* [34]. For any matrix $A := (a_{mn})_{m,n\in\mathbb{N}}$, the associated Hadamard map $H^{(A)}$ is defined by

$$H^{(A)}(|m\rangle\langle n|) = a_{mn}|m\rangle\langle n| \tag{5}$$

for all m, n [34]. In practice, $H^{(A)}$ acts on the input density matrix by multiplying each (m, n) entry by the corresponding coefficient a_{mn}. Importantly, $H^{(A)}$ is a quantum channel if and only if A is Hermitian, positive semidefinite, and has all 1's on the main diagonal [34]. The crucial observation is that it is always possible to find an infinite matrix $A_{\lambda,\gamma}$ (possibly not positive semidefinite), which is real, symmetric, and has all 1's on the main diagonal, such that the operator $\mathrm{id}_A \otimes H^{(A_{\lambda,\gamma})}_{B_1} \otimes H^{(A_{\lambda,\gamma})}_{B_2}(\tau_{AB_1B_2})$ coincides with $C^{(r)}_{AB}(\mathcal{N}_{\lambda,\gamma})$ when tracing out either B_1 or B_2.

This, however, does not mean that we have found a two-extension of $C^{(r)}_{AB}$, because the above operator is not necessarily a state—it may fail to be positive semidefinite. It is a state, however, whenever $H^{(A_{\lambda,\gamma})}$ is a quantum channel, i.e., when the infinite matrix $A_{\lambda,\gamma}$ is positive semidefinite, in formula $A_{\lambda,\gamma} \geqslant 0$. Therefore, a sufficient condition on the antidegradability of $\mathcal{N}_{\lambda,\gamma}$ is that $A_{\lambda,\gamma} \geqslant 0$.

The rest of the proof consists in showing that under condition (ii) one indeed finds $A_{\lambda,\gamma} \geqslant 0$. This is not straightforward to check, because $A_{\lambda,\gamma}$ is an *infinite* matrix, and it cannot be diagonalized analytically nor numerically. To bypass this last hurdle we employ the theory of diagonally dominant matrices, and in particular the statement that if a matrix A is such that $a_{nn} - \sum_{m:\, m\neq n} |a_{mn}| \geqslant 0$ for all n, then necessarily $A \geqslant 0$ [[43], Chapter 6]. We demonstrate that if λ and γ satisfy condition (ii), then $A_{\lambda,\gamma}$ satisfies this condition, which establishes that $A_{\lambda,\gamma} \geqslant 0$ and hence concludes the proof. For a more detailed proof, refer to Theorem 30 in the Appendixes.■

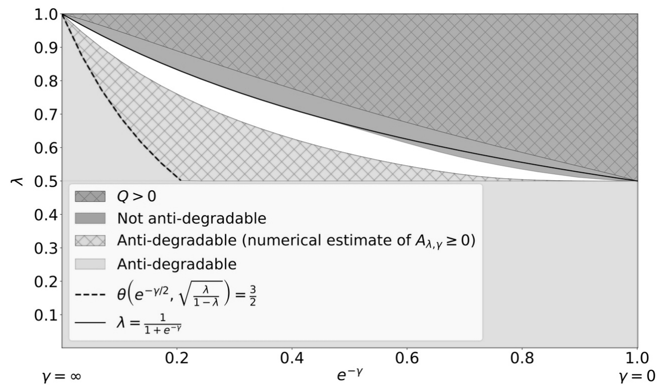

FIG. 1. Summary of results on the antidegradability of the bosonic loss-dephasing channel $\mathcal{N}_{\lambda,\gamma}$. The vertical axis represents the transmissivity λ, and the horizontal axis corresponds to $e^{-\gamma}$, where γ is the dephasing parameter. In the light gray region $\mathcal{N}_{\lambda,\gamma}$ is not antidegradable, and in the dark gray region it is antidegradable. In the crossed light gray region, the quantum capacity of $\mathcal{N}_{\lambda,\gamma}$ is strictly positive. The crossed dark gray region is a numerical estimate of the region where the infinite matrix $A_{\lambda,\gamma}$ is positive semidefinite, a condition implying that $\mathcal{N}_{\lambda,\gamma}$ is antidegradable, as explained in the proof sketch of Theorem 1. Such an estimate can be obtained by examining the positive semidefiniteness of the $d \times d$ top left corner of $A_{\lambda,\gamma}$ for large values of d (here we employ $d = 30$, but increasing d already beyond $d \geqslant 20$ yields no discernible change in the plot). The restriction $\mathcal{N}^{(6)}_{\lambda,\gamma}$ is antidegradable if and only if λ and γ fall within the light gray region, and this is the reason why $\mathcal{N}_{\lambda,\gamma}$ is not antidegradable in the light gray region. Below the curve $\theta(e^{-\gamma/2}, \sqrt{\frac{\lambda}{1-\lambda}}) = \frac{3}{2}$, the channel $\mathcal{N}_{\lambda,\gamma}$ is antidegradable, as stated in Theorem 1. Above the curve $\lambda = \frac{1}{1+e^{-\gamma}}$, the channel is not antidegradable, as guaranteed by Theorem 2.

Theorem 1 identifies a region of the parameter space (λ, γ), with λ identifying the transmissivity and γ the dephasing, where the channel is antidegradable and thus its quantum capacity vanishes, thereby implying the absence of viable error correcting codes for quantum data transfer and storage. This region is illustrated in Fig. 1. Interestingly, Theorem 1 implies that even if $\lambda > \frac{1}{2}$ one can pick γ large enough so that there exists an antidegrading map achieving the transformation $\mathcal{N}^{c}_{\lambda,\gamma}(|n\rangle\langle n|_F) \longrightarrow \mathcal{N}_{\lambda,\gamma}(|n\rangle\langle n|_F)$, which can be expressed as (see Lemma 27 in the Appendixes)

$$\mathcal{E}_{1-\lambda}(|n\rangle\langle n|_F) \otimes |\sqrt{\gamma}n\rangle\langle\sqrt{\gamma}n|_C \longrightarrow \mathcal{E}_{\lambda}(|n\rangle\langle n|_F), \tag{6}$$

where $|n\rangle_F$ denotes the nth Fock state and $|\sqrt{\gamma}n\rangle_C$ denotes a coherent state [2]. In Theorem 32 in the Appendixes, we provide an explicit construction of such an antidegrading map. This entails the following remarkable fact: for $\lambda > 1/2$ and large enough γ there exists an *n-independent* strategy to convert the lossy Fock state $\mathcal{E}_{1-\lambda}(|n\rangle\langle n|_F)$ into the less lossy Fock state $\mathcal{E}_{\lambda}(|n\rangle\langle n|_F)$ using the coherent state $|\sqrt{\gamma}n\rangle_C$ as a resource. In other words, one can undo part of the loss on $|n\rangle_F$ if one has a coherent state that contains some information on n, sufficiently amplified so that that information is accessible enough. The nontrivial and somewhat surprising nature of this exact conversion strategy arises from the fact that the coherent states $\{|\sqrt{\gamma}n\rangle_C\}_{n\in\mathbb{N}}$ are not orthogonal, meaning that the strategy that consists in measuring the coherent state, guessing n, and repreparing $\mathcal{E}_\lambda(|n\rangle\langle n|_F)$ cannot succeed with probability 1.

Theorem 1 does not identify the entire antidegradability region of $\mathcal{N}_{\lambda,\gamma}$, but only a subset of it. One way to improve this approximation is to determine numerically the region where the infinite matrix $A_{\lambda,\gamma}$ introduced in the proof sketch of Theorem 1 is positive semidefinite. In Fig. 1 we depict a numerical estimate of this region (see the crossed dark gray part of the plot).

So far we have been concerned with inner approximations of the antidegradability region. To obtain outer approximations, instead, one can start by observing that the action of $\mathcal{N}_{\lambda,\gamma}$ can only subtract and never add any photons. Mathematically, if the input state to $\mathcal{N}_{\lambda,\gamma}$ is supported on the span of the first d Fock states, so is the output state. This restriction defines a qudit-to-qudit channel $\mathcal{N}^{(d)}_{\lambda,\gamma}$, analyzing which can yield some insights into $\mathcal{N}_{\lambda,\gamma}$ itself. First, if $\mathcal{N}^{(d)}_{\lambda,\gamma}$ is *not* antidegradable, then the same is true of $\mathcal{N}_{\lambda,\gamma}$ (see Corollary 35 in the Appendixes); second, as discussed above the antidegradability of $\mathcal{N}^{(d)}_{\lambda,\gamma}$ is equivalent to the two-extendibility of the corresponding Choi state [41], and for moderate values of d this latter condition can be efficiently checked numerically via *semidefinite programming* [34,44]. In this way, as discussed in Sec. B 2 of Appendix B, we can numerically determine a parameter region (see light gray region of Fig. 1) where $\mathcal{N}_{\lambda,\gamma}$ is not antidegradable.

Interestingly, already the qubit restriction $\mathcal{N}^{(2)}_{\lambda,\gamma}$, which coincides with the composition between the amplitude damping channel and the qubit dephasing channel [34], yields the necessary condition $\lambda \leqslant \frac{1}{1+e^{-\gamma}}$ on the antidegradability of $\mathcal{N}_{\lambda,\gamma}$, as shown in the forthcoming Theorem 2. Based on the analysis of the qudit restrictions, we conjecture that, if γ is sufficiently large, the latter condition $\lambda \leqslant \frac{1}{1+e^{-\gamma}}$ is not only necessary but also sufficient.

Theorem 2. If $\lambda > \frac{1}{1+e^{-\gamma}}$, then $\mathcal{N}_{\lambda,\gamma}$ is not antidegradable.

Proof. A qubit channel $\mathcal{N}$ is antidegradable if and only if

$$\tfrac{1}{4}\mathrm{Tr}[\mathcal{N}(\mathbb{1}_2)^2] \geqslant \mathrm{Tr}[C(\mathcal{N})^2] - 4\sqrt{\det[C(\mathcal{N})]}, \tag{7}$$

where $C(\mathcal{N})$ is the Choi state [41,45,46]. By employing the condition in (7) together with the expression of the Choi state of the qubit restriction $\mathcal{N}^{(2)}_{\lambda,\gamma}$ provided in (B29) in the Appendixes, one can show that $\mathcal{N}^{(2)}_{\lambda,\gamma}$ is antidegradable if and only if $\lambda \leqslant \frac{1}{1+e^{-\gamma}}$. Hence, we conclude that the bosonic loss-dephasing channel $\mathcal{N}_{\lambda,\gamma}$ is not antidegradable if $\lambda > \frac{1}{1+e^{-\gamma}}$. ■

We are interested in the antidegradability of the loss-dephasing channel because it implies that the quantum capacity vanishes, entailing the impossibility of quantum communication. We now look at the complementary question: when is the quantum capacity $Q(\mathcal{N}_{\lambda,\gamma})$ strictly positive? A simple sufficient condition can be obtained by optimizing the coherent information [33,34] of $\mathcal{N}_{\lambda,\gamma}$ over input states of the form $\rho_p := p|0\rangle\langle 0| + (1-p)|1\rangle\langle 1|$. By doing so we identify a region of the (λ,γ) parameter space where $Q(\mathcal{N}_{\lambda,\gamma}) > 0$ (see the crossed light gray region in Fig. 1). In this region quantum communication and quantum error correction become feasible.

IV. TWO-WAY QUANTUM COMMUNICATION

As we have just seen, (unassisted) quantum communication is not possible when the combined effects of loss and dephasing are too strong. However, in the forthcoming Theorem 3 we show that if Alice (the sender) and Bob (the receiver) have access to a *two-way* classical communication line, then quantum communication, entanglement distribution, and quantum-key distribution [47] become again achievable for any value of loss and dephasing, even when Alice's input signals are constrained to have limited energy.

In this two-way communication setting the relevant notion of capacity is the *two-way quantum capacity* $Q_2(\mathcal{N})$ [33,34], defined as the maximum achievable rate of qubits that can be reliably transmitted across $\mathcal{N}$ with the aid of two-way classical communication. Since in practice Alice has only a limited amount of energy to produce her input signals, one usually defines the so-called *energy-constrained* two-way quantum capacity [48,49], denoted as $Q_2(\mathcal{N}, N_s)$. Here N_s denotes the mean photon number constraint at the input of the channel.

Theorem 3. For all $N_s > 0$, $\lambda \in (0,1]$, and $\gamma \geqslant 0$, the energy-constrained two-way quantum capacity of the bosonic loss-dephasing channel is strictly positive, i.e., $Q_2(\mathcal{N}_{\lambda,\gamma}, N_s) > 0$. In particular, $\mathcal{N}_{\lambda,\gamma}$ is not entanglement breaking. An explicit lower bound is

$$Q_2(\mathcal{N}_{\lambda,\gamma}, N_s) \geqslant \frac{\lambda^N}{k}\left[\log_2\binom{N+k-1}{N} - S(\rho_{N,k,\gamma})\right] \tag{8}$$

for any $N, k \in \mathbb{N}_+$ satisfying $\frac{N}{k} \leqslant N_s$. Here $S(\cdot)$ is the von Neumann entropy, $\rho_{N,k,\gamma}$ is a $\binom{N+k-1}{N}$-dimensional state defined by

$$\rho_{N,k,\gamma} := \binom{N+k-1}{N}^{-1} \sum_{p,q\in\Pi(N,k)} e^{-\frac{\gamma}{2}\|p-q\|_2^2}|p\rangle\langle q|, \tag{9}$$

where $\Pi(N,k) := \{p \in \mathbb{N}^k : \sum_{i=1}^k p_i = N\}$ represents the set of partitions of a set of N elements into k parts, and the vectors $\{|p\rangle\}_{p\in\Pi(N,k)}$ are orthonormal.

Here we provide a sketch of the proof, and in Theorem 44 in the Appendixes we provide a detailed proof.

Proof sketch. The proof exploits entanglement transmission protected by a particular error-correction technique, *rail encoding*. In a k-mode bosonic system, consider the subspace $V_{N,k}$ corresponding to a total photon number N. We can use this subspace, whose dimension is $d_{N,k} := \dim V_{N,k} = \binom{N+k-1}{N}$, as an error correction code that protects against the detrimental action of $\mathcal{N}_{\lambda,\gamma}$. To this end, we prepare a maximally entangled state of dimension $d_{N,k}$ and send one share of it through k copies of the channel $\mathcal{N}_{\lambda,\gamma}$, one per mode. Since under the action of $\mathcal{N}_{\lambda,\gamma}$ photons can only be lost and never added, and each photon has a probability λ of being transmitted, the probability that an N-photon state will retain all of its photons at the output of the channel is exactly λ^N. If this happens to be the case, which—crucially—can be certified by a total photon number measurement at the output, then the input state has been subjected to no loss and only dephasing. The entanglement of the resulting, maximally correlated state can be distilled via an explicit protocol known as the hashing protocol [34,50], resulting in $\log_2 d_{N,k} - S(\rho_{N,k,\gamma}) > 0$ singlet (a.k.a. ebit, i.e., unit of entanglement) yield. The strict positivity of this yield follows by observing that $\rho_{N,k,\gamma}$ is a $d_{N,k}$-dimensional mixed state that is not maximally mixed. ■

V. DEGRADABILITY

The bosonic loss-dephasing channel $\mathcal{N}_{\lambda,\gamma}$ is never degradable, except in the simple cases when either $\lambda = 1$ or $\gamma = 0$ and $\lambda \geqslant 1/2$ [32]. This in turn implies that no single-letter formula for its quantum capacity is known outside of the antidegradability region studied here, where the capacity vanishes. The failure of degradability has been demonstrated in [32] through a lengthy proof; we are now in position to provide an alternative, much simpler argument. The key ideas are as follows: (i) If the qubit restriction $\mathcal{N}^{(2)}_{\lambda,\gamma}$ is not degradable, then $\mathcal{N}_{\lambda,\gamma}$ is not degradable either (see Corollary 35 in the Appendixes); and (ii) if the rank of the Choi state of a qubit channel is greater or equal to 3 than such channel is not degradable [[51], Theorem 4]. The result then follows by observing that the Choi state of the qubit channel $\mathcal{N}^{(2)}_{\lambda,\gamma}$, as provided in (B29) in the Appendixes, has a rank exactly equal to 3. For a detailed proof see Theorem 37 in the Appendixes.

VI. CONCLUSION

In this paper we have provided an analytical investigation of the quantum communication capabilities of the bosonic loss-dephasing channel, a much more realistic model of noise than dephasing and loss treated separately. Refuting a conjecture put forth in [32], we showed that the bosonic loss-dephasing channel is antidegradable in a large region of the loss-dephasing parameter space, entailing that neither quantum communication nor quantum error correcting codes are possible in this region. On the positive side, we also showed that if two-way classical communication is suitably exploited, then quantum communication is always achievable, even in scenarios characterized by high levels of loss and dephasing, and even in the presence of stringent energy constraints.

Two fundamental technical innovations are key to our approach. First, an alternative method to analyze antidegradability of bosonic channels, based on a two-stage construction of a symmetric extension of the Choi state. The introduction of this technique is crucial here also on the conceptual level, as *all* other known tools to analyze quantum capacities (e.g., degradability [34], PPT-ness [52], or teleportation simulability [53,54]) fail completely for the loss-dephasing channel [32]. In Appendix C we envision that our technique could also be applied to other cases, e.g., to analyze the antidegradability of the composition between the pure-loss channel and a *general* bosonic dephasing channel. The second innovation we introduce is based on the use of rail encoding to investigate two-way assisted entanglement generation on the loss-dephasing channel. This technique, which we anticipate may be used to study general processes where photon loss is involved, has the additional benefit of yielding an explicit entanglement generation protocol.

Although the capacities of the dephasing channel and the pure-loss channel (separately) have been determined [26,28,38,39,49,53,55,56], the capacities of the channel resulting from their combined action remain unknown. An intriguing open problem is to calculate or approximate these capacities.

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APPENDIX A: PRELIMINARIES AND NOTATION

1. Quantum states and channels

In this subsection, we present a summary of the notation and fundamental properties used in the paper, drawing from the conventions established in standard quantum information theory textbooks [33,34,44,57]. Every quantum system is associated with a separable complex Hilbert space $\mathcal{H}$ whose dimension is denoted by $|\mathcal{H}|$. We use subscripts to denote the system associated to a Hilbert space and also systems on which the operators act. The composite quantum systems A and B exist within the tensor product of their individual Hilbert spaces $\mathcal{H}_A \otimes \mathcal{H}_B$, which is also denoted by $\mathcal{H}_{AB}$.

We use $\mathbb{1}$ to denote the identity operator on $\mathcal{H}$. The operator norm of a linear operator $\Theta : \mathcal{H} \to \mathcal{H}$ is defined as

$$\|\Theta\|_\infty := \sup_{|\psi\rangle\in\mathcal{H}:\, \langle\psi|\psi\rangle=1} \sqrt{\langle\psi|\Theta^\dagger\Theta|\psi\rangle}. \tag{A1}$$

An alternative (but equivalent) definition of the operator norm is as follows:

$$\|\Theta\|_\infty := \sup_{|v\rangle,|w\rangle\in\mathcal{H},\, \langle v|v\rangle=\langle w|w\rangle=1} |\langle v|\Theta|w\rangle|. \tag{A2}$$

An operator is called bounded if its operator norm is bounded, i.e., $\|\Theta\|_\infty < \infty$. A bounded operator Θ is positive semidefinite if $\langle\psi|\Theta|\psi\rangle \geqslant 0, \forall\, |\psi\rangle \in \mathcal{H}$, while it is positive definite if $\langle\psi|\Theta|\psi\rangle > 0, \forall\, |\psi\rangle \in \mathcal{H}$. The trace norm of a linear operator $\Theta : \mathcal{H} \to \mathcal{H}$ is defined as $\|\Theta\|_1 := \mathrm{Tr}\sqrt{\Theta^\dagger\Theta}$. The set of trace class operators, denoted as $\mathcal{T}(\mathcal{H})$, is the set of all the linear operators on $\mathcal{H}$ such that their trace norm is finite, i.e., $\|\Theta\|_1 < \infty$. The operator and trace norm satisfy $\|\Theta\|_\infty \leqslant \|\Theta\|_1$. The set of quantum states (density operators), denoted as $\mathcal{P}(\mathcal{H})$, is the set of positive semidefinite trace class operators on $\mathcal{H}$ with unit trace. The *fidelity* between two quantum states $\rho, \sigma \in \mathcal{P}(\mathcal{H})$ is defined as $F(\rho, \sigma) := \mathrm{Tr}[\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}]$.

A superoperator is a linear map between spaces of linear operators. The identity superoperator will be denoted as id. Quantum channels are completely positive trace-preserving (cptp) superoperators. In this paper we will use two different representations of a quantum channel that are known as Stinespring and Choi-Jamiołkowski representation. A quantum channel $\mathcal{N}_{A'\to B}$ can be represented in Stinespring representation as

$$\mathcal{N}_{A'\to B}(\cdot) = \mathrm{Tr}_E[U_{A'E\to BE}(\cdot \otimes |0\rangle\langle 0|_E)U^\dagger_{A'E\to BE}].$$

Here E is an environment system, $|0\rangle_E$ is a pure state of the environment, and $U_{A'E\to BE}$ is an isometry that takes as input the systems A' and E and outputs the systems B, E. The associated complementary channel $\mathcal{N}^{c}_{A'\to B}$ is given by

$$\mathcal{N}^{c}_{A'\to E}(\cdot) = \mathrm{Tr}_B[U_{A'E\to BE}(\cdot \otimes |0\rangle\langle 0|_E)U^{\dagger}_{A'E\to BE}].$$

A channel $\mathcal{N}$ is called degradable if there exist a quantum channel $\mathcal{J}$, such that when is used after $\mathcal{N}$, we get back to the complementary channel $\mathcal{N}^{c}$, i.e., $\mathcal{J}\circ\mathcal{N} = \mathcal{N}^{c}$. On the other hand, a channel is called antidegradable if there is another quantum channel $\mathcal{A}$, such that using it after the complementary channel, gives back the original channel, i.e., $\mathcal{A}\circ\mathcal{N}^{c} = \mathcal{N}$. The channels $\mathcal{J}$ and $\mathcal{A}$ are usually called the degrading map and the antidegrading map of $\mathcal{N}$, respectively.

The Choi-Jamiołkowski representation of the channel $\mathcal{N}_{A'\to B}$ is the operator $C(\mathcal{N}) \in \mathcal{T}(\mathcal{H}_A \otimes \mathcal{H}_B)$ that is defined as

$$C(\mathcal{N}) := \mathrm{id}_A \otimes \mathcal{N}_{A'\to B}(|\Phi\rangle\langle\Phi|_{AA'}), \tag{A3}$$

where $|\Phi\rangle = \frac{1}{\sqrt{|\mathcal{H}_A|}}\sum_{i=0}^{|\mathcal{H}_A|-1}|i\rangle_A \otimes |i\rangle_{A'}$ is a maximally entangled state of Schmidt rank $|\mathcal{H}_A|$, the set of states $\{|i\rangle\}_{i=0,\dots,|\mathcal{H}_A|-1}$ forms a basis for $\mathcal{H}_A$, and $\mathcal{H}_A = \mathcal{H}_{A'}$. It is a well-established fact that the superoperator $\mathcal{N}$ is a quantum channel if and only if $C(\mathcal{N}) \geqslant 0$ and $\mathrm{Tr}_B C(\mathcal{N}) = \mathbb{1}_A/|\mathcal{H}_A|$ [33].

Definition 4. A bipartite state ρ_{AB} is symmetric two-extendible on B if there exists a tripartite state $\tau_{AB_1B_2}$ such that

$$F_{B_1B_2}\tau_{AB_1B_2}F^{\dagger}_{B_1B_2} = \tau_{AB_1B_2},$$

$$\mathrm{Tr}_{B_1}\tau_{AB_1B_2} = \rho_{AB},$$

where B_1 and B_2 are two copies of the system B, the operator $F_{B_1B_2} := \sum_{i,j}|i\rangle\langle j|_{B_1}\otimes|j\rangle\langle i|_{B_2}$ denotes the swap unitary, and $\{|i\rangle_{B_1}\}_i$ and $\{|i\rangle_{B_2}\}_i$ form an orthonormal basis. The state $\tau_{AB_1B_2}$ is called a symmetric two-extension of ρ_{AB} on B.

Definition 5. A bipartite state ρ_{AB} is called two-extendible on B if there exists a tripartite state $\tau_{AB_1B_2}$ such that

$$\mathrm{Tr}_{B_1}\tau_{AB_1B_2} = \mathrm{Tr}_{B_2}\tau_{AB_1B_2} = \rho_{AB}, \tag{A4}$$

where B_1 and B_2 are two copies of the system B.

Lemma 6 ([58]). A bipartite state ρ_{AB} is two-extendible on B if and only if it is symmetric two-extendible on B.

Proof. First, assume ρ_{AB} is symmetric two-extendible on B. Since $F_{B_1B_2}\tau_{AB_1B_2}F^{\dagger}_{B_1B_2} = \tau_{AB_1B_2}$, it holds that $\mathrm{Tr}_{B_2}\tau_{AB_1B_2} = \mathrm{Tr}_{B_1}\tau_{AB_1B_2}$. This implies that ρ_{AB} is two-extendible on B. Second, let ρ_{AB} be two-extendible on B. One can easily check that the state $1/2(\tau_{AB_1B_2} + F_{B_1B_2}\tau_{AB_1B_2}F^{\dagger}_{B_1B_2})$ is a symmetric two-extension of ρ_{AB}. ■

It has been demonstrated that a quantum channel is antidegradable if and only if its Choi state is two-extendible on the output system [41]. This equivalence leads to a simple necessary and sufficient condition for the antidegradability of qubit channels:

Lemma 7. [[45], Corollary 4] (see also [41,46]) Any qubit quantum channel $\mathcal{N}$ is antidegradable if and only if it satisfies

$$\mathrm{Tr}\left\{\left[\mathcal{N}\left(\frac{\mathbb{1}_2}{2}\right)\right]^2\right\} \geqslant \mathrm{Tr}[(C(\mathcal{N}))^2] - 4\sqrt{\det(C(\mathcal{N}))},$$

where $\mathbb{1}_2$ denotes the identity operator on the qubit Hilbert space.

2. Bosonic systems

In this subsection, we will provide an overview of relevant definitions and properties concerning quantum information with continuous variable systems; refer to [2] for detailed explanations. A single-mode of electromagnetic radiation with definite frequency and polarization is represented by the Hilbert space $L^2(\mathbb{R})$, which comprises all square-integrable complex-valued functions over $\mathbb{R}$. Let $\mathbb{N}_+$ be the set of strictly positive integers and let $\mathbb{N} := \{0\}\cup\mathbb{N}_+$. For any $n \in \mathbb{N}$, the construction of the *Fock state* $|n\rangle$ (the quantum state with n photons) involves the application of the bosonic creation operator $\hat{a}^{\dagger}$ to the *vacuum state* $|0\rangle$:

$$|n\rangle := \frac{1}{\sqrt{n!}}(\hat{a}^{\dagger})^n|0\rangle. \tag{A5}$$

The Fock states $\{|n\rangle\}_{n\in\mathbb{N}}$ form an orthonormal basis of $L^2(\mathbb{R})$. The bosonic annihilation operator $\hat{a}$ and creation operator $\hat{a}^{\dagger}$ satisfy the well-known canonical commutation relation $[\hat{a}, \hat{a}^{\dagger}] = \mathbb{1}$.

Let $\mathbb{C}$ be the set of complex numbers. For any $\alpha \in \mathbb{C}$, let $D(\alpha) := e^{\alpha\hat{a}^{\dagger}-\alpha^*\hat{a}}$ be the displacement operator. A coherent state of parameter α, denoted by $|\alpha\rangle$, is defined by applying the displacement operator $D(\alpha)$ to the vacuum state, i.e., $|\alpha\rangle := D(\alpha)|0\rangle$. The overlap between coherent states is given by

$$\langle\alpha|\beta\rangle = e^{-\frac{1}{2}(|\alpha|^2+|\beta|^2-2\alpha^*\beta)}. \tag{A6}$$

Quantum channels acting on bosonic systems are sometimes called bosonic channels. Similar to finite-dimensional channels, bosonic channels admit a Choi-Jamiołkowski representation, usually referred to as *generalized* Choi-Jamiołkowski representation [59]. Consider isomorphic Hilbert spaces $\mathcal{H}_A, \mathcal{H}_{A'}$ which are possibly infinite dimensional. Let $|\psi\rangle_{AA'}$ be a pure state satisfying $\mathrm{Tr}_{A'}[|\psi\rangle\langle\psi|_{AA'}] > 0$ (See [[60], Lemma 26]). The generalized Choi state is constructed by applying the channel to the subsystem A' of $|\psi\rangle_{A'A}$ (see Lemma 53 in the Appendixes): $\mathrm{id}_A \otimes \mathcal{N}_{A'\to B}(|\psi\rangle\langle\psi|_{AA'})$. The construction of the generalized Choi state usually utilises the two-mode squeezed vacuum state with squeezing parameter $r > 0$, defined as follows [2]:

$$|\psi(r)\rangle_{AA'} := \frac{1}{\cosh(r)}\sum_{n=0}^{\infty}\tanh^n(r)|n\rangle_A|n\rangle_{A'}. \tag{A7}$$

The equivalence between antidegradability of a channel and two-extendibility of its Choi state extends to the infinite-dimensional channels [42]. We provide a detailed proof of this equivalence in Lemma 54 in these Appendixes as it helps us in developing our intuition in inventing an explicit example of an antidegrading map of the bosonic loss-dephasing channel.

3. Hadamard maps

Let $A = (a_{mn})_{m,n\in\mathbb{N}}$, $a_{mn} \in \mathbb{C}$, be an infinite matrix of complex numbers. Consider the superoperator H on $\mathcal{T}(L^2(\mathbb{R}))$, recognized as the *Hadamard map*, whose action on rank one operator $|m\rangle\langle n|$ is defined as follows:

$$H(|m\rangle\langle n|) = a_{mn}|m\rangle\langle n|, \quad \forall\, m, n \in \mathbb{N}.$$

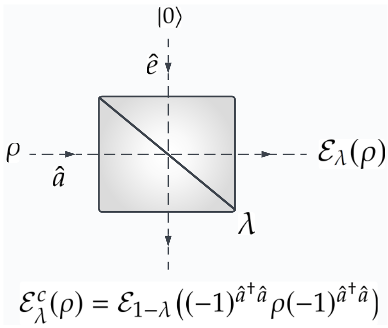

FIG. 2. Stinespring representation of the pure-loss channel $\mathcal{E}_\lambda$. The pure-loss channel $\mathcal{E}_\lambda$ acts on the input state ρ by mixing it in a beam splitter of transmissivity λ (represented by the gray box) with an environmental vacuum state $|0\rangle$. Moreover, $\hat{a}$ and $\hat{e}$ are the annihilation operators of the input mode and the environmental mode, respectively. The complementary channel of the pure-loss channel is given by $\mathcal{E}^c_\lambda(\rho) = \mathcal{E}_{1-\lambda}[(-1)^{\hat{a}^\dagger\hat{a}}\rho(-1)^{\hat{a}^\dagger\hat{a}}]$.

In Sec. E 1 in Appendix E, we provide an overview of relevant properties of Hadamard maps. In particular, by combining known results about Hadamard maps and matrix analysis, in Lemma 51 in the Appendixes, we show that given an infinite matrix $A = (a_{mn})_{m,n\in\mathbb{N}}$, $a_{mn} \in \mathbb{C}$, the associated Hadamard map is a quantum channel if

A is Hermitian.

$a_{nn} = 1, \quad \forall n \in \mathbb{N}.$

A is diagonally dominant, i.e., $\sum_{m=0 m\neq n}^{\infty} |a_{mn}| \leqslant 1,\ \forall n \in \mathbb{N}.$

4. Beam splitter

A beam splitter serves as a linear optical tool employed for creating quantum entanglement between two modes, referred to as the *system* mode (denoted as S) and the *environment* mode (denoted as E). A depiction of a beam splitter is reported in Fig. 2.

Definition 8. Let $\mathcal{H}_S, \mathcal{H}_E := L^2(\mathbb{R})$. Let $\hat{a}$ and $\hat{e}$ denote the annihilation operator of $\mathcal{H}_S$ and $\mathcal{H}_E$, respectively. For all $\lambda \in [0, 1]$, the beam splitter unitary of transmissivity λ is given by

$$U_\lambda^{SE} := \exp[\arccos\sqrt{\lambda}(\hat{a}^\dagger\hat{e} - \hat{a}\,\hat{e}^\dagger)]. \tag{A8}$$

Lemma 9. For all $\lambda \in [0, 1]$, it holds that

$$\begin{aligned}
\left(U_\lambda^{SE}\right)^\dagger \hat{a}\, U_\lambda^{SE} &= \sqrt{\lambda}\,\hat{a} + \sqrt{1-\lambda}\,\hat{e},\\
U_\lambda^{SE} \hat{a}\left(U_\lambda^{SE}\right)^\dagger &= \sqrt{\lambda}\,\hat{a} - \sqrt{1-\lambda}\,\hat{e},\\
\left(U_\lambda^{SE}\right)^\dagger \hat{e}\, U_\lambda^{SE} &= -\sqrt{1-\lambda}\,\hat{a} + \sqrt{\lambda}\,\hat{e},\\
U_\lambda^{SE} \hat{e}\left(U_\lambda^{SE}\right)^\dagger &= \sqrt{1-\lambda}\,\hat{a} + \sqrt{\lambda}\,\hat{e}.
\end{aligned} \tag{A9}$$

Proof. These identities can be readily proved by applying the Baker-Campbell-Hausdorff formula. For an alternative proof see [[21], Lemma A.2]. ■

Lemma 10. For all $\lambda \in [0, 1]$ and all $n \in \mathbb{N}$, it holds that

$$U_\lambda^{SE}|n\rangle_S \otimes |0\rangle_E = \sum_{l=0}^{n}(-1)^l\sqrt{\binom{n}{l}}\lambda^{\frac{n-l}{2}}(1-\lambda)^{\frac{l}{2}}|n-l\rangle_S \otimes |l\rangle_E, \tag{A10}$$

$$U_\lambda^{SE}|0\rangle_S \otimes |n\rangle_E = \sum_{l=0}^{n}\sqrt{\binom{n}{l}}(1-\lambda)^{\frac{l}{2}}\lambda^{\frac{n-l}{2}}|l\rangle_S \otimes |n-l\rangle_E. \tag{A11}$$

Proof. Due to Lemma (9), we have that $U_\lambda^{SE}\hat{a}(U_\lambda^{SE})^\dagger = \sqrt{\lambda}\hat{a} - \sqrt{1-\lambda}\hat{e}$. Consequently, it holds that

$$\begin{aligned}
U_\lambda^{SE}|n\rangle_S \otimes |0\rangle_E &= \frac{1}{\sqrt{n!}}U_\lambda^{SE}(a^\dagger)^n|0\rangle_S \otimes |0\rangle_E\\
&= \frac{1}{\sqrt{n!}}\left(U_\lambda^{SE}a^\dagger\left(U_\lambda^{SE}\right)^\dagger\right)^n|0\rangle_S \otimes |0\rangle_E\\
&= \frac{1}{\sqrt{n!}}\left(\sqrt{\lambda}a^\dagger - \sqrt{1-\lambda}b^\dagger\right)^n|0\rangle_S \otimes |0\rangle_E\\
&= \frac{1}{\sqrt{n!}}\sum_{l=0}^{n}(-1)^l\binom{n}{l}\lambda^{\frac{n-l}{2}}(1-\lambda)^{\frac{l}{2}}(a^\dagger)^{n-l}\\
&\quad \times |0\rangle_S \otimes (b^\dagger)^l|0\rangle_E\\
&= \sum_{l=0}^{n}(-1)^l\sqrt{\binom{n}{l}}\lambda^{\frac{n-l}{2}}(1-\lambda)^{\frac{l}{2}}|n-l\rangle_S\\
&\quad \otimes |l\rangle_E.
\end{aligned} \tag{A12}$$

Analogously, one can show the validity of (A11) by exploiting $U_\lambda^{SE}\hat{e}\,(U_\lambda^{SE})^\dagger = \sqrt{1-\lambda}\,\hat{a} + \sqrt{\lambda}\,\hat{e}$. ■

Remark 11. It is easily seen that Eq. (A10) is equivalent to

$$U_\lambda^{SE}|n\rangle_S \otimes |0\rangle_E = (-1)^n\sum_{l=0}^{n}(-1)^l\sqrt{\binom{n}{l}}\lambda^{\frac{l}{2}}(1-\lambda)^{\frac{n-l}{2}}|l\rangle_S \otimes |n-l\rangle_E.$$

5. Pure-loss channel

In optical platforms, the most common source of noise is photon loss, which is modeled by the *pure-loss channel* [2]. For any $\lambda \in [0, 1]$, the pure-loss channel $\mathcal{E}_\lambda$ of transmissivity λ is a single-mode bosonic channel which acts on the input system by mixing it in a beam splitter of transmissivity λ with an environmental vacuum state; see Fig. 2. In this model, the input signal is partially transmitted and partially reflected by the beam splitter, representing the loss of photons (or energy) in the channel. When $\lambda = 1$, the pure-loss channel is noiseless (it equals the identity superoperator). Conversely, when $\lambda = 0$, the pure-loss channel is completely noisy (specifically, it is a constant channel that maps any state in $|0\rangle\langle 0|$).

Definition 12. Let $\mathcal{H}_S, \mathcal{H}_E := L^2(\mathbb{R})$. For all $\lambda \in [0, 1]$, the pure-loss channel of transmissivity λ is a quantum channel $\mathcal{E}_\lambda : \mathcal{T}(\mathcal{H}_S) \to \mathcal{T}(\mathcal{H}_S)$ defined as follows:

$$\mathcal{E}_\lambda(\rho) := \mathrm{Tr}_E\left[U_\lambda^{SE}(\rho_S \otimes |0\rangle\langle 0|_E)\left(U_\lambda^{SE}\right)^\dagger\right], \quad \forall\,\rho \in \mathcal{T}(\mathcal{H}_S),$$

where $|0\rangle\langle 0|_E$ denotes the vacuum state of $\mathcal{H}_E$ and U_λ^{SE} denotes the beam splitter unitary defined in (A8).

Lemma 13. For all $\lambda \in [0, 1]$ and all $n, m \in \mathbb{N}$, it holds that

$$\mathcal{E}_\lambda(|m\rangle\langle n|) = \sum_{\ell=0}^{\min(n,m)} \sqrt{\binom{m}{\ell}\binom{n}{\ell}}\,(1-\lambda)^\ell \lambda^{\frac{n+m}{2}-\ell} \times |m-\ell\rangle\langle n-\ell|.$$

Proof. This is a direct consequence of (A10) and of the definition of pure-loss channel. ■

Lemma 14. For all $\lambda \in [0, 1]$, a complementary channel $\mathcal{E}_\lambda^{\rm c}:\mathcal{T}(\mathcal{H}_S) \to \mathcal{T}(\mathcal{H}_E)$ of the pure-loss channel $\mathcal{E}_\lambda : \mathcal{T}(\mathcal{H}_S) \to \mathcal{T}(\mathcal{H}_S)$ is given by

$$\mathcal{E}_\lambda^{\rm c}(\rho) := \mathrm{Tr}_S\big[U_\lambda^{SE}(\rho_S \otimes |0\rangle\langle 0|_E)\big(U_\lambda^{SE}\big)^\dagger\big] = \mathcal{E}_{1-\lambda}\circ\mathcal{R}(\rho), \quad \forall\,\rho\in\mathcal{T}(\mathcal{H}_S), \tag{A13}$$

where $\mathcal{R}(\cdot) := V \cdot V^\dagger$ with $V := (-1)^{\hat a^\dagger \hat a}$.

Proof. By linearity, it suffices to show the identity in (A13) for rank-one Fock operators of the form $|m\rangle\langle n|$ for any $n, m \in \mathbb{N}$,

$$\mathrm{Tr}_S\big[U_\lambda^{SE}(|m\rangle\langle n|_S \otimes |0\rangle\langle 0|_E)\big(U_\lambda^{SE}\big)^\dagger\big] = \mathcal{E}_{1-\lambda}\circ\mathcal{R}(|m\rangle\langle n|). \tag{A14}$$

This follows directly from (A10). ■

Proposition 15. [38,39] The pure-loss channel $\mathcal{E}_\lambda$ is degradable if and only if $\lambda \in [\frac{1}{2}, 1]$, and it is antidegradable if and only if $\lambda \in [0, \frac{1}{2}]$.

Lemma 16. [[21], Lemma A.7] For all $\lambda_1, \lambda_2 \in [0, 1]$ it holds that $\mathcal{E}_{\lambda_1}\circ\mathcal{E}_{\lambda_2} = \mathcal{E}_{\lambda_1\lambda_2}$.

6. Bosonic dephasing channel

Another main source of noise in optical platforms is bosonic dephasing, which serves as a prominent example of a non-Gaussian channel [28,40].

Definition 17. Let $\mathcal{H}_S := L^2(\mathbb{R})$ and let $\hat a$ be the corresponding annihilation operator. For all $\gamma \geqslant 0$, the bosonic dephasing channel $\mathcal{D}_\gamma : \mathcal{T}(\mathcal{H}_S) \to \mathcal{T}(\mathcal{H}_S)$ is a quantum channel defined as follows:

$$\mathcal{D}_\gamma(\rho) := \frac{1}{\sqrt{2\pi\gamma}}\int_{-\infty}^{\infty} \mathrm{d}\phi\, e^{-\frac{\phi^2}{2\gamma}}\, e^{i\phi\hat a^\dagger\hat a}\,\rho\, e^{-i\phi\hat a^\dagger\hat a}, \quad \forall\,\rho\in\mathcal{T}(\mathcal{H}_S).$$

In other words, $\mathcal{D}_\gamma$ is a convex combination of phase space rotations $\rho \longmapsto e^{i\phi\hat a^\dagger\hat a}\,\rho\,e^{-i\phi\hat a^\dagger\hat a}$, where the random variable ϕ follows a centered Gaussian distribution with a variance of γ.

Lemma 18. For all $\gamma \geqslant 0$ and all $n, m \in \mathbb{N}$, it holds that

$$\mathcal{D}_\gamma(|m\rangle\langle n|) = e^{-\frac{1}{2}\gamma(n-m)^2}|m\rangle\langle n|.$$

Proof. This result follows from the Fourier transform of the Gaussian function:

$$\frac{1}{\sqrt{2\pi\gamma}}\int_{-\infty}^{\infty}\mathrm{d}\phi\, e^{-\frac{\phi^2}{2\gamma}}\, e^{i\phi k} = e^{-\frac{1}{2}\gamma k^2}, \quad \forall\,k\in\mathbb{R}. \tag{A15}$$

■

When $\gamma = 0$, the bosonic dephasing channel is noiseless. In contrast, when $\gamma \to \infty$ it annihilates all off-diagonal components of the input density matrix, reducing it to an incoherent probabilistic mixture of Fock states. We now present a Stinespring dilation of the bosonic dephasing channel.

Lemma 19. Let $\mathcal{H}_S = \mathcal{H}_E := L^2(\mathbb{R})$ and let $\hat a$ and $\hat e$ be annihilation operators on $\mathcal{H}_S$ and $\mathcal{H}_E$, respectively. For all $\gamma > 0$, the bosonic dephasing channel $\mathcal{D}_\gamma : \mathcal{T}(\mathcal{H}_S) \to \mathcal{T}(\mathcal{H}_S)$ can be expressed in Stinespring representation as

$$\mathcal{D}_\gamma(\rho) = \mathrm{Tr}_E\big[V_\gamma^{SE}(\rho_S\otimes|0\rangle\langle 0|_E)\big(V_\gamma^{SE}\big)^\dagger\big], \quad \forall\,\rho\in\mathcal{T}(\mathcal{H}_S), \tag{A16}$$

where V_γ^{SE} denotes the conditional displacement unitary defined by

$$V_\gamma^{SE} := \exp[\sqrt{\gamma}\,\hat a^\dagger\hat a\otimes(\hat e^\dagger - \hat e)] = \sum_{n=0}^{\infty}|n\rangle\langle n|_S\otimes D(\sqrt{\gamma}n). \tag{A17}$$

Proof. For any $n, m \in \mathbb{N}$ it holds that

$$\begin{aligned}
&\mathrm{Tr}_E\big[V_\gamma^{SE}(|m\rangle\langle n|_S\otimes|0\rangle\langle 0|_E)\big(V_\gamma^{SE}\big)^\dagger\big]\\
&\overset{\rm (i)}{=} \mathrm{Tr}_E[|m\rangle\langle n|_S\otimes|\sqrt{\gamma}n\rangle\langle\sqrt{\gamma}m|_E]\\
&\overset{\rm (ii)}{=} e^{-\frac{\gamma}{2}(n-m)^2}|m\rangle\langle n|\\
&\overset{\rm (iii)}{=} \mathcal{D}_\gamma(|m\rangle\langle n|).
\end{aligned} \tag{A18}$$

Here in (i) we used that $V_\gamma^{SE}|n\rangle_S\otimes|0\rangle_E = |n\rangle_S\otimes|\sqrt{\gamma}n\rangle_E$, where $|\sqrt{\gamma}n\rangle_E$ denotes the coherent state with parameter $\sqrt{\gamma}n$. In (ii) we exploited the formula for the overlap between coherent states provided in (A6), and in (iii) we used Lemma 18. The proof is completed by linearity. ■

Remark 20. A different approach to dilating the bosonic dephasing channel, as outlined in the existing literature [40,61,62], is as follows:

$$\tilde V_\gamma^{\rm SE} = \exp[-i\sqrt{\gamma}\,\hat a^\dagger\hat a\,(\hat e^\dagger + \hat e)].$$

This unitary transformation is achieved by rotating the environmental mode of the unitary operator $V_\gamma^{\rm SE}$ in (A17) by $\frac{\pi}{2}$, that is, $\tilde V_\gamma^{\rm SE} = e^{i\frac{\pi}{2}\hat e^\dagger\hat e}V_\gamma^{\rm SE}e^{-i\frac{\pi}{2}\hat e^\dagger\hat e}$. These dilations yield the same dephasing channel, as all dilations are equivalent up to unitary transformations.

Proposition 21 ([32]). The bosonic dephasing channel $\mathcal{D}_\gamma$ is degradable for all $\gamma \geqslant 0$.

Proposition 22 ([32]). The bosonic dephasing channel $\mathcal{D}_\gamma$ is never antidegradable.

Lemma 23. For all $\gamma_1, \gamma_2 \geqslant 0$, the composition of bosonic dephasing channels is given by $\mathcal{D}_{\gamma_1}\circ\mathcal{D}_{\gamma_2} = \mathcal{D}_{\gamma_1+\gamma_2}$.

Proof. This can be shown by leveraging Lemma 18. ■

7. Bosonic loss-dephasing channel

Consider an optical system undergoing simultaneous loss and dephasing over a finite time interval. At each instant, the system is susceptible to both an infinitesimal pure-loss channel and an infinitesimal bosonic dephasing channel. Hence, the overall channel describing the simultaneous effect of loss and dephasing results in a suitable composition of numerous concatenations between infinitesimal pure-loss and bosonic dephasing channels. However, given that

The pure-loss channel and the bosonic dephasing channel commute, i.e., $\mathcal{E}_\lambda\circ\mathcal{D}_\gamma = \mathcal{D}_\gamma\circ\mathcal{E}_\lambda$, as implied by Lemma 13 and Lemma 18

The composition of pure-loss channels is a pure-loss channel, $\mathcal{E}_{\lambda_1}\circ\mathcal{E}_{\lambda_2} = \mathcal{E}_{\lambda_1\lambda_2}$ (Lemma 16) and

The composition of bosonic dephasing channels is a bosonic dephasing channel, $\mathcal{D}_{\gamma_1} \circ \mathcal{D}_{\gamma_2} = \mathcal{D}_{\gamma_1+\gamma_2}$ (Lemma 23) it follows that the combined effect of loss and dephasing can be modeled by the composition between pure-loss channel and bosonic dephasing channel,

$$\mathcal{N}_{\lambda,\gamma} := \mathcal{E}_\lambda \circ \mathcal{D}_\gamma, \tag{A19}$$

dubbed the *bosonic loss-dephasing channel*.

Definition 24. For all $\gamma \geqslant 0$ and $\lambda \in [0, 1]$, the bosonic loss-dephasing channel $\mathcal{N}_{\lambda,\gamma}$ is a quantum channel defined by the composition between pure-loss channel and bosonic dephasing channel: $\mathcal{N}_{\lambda,\gamma} := \mathcal{D}_\gamma \circ \mathcal{E}_\lambda$.

Lemma 25. For all $\lambda \in [0, 1]$ and $\gamma \geqslant 0$, it holds that $\mathcal{N}_{\lambda,\gamma} := \mathcal{D}_\gamma \circ \mathcal{E}_\lambda = \mathcal{E}_\lambda \circ \mathcal{D}_\gamma$. Moreover, for all $n, m \in \mathbb{N}$ it holds that

$$\begin{aligned}\mathcal{N}_{\lambda,\gamma}(|m\rangle\langle n|) &= e^{-\frac{1}{2}\gamma(n-m)^2}\mathcal{E}_\lambda(|m\rangle\langle n|)\\ &= e^{-\frac{1}{2}\gamma(n-m)^2}\sum_{\ell=0}^{\min(n,m)}\sqrt{\binom{n}{\ell}\binom{m}{\ell}}(1-\lambda)^\ell\\ &\quad\times \lambda^{\frac{n+m}{2}-\ell}|m-\ell\rangle\langle n-\ell|.\end{aligned}$$

Proof. It follows from Lemma 13 and Lemma 18. ■

Lemma 26. Let $\mathcal{H}_S, \mathcal{H}_{E_1}, \mathcal{H}_{E_2} := L^2(\mathbb{R})$. For all $\lambda \in [0, 1]$ and all $\gamma \geqslant 0$, the bosonic loss-dephasing channel $\mathcal{N}_{\lambda,\gamma} : \mathcal{T}(\mathcal{H}_S) \to \mathcal{T}(\mathcal{H}_S)$ admits the following Stinespring representation:

$$\begin{aligned}\mathcal{N}_{\lambda,\gamma}(\rho) = \mathrm{Tr}_{E_1E_2}\big[&U_\lambda^{SE_1}V_\gamma^{SE_2}(\rho_S \otimes |0\rangle\langle 0|_{E_1} \otimes |0\rangle\langle 0|_{E_2})\\ &\times \big(U_\lambda^{SE_1}V_\gamma^{SE_2}\big)^\dagger\big],\\ \forall \rho \in \mathcal{T}&(\mathcal{H}_S),\end{aligned}$$

The associated complementary channel $\mathcal{N}^{\mathrm{c}}_{\lambda,\gamma} : \mathcal{T}(\mathcal{H}_S) \to \mathcal{T}(\mathcal{H}_{E_1} \otimes \mathcal{H}_{E_2})$ is defined as follows:

$$\begin{aligned}\mathcal{N}^{\mathrm{c}}_{\lambda,\gamma}(\rho) := \mathrm{Tr}_S\big[&U_\lambda^{SE_1}V_\gamma^{SE_2}(\rho_S \otimes |0\rangle\langle 0|_{E_1} \otimes |0\rangle\langle 0|_{E_2})\\ &\times \big(U_\lambda^{SE_1}V_\gamma^{SE_2}\big)^\dagger\big],\\ \forall \rho \in \mathcal{T}&(\mathcal{H}_S).\end{aligned}$$

In particular,

$$\begin{aligned}\mathcal{N}^{\mathrm{c}}_{\lambda,\gamma}(|m\rangle\langle n|) &= (-1)^{m-n}\mathcal{E}_{1-\lambda}(|m\rangle\langle n|_{E_1}) \otimes |\sqrt{\gamma}m\rangle\langle\sqrt{\gamma}n|_{E_2},\\ &\forall m, n \in \mathbb{N},\end{aligned} \tag{A20}$$

where $|\sqrt{\gamma}n\rangle_{E_2}$ denotes the coherent state with parameter $\sqrt{\gamma}n$.

Proof. The Eq. (A20) is derived by first applying (A13) and subsequently utilizing the dilation of the bosonic dephasing channel. Finally, the nonenvironment mode of the bosonic dephasing channel is traced out. ■

Lemma 27. Consider the Hilbert spaces $\mathcal{H}_S$, $\mathcal{H}_{E_1}$, and $\mathcal{H}_{E_2}$, all isomorphic to $L^2(\mathbb{R})$. Let $\lambda \in [0, 1]$ and $\gamma \geqslant 0$. The bosonic loss-dephasing channel $\mathcal{N}_{\lambda,\gamma} : \mathcal{T}(\mathcal{H}_S) \to \mathcal{T}(\mathcal{H}_S)$ exhibits antidegradability if and only if there exists a quantum channel $\mathcal{A}_{\lambda,\gamma} : \mathcal{T}(\mathcal{H}_{E_{\mathrm{out}}}) \to \mathcal{T}(\mathcal{H}_S)$ satisfying the following condition:

$$\mathcal{A}_{\lambda,\gamma} \circ \mathcal{N}^{\mathrm{c}}_{\lambda,\gamma}(|m\rangle\langle n|) = \mathcal{N}_{\lambda,\gamma}(|m\rangle\langle n|), \quad \forall m, n \in \mathbb{N}, \tag{A21}$$

where $\mathcal{N}^{\mathrm{c}}_{\lambda,\gamma} : \mathcal{T}(\mathcal{H}_S) \to \mathcal{T}(\mathcal{H}_{E_{\mathrm{out}}})$ denotes the complementary channel reported in (A20), and $\mathcal{H}_{E_{\mathrm{out}}} \subset \mathcal{H}_{E_1} \otimes \mathcal{H}_{E_2}$ is defined in (B18). Moreover, the condition in (A21) is equivalent to

$$\begin{aligned}\mathcal{A}_{\lambda,\gamma}(\mathcal{E}_{1-\lambda}(|m\rangle\langle n|) \otimes |\sqrt{\gamma}m\rangle\langle\sqrt{\gamma}n|) &= (-1)^{m-n}e^{-\frac{\gamma}{2}(m-n)^2}\\ &\quad\times \mathcal{E}_\lambda(|m\rangle\langle n|),\\ &\forall m, n \in \mathbb{N},\end{aligned} \tag{A22}$$

where $|\sqrt{\gamma}n\rangle$ denotes the coherent state with parameter $\sqrt{\gamma}n$.

Proof. $\mathcal{N}_{\lambda,\gamma}$ is antidegradable if and only if there exists a quantum channel $\mathcal{A}_{\lambda,\gamma} : \mathcal{T}(\mathcal{H}_{E_{\mathrm{out}}}) \to \mathcal{T}(\mathcal{H}_S)$ such that

$$\mathcal{A}_{\lambda,\gamma} \circ \mathcal{N}^{\mathrm{c}}_{\lambda,\gamma}(\rho) = \mathcal{N}_{\lambda,\gamma}(\rho), \quad \forall \rho \in \mathcal{T}(\mathcal{H}_S). \tag{A23}$$

By linearity, it suffices to show the condition in (A21), i.e., which corresponds to the condition in (A23) restricted to rank-one Fock operators of the form $\rho = |m\rangle\langle n|$ with $m, n \in \mathbb{N}$. Moreover, by exploiting Lemma 25 and (A20), the condition in (A21) is equivalent to (A22). ■

Lemma 28. For all $\gamma_1, \gamma_2 \geqslant 0$ and all λ_1, λ_2 it holds that $\mathcal{N}_{\lambda_1,\gamma_1} \circ \mathcal{N}_{\lambda_2,\gamma_2} = \mathcal{N}_{\lambda_1\lambda_2,\,\gamma_1+\gamma_2}$.

Proof. This can be shown by exploiting Lemma 16 and Lemma 23. ■

8. Preliminaries on capacities of quantum channels

Quantum channels can be suitably exploited in order to transfer information from their input port to a possibly distant output port, a crucial task in quantum information theory [44,57]. In particular, quantum Shannon theory [33,34] primarily helps us understand the fundamental limits of quantum communication using a quantum channel $\mathcal{N}$. These limits are called *capacities* and tell us the ultimate amount of information we can send through the channel when we use it many times [33,34]. Different capacities are defined based on the type of information that is being sent down the channel. For example, classical and quantum capacities of a quantum channel correspond to its ultimate capability of transmission of classical and quantum information, respectively. A channel can also be used to generate secret bits and the relevant capacity in this context is the so-called secret-key capacity. Each of the above-mentioned capacities might be endowed with other resources such as initial shared entanglement between the sender and the receiver or (possibly interactive) classical communication over a noiseless but public channel. This latter scenario gives rise to the notion of two-way capacities.

Specifically, the *quantum capacity* $Q(\mathcal{N})$ of a quantum channel $\mathcal{N}$ is the maximum rate at which qubits can be reliably transmitted through $\mathcal{N}$ [34]. We can further assume that both the sender, Alice, and the receiver, Bob, have free access to a public, noiseless two-way classical channel. In this two-way communication setting, the relevant notion of capacities are the *two-way quantum capacity* $Q_2(\mathcal{N})$ and the *secret-key capacity* $K(\mathcal{N})$ [33,34], defined as the maximum achievable rate of qubits and secret-key bits, respectively, that can be reliably transmitted across $\mathcal{N}$ with the aid of two-way classical communication. Since an ebit (i.e., a maximally entangled state of Schmidt rank 2) can always be used to generate one bit of secret key [47], a trivial bound relates these capacities: $Q_2(\mathcal{N}) \leqslant K(\mathcal{N})$.

In practical scenarios, it is important to consider that the input state prepared by Alice can not have unlimited energy and it adheres to specific energy constraints. In bosonic systems, it is common to limit the average photon number of any input state ρ as $\mathrm{Tr}[\hat{a}^\dagger\hat{a}\rho] \leqslant N_s$, where $N_s > 0$ is a given energy constraint. For any $N_s > 0$, the energy-constrained (EC) two-way capacities for transmitting qubits and secret-key bits, denoted as $Q_2(\mathcal{N}, N_s)$ and $K(\mathcal{N}, N_s)$ respectively, are defined similarly to the unconstrained capacities with the difference that now the optimization is performed over those strategies that exploit input states that adhere to the specified energy constraint. As in the unconstrained scenario, the relation between EC two-way capacities continue to hold, i.e., $Q_2(\mathcal{N}, N_s) \leqslant K(\mathcal{N}, N_s)$. Moreover, the unconstrained capacities are upper bounds for the corresponding energy constrained capacities and they become equal in the limit $N_s \to \infty$.

Two-way capacities of a quantum channel are closely related to another important information-processing task, namely, entanglement distillation over a quantum channel. Suppose Alice generates n copies of a state $\rho_{A'A}$ and sends the A' subsystems to Bob using the channel $\mathcal{N}$ for n times. Now, Alice and Bob share n copies of the state $\rho'_{AB} := \mathrm{id}_A \otimes \mathcal{N}_{A'\to B}(\rho_{AA'})$. The task of an entanglement distillation protocol concerns identifying the largest number m of ebits that can be extracted using n copies of ρ'_{AB} via LOCC (Local Operations and Classical Communication) operations. The rate of an entanglement distillation protocol is defined by the ratio m/n. The distillable entanglement $E_d(\rho'_{AB})$ of ρ'_{AB} is defined as the maximum rate over all the possible entanglement distillation protocols [63] [[34], Chapter 8]. Note that without extra classical communication, entanglement distillation is not possible [64]. The following lemma establishes a link between the two-way quantum capacity, secret-key capacity, and distillable entanglement.

Lemma 29. Let $\mathcal{H}_A, \mathcal{H}_{A'}, \mathcal{H}_B := L^2(\mathbb{R})$. Let $\mathcal{N} : \mathcal{H}_{A'} \to \mathcal{H}_B$ be a quantum channel. Let $N_s > 0$ be the energy constaint, and let $\rho_{A'A} \in \mathcal{P}(\mathcal{H}_A \otimes \mathcal{H}_{A'})$ be a two-mode state satisfying the energy constraint $\mathrm{Tr}[(\hat{a}^\dagger\hat{a} \otimes \mathrm{id}_A)\rho_{A'A}] \leqslant N_s$, where $\hat{a}$ denotes the annihilation operator on $\mathcal{H}_{A'}$. Then it holds that

$$K(\mathcal{N}, N_s) \geqslant Q_2(\mathcal{N}, N_s) \geqslant E_d(\mathrm{id}_A \otimes \mathcal{N}(\rho_{AA'})), \qquad \text{(A24)}$$

where $K(\mathcal{N}, N_s)$ denotes the energy-constrained secret-key capacity of $\mathcal{N}$, $Q_2(\mathcal{N}, N_s)$ denotes the energy-constrained two-way quantum capacity of $\mathcal{N}$, and $E_d(\mathrm{id}_A \otimes \mathcal{N}(\rho_{AA'}))$ denotes the distillable entanglement of the state $\mathrm{id}_A \otimes \mathcal{N}(\rho_{AA'})$.

The proof idea of the above lemma is the following. Suppose that Alice produces n copies of a state $\rho_{AA'}$ such that the energy constraint is satisfied. Then she can use the channel n times to send all subsystems A' to Bob. Then Alice and Bob share n copies of $\mathrm{id}_A \otimes \mathcal{N}(\rho_{AA'})$, which can now be used to generate $\approx n\,E_d(\mathrm{id}_A \otimes \mathcal{N}(\rho_{AA'}))$ ebits by means of a suitable entanglement distillation protocol. The ebit rate of this protocol is thus $E_d(\mathrm{id}_A \otimes \mathcal{N}(\rho_{AA'}))$, which provides a lower bound on $Q_2(\mathcal{N}, N_s)$ thanks to quantum teleportation [65]. In addition, it holds that $K(\mathcal{N}, N_s) \geqslant Q_2(\mathcal{N}, N_s)$ because an ebit can generate a secret-key bit [47]. Consequently, (A24) holds.

APPENDIX B: ANTIDEGRADABILITY AND DEGRADABILITY OF BOSONIC LOSS-DEPHASING CHANNEL

This section is split into two parts based on the observation that if the input state to the bosonic loss-dephasing channel is chosen from a finite-dimensional subspace, the bosonic loss-dephasing channel effectively becomes a finite-dimensional channel, a fact we show in Lemma 36. This property allows us to apply established insights about the finite-dimensional channels to the bosonic loss-dephasing channel. In Sec. B 1 we present our study of the bosonic loss-dephasing channel when the input resides in the entire infinite-dimensional space, while Sec. B 2 is dedicated to the findings resulting from analysis of finite-dimensional restrictions of the bosonic loss-dephasing channel.

1. Sufficient condition on antidegradability

It is known that the bosonic dephasing channel $\mathcal{D}_\gamma$ is degradable across all dephasing parameter range $\gamma \geqslant 0$ and also it is never antidegradable [40]. The pure-loss channel $\mathcal{E}_\lambda$ displays the peculiar characteristic of being antidegradable for transmissivity values within the range $\lambda \in [0, \frac{1}{2}]$ and degradable for $\lambda \in [\frac{1}{2}, 1]$ [38,39]. It turns out that when an antidegradable channel is concatenated with another channel, the resulting channel inherits the property of being antidegradable (see Lemma 55). This implies the following: if $\lambda \in [0, \frac{1}{2}]$ and $\gamma \geqslant 0$, then the bosonic loss-dephasing channel $\mathcal{N}_{\lambda,\gamma}$ is antidegradable [32]. The authors of [32] left as an open question to understand whether or not the bosonic loss-dephasing channel $\mathcal{N}_{\lambda,\gamma}$ is antidegradable in the region $\lambda \in (\frac{1}{2}, 1]$. In particular, they conjecture that $\mathcal{N}_{\lambda,\gamma}$ is not antidegradable for all transmissivity values $\lambda \in (\frac{1}{2}, 1]$ and for all $\gamma \geqslant 0$. In the following theorem, we refute this conjecture by explicitly finding values of $\lambda \in (\frac{1}{2}, 1]$ and $\gamma \geqslant 0$ where the channel is antidegradable. Our approach also yields an explicit expression for an antidegrading map of the bosonic loss-dephasing channel.

Theorem 30. Each of the following is a sufficient condition for the bosonic loss-dephasing channel $\mathcal{N}_{\lambda,\gamma}$ to exhibit antidegradability:

(1) $\lambda \in [0, \frac{1}{2}]$ and $\gamma \geqslant 0$.

(2) $\lambda \in (\frac{1}{2}, 1)$ and $\theta(e^{-\gamma/2}, \sqrt{\frac{\lambda}{1-\lambda}}) \leqslant \frac{3}{2}$, where θ is defined as $\theta(x, y) := \sum_{n=0}^{\infty} x^{n^2} y^n$, $\forall x, y \in [0, 1)$.

In particular, $\mathcal{N}_{\lambda,\gamma}$ is antidegradable if $\lambda \leqslant \max(\frac{1}{2}, \frac{1}{1+9e^{-\gamma}})$.

Proof. The proof of the sufficient condition (i) follows directly from the observation that the composition of a pure-loss channel with transmissivity $\lambda \in [0, \frac{1}{2}]$ with any other channel inherits the antidegradability from the pure-loss channel (see Lemma 55).

The proof of the sufficient condition (ii) is more involved, and it is the main technical contribution of our work. We rely on the equivalence between antidegradability of a quantum channel and the two-extendibility of its Choi state [41,42]. To provide a comprehensive and intuitive understanding of this idea and to aid in the construction of an antidegrading map of the bosonic loss-dephasing channel, we present this equivalence in Lemma 54 in the Appendixes.

Assume $\lambda \in (\frac{1}{2}, 1)$ and $\theta(e^{-\gamma/2}, \sqrt{\frac{\lambda}{1-\lambda}}) \leqslant \frac{3}{2}$. Let $\mathcal{H}_A, \mathcal{H}_B, \mathcal{H}_{B_1}, \mathcal{H}_{B_2} = L^2(\mathbb{R})$ and suppose that the bosonic loss-dephasing channel $\mathcal{N}_{\lambda,\gamma}$ is a quantum channel from the system A' to B. We want to show that the generalized Choi state of $\mathcal{N}_{\lambda,\gamma}$ is two-extendible on B. In other words, we want to show that there exists a tripartite state $\rho_{AB_1B_2}$ such that

$$\begin{aligned} \mathrm{Tr}_{B_2}\left[\rho_{AB_1B_2}\right] &= \tau_{AB_1}, \\ \mathrm{Tr}_{B_1}\left[\rho_{AB_1B_2}\right] &= \tau_{AB_2}, \end{aligned} \tag{B1}$$

where $\tau_{AB} := \mathrm{id}_A \otimes \mathcal{N}_{\lambda,\gamma}(|\psi(r)\rangle\langle\psi(r)|_{AA'})$ is the generalized Choi state of $\mathcal{N}_{\lambda,\gamma}$, and $|\psi(r)\rangle_{AA'}$ is the two-mode squeezed vacuum state with squeezing parameter $r > 0$ defined in (A7). By Lemma (25), the generalized Choi state of the bosonic loss-dephasing channel can be expressed as follows:

$$\tau_{AB} = \frac{1}{\cosh^2(r)} \sum_{m,n=0}^{\infty} \sum_{\ell=0}^{\min(m,n)} (\tanh(r))^{m+n} e^{-\frac{\gamma}{2}(m-n)^2} \sqrt{\mathcal{B}_\ell(m, 1-\lambda)\, \mathcal{B}_\ell(n, 1-\lambda)} |m\rangle\langle n|_A \otimes |m-\ell\rangle\langle n-\ell|_B, \tag{B2}$$

where $\mathcal{B}_\ell(n,\lambda) := \binom{n}{\ell}\lambda^\ell(1-\lambda)^{n-\ell}$. We observe that τ_{AB} is a linear combination of $|m\rangle\langle n| \otimes |j_1\rangle\langle j_2|$, where $m, n, j_1, j_2 \in \mathbb{N}$, $j_1 \leqslant n_1$, $j_2 \leqslant n$, and $m - n = j_1 - j_2$. This insight leads to the following educated guess about the structure of a potential two-extension:

$$\tilde{\rho}_{AB_1B_2} = \sum_{m,n=0}^{\infty} \sum_{\ell_1=0}^{m} \sum_{\ell_2=0}^{n} \sum_{k=0}^{\min(m-\ell_1, n-\ell_2)} c(m,n,\ell_1,\ell_2,k) |m\rangle\langle n|_A \otimes |m-\ell_1\rangle\langle n-\ell_2|_{B_1} \otimes |\ell_1+k\rangle\langle \ell_2+k|_{B_2}, \tag{B3}$$

where $\{c(m,n,\ell_1,\ell_2,k)\}_{m,n,\ell_1,\ell_2,k}$ are some suitable coefficients. The soundness of this guess is confirmed by the fact that both $\mathrm{Tr}_{B_2}[\tilde{\rho}_{AB_1B_2}]$ and $\mathrm{Tr}_{B_1}[\tilde{\rho}_{AB_1B_2}]$ are linear combinations of $|m\rangle\langle n| \otimes |j_1\rangle\langle j_2|$ with $m, n, j_1, j_2 \in \mathbb{N}$, $j_1 \leqslant m$, $j_2 \leqslant n$, and $m - n = j_1 - j_2$, similar to the generalized Choi state τ_{AB} in (B2).

At this point, one could try to define the coefficients $\{c(m,n,\ell_1,\ell_2,k)\}_{m,n,\ell_1,\ell_2,k}$ so as to satisfy the required conditions of the extendibility. However, the resulting tripartite operator may not qualify as a quantum state. In order to ensure that we obtain a quantum state, our approach consists in producing the operator $\tilde{\rho}_{AB_1B_2}$ via a physical process that consists in applying a sequence of quantum channels to a quantum state.

We begin by constructing a quantum state that has the same operator structure as the operator $\tilde{\rho}_{AB_1B_2}$ in (B3). This means that at this initial stage, we only aim to build a tripartite state consisting of a linear combination of operators $|m\rangle\langle n|_A \otimes |m-\ell_1\rangle\langle n-\ell_2|_{B_1} \otimes |\ell_1+k\rangle\langle \ell_2+k|_{B_2}$ with the summation limits identical to those in (B3). This ensures that the coefficients equal to zero coincide in the two operators.

We now illustrate on each step of this construction. For a fixed $n \in \mathbb{N}$, consider the state $|n\rangle_A \otimes |n\rangle_{B_1}$. We introduce two auxiliary single-mode systems B_2 and C initially in vacuum states: $|n\rangle_A \otimes |n\rangle_{B_1} \otimes |0\rangle_{B_2} \otimes |0\rangle_C$. We next send the systems B_2 and B_1 through the ports of a beam splitter, resulting in a superposition of $|n\rangle_A \otimes |n-\ell\rangle_{B_1} \otimes |\ell\rangle_{B_2} \otimes |0\rangle_C$ for $\ell = 0, 1, \ldots, n$, as implied by Lemma (10). We repeat this for systems B_1 and C, thus obtaining a superposition of $|n\rangle_A \otimes |n-\ell-k\rangle_{B_1} \otimes |\ell\rangle_{B_2} \otimes |k\rangle_C$, with $k = 0, 1, \ldots, n-\ell$ and $\ell = 0, 1, \ldots, n$. Consider now the isometry $W^{CB_1B_2}$ defined by

$$W^{CB_1B_2}|n\rangle_{B_1} \otimes |m\rangle_{B_2} \otimes |k\rangle_C = |n+k\rangle_{B_1} \otimes |m+k\rangle_{B_2} \otimes |k\rangle_C, \quad \forall\, n, m, k \in \mathbb{N}, \tag{B4}$$

dubbed *controlled-add-add isometry* (mode C is the control mode). By applying this isometry to the superposition we created by using beam splitters, we obtain a superposition of $|n\rangle_A \otimes |n-\ell\rangle_{B_1} \otimes |\ell+k\rangle_{B_2} \otimes |k\rangle_C$ with $k = 0, 1, \ldots, n-\ell$ and $\ell = 0, 1, \ldots, n$. Finally, by tracing out system C, we obtain the same operator structure of the operator $\tilde{\rho}_{AB_1B_2}$ in (B3). Having focused on the operator structure, we have not considered the transmissivities of the two beam splitters so far. We will see that these transmissivities can be chosen carefully such that the diagonal elements of the operator at hand becomes equal to those of the Choi state.

We now apply the outlined construction to the two-mode squeezed vacuum state with two vacuum states appended to it, i.e., $|\psi(r)\rangle_{AB_1} \otimes |0\rangle_{B_2} \otimes |0\rangle_C$. For reasons that will become clear in a moment, we choose the two beam splitter transmissivities to be λ (for the beam splitter acting on B_2B_1) and $\frac{1-\lambda}{\lambda}$ (for the one acting on CB_1). By doing so we obtain the state

$$\begin{aligned} |\phi\rangle_{AB_1B_2C} &:= W^{C,B_1B_2} U^{CB_1}_{\frac{1-\lambda}{\lambda}} U^{B_2B_1}_{\lambda} |\psi(r)\rangle_{AB_1} \otimes |0\rangle_{B_2} \otimes |0\rangle_C \\ &= \frac{1}{\cosh(r)} \sum_{n=0}^{\infty} \sum_{\ell=0}^{n} \sum_{k=0}^{n-\ell} \tanh^n(r) \sqrt{\mathcal{B}_\ell(n, 1-\lambda)\, \mathcal{B}_k\!\left(n-\ell, \frac{2\lambda-1}{\lambda}\right)} |n\rangle_A \otimes |n-\ell\rangle_{B_1} \otimes |\ell+k\rangle_{B_2} \otimes |k\rangle_C, \end{aligned} \tag{B5}$$

where we used (A7) and Lemma 10. Note that the transmissivities of the beam splitters $U^{CB_1}_{\frac{1-\lambda}{\lambda}}$ and $U^{B_2B_1}_{\lambda}$ are chosen such that the diagonal elements of $\mathrm{Tr}_{B_2C}[|\phi\rangle\langle\phi|_{AB_1B_2C}]$ and $\mathrm{Tr}_{B_1C}[|\phi\rangle\langle\phi|_{AB_1B_2C}]$ both coincide with those of the Choi state τ_{AB} in (B2). To verify this, let us calculate the state $\mathrm{Tr}_C[|\phi\rangle\langle\phi|_{AB_1B_2C}]$:

$$\mathrm{Tr}_C\big[|\phi\rangle\langle\phi|_{AB_1B_2C}\big] = \frac{1}{\cosh^2(r)}\sum_{m,n=0}^{\infty}\sum_{\ell_1=0}^{m}\sum_{\ell_2=0}^{n}\sum_{k=0}^{\min(m-\ell_1,n-\ell_2)}(\tanh(r))^{m+n}\sqrt{\mathcal{B}_{\ell_1}(m,1-\lambda)\,\mathcal{B}_{\ell_2}(n,1-\lambda)}$$
$$\times\left[\sqrt{\mathcal{B}_k\!\left(m-\ell_1,\frac{2\lambda-1}{\lambda}\right)\mathcal{B}_k\!\left(n-\ell_2,\frac{2\lambda-1}{\lambda}\right)}\right]|m\rangle\langle n|_A\otimes|m-\ell_1\rangle\langle n-\ell_2|_{B_1}\otimes|\ell_1+k\rangle\langle\ell_2+k|_{B_2}.$$

Notably, the structure of the state $\mathrm{Tr}_C[|\phi\rangle\langle\phi|_{AB_1B_2C}]$ mirrors that of (B3) with specific coefficients $c(m,n,\ell_1,\ell_2,k)$. Moreover, it holds that

$$\mathrm{Tr}_{B_2C}\big[|\phi\rangle\langle\phi|_{AB_1B_2C}\big] = \frac{1}{\cosh^2(r)}\sum_{m,n=0}^{\infty}\sum_{\ell=0}^{\min(m,n)}(\tanh(r))^{m+n}\sqrt{\mathcal{B}_{\ell}(m,1-\lambda)\,\mathcal{B}_{\ell}(n,1-\lambda)}$$
$$\left[\sum_{k=0}^{\min(m-\ell,n-\ell)}\sqrt{\mathcal{B}_k\!\left(m-\ell,\frac{2\lambda-1}{\lambda}\right)\mathcal{B}_k\!\left(n-\ell,\frac{2\lambda-1}{\lambda}\right)}\right]|m\rangle\langle n|_A\otimes|m-\ell\rangle\langle n-\ell|_{B_1} \tag{B6}$$

and that

$$\mathrm{Tr}_{B_1C}\big[|\phi\rangle\langle\phi|_{AB_1B_2C}\big] = \mathrm{Tr}_{B_2C}\big[|\phi\rangle\langle\phi|_{AB_1B_2C}\big]. \tag{B7}$$

In order to prove (B7), observe that

$$\mathrm{Tr}_{B_1C}\big[|\phi\rangle\langle\phi|_{AB_1B_2C}\big] = \frac{1}{\cosh^2(r)}\sum_{m,n=0}^{\infty}\sum_{\ell_1=\max(m-n,0)}^{m}\sum_{k=0}^{m-\ell_1}[\tanh(r)]^{m+n}\sqrt{\mathcal{B}_{\ell_1}(m,1-\lambda)\,\mathcal{B}_{n-m+\ell_1}(n,1-\lambda)}$$
$$\mathcal{B}_k\!\left(m-\ell_1,\frac{2\lambda-1}{\lambda}\right)|m\rangle\langle n|_A\otimes|\ell_1+k\rangle\langle n-m+\ell_1+k|_{B_2}$$
$$= \frac{1}{\cosh^2(r)}\sum_{m,n=0}^{\infty}\sum_{\ell_1=\max(m-n,0)}^{m}\sum_{\ell=0}^{m-\ell_1}[\tanh(r)]^{m+n}\sqrt{\mathcal{B}_{\ell_1}(m,1-\lambda)\,\mathcal{B}_{n-m+\ell_1}(n,1-\lambda)}$$
$$\mathcal{B}_{m-\ell_1-\ell}\!\left(m-\ell_1,\frac{2\lambda-1}{\lambda}\right)|m\rangle\langle n|_A\otimes|m-\ell\rangle\langle n-\ell|_{B_2}$$
$$= \frac{1}{\cosh^2(r)}\sum_{m,n=0}^{\infty}\sum_{\ell=0}^{\min(m,n)}[\tanh(r)]^{m+n}\sum_{\ell_1=\max(m-n,0)}^{m-\ell}\sqrt{\mathcal{B}_{\ell_1}(m,1-\lambda)\,\mathcal{B}_{n-m+\ell_1}(n,1-\lambda)}$$
$$\mathcal{B}_{m-\ell_1-\ell}\!\left(m-\ell_1,\frac{2\lambda-1}{\lambda}\right)|m\rangle\langle n|_A\otimes|m-\ell\rangle\langle n-\ell|$$
$$= \frac{1}{\cosh^2(r)}\sum_{m,n=0}^{\infty}\sum_{\ell=0}^{\min(m,n)}[\tanh(r)]^{m+n}\sum_{k=0}^{\min(n-\ell,m-\ell)}\sqrt{\mathcal{B}_{m-\ell-k}(m,1-\lambda)\,\mathcal{B}_{n-\ell-k}(n,1-\lambda)}$$
$$\mathcal{B}_k\!\left(k+\ell,\frac{2\lambda-1}{\lambda}\right)|m\rangle\langle n|_A\otimes|m-\ell\rangle\langle n-\ell|$$
$$\overset{\text{(i)}}{=} \frac{1}{\cosh^2(r)}\sum_{m,n=0}^{\infty}\sum_{\ell=0}^{\min(m,n)}[\tanh(r)]^{m+n}\sqrt{\mathcal{B}_{\ell}(m,1-\lambda)\,\mathcal{B}_{\ell}(n,1-\lambda)}$$
$$\left[\sum_{k=0}^{\min(m-\ell,n-\ell)}\sqrt{\mathcal{B}_k\!\left(m-\ell,\frac{2\lambda-1}{\lambda}\right)\mathcal{B}_k\!\left(n-\ell,\frac{2\lambda-1}{\lambda}\right)}\right]|m\rangle\langle n|_A\otimes|m-\ell\rangle\langle n-\ell|_{B_2}$$
$$\overset{\text{(ii)}}{=} \mathrm{Tr}_{B_2C}[|\phi\rangle\langle\phi|_{AB_1B_2C}]. \tag{B8}$$

Here in (i) we used the identity

$$\sqrt{\mathcal{B}_{m-\ell-k}(m,1-\lambda)\,\mathcal{B}_{n-\ell-k}(n,1-\lambda)}\mathcal{B}_k\!\left(k+\ell,\frac{2\lambda-1}{\lambda}\right)=\sqrt{\mathcal{B}_\ell(m,1-\lambda)\mathcal{B}_\ell(n,1-\lambda)}\times\sqrt{\mathcal{B}_k\!\left(m-\ell,\frac{2\lambda-1}{\lambda}\right)\mathcal{B}_k\!\left(n-\ell,\frac{2\lambda-1}{\lambda}\right)},\tag{B9}$$

which can be easily proved by substituting the definition $\mathcal{B}_\ell(n,\lambda):=\binom{n}{\ell}\lambda^\ell(1-\lambda)^{n-\ell}$ and by leveraging the binomial identity

$$\binom{n}{l+k}\binom{l+k}{k}=\binom{n}{l}\binom{n-l}{k}.\tag{B10}$$

Moreover, in (ii) we exploited (B6).

Note that the off-diagonal terms of the state in (B6) are not equal to those of the Choi state τ_{AB} in (B2). Specifically, the points of difference with the Choi state τ_{AB} are the presence of the term inside the square brackets and the absence of the dephasing exponent. To address these additional terms, let us use the toolbox of *Hadamard maps*. Let H be the Hadamard map, introduced in Sec. A 3, associated with the infinite matrix $A:=(a_{mn})_{m,n\in\mathbb{N}}$ defined as follows:

$$a_{mn}:=\frac{e^{-\frac{\gamma}{2}(n-m)^2}}{\sum_{j=0}^{\min(n,m)}\sqrt{\mathcal{B}_j\big(n,\frac{2\lambda-1}{\lambda}\big)\,\mathcal{B}_j\big(m,\frac{2\lambda-1}{\lambda}\big)}},\qquad\forall\,n,m\in\mathbb{N}.\tag{B11}$$

By construction, we have that

$$\begin{aligned}\mathrm{id}_A\otimes H_{B_1}\big(\mathrm{Tr}_{B_2C}\big[|\phi\rangle\langle\phi|_{AB_1B_2C}\big]\big)&=\frac{1}{\cosh^2(r)}\sum_{m,n=0}^{\infty}\sum_{\ell=0}^{\min(m,n)}[\tanh(r)]^{m+n}\sqrt{\mathcal{B}_\ell(m,1-\lambda)\,\mathcal{B}_\ell(n,1-\lambda)}\\&\quad\left[\sum_{k=0}^{\min(m-\ell,n-\ell)}\sqrt{\mathcal{B}_k\!\left(m-\ell,\frac{2\lambda-1}{\lambda}\right)\mathcal{B}_k\!\left(n-\ell,\frac{2\lambda-1}{\lambda}\right)}\right]a_{m-\ell,\,n-\ell}\,|m\rangle\langle n|_A\\&\quad\otimes|m-\ell\rangle\langle n-\ell|_{B_1}\\&=\frac{1}{\cosh^2(r)}\sum_{m,n=0}^{\infty}\sum_{\ell=0}^{\min(m,n)}[\tanh(r)]^{m+n}e^{-\frac{\gamma}{2}(n-m)^2}\sqrt{\mathcal{B}_\ell(m,1-\lambda)\,\mathcal{B}_\ell(n,1-\lambda)}|m\rangle\langle n|_A\\&\quad\otimes|m-\ell\rangle\langle n-\ell|_{B_1}\\&=\tau_{AB_1}.\end{aligned}\tag{B12}$$

This means that the operator

$$\rho_{AB_1B_2}:=\mathrm{id}_A\otimes H_{B_1}\otimes H_{B_2}(\mathrm{Tr}_C[|\phi\rangle\langle\phi|_{AB_1B_2C}])\tag{B13}$$

satisfies the extendibility conditions in (B1). All that remains to prove is that $\rho_{AB_1B_2}$ is in fact a quantum state. We will do this by showing that the superoperator H is a quantum channel. In Sec. A 3 we establish that a Hadamard map is a quantum channel if its defining infinite matrix is Hermitian, has diagonal elements equal to one, and is diagonally dominant. The first two properties are trivially satisfied by the infinite matrix A defined in (B11). We only need to demonstrate that for the parameter region $\lambda>\frac{1}{2}$ and $\theta(e^{-\gamma/2},\sqrt{\frac{\lambda}{1-\lambda}})\leqslant\frac{3}{2}$, the infinite matrix A is diagonally dominant. We recall that, by definition, A is diagonally dominant if it holds that $\sum_{\substack{m=0\\m\neq n}}^{\infty}|a_{mn}|\leqslant1,\ \forall\,n\in\mathbb{N}$. Note that for any $n,m\in\mathbb{N}$ we have that

$$\begin{aligned}|a_{nm}|&=\frac{e^{-\frac{\gamma}{2}(n-m)^2}}{\sum_{j=0}^{\min(n,m)}\sqrt{\binom{n}{j}\binom{m}{j}\big(\frac{1-\lambda}{\lambda}\big)^{n+m-2j}\big(\frac{2\lambda-1}{\lambda}\big)^{2j}}}\leqslant\frac{e^{-\frac{\gamma}{2}(n-m)^2}}{\sum_{j=0}^{\min(n,m)}\sqrt{\binom{\min(n,m)}{j}^2\big(\frac{1-\lambda}{\lambda}\big)^{n+m-2j}\big(\frac{2\lambda-1}{\lambda}\big)^{2j}}}\\&=e^{-\frac{\gamma}{2}(n-m)^2}\left(\frac{\lambda}{1-\lambda}\right)^{\frac{|n-m|}{2}}.\end{aligned}\tag{B14}$$

Consequently, if λ and γ are such that $\lambda > \frac{1}{2}$ and $\theta(e^{-\gamma/2}, \sqrt{\frac{\lambda}{1-\lambda}}) \leqslant \frac{3}{2}$, for any $n \in \mathbb{N}$ we have that

$$\sum_{\substack{m=0\\ m\neq n}}^{\infty} |a_{mn}| \leqslant \sum_{\substack{m=0\\ m\neq n}}^{\infty} e^{-\frac{\gamma}{2}(m-n)^2}\left(\frac{\lambda}{1-\lambda}\right)^{\frac{|m-n|}{2}} = \sum_{k=1}^{\infty} e^{-\frac{\gamma}{2}k^2}\left(\frac{\lambda}{1-\lambda}\right)^{\frac{k}{2}} + \sum_{k=1}^{n} e^{-\frac{\gamma}{2}k^2}\left(\frac{\lambda}{1-\lambda}\right)^{\frac{k}{2}}$$
$$\leqslant 2\sum_{k=1}^{\infty} e^{-\frac{\gamma}{2}k^2}\left(\frac{\lambda}{1-\lambda}\right)^{\frac{k}{2}} = 2\,\theta\left(e^{-\gamma/2}, \sqrt{\frac{\lambda}{1-\lambda}}\right) - 2 \leqslant 1. \tag{B15}$$

Therefore, the infinite matrix A is diagonally dominant if $\lambda > \frac{1}{2}$ and $\theta(e^{-\gamma/2}, \sqrt{\frac{\lambda}{1-\lambda}}) \leqslant \frac{3}{2}$. This establishes that H_{B_1} and H_{B_2} are valid quantum channels in this parameter range, implying that $\rho_{AB_1B_2}$ is a valid two-extension of the Choi state of $\mathcal{N}_{\lambda,\gamma}$, and in turn entailing that $\mathcal{N}_{\lambda,\gamma}$ is antidegradable. Finally, note that if $\lambda > \frac{1}{2}$ the condition

$$\theta\left(e^{-\gamma/2}, \sqrt{\frac{\lambda}{1-\lambda}}\right) \leqslant \frac{3}{2} \tag{B16}$$

is implied by $\lambda \leqslant \frac{1}{1+9e^{-\gamma}}$. Indeed,

$$\theta\left(e^{-\gamma/2}, \sqrt{\frac{\lambda}{1-\lambda}}\right) := \sum_{k=0}^{n} e^{-\frac{\gamma}{2}k^2}\left(\sqrt{\frac{\lambda}{1-\lambda}}\right)^{k} \leqslant \sum_{k=0}^{\infty}\left(\frac{e^{-\gamma}\lambda}{1-\lambda}\right)^{k/2} = \frac{1}{1-\sqrt{\frac{e^{-\gamma}\lambda}{1-\lambda}}} \leqslant \frac{3}{2},$$

where the last inequality follows from $\frac{e^{-\gamma}\lambda}{1-\lambda} \leqslant \frac{1}{9}$, which is implied by $\lambda \leqslant \frac{1}{1+9e^{-\gamma}}$. ■

a. Expanding the antidegradability region numerically

Note that Theorem 30 does not identify the entire antidegradability region of the bosonic loss-dephasing channel $\mathcal{N}_{\lambda,\gamma}$. In fact, from the above argument it becomes clear that a way to obtain a better inner approximation of this region is to check for which values of the parameters the infinite matrix A defined by (B11) is positive semidefinite. This is established in the following theorem.

Theorem 31. Let $\ell^2(\mathbb{N})$ be the space of square-summable complex-valued sequences (defined by (E1) below). For any $\lambda \in (\frac{1}{2}, 1)$ and $\gamma > 0$, let $A = (a_{mn})_{m,n\in\mathbb{N}}$ be the infinite matrix whose components are defined by (B11). If $A \geqslant 0$ is positive semidefinite as an operator on $\ell^2(\mathbb{N})$, then the bosonic loss-dephasing channel $\mathcal{N}_{\lambda,\gamma}$ is antidegradable.

Proof. In the proof of Theorem 30 we have seen that the bosonic loss-dephasing channel is antidegradable if the Hadamard map associated with the infinite matrix A [given in (B11)] is a quantum channel. Since the diagonal elements of A are equal to one, from Lemma 50 we deduce that the Hadamard map associated to A is a quantum channel if and only if A is positive semidefinite. This concludes the proof. ■

In Theorem 30, we showed that the above-mentioned infinite matrix A is positive semidefinite if $\theta(e^{-\gamma/2}, \sqrt{\lambda/(1-\lambda)}) \leqslant \frac{3}{2}$, where $\theta(x, y) := \sum_{n=0}^{\infty} x^{n^2} y^n$. This identifies just a portion of the full region of parameters of λ and γ where the infinite matrix A is positive semidefinite.

To analyze the positive semidefiniteness of the infinite matrix A further, let $A^{(d)}$ denote its $d \times d$ top left corner. Note that it is well known that an infinite matrix is positive semidefinite if and only if its $d \times d$ top left corner is positive semidefinite for all $d \in \mathbb{N}$. For modest values of d, we can numerically determine the parameter region where $A^{(d)}$ is positive semidefinite. To achieve this, we plot in Fig. 3 the quantity

$$\eta_d(\gamma) := \max\left(\lambda \in \left(\tfrac{1}{2}, 1\right] : A^{(d)} \text{ is positive semidefinite}\right), \tag{B17}$$

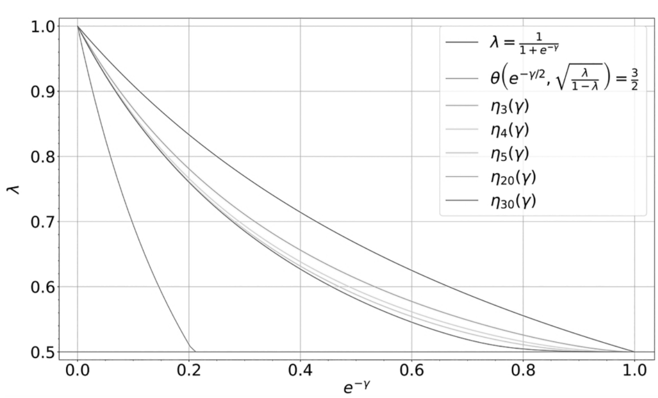

FIG. 3. Numerical estimation of the antidegradability of the loss-dephasing channel. The horizontal axis shows the quantity $e^{-\gamma}$, varying from 0 to 1 as dephasing parameter γ decreases from ∞ to 0, while the vertical axis corresponds to the transmissivity λ. Theorem 30 establishes that below the region defined by the curve corresponding to the condition $\theta[e^{-\gamma/2}, \sqrt{\lambda/(1-\lambda)}] = 3/2$, the bosonic loss-dephasing channel is antidegradable. Moreover, Theorem 38 establishes that above the region defined by the curve corresponding to $\lambda = \frac{1}{1+e^{-\gamma}}$, the bosonic loss-dephasing channel is not antidegradable. The other curves depict the quantity $\eta_d(\gamma)$, which is defined in (B17) as the maximum value of the transmissivity where $A^{(d)}$ is positive semidefinite, for various values of d. Our numerical analysis seems to indicate that in the region below the curve corresponding to $\lambda \leqslant \eta_{30}(\gamma)$, the bosonic loss-dephasing channel is antidegradable. Here we employ $d = 30$, as increasing d beyond $d \geqslant 20$ yields no discernible change in the plot.

with respect to $e^{-\gamma}$ for various values of d. This quantity is relevant because $A^{(d)}$ is positive semidefinite if and only if $\lambda \leqslant \eta_d(\gamma)$. Moreover, the quantity $\eta_d(\gamma)$ monotonically decreases in d and converges to some $\bar{\eta}(\gamma)$ as $d \to \infty$. Notably, the condition $\lambda \leqslant \bar{\eta}(\gamma)$ is necessary and sufficient for positive semidefiniteness of the infinite matrix A, and also a sufficient condition for the antidegradability of the the bosonic loss-dephasing channel $\mathcal{N}_{\lambda,\gamma}$. Our numerical investigation seems to suggest that when d is approximately 20, the quantity $\eta_d(\gamma)$ has already reached its limiting value $\bar{\eta}(\gamma)$, which can be approximated, for instance, by considering, the curve $\eta_{30}(\gamma)$.

b. Antidegrading maps

Theorem 30 discovers parameter regions of transmissivity λ and dephasing γ in which the bosonic loss-dephasing channel $\mathcal{N}_{\lambda,\gamma}$ is antidegradable. Although the proof of this theorem ensures the existence of antidegrading maps for $\mathcal{N}_{\lambda,\gamma}$ within these parameter regions, it does not offer explicit constructions of such antidegrading maps. In the forthcoming Theorem 32, we present such explicit constructions. Note that, due to Lemma 26, the output operators of the complementary channel $\mathcal{N}^{c}_{\lambda,\gamma}$ reside within the space $\mathcal{T}(\mathcal{H}_{E_{\rm out}})$, where $\mathcal{H}_{E_{\rm out}}$ is the following subspace of the two-mode Hilbert space $\mathcal{H}_{E_1}\otimes\mathcal{H}_{E_2} = L^2(\mathbb{R})\otimes L^2(\mathbb{R})$:

$$\mathcal{H}_{E_{\rm out}} := {\rm Span}\{|\ell\rangle_{E_1}\otimes|\sqrt{\gamma}n\rangle_{E_2} : \ \ell \leqslant n \text{ with } \ell, n\in\mathbb{N}\}, \tag{B18}$$

where $|n\rangle$ represents the nth Fock state, and $|\sqrt{\gamma}n\rangle$ denotes the coherent state with a parameter of $\sqrt{\gamma}n$. These states correspond to the environmental modes of the pure-loss and dephasing channels, respectively (we shall maintain this notation throughout).

Theorem 32. Anti-degrading maps corresponding to each parameter region in Theorem 30 can be defined as follows. In the region (i), i.e., $\lambda\in[0,\frac{1}{2}]$ and $\gamma \geqslant 0$, an antidegrading map is given by

$$\mathcal{A}_{\lambda,\gamma} = \left(\mathcal{E}_{\frac{\lambda}{1-\lambda}}\circ\mathcal{R}_{E_1}\right)\otimes{\rm Tr}_{E_2}, \tag{B19}$$

where $\mathcal{R}_{E_1}(\cdot) := (-1)^{\hat{e}_1^\dagger\hat{e}_1}\cdot(-1)^{\hat{e}_1^\dagger\hat{e}_1}$, with $\hat{e}_1$ as the annihilation operator of the output mode of the pure-loss channel E_1.

In the region (ii), i.e., $\lambda\in(\frac{1}{2},1)$ and γ such that $\theta(e^{-\gamma/2},\sqrt{\frac{\lambda}{1-\lambda}})\leqslant\frac{3}{2}$, an antidegrading map $\mathcal{A}_{\lambda,\gamma} : \mathcal{T}(\mathcal{H}_{E_{\rm out}}) \to \mathcal{T}(\mathcal{H}_{\rm B})$, with $\mathcal{H}_{E_{\rm out}}$ given by (B18), is defined as follows. For all $\ell_1,\ell_2,n_1,n_2\in\mathbb{N}$ with $\ell_1\leqslant n_1$ and $\ell_2\leqslant n_2$, it holds that

$$\mathcal{A}_{\lambda,\gamma}\big(|\ell_1\rangle\langle\ell_2|\otimes|\sqrt{\gamma}n_1\rangle\langle\sqrt{\gamma}n_2|\big) := \sum_{k=0}^{\min(n_1-\ell_1,n_2-\ell_2)} c_k^{(\ell_1,\ell_2,n_1,n_2)}|k+\ell_1\rangle\langle k+\ell_2|, \tag{B20}$$

where for all $k\in\{0,1,\ldots,\min(n_1-\ell_1,n_2-\ell_2)\}$ the coefficients $c_k^{(\ell_1,\ell_2,n_1,n_2)}$ are defined as

$$c_k^{(\ell_1,\ell_2,n_1,n_2)} := (-1)^{\ell_1-\ell_2}\sqrt{\mathcal{B}_k\!\left(n_1-\ell_1,\frac{2\lambda-1}{\lambda}\right)\mathcal{B}_k\!\left(n_2-\ell_2,\frac{2\lambda-1}{\lambda}\right)}\,a_{n_1-\ell_1 n_2-\ell_2}\,a_{k+\ell_1 k+\ell_2}, \tag{B21}$$

where $\mathcal{B}_l(n,\lambda) := \binom{n}{l}\lambda^l(1-\lambda)^{n-l}$, and a_{mn} is defined in (B11).

Proof of Theorem 32. Let us suppose that λ and γ fall within the parameter region (i). A complementary channel of the pure-loss channel $\mathcal{E}_\lambda$ is given by $\mathcal{E}_\lambda^c = \mathcal{E}_{1-\lambda}\circ\mathcal{R}$. Consequently, Lemma 16 implies that $(\mathcal{E}_{\frac{\lambda}{1-\lambda}}\circ\mathcal{R})\circ\mathcal{E}_\lambda^c = \mathcal{E}_\lambda$, i.e., the channel $\mathcal{E}_{\frac{\lambda}{1-\lambda}}\circ\mathcal{R}$ is an antidegrading map of the pure-loss channel. The general construction detailed in the proof of Lemma 55 for the antidegrading map of the composition between an antidegradable channel and another channel demonstrates that the map given in (B19) is an antidegrading map of $\mathcal{N}_{\lambda,\gamma}$.

Let us now suppose that λ and γ fall within the parameter region (ii). To come up with the antidegrading map defined in (B20), we drew intuition from the proof of Lemma 54, which demonstrates the equivalence between two-extendibility of the Choi state and the existence of an antidegrading map, while also considering the two-extension of the Choi state of $\mathcal{N}_{\lambda,\gamma}$ explicitly found in (B13). In order to show that the map $\mathcal{A}_{\lambda,\gamma}$ in (B20) is an antidegrading map of $\mathcal{N}_{\lambda,\gamma}$, we need to show that it is a quantum channel satisfying $\mathcal{A}_{\lambda,\gamma}\circ\mathcal{N}^c_{\lambda,\gamma} = \mathcal{N}_{\lambda,\gamma}$. We begin by proving that $\mathcal{A}_{\lambda,\gamma}$ is trace preserving. By linearity, it suffices to show that for any $\ell_1,\ell_2,n_1,n_2\in\mathbb{N}$ with $\ell_1\leqslant n_1$ and $\ell_2\leqslant n_2$ it holds that

$${\rm Tr}\big[\mathcal{A}_{\lambda,\gamma}\big(|\ell_1\rangle\langle\ell_2|_{E_1}\otimes|\sqrt{\gamma}n_1\rangle\langle\sqrt{\gamma}n_2|_{E_2}\big)\big] = {\rm Tr}\big[|\ell_1\rangle\langle\ell_2|_{E_1}\otimes|\sqrt{\gamma}n_1\rangle\langle\sqrt{\gamma}n_2|_{E_2}\big]. \tag{B22}$$

Indeed, we obtain

$$\begin{aligned}{\rm Tr}\big[\mathcal{A}_{\lambda,\gamma}\big(|\ell_1\rangle\langle\ell_2|_{E_1}\otimes|\sqrt{\gamma}m\rangle\langle\sqrt{\gamma}n|_{E_2}\big)\big] &= \delta_{\ell_1,\ell_2}\sum_{k=0}^{\min(m-\ell_1,n-\ell_1)} c_k^{(\ell_1,\ell_1,m,n)}\\ &= \delta_{\ell_1,\ell_2}\sum_{k=0}^{\min(m-\ell_1,n-\ell_1)}\sqrt{\mathcal{B}_k\!\left(m-\ell_1,\frac{2\lambda-1}{\lambda}\right)\mathcal{B}_k\!\left(n-\ell_1,\frac{2\lambda-1}{\lambda}\right)}a_{m-\ell_1,n-\ell_1}\,a_{k+\ell_1,k+\ell_1}\end{aligned}$$

$$= \delta_{\ell_1,\ell_2} e^{-\frac{\gamma}{2}(m-n)^2}$$
$$= \mathrm{Tr}\big[|\ell_1\rangle\langle\ell_2|_{E_1} \otimes |\sqrt{\gamma}n_1\rangle\langle\sqrt{\gamma}n_2|_{E_2}\big],$$

where δ_{ℓ_1,ℓ_2} denotes the Kronecker delta and where we have exploited the formula for the overlap between coherent states provided in (A6). Now, let us show that $\mathcal{A}_{\lambda,\gamma}$ is completely positive. To achieve this, we need to find a pure state Ψ on $\mathcal{H}_{\mathrm{anc}} \otimes \mathcal{H}_{E_{\mathrm{out}}}$, where $\mathcal{H}_{\mathrm{anc}}$ in an auxiliary reference, such that $\mathrm{Tr}_{\mathrm{anc}}[|\Psi\rangle\langle\Psi|] > 0$ and $(\mathrm{id}_{\mathrm{anc}} \otimes \mathcal{A}_{\lambda,\gamma})(|\Psi\rangle\langle\Psi|) \geqslant 0$ [59,66]. Let $\mathcal{H}_{\mathrm{anc}} := \mathcal{H}_A \otimes \mathcal{H}_{B_1} = L^2(\mathbb{R}) \otimes L^2(\mathbb{R})$ and let us construct the pure state $|\Psi\rangle_{AB_1E_1E_2} \in \mathcal{H}_{\mathrm{anc}} \otimes \mathcal{H}_{E_{\mathrm{out}}}$ as follows:

$$|\Psi\rangle_{AB_1E_1E_2} := U_\lambda^{B_1E_1} V_\gamma^{B_1E_2} |\psi(r)\rangle_{AB_1} |0\rangle_{E_1} |0\rangle_{E_2}$$
$$= \frac{1}{\cosh(r)} \sum_{n=0}^{\infty} \sum_{\ell=0}^{n} (-1)^\ell \tanh^n(r) \sqrt{\mathcal{B}_l(n, 1-\lambda)} |n\rangle_A |n-\ell\rangle_{B_1} |\ell\rangle_{E_1} |\sqrt{\gamma}n\rangle_{E_2}, \tag{B23}$$

where $U_\lambda^{B_1E_1}$ is the beam splitter unitary, $V_\gamma^{B_1E_2}$ is the conditional displacement unitary, and $|\psi(r)\rangle_{AB_1}$ is the two-mode squeezed vacuum state with squeezing $r > 0$. Let us now show that $\mathrm{Tr}_{AB_1}[|\Psi\rangle\langle\Psi|_{AB_1E_1E_2}]$ is positive definite on $\mathcal{H}_{E_{\mathrm{out}}}$. Let $|\phi\rangle_{E_1E_2} \in \mathcal{H}_{E_{\mathrm{out}}}$. Since there exists $\bar{\ell}, \bar{n} \in \mathbb{N}$ with $\bar{l} \leqslant \bar{n}$ such that $\langle\phi|_{E_1E_2} |\bar{\ell}\rangle_{E_1} \otimes |\sqrt{\gamma}\bar{n}\rangle_{E_2} \neq 0$, (B23) implies that

$$\langle\phi|_{E_1E_2} \mathrm{Tr}_{AB_1}[|\Psi\rangle\langle\Psi|_{AB_1E_1E_2}]|\phi\rangle_{E_1E_2} = \frac{1}{\cosh^2(r)} \sum_{n=0}^{\infty} \sum_{l=0}^{n} \tanh^{2n}(r)\, \mathcal{B}_\ell(n, 1-\lambda) \big|\langle\phi|_{E_1E_2} |\ell\rangle_{E_1} \otimes |\sqrt{\gamma}n\rangle_{E_2}\big|^2$$
$$\geqslant \frac{1}{\cosh^2(r)} \tanh^{2\bar{n}}(r)\, \mathcal{B}_{\bar{l}}(\bar{n}, 1-\lambda) \big|\langle\phi|_{E_1E_2} |\bar{\ell}\rangle_{E_1} \otimes |\sqrt{\gamma}\bar{n}\rangle_{E_2}\big|^2$$
$$> 0.$$

We next show that $\mathrm{id}_{AB_1} \otimes \mathcal{A}_{\lambda,\gamma}(|\Psi\rangle\langle\Psi|)$ is positive semidefinite. Let B_2 denote the output system of $\mathcal{A}_{\lambda,\gamma}$. Note that

$$\mathrm{id}_{AB_1} \otimes \mathcal{A}_{\lambda,\gamma}(|\Psi\rangle\langle\Psi|_{AB_1E_1E_2}) \overset{\text{(i)}}{=} \frac{1}{\cosh^2(r)} \sum_{m,n=0}^{\infty} \sum_{\ell_1=0}^{m} \sum_{\ell_2=0}^{n} (-1)^{\ell_1+\ell_2} [\tanh(r)]^{m+n} \sqrt{\mathcal{B}_{\ell_1}(m, 1-\lambda)\mathcal{B}_{\ell_2}(n, 1-\lambda)}$$
$$|m\rangle\langle n|_A \otimes |m-\ell_1\rangle\langle n-\ell_2|_{B_1} \otimes \mathcal{A}_{\lambda,\gamma}\big(|\ell_1\rangle\langle\ell_2|_{E_1} \otimes |\sqrt{\gamma}m\rangle\langle\sqrt{\gamma}n|_{E_2}\big)$$
$$\overset{\text{(ii)}}{=} \frac{1}{\cosh^2(r)} \sum_{m,n=0}^{\infty} \sum_{\ell_1=0}^{m} \sum_{\ell_2=0}^{n} \sum_{k=0}^{\min(m-\ell_1, n-\ell_2)} [\tanh(r)]^{m+n}$$
$$\times \sqrt{\mathcal{B}_{\ell_1}(m, 1-\lambda)\, \mathcal{B}_{\ell_2}(n, 1-\lambda)\, \mathcal{B}_k\left(m-\ell_1, \frac{2\lambda-1}{\lambda}\right) \mathcal{B}_k\left(n-\ell_2, \frac{2\lambda-1}{\lambda}\right)}$$
$$a_{m-\ell_1, n-\ell_2}\, a_{k+\ell_1, k+\ell_2} |n_1\rangle\langle n|_A \otimes |m-\ell_1\rangle\langle n-\ell_2|_{B_1} \otimes |k+\ell_1\rangle\langle k+\ell_2|_{B_2}$$
$$\overset{\text{(iii)}}{=} \rho_{AB_1B_2}. \tag{B24}$$

Here in (i) we used the definition of $|\Psi\rangle_{AB_1E_1E_2}$ given in (B23); in (ii) we utilized the definition of the map $\mathcal{A}_{\lambda,\gamma}$ from (B20); and in (iii) we recognized the tripartite operator $\rho_{AB_1B_2}$ defined in (B13), which is a quantum state provided that λ and γ satisfy $\theta(e^{-\gamma/2}, \sqrt{\frac{\lambda}{1-\lambda}}) \leqslant \frac{3}{2}$. Therefore, in such a parameter region, $\mathrm{id}_{AB_1} \otimes \mathcal{A}_{\lambda,\gamma}(|\Psi\rangle\langle\Psi|_{AB_1E_1E_2})$ is positive semidefinite, and thus $\mathcal{A}_{\lambda,\gamma}$ is a quantum channel. Let us now verify that $\mathcal{A}_{\lambda,\gamma} \circ \mathcal{N}^{\mathrm{c}}_{\lambda,\gamma} = \mathcal{N}_{\lambda,\gamma}$. To show this, note that

$$\mathrm{id}_A \otimes \big(\mathcal{A}_{\lambda,\gamma} \circ \mathcal{N}^{\mathrm{c}}_{\lambda,\gamma}\big)(|\psi(r)\rangle\langle\psi(r)|_{AB_1}) \overset{\text{(iv)}}{=} \mathrm{Tr}_{B_1}\big[\mathrm{id}_{AB_1} \otimes \mathcal{A}_{\lambda,\gamma}(|\Psi\rangle\langle\Psi|_{AB_1E_1E_2})\big] \overset{\text{(v)}}{=} \mathrm{Tr}_{B_1}[\rho_{AB_1B_2}]$$
$$\overset{\text{(vi)}}{=} \mathrm{id}_A \otimes \mathcal{N}_{\lambda,\gamma}[|\psi(r)\rangle\langle\psi(r)|_{AB_2}], \tag{B25}$$

Here in (iv) we employed (B23); in (v) we exploited (B24); and in (vi) we used that $\rho_{AB_1B_2}$ is a two-extension of $\mathrm{id}_A \otimes \mathcal{N}_{\lambda,\gamma}(|\psi(r)\rangle\langle\psi(r)|)$, as established in the proof of Theorem 30. Finally, since the two-mode squeezed vacuum state $|\psi(r)\rangle_{AB}$ satisfies $\mathrm{Tr}_B[|\psi(r)\rangle\langle\psi(r)|_{AB}] > 0$, we conclude $\mathcal{A}_{\lambda,\gamma} \circ \mathcal{N}^{\mathrm{c}}_{\lambda,\gamma} = \mathcal{N}_{\lambda,\gamma}$. ■

2. Analysis of the bosonic loss-dephasing channel via its finite-dimensional restrictions

Definition 33. Let $d \in \mathbb{N}$ and let $\mathcal{H}_d := \mathrm{Span}(\{|n\rangle\}_{n=0,\ldots,d-1})$ be the subspace spanned by the first d Fock states. The qudit restriction of the bosonic loss-dephasing channel $\mathcal{N}^{(d)}_{\lambda,\gamma}$ is a quantum channel defined by

$$\mathcal{N}^{(d)}_{\lambda,\gamma}(\Theta) := \mathcal{N}_{\lambda,\gamma}(\Theta), \quad \forall\, \Theta \in \mathcal{T}(\mathcal{H}_d) \tag{B26}$$

Lemma 34. Let $\mathcal{H}_A, \mathcal{H}_B := L^2(\mathbb{R})$. Let $\mathcal{N} : \mathcal{T}(\mathcal{H}_A) \to \mathcal{T}(\mathcal{H}_B)$ be a quantum channel and let $\mathcal{N}^{(d)}$ be its qudit restriction, defined by

$$\mathcal{N}^{(d)}(\Theta) := \mathcal{N}(\Theta), \quad \forall\, \Theta \in \mathcal{T}(\mathcal{H}_d). \tag{B27}$$

If $\mathcal{N}$ is (anti-)degradable, then its qudit restriction $\mathcal{N}^{(d)}$ is (anti-)degradable.

Proof. Let $\mathcal{N}(\cdot) = \mathrm{Tr}_E[V_{A\to BE}(\cdot)V_{A\to BE}^\dagger]$ be a Stinespring representation of $\mathcal{N}$, and let $\mathcal{N}^c(\cdot) = \mathrm{Tr}_B[V_{A\to BE}(\cdot)V_{A\to BE}^\dagger]$ be the associated complementary channel. Note that the isometry $V_{A\to BE}$ provides a Stinespring representation also for the qudit restriction $\mathcal{N}^{(d)}$. Hence, the qudit restriction of the complementary channel $\mathcal{N}^c$ is a complementary channel of the qudit restriction $\mathcal{N}^{(d)}$, i.e.,

$$(\mathcal{N}^{(d)})^c(\Theta) = \mathcal{N}^c(\Theta), \quad \forall\, \Theta \in \mathcal{T}(\mathcal{H}_d). \tag{B28}$$

Moreover, note that any degrading or antidegrading map of $\mathcal{N}$ is effective for all input states, including those restricted to $\mathcal{H}_d$. Consequently, an (anti-)degrading map of $\mathcal{N}$ is also an (anti-)degrading map of its qudit restriction $\mathcal{N}^{(d)}$. ■

Corollary 35. If the qudit restriction $\mathcal{N}_{\lambda,\gamma}^{(d)}$ is not (anti-)degradable, then the bosonic loss-dephasing channel $\mathcal{N}_{\lambda,\gamma}$ is also not (anti-)degradable.

The following lemma shows that qudit restriction $\mathcal{N}_{\lambda,\gamma}^{(d)}$ is a qudit-to-qudit channel, mapping the space spanned by $\{|n\rangle\}_{n=0,\ldots,d-1}$ into itself.

Lemma 36. If the input state to the bosonic loss-dephasing channel is confined into the finite-dimensional subspace spanned by $\{|n\rangle\}_{n=0,\ldots,d-1}$, the resulting output state will similarly be confined to this subspace.

Proof. Examining Lemma 25 reveals that the operator $|m\rangle\langle n|$, when acted on by the bosonic loss-dephasing channel, is transformed into linear combinations of operators $\{|\ell\rangle\langle k|\}_{\ell\leqslant m, k\leqslant n}$. This means that if the input state to the bosonic loss-dephasing channel is restricted to the d-dimensional subspace $\{|n\rangle\}_{n=0,\ldots,d-1}$, the output of the channel will reside within the same subspace. ■

The qubit restriction $\mathcal{N}_{\lambda,\gamma}^{(2)}$ of the bosonic loss-dephasing channel coincides with the composition between the amplitude damping channel and the qubit dephasing channel [33,34], which we dub *amplitude-phase damping channel*. Theorems 37 and 38 utilize the amplitude-phase damping channel $\mathcal{N}_{\lambda,\gamma}^{(2)}$ to find that the bosonic loss-dephasing channel $\mathcal{N}_{\lambda,\gamma}$ is never degradable and, additionally, it is not antidegradable for $\lambda > \frac{1}{1+e^{-\gamma}}$, respectively.

a. The bosonic loss-dephasing channel is never degradable

The bosonic loss-dephasing channel $\mathcal{N}_{\lambda,\gamma}$ is never degradable, except when it coincides with either the bosonic dephasing channel ($\gamma > 0$ and $\lambda = 1$) or the degradable pure-loss channel ($\gamma = 0$ and $\lambda \geqslant \frac{1}{2}$), thereby complicating the derivation of its quantum capacity [32]. This result has been previously demonstrated in [32] through a pages-long proof; however, here we provide a significantly simpler proof of this result.

Theorem 37. Let $\lambda \in [0, 1]$ and $\gamma \geqslant 0$. The bosonic loss-dephasing channel $\mathcal{N}_{\lambda,\gamma}$ is degradable if and only if one of the following conditions is satisfied:

$$\gamma = 0 \text{ and } \lambda \in \left[\tfrac{1}{2}, 1\right]$$
$$\gamma \geqslant 0 \text{ and } \lambda = 1$$

Proof. Due to Corollary 35, a necessary condition for $\mathcal{N}_{\lambda,\gamma}$ to be degradable is the degradability of the amplitude-phase damping channel $\mathcal{N}_{\lambda,\gamma}^{(2)}$. We now apply [[51], Theorem 4], which establishes a necessary condition on the degradability of any qubit channel. Specifically, the rank of the Choi state of a degradable qubit channel is necessarily less or equal to 2. By using the notation used in (A3), the matrix associated with the Choi state of the amplitude-phase damping channel $C(\mathcal{N}_{\lambda,\gamma}^{(2)})$ in the computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ is given by

$$\frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & \sqrt{e^{-\gamma}\lambda} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1-\lambda & 0 \\ \sqrt{e^{-\gamma}\lambda} & 0 & 0 & \lambda \end{pmatrix}. \tag{B29}$$

One can easily see that its rank is equal to 3 for $\gamma > 0$ and $\lambda \in (0, 1)$. In addition, for $\lambda = 1$ the bosonic loss-dephasing channel coincides with the bosonic dephasing channel, $\mathcal{N}_{1,\gamma} = \mathcal{D}_\gamma$, which is degradable for any value of $\gamma \geqslant 0$ [40]. Finally, for $\gamma = 0$ the bosonic loss-dephasing channel coincides with the pure-loss channel, $\mathcal{N}_{\lambda,0} = \mathcal{E}_\lambda$, which is degradable if and only if $\lambda \in [\frac{1}{2}, 1]$ [38,39]. ■

b. Necessary condition on antidegradability via qubit restriction

The next theorem establishes the parameter range where the bosonic loss-dephasing channel is not antidegradable. We provide three different proofs for this theorem.

Theorem 38. Let $\gamma \geqslant 0$. If $\lambda > \frac{1}{1+e^{-\gamma}}$, then the bosonic loss-dephasing channel $\mathcal{N}_{\lambda,\gamma}$ is not antidegradable.

Proof 1. Assume that $\mathcal{N}_{\lambda,\gamma}$ is antidegradable; then substituting $m = 0$ and $n = 1$ in (A22) yields

$$\mathcal{A}_{\lambda,\gamma}\left(\mathcal{E}_{1-\lambda}(|0\rangle\langle 1|) \otimes |0\rangle\langle\sqrt{\gamma}|\right) = -e^{-\frac{1}{2}\gamma}\mathcal{E}_\lambda(|0\rangle\langle 1|). \tag{B30}$$

By exploiting $\mathcal{E}_\lambda(|0\rangle\langle 1|) = \sqrt{\lambda}|0\rangle\langle 1|$, we have

$$\mathcal{A}_{\lambda,\gamma}\left(|0\rangle\langle 1| \otimes |0\rangle\langle\sqrt{\gamma}|\right) = -\sqrt{\frac{e^{-\gamma}\lambda}{1-\lambda}}|0\rangle\langle 1|.$$

Using Lemma 52 in the Appendixes, we find $\sqrt{\frac{e^{-\gamma}\lambda}{1-\lambda}} \leqslant 1$, or $\lambda \leqslant \frac{1}{1+e^{-\gamma}}$. ■

Proof 2. Assume that $\mathcal{N}_{\lambda,\gamma}$ is antidegradable. As a consequence of (A22) and of the data-processing inequality for the fidelity [57], we find

$$F(\mathcal{E}_\lambda(|0\rangle\langle 0|), \mathcal{E}_\lambda(|1\rangle\langle 1|)) \geqslant F(\mathcal{E}_{1-\lambda}(|0\rangle\langle 0|) \otimes |0\rangle\langle 0|, \mathcal{E}_{1-\lambda}(|1\rangle\langle 1|) \otimes |\sqrt{\gamma}\rangle\langle\sqrt{\gamma}|). \tag{B31}$$

Furthermore, we obtain

$$\begin{aligned}\sqrt{1-\lambda} &= F\left(|0\rangle\langle 0|, \lambda|1\rangle\langle 1| + (1-\lambda)|0\rangle\langle 0|\right) \\ &\overset{\text{(i)}}{=} F\left(\mathcal{E}_\lambda(|0\rangle\langle 0|), \mathcal{E}_\lambda(|1\rangle\langle 1|)\right) \\ &\overset{\text{(ii)}}{\geqslant} F\left(|0\rangle\langle 0|, |\sqrt{\gamma}\rangle\langle\sqrt{\gamma}|\right) F\left(\mathcal{E}_{1-\lambda}(|0\rangle\langle 0|), \mathcal{E}_{1-\lambda}(|1\rangle\langle 1|)\right) \\ &\overset{\text{(iii)}}{\geqslant} F\left(|0\rangle\langle 0|, |\sqrt{\gamma}\rangle\langle\sqrt{\gamma}|\right) F\left(|0\rangle\langle 0|, \lambda|0\rangle\langle 0| \right. \\ &\quad \left. + (1-\lambda)|1\rangle\langle 1|\right) \overset{\text{(iv)}}{=} \sqrt{e^{-\gamma}\lambda}.\end{aligned} \tag{B32}$$

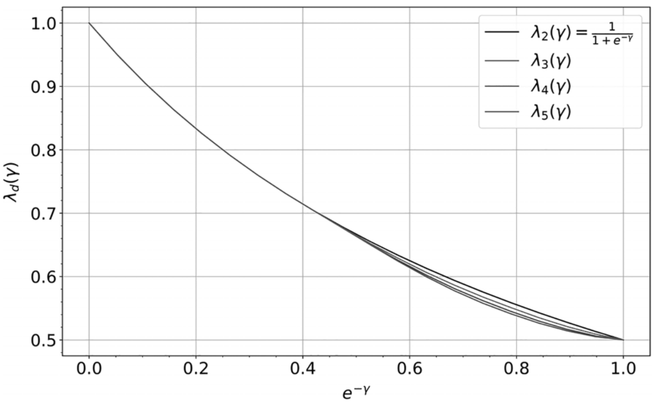

FIG. 4. Each curve indicates necessary and sufficient conditions where the qudit restriction $\mathcal{N}^{(d)}_{\lambda,\gamma}$ of the bosonic loss-dephasing channel $\mathcal{N}_{\lambda,\gamma}$ is antidegradable. In the region above the curve $\lambda_5(\gamma)$, the bosonic loss-dephasing channel is never antidegradable.

Here (i) follows from Lemma 13, (ii) follows from (B31) and from the fact that the fidelity is multiplicative under tensor product [57], (iii) uses Lemma 13 again, and in (iv) we exploited that $|\langle 0|\sqrt{\gamma}\rangle| = \sqrt{e^{-\gamma}}$. This yields $\sqrt{1-\lambda} \geqslant \sqrt{e^{-\gamma}\lambda}$, or $\lambda \leqslant \frac{1}{1+e^{-\gamma}}$. ■

Proof 3. By exploiting Lemma 7 and the Choi matrix of the qubit channel $\mathcal{N}^{(2)}_{\lambda,\gamma}$ reported in (B29), one can easily obtain that $\mathcal{N}^{(2)}_{\lambda,\gamma}$ is antidegradable if and only if $\lambda \leqslant \frac{1}{1+e^{-\gamma}}$. Consequently, thanks to Corollary 35, the bosonic loss-dephasing channel $\mathcal{N}_{\lambda,\gamma}$ is not antidegradable for $\lambda > \frac{1}{1+e^{-\gamma}}$. ■

c. Necessary condition on antidegradability via qudit restrictions

Let us introduce the following quantity for any $d \in \mathbb{N}$ and $\gamma \geqslant 0$:

$$\lambda_d(\gamma) := \max\left(\lambda \in [0,1] : \ \mathcal{N}^{(d)}_{\lambda,\gamma} \text{ is antidegradable}\right). \quad \text{(B33)}$$

This quantity is relevant since it allows us to find parameter region where the bosonic loss-dephasing channel is not antidegradable. Specifically, for $\lambda > \lambda_d(\gamma)$ the bosonic loss-dephasing channel $\mathcal{N}_{\lambda,\gamma}$ is not antidegradable, as established by Corollary 35. Due to the Proof 3 of Theorem 38, it follows that for $d = 2$ we have that $\lambda_2(\gamma) = \frac{1}{1+e^{-\gamma}}$, thereby establishing that $\mathcal{N}_{\lambda,\gamma}$ is not antidegradable for $\lambda > \frac{1}{1+e^{-\gamma}}$. Through an examination of larger values of d, we aim to identify an extended parameter region where the channel is not antidegradable (see Fig. 4). We begin by proving some useful properties of the quantity $\lambda_d(\gamma)$.

Lemma 39. For any $\gamma \geqslant 0$ and $d \in \mathbb{N}, d \geqslant 2$, the following facts hold:

Fact 1: The qudit restriction of the bosonic loss-dephasing channel $\mathcal{N}^{(d)}_{\lambda,\gamma}$ is antidegradable if and only if $\lambda \leqslant \lambda_d(\gamma)$

Fact 2: The quantity $\lambda_d(\gamma)$ is monotonically increasing in γ

Fact 3: For $d = 2$, it precisely holds that $\lambda_2(\gamma) = \frac{1}{1+e^{-\gamma}}$

Fact 4: The quantity $\lambda_d(\gamma)$ is monotonically nonincreasing in d

Fact 5: It holds that $\frac{1}{2} \leqslant \lambda_d(\gamma) \leqslant \frac{1}{1+e^{-\gamma}}$

Fact 6: When $d = 3$ and $e^{-\gamma} \leqslant \sqrt{2} - 1$ (or $\gamma \geqslant 0.881$), it exactly holds that $\lambda_3(\gamma) = \frac{1}{1+e^{-\gamma}}$

Proof. Fact 1. It suffices to show that for any $\gamma \geqslant 0$ and $\lambda, \lambda' \in [0,1]$ with $\lambda' < \lambda$, if $\mathcal{N}^{(d)}_{\lambda,\gamma}$ is antidegradable, then so is $\mathcal{N}^{(d)}_{\lambda',\gamma}$. To show this, we exploit the composition rule

$$\mathcal{E}_{\lambda_1} \circ \mathcal{N}_{\lambda_2,\gamma} = \mathcal{N}_{\lambda_1\lambda_2,\gamma}, \quad \forall\, \lambda_1, \lambda_2 \in [0,1], \quad \text{(B34)}$$

as established by Lemma 28, implying that the channel $\mathcal{N}^{(d)}_{\lambda',\gamma}$ can be written as the composition between $\mathcal{N}^{(d)}_{\lambda,\gamma}$ and another channel. Consequently, Lemma 55 concludes the proof.

Fact 2. Analogously to Fact 1, it suffices to show that for any $\lambda \in [0,1]$ and for any $\gamma' \geqslant \gamma \geqslant 0$, if $\mathcal{N}^{(d)}_{\lambda,\gamma}$ is antidegradable, then so is $\mathcal{N}^{(d)}_{\lambda,\gamma'}$. This follows from the composition rule

$$\mathcal{D}_{\gamma_1} \circ \mathcal{N}_{\lambda,\gamma_2} = \mathcal{N}_{\lambda,\gamma_1+\gamma_2}, \quad \forall\, \gamma_1, \gamma_2 \geqslant 0, \quad \text{(B35)}$$

as proved in Lemma 28. Furthermore, Lemma 55 concludes the proof.

Fact 3. This has already been proved in the Proof 3 of Theorem 38.

Fact 4. This follows from the observation that for all $d' \geqslant d$, if $\mathcal{N}^{(d')}_{\lambda,\gamma}$ is antidegradable, then so is $\mathcal{N}^{(d)}_{\lambda,\gamma}$.

Fact 5. The upper bound $\lambda_d(\gamma) \leqslant \frac{1}{1+e^{-\gamma}}$ follows from Fact 3 and Fact 4. Moreover, since the pure-loss channel $\mathcal{E}_\lambda$ is antidegradable if and only if $\lambda \leqslant \frac{1}{2}$, Fact 4 implies that $\lambda_d(0) \geqslant \frac{1}{2}$ [more specifically, one can also show that $\lambda_d(0) = \frac{1}{2}$]. Consequently, Fact 2 concludes the proof.

Fact 6. This proof relies on the equivalence between antidegradability of a channel and two-extendibility of its Choi state, as established in Lemma 54. Let λ and γ be such that $\lambda = \frac{1}{1+e^{-\gamma}}$ and $e^{-\gamma} \leqslant \sqrt{2} - 1$, implying that $\lambda \geqslant \frac{1}{\sqrt{2}}$. By using Lemma 25, we obtain the Choi state of $\mathcal{N}^{(3)}_{\lambda,\gamma}$ as follows:

$$\mathrm{id}_A \otimes \mathcal{N}_{\lambda,\gamma}(|\Phi_3\rangle\langle\Phi_3|) = \frac{1}{3}\sum_{m=0}^{2}\sum_{n=0}^{2}\sum_{\ell=0}^{\min(m,n)} e^{-\frac{\gamma}{2}(m-n)^2}\sqrt{\binom{m}{\ell}\binom{n}{\ell}}\lambda^{\frac{m+n}{2}-\ell}(1-\lambda)^l |m\rangle\langle n|_A \otimes |m-\ell\rangle\langle n-\ell|_B,$$

where Φ_3 is the maximally entangled state of Schmidt rank 3. We define a two-extension $\tilde{\rho}_{AB_1B_2}$ of the Choi state by the following conditions. First, $\tilde{\rho}_{AB_1B_2}$ satisfies the following $B_1 \leftrightarrow B_2$ symmetry for all $i_1, i_2, i_3, j_1, j_2, j_3 \in \{0,1,2\}$:

$$\begin{aligned}
\langle j_1|_A\langle j_2|_{B_1}\langle j_3|_{B_2}\tilde{\rho}_{AB_1B_2}|i_1\rangle_A|i_2\rangle_{B_1}|i_3\rangle_{B_2} &= \langle j_1|_A\langle j_3|_{B_1}\langle j_2|_{B_2}\tilde{\rho}_{AB_1B_2}|i_1\rangle_A|i_2\rangle_{B_1}|i_3\rangle_{B_2},\\
\langle j_1|_A\langle j_2|_{B_1}\langle j_3|_{B_2}\tilde{\rho}_{AB_1B_2}|i_1\rangle_A|i_2\rangle_{B_1}|i_3\rangle_{B_2} &= \langle j_1|_A\langle j_2|_{B_1}\langle j_3|_{B_2}\tilde{\rho}_{AB_1B_2}|i_1\rangle_A|i_3\rangle_{B_1}|i_2\rangle_{B_2}.
\end{aligned} \quad \text{(B36)}$$

Furthermore, if $i_1 < \max(i_2, i_3)$ or $j_1 < \max(j_2, j_3)$, or if $i_1 > i_2 + i_3$ or $j_1 > j_2 + j_3$, then $\langle j_1|_A \langle j_2|_{B_1} \langle j_3|_{B_2} \tilde{\rho}_{AB_1B_2} |i_1\rangle_A |i_2\rangle_{B_1} |i_3\rangle_{B_2} = 0$. We can thus define $\tilde{\rho}_{AB_1B_2}$ by writing only the matrix elements with respect the set $\{|i_1\rangle_A|i_2\rangle_{B_1}|i_3\rangle_{B_2}\}$ with $2 \geqslant i_1 \geqslant i_2 \geqslant i_3 \geqslant 0$ such that $i_2 + i_3 \geqslant i_1$. Hence, in order to fully define $\tilde{\rho}_{AB_1B_2}$, it suffices to write the matrix elements of $\tilde{\rho}_{AB_1B_2}$ with respect to the set $\{ |0\rangle_A|0\rangle_{B_1}|0\rangle_{B_2}, |1\rangle_A|0\rangle_{B_1}|0\rangle_{B_2}, |1\rangle_A|1\rangle_{B_1}|1\rangle_{B_2}, |2\rangle_A|1\rangle_{B_1}|1\rangle_{B_2}, |2\rangle_A|2\rangle_{B_1}|0\rangle_{B_2}, |2\rangle_A|2\rangle_{B_1}|1\rangle_{B_2}, |2\rangle_A|2\rangle_{B_1}|2\rangle_{B_2} \}$. This gives rise to the following 7×7 matrix:

	$0_A0_{B_1}0_{B_2}$	$1_A1_{B_1}0_{B_2}$	$1_A1_{B_1}1_{B_2}$	$2_A1_{B_1}1_{B_2}$	$2_A2_{B_1}0_{B_2}$	$2_A2_{B_1}1_{B_2}$	$2_A2_{B_1}2_{B_2}$
$0_A0_{B_1}0_{B_2}$	1	$\sqrt{1-\lambda}$	0	$\sqrt{2}(1-\lambda)$	$\frac{(1-\lambda)^2}{\lambda}$	0	0
$1_A1_{B_1}0_{B_2}$	$\sqrt{1-\lambda}$	$1-\lambda$	0	$\sqrt{2(1-\lambda)^3}$	$\sqrt{\frac{(1-\lambda)^5}{\lambda^2}}$	0	0
$1_A1_{B_1}1_{B_2}$	0	0	$2\lambda-1$	0	0	$\frac{2\lambda-1}{\lambda}\sqrt{1-\lambda}$	0
$2_A1_{B_1}1_{B_2}$	$\sqrt{2}(1-\lambda)$	$\sqrt{2(1-\lambda)^3}$	0	$2(1-\lambda)^2$	$\sqrt{2}\frac{(1-\lambda)^3}{\lambda}$	0	0
$2_A2_{B_1}0_{B_2}$	$\frac{(1-\lambda)^2}{\lambda}$	$\sqrt{\frac{(1-\lambda)^5}{\lambda^2}}$	0	$\sqrt{2}\frac{(1-\lambda)^3}{\lambda}$	$(1-\lambda)^2$	0	0
$2_A2_{B_1}1_{B_2}$	0	0	$\frac{2\lambda-1}{\lambda}\sqrt{1-\lambda}$	0	0	$2(1-\lambda)(2\lambda-1)$	0
$2_A2_{B_1}2_{B_2}$	0	0	0	0	0	0	$(2\lambda-1)^2$

One can show by direct calculation that this matrix is positive semidefinite if and only $\lambda \geqslant \frac{1}{\sqrt{2}}$. Note that the matrix is positive semidefinite if and only if $\tilde{\rho}_{AB_1B_2}$ is positive semidefinite. The latter follows from the following two simple facts: (i) A $n \times n$ symmetric matrix with a duplicate column is positive semidefinite if and only if the $(n-1)\times(n-1)$ matrix obtained by deleting one of the two equal column and the corresponding row is positive semidefinite, and (ii) A $n \times n$ symmetric matrix with a zero column is positive semidefinite if and only if the $(n-1)\times(n-1)$ matrix obtained by deleting such a column and the corresponding row is positive semidefinite. One can also verify $\mathrm{Tr}_{B_1}\tilde{\rho}_{AB_1B_2} = \mathrm{Tr}_{B_2}\tilde{\rho}_{AB_1B_2}$, and they are equal to the Choi state of the qutrit restriction with $\lambda = \frac{1}{1+e^{-\gamma}}$. We therefore conclude that the curve $\lambda = \frac{1}{1+e^{-\gamma}}$ provides a necessary and sufficient condition for the antidegradability of the qutrit channel $\mathcal{N}^{(3)}_{\lambda,\gamma}$ when $\lambda \geqslant \frac{1}{\sqrt{2}}$, or equivalently when $e^{-\gamma} \leqslant \sqrt{2}-1$. ■

To numerically compute the quantity $\lambda_d(\gamma)$ given in (B33), we utilize the equivalence between antidegradability of a channel and two-extendibility of its Choi state [41]. Specifically, for small values of d, we can determine necessary and sufficient conditions for the antidegradability of $\mathcal{N}^{(d)}_{\lambda,\gamma}$ by numerically solving the following *semidefinite program*:

$$\begin{aligned} \min_{\rho_{AB_1B_2}} \; & 1 \\ \text{s.t. } & \rho_{AB_1B_2} \geqslant 0, \\ & \mathrm{Tr}[\rho_{AB_1B_2}] = 1, \\ & \mathrm{Tr}_{B_2}[\rho_{AB_1B_2}] = \mathrm{id}_A \otimes \mathcal{N}^{(d)}_{\lambda,\gamma}(|\Phi_d\rangle\langle\Phi_d|_{AA'}), \\ & \mathrm{Tr}_{B_1}[\rho_{AB_1B_2}] = \mathrm{id}_A \otimes \mathcal{N}^{(d)}_{\lambda,\gamma}(|\Phi_d\rangle\langle\Phi_d|_{AA'}), \end{aligned} \tag{B37}$$

where $|\Phi_d\rangle$ is the maximally entangled state of schmidt rank d. The channel $\mathcal{N}^{(d)}_{\lambda,\gamma}$ is antidegradable if and only if the semidefinite program admits a feasible solution. We compute the quantity $\lambda_d(\gamma)$ defined in (B33) by numerically solving the semidefinite program. The results are plotted with respect to $e^{-\gamma}$ for various values of d in Fig. 4, showcasing the dependence of $\lambda_d(\gamma)$ on γ for small values of d. Our numerical analysis reveals that when γ is sufficiently large, i.e., $e^{-\gamma} \lesssim 0.41$ or $\gamma \gtrsim 0.89$, the value of $\lambda_d(\gamma)$ consistently equals $\frac{1}{1+e^{-\gamma}}$ for all examined values of d. In particular, this seems to suggest that within this range of the dephasing parameter, $\mathcal{N}_{\lambda,\gamma}$ is antidegradable if and only if $\lambda \leqslant \frac{1}{1+e^{-\gamma}}$. Based on this numerical exploration, we propose the following conjecture.

Conjecture 40. If γ is sufficiently large ($\gamma \gtrsim 0.89$), then the bosonic loss-dephasing channel $\mathcal{N}_{\lambda,\gamma}$ is antidegradable if and only if $\lambda \leqslant \frac{1}{1+e^{-\gamma}}$.

Notably, from Fig. 4 we observe that if λ and γ satisfy $\lambda = \frac{1}{1+e^{-\gamma}}$ with $e^{-\gamma} \gtrsim 0.41$ (or $\gamma \lesssim 0.89$), then $\mathcal{N}_{\lambda,\gamma}$ is not antidegradable.

APPENDIX C: GENERALIZATION OF OUR METHODS TO GENERAL BOSONIC DEPHASING CHANNELS

In Theorem 30 we introduced a method to analyze antidegradability of the bosonic loss-dephasing channel. In this section, we show that this method can be applied also to analyze the antidegradability of the composition between a *general* bosonic dephasing channel and the pure-loss channel channel.

Given a probability distribution $p(\cdot)$ over $\mathbb{R}$, the associated *general bosonic dephasing channel* is given by

$$\mathcal{D}^{(p)}(X) := \int_{-\infty}^{\infty} \mathrm{d}\phi \, p(\phi)\, e^{i\phi \hat{a}^\dagger \hat{a}} X e^{-i\phi \hat{a}^\dagger \hat{a}}. \tag{C1}$$

If $p(\phi)$ is the Gaussian distribution $p(\phi) := \frac{1}{\sqrt{2\pi\gamma}} e^{-\frac{\phi^2}{2\gamma}}$, the general bosonic dephasing channel $\mathcal{D}^{(p)}$ exactly coincides with the bosonic dephasing channel $\mathcal{D}_\gamma$ analyzed in this work. The action of $\mathcal{D}^{(p)}$ on operators of the form $|n\rangle\langle m|$ is given by

$$\mathcal{D}^{(p)}(|n\rangle\langle m|) = \tilde{p}(n-m)|n\rangle\langle m|, \tag{C2}$$

where $\tilde{p}$ is the Fourier transform of the probability distribution p,

$$\tilde{p}(k) := \int_{-\infty}^{\infty} \mathrm{d}\phi\, p(\phi)\, e^{ik\phi}. \tag{C3}$$

Let $\mathcal{N}_\lambda^{(p)}$ be the composition between such a general bosonic dephasing channel $\mathcal{D}^{(p)}$ and the pure-loss channel,

$$\mathcal{N}_\lambda^{(p)} := \mathcal{D}^{(p)} \circ \mathcal{E}_\lambda = \mathcal{E}_\lambda \circ \mathcal{D}^{(p)}. \tag{C4}$$

We can apply the exact same method that we have introduced in the proof of Thereom 30 in order to analyze the antidegradability of $\mathcal{N}_\lambda^{(p)}$. The key observation is that in the proof of Theorem 30 we did not use the explicit expression of the channel $\mathcal{D}^{(p)}$ before stating (B13). This simple observation allows us to generalise our results to arbitrary bosonic dephasing channels, as stated in the following theorem.

Theorem 41 (Sufficient condition on the antidegradability of the composition between a general bosonic dephasing channel and pure-loss channel). Let $\lambda \in [0, 1)$ and let $p(\cdot)$ be a probability distribution over $\mathbb{R}$. Let $A = (a_{mn})_{m,n\in\mathbb{N}}$ be the infinite matrix whose components are defined by

$$a_{mn} := \frac{\tilde{\phi}(n-m)}{\sum_{j=0}^{\min(n,m)} \sqrt{\mathcal{B}_j\left(n, \frac{2\lambda-1}{\lambda}\right) \mathcal{B}_j\left(m, \frac{2\lambda-1}{\lambda}\right)}}, \quad \forall\, n, m \in \mathbb{N}. \tag{C5}$$

The channel $\mathcal{N}_\lambda^{(p)}$ is antidegradable as long as either $\lambda \in [0, \frac{1}{2}]$ or the infinite matrix A is positive semidefinite.

APPENDIX D: COHERENCE PRESERVATION OF THE BOSONIC LOSS-DEPHASING CHANNEL

In this section we use the notation introduced in Sec. A 8.

Theorem 42. Let $\gamma \geqslant 0$ and $\lambda \in (0, 1]$. For any energy constraint $N_s > 0$, the energy-constrained two-way quantum and secret-key capacities of the bosonic loss-dephasing channel are strictly positive, $K(\mathcal{N}_{\lambda,\gamma}, N_s) \geqslant Q_2(\mathcal{N}_{\lambda,\gamma}, N_s) > 0$.

Proof. We begin by assuming $N_s \in (0, 1)$ and defining the two-mode state

$$|\Psi_{N_s}\rangle_{AA'} := \sqrt{1-N_s}|00\rangle_{AA'} + \sqrt{N_s}|11\rangle_{AA'}, \tag{D1}$$

where the mean photon number of A' system is equal to N_s. By exploiting Lemma 25, one can observe that the state

$$\rho_{AB} := \mathrm{id}_A \otimes \mathcal{N}_{\lambda,\gamma}(|\Psi_{N_s}\rangle\langle\Psi_{N_s}|_{AA'}) \tag{D2}$$

is effectively a two-qubit state and its matrix with respect to the computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ is given by

$$\rho_{AB} = \begin{pmatrix} 1-N_s & 0 & 0 & \sqrt{(1-N_s)N_s e^{-\gamma}\lambda} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & (1-\lambda)N_s & 0 \\ \sqrt{(1-N_s)N_s e^{-\gamma}\lambda} & 0 & 0 & \lambda N_s \end{pmatrix}.$$

If we perform partial transpose with respect to the system B, we find the matrix

$$(\rho_{AB})^{\intercal_B} = \begin{pmatrix} 1-N_s & 0 & 0 & 0 \\ 0 & 0 & \sqrt{(1-N_s)N_s e^{-\gamma}\lambda} & 0 \\ 0 & \sqrt{(1-N_s)N_s e^{-\gamma}\lambda} & (1-\lambda)N_s & 0 \\ 0 & 0 & 0 & \lambda N_s \end{pmatrix},$$

whose eigenvalues are not all positive, i.e., the state ρ_{AB} is not PPT [67]. By exploiting the fact that any two-qubit state is distillable if and only if it is not PPT [67], it follows that $E_d(\mathrm{id}_A \otimes \mathcal{N}_{\lambda,\gamma}(|\Psi_{N_s}\rangle\langle\Psi_{N_s}|_{AA'})) > 0$, where E_d is the distillable entanglement. On the other hand, from Lemma 29 we have that

$$\begin{aligned} K(\mathcal{N}_{\lambda,\gamma}, N_s) &\geqslant Q_2(\mathcal{N}_{\lambda,\gamma}, N_s) \\ &\geqslant E_d\big(\mathrm{id}_A \otimes \mathcal{N}_{\lambda,\gamma}(|\Psi_{N_s}\rangle\langle\Psi_{N_s}|_{AA'})\big). \end{aligned} \tag{D3}$$

This concludes the proof for $N_s \in (0, 1)$. Since the energy-constrained capacities are monotonically nondecreasing in the energy constraint N_s, the proof follows for any $N_s > 0$. ■

Since the state $\mathrm{id}_A \otimes \mathcal{N}_{\lambda,\gamma}(|\Psi_{N_s}\rangle\langle\Psi_{N_s}|_{AA'})$ in (D2) is always entangled, it follows that the bosonic loss-dephasing channel $\mathcal{N}_{\lambda,\gamma}$ is never entanglement breaking [33]. We state this formally in the following theorem.

Theorem 43. For all $\gamma \geqslant 0$ and all $\lambda \in (0, 1]$, the bosonic loss-dephasing channel $\mathcal{N}_{\lambda,\gamma}$ is not entanglement breaking.

In the following subsection we will find an explicit strictly positive lower bound on the two-way capacities of the bosonic loss-dephasing channel.

1. Multirail multiphoton encoding

Let $\Pi(N, k)$ be the set of partitions of N objects into k (possibly empty) parts. It is well known that $|\Pi(N, k)| = \binom{N+k-1}{k-1} = \binom{N+k-1}{N}$. Clearly, we can think of each $p \in \Pi(N, k)$ as a vector in $\mathbb{N}_+^k$, also denoted by p, with the constraint that $\sum_{\ell=1}^{k} p_\ell = N$. For each $p \in \Pi(N, k)$, define the associate

k-mode Fock state as

$$|\psi_p\rangle := |p_1\rangle \dots |p_k\rangle. \tag{D4}$$

Note that for $p, q \in \Pi(N,k)$, we have that $\langle\psi_p|\psi_q\rangle = \delta_{p,q}$. Let us call

$$P_{N,k} := \sum_{p\in\Pi(N,k)} \psi_p \tag{D5}$$

the projector onto the k-mode subspace of total photon number N. Note that the bosonic dephasing channel satisfies that

$$\begin{aligned}\mathcal{D}_\gamma^{\otimes k}(|\psi_p\rangle\langle\psi_q|) &= e^{-\frac{\gamma}{2}\sum_\ell (p_\ell - q_\ell)^2}|\psi_p\rangle\langle\psi_q| = e^{-\frac{\gamma}{2}\|p-q\|^2}|\psi_p\rangle\langle\psi_q| \\ &= (K_{N,k,\gamma})_{pq}|\psi_p\rangle\langle\psi_q|,\end{aligned} \tag{D6}$$

where $K_{N,k,\gamma}$ is the $\binom{N+k-1}{N} \times \binom{N+k-1}{N}$ matrix with entries

$$(K_{N,k,\gamma})_{pq} := e^{-\frac{\gamma}{2}\|p-q\|^2}. \tag{D7}$$

For any $\binom{N+k-1}{N}$-dimensional state σ, let us denote as $\overline{\sigma}$ the following isometrically equivalent state:

$$\overline{\sigma} := \sum_{p,q\in\Pi(N,k)} \sigma_{pq}|\psi_p\rangle\langle\psi_q|. \tag{D8}$$

The state $\overline{\sigma}$, which is termed as the *rail encoding* of σ, is supported on the subspace of k modes with total photon number equal to N. Now, let ρ be a $\binom{N+k-1}{N}$-dimensional state and let us calculate the output of $\mathcal{D}_\gamma^{\otimes k}$ when the input is $\overline{\rho}$:

$$\begin{aligned}\mathcal{D}_\gamma^{\otimes k}(\overline{\rho}) &= \sum_{p,q\in\Pi(N,k)} \rho_{pq}(K_{N,k,\gamma})_{pq}|\psi_p\rangle\langle\psi_q| \\ &= \overline{K_{N,k,\gamma}\circ\rho} = \overline{\Theta_{N,k,\gamma}(\rho)},\end{aligned} \tag{D9}$$

where the operation $\circ$ denotes the element-wise product between matrices and where we have introduced the following Hadamard channel:

$$\Theta_{N,k,\gamma}(X) := K_{N,k,\gamma}\circ X. \tag{D10}$$

Since $\mathcal{D}_\gamma^{\otimes k}$ is a (completely) positive map, this in particular shows that $K_{N,k,\gamma} \geqslant 0$. (This latter statement can also be proved directly with techniques similar to that in the proof of [[68], Lemma 15].) In practice, the k-fold application of the bosonic dephasing channel on the k-mode N-photon code space behaves as a new Hadamard channel $\Theta_{N,k,\gamma}$ with associated matrix $K_{N,k,\gamma}$.

a. Lower bound on the two-way capacity of the bosonic loss-dephasing channel

Since under the action of $\mathcal{N}_{\lambda,\gamma}$ photons can only be lost and never added, and each photon has a probability λ of being transmitted, the probability that an N-photon state will retain N photons at the output of the channel is exactly λ^N. *If that happens*, then the state in the code space is effectively left untouched by the loss and only dephased under the action of the Hadamard channel $\Theta_{N,k,\gamma}$.

More formally, from the Kraus representation

$$\mathcal{E}_\lambda(X) = \sum_{n=0}^{\infty}\frac{1}{n!}(1-\lambda)^n\lambda^{\frac{a^\dagger a}{2}}a^n X (a^\dagger)^n\lambda^{\frac{a^\dagger a}{2}} \tag{D11}$$

it is easy to deduce the handy identity

$$\mathcal{E}_\lambda^{\otimes k}(\overline{\rho}\,) = \lambda^N\overline{\rho} + (1-\lambda^N)\delta_{N,k,\lambda}, \tag{D12}$$

valid for all $\binom{N+k-1}{N}$-dimensional states ρ, with the notation of (D8). Here, $\delta_{N,k,\lambda}$ is a suitable k-mode state supported on the subspace of total photon number at most $N-1$, and thus $\overline{\rho}\delta_{N,k,\lambda} = \delta_{N,k,\lambda}\overline{\rho} = 0$. In turn, the above identity implies that

$$\begin{aligned}\mathcal{N}_{\lambda,\gamma}^{\otimes k}(\overline{\rho}) &= \lambda^N\,\overline{K_{N,k,\gamma}\circ\rho} + (1-\lambda^N)\delta'_{N,k,\lambda,\gamma} \\ &= \lambda^N\,\overline{\Theta_{N,k,\gamma}(\rho)} + (1-\lambda^N)\delta'_{N,k,\lambda,\gamma},\end{aligned} \tag{D13}$$

where once again $\delta'_{N,k,\lambda,\gamma}$ is a suitable k-mode state supported on the subspace of total photon number at most $N-1$.

Therefore, we can use the channel $\mathcal{N}_{\lambda,\gamma}^{\otimes k}$ to simulate $\Theta_{N,k,\gamma}$ *probabilistically*, with probability λ^N. The simulation works as follows:

(i) The input state ρ is encoded in the k-mode N-photon subspace according to the mapping $\rho \mapsto \overline{\rho}$.

(ii) The k-mode state $\overline{\rho}$ is sent across $\mathcal{N}_{\lambda,\gamma}^{\otimes k}$, via k uses of the bosonic loss-dephasing channel.

(iii) The total photon number is measured at the output. If N photons are found then the simulation is successful, otherwise the protocol is aborted.

A wealth of operational resource inequalities can be deduced from the above considerations. Here we limit ourselves to the observation that the two-way quantum capacity must satisfy

$$Q_2(\mathcal{N}_{\lambda,\gamma}) \overset{\text{(i)}}{\geqslant} \frac{\lambda^N}{k}\,Q_2(\Theta_{N,k,\gamma}). \tag{D14}$$

Consequently, it holds that

$$\begin{aligned}&Q_2(\mathcal{N}_{\lambda,\gamma}) \\ &\geqslant \frac{\lambda^N}{k}\,Q_2(\Theta_{N,k,\gamma}) \overset{\text{(i)}}{\geqslant} \frac{\lambda^N}{k}\,I_{\text{coh}}[\text{id}\otimes\Theta_{N,k,\gamma}(|\Psi\rangle\langle\Psi|)] \\ &\overset{\text{(iii)}}{=} \frac{\lambda^N}{k}\left\{\log_2\binom{N+k-1}{N} - S\left[\binom{N+k-1}{N}^{-1}K_{N,k,\gamma}\right]\right\}.\end{aligned} \tag{D15}$$

Here in (ii) we used the fact that the two-way quantum capacity of a channel can be lower bounded in terms of the coherent information [33,34], and we introduced the two-qu$d$it maximally entangled state $|\Psi\rangle$ of dimension $d = \binom{N+k-1}{N}$. In (iii) we used the definition of coherent information $I_{\text{coh}}(\rho_{AB}) := S(\rho_B) - S(\rho_{AB})$, with $S(\cdot)$ being the von Neumann entropy, and the fact that

$$\begin{aligned}&\text{id}\otimes\Theta_{N,k,\gamma}(|\Psi\rangle\langle\Psi|) \\ &\quad= \frac{1}{\binom{N+k-1}{N}}\sum_{p,q\in\Pi(N,k)}|p\rangle\langle q|\otimes\Theta_{N,k,\gamma}(|p\rangle\langle q|) \\ &\quad= \frac{1}{\binom{N+k-1}{N}}\sum_{p,q\in\Pi(N,k)}(K_{N,k,\gamma})_{pq}|p\rangle\langle q|\otimes|p\rangle\langle q|,\end{aligned} \tag{D16}$$

which implies that the spectrum of $\text{id}\otimes\Theta_{N,k,\gamma}(|\Psi\rangle\langle\Psi|)$ coincides with the spectrum of the matrix $\binom{N+k-1}{N}^{-1}K_{N,k,\gamma}$.

Consequently, we have that

$$Q_2(\mathcal{N}_{\lambda,\gamma}) \geqslant \max_{N,k\in\mathbb{N}_+} \frac{\lambda^N}{k}\left\{\log_2\binom{N+k-1}{N} - S\left[\binom{N+k-1}{N}^{-1} K_{N,k,\gamma}\right]\right\}. \tag{D17}$$

One can obtain a lower bound on the energy-constrained two-way quantum capacity $Q_2(\mathcal{N}_{\lambda,\gamma}, N_s)$ by restricting the optimization to the values of N and k such that $\frac{N}{k} \leqslant N_s$. Indeed, note that the rail-encoded state $\bar{\rho}$ satisfies the energy constraint as its mean photon number per mode is $\frac{N}{k}$. In formula, we have that

$$Q_2(\mathcal{N}_{\lambda,\gamma}, N_s) \geqslant \max_{N,k\in\mathbb{N}_+:\,\frac{N}{k}\leqslant N_s} \frac{\lambda^N}{k}\left\{\log_2\binom{N+k-1}{N} - S\left[\binom{N+k-1}{N}^{-1} K_{N,k,\gamma}\right]\right\}. \tag{D18}$$

Note that $\log_2\binom{N+k-1}{N} - S(\binom{N+k-1}{N}^{-1}K_{N,k,\gamma})$ is always positive because $\binom{N+k-1}{N}^{-1}K_{N,k,\gamma}$ is a $\binom{N+k-1}{N}$-dimensional, nonmaximally mixed, state, and thus its von Neumann entropy is strictly smaller than by $\log_2\binom{N+k-1}{N}$. Consequently, we have the following theorem.

Theorem 44. Let $\gamma \geqslant 0$ and $\lambda \in (0, 1]$. For any energy constraint $N_s > 0$, the energy-constrained two-way quantum and secret-key capacities of the bosonic loss-dephasing channel are lower bounded by

$$K(\mathcal{N}_{\lambda,\gamma}, N_s) \geqslant Q_2(\mathcal{N}_{\lambda,\gamma}, N_s) \geqslant \max_{\substack{N,k\in\mathbb{N}_+\\ \frac{N}{k}\leqslant N_s}} \frac{\lambda^N}{k}\left[\log_2\binom{N+k-1}{N} - S(\rho_{N,k,\gamma})\right] > 0. \tag{D19}$$

Here $S(\cdot)$ is the von Neumann entropy, $\rho_{N,k,\gamma}$ is a $\binom{N+k-1}{N}$-dimensional state defined by

$$\rho_{N,k,\gamma} := \binom{N+k-1}{N}^{-1} \sum_{p,q\in\Pi(N,k)} e^{-\frac{\gamma}{2}\|p-q\|_2^2} |p\rangle\langle q|, \tag{D20}$$

where $\Pi(N,k) := \{p \in \mathbb{N}^k : \sum_{i=1}^k p_i = N\}$ represents the set of partitions of a set of N elements into k parts, and the vectors $\{|p\rangle\}_{p\in\Pi(N,k)}$ are orthonormal. In particular,

$$K(\mathcal{N}_{\lambda,\gamma}, N_s) \geqslant Q_2(\mathcal{N}_{\lambda,\gamma}, N_s) > \max_{N,k\in\mathbb{N}_+} \frac{\lambda^N}{k} \times\left[\log_2\binom{N+k-1}{N} - St(\rho_{N,k,\gamma})\right] > 0. \tag{D21}$$

APPENDIX E: TECHNICAL LEMMAS

1. Hadamard maps

Given an infinite matrix $A = (a_{mn})_{m,n\in\mathbb{N}}, a_{mn} \in \mathbb{C}$, we can introduce a superoperator H, recognized as the *Hadamard map*, whose action is defined as $H(|m\rangle\langle n|) = a_{n,m}|m\rangle\langle n|$ for all $n, m \in \mathbb{N}$. We are interested in establishing requirements for an infinite matrix A to ensure that the associated Hadamard map H is a quantum channel. We begin with some preliminaries. Let $\ell^2(\mathbb{N})$ be the space of square-summable complex-valued sequences defined as

$$\ell^2(\mathbb{N}) := \left\{x := \{x_n\}_{n\in\mathbb{N}}, x_n \in \mathbb{C} : \|x\| := \sqrt{\sum_{n=0}^{\infty}|x_n|^2} < \infty\right\}. \tag{E1}$$

An infinite matrix $A := (a_{mn})_{m,n\in\mathbb{N}}, a_{mn} \in \mathbb{C}$ defines a linear operator on $\ell^2(\mathbb{N})$. The operator norm of A is defined as follows:

$$\|A\|_\infty := \sup_{\substack{x\in\ell^2(\mathbb{N})\\ \|x\|=1}} \|Ax\| = \sup_{\substack{\{x_n\}_{n\in\mathbb{N}},x_n\in\mathbb{C}\\ \sum_{n=0}^{\infty}|x_n|^2=1}} \sqrt{\sum_{m=0}^{\infty}\left|\sum_{n=0}^{\infty} a_{mn}x_n\right|^2}.$$

A is said to be bounded if $\|A\|_\infty < \infty$. The following lemma, referred to as *Schur test*, gives a sufficent condition for an infinite matrix to be bounded (e.g., [[69], p. 24, Problem 45]).

Lemma 45. Let $A := (a_{mn})_{m,n\in\mathbb{N}}, a_{mn} \in \mathbb{C}$, be an infinite matrix. Suppose that there exist $\{p_n\}_{n\in\mathbb{N}}, p_n \in \mathbb{R}_{>0}$ and $\{q_m\}_{m\in\mathbb{N}}, q_m \in \mathbb{R}_{>0}$, and $\beta > 0$, and $\gamma > 0$ such that

$$\sum_{m=0}^{\infty}|a_{mn}|p_m \leqslant \beta q_n \quad\text{and}\quad \sum_{n=0}^{\infty}|a_{mn}|q_n \leqslant \gamma p_m, \quad \forall\, m, n \in \mathbb{N}.$$

Then the matrix A satisfies $\|A\|_\infty \leqslant \beta\gamma$. In particular, A is bounded.

By choosing $p_n = q_n = 1$ and $\gamma = \beta = \sup_{n\in\mathbb{N}}\sum_{m=0}^{\infty}|a_{mn}|$, we obtain the following corollary.

Corollary 46. Let $A = (a_{mn})_{m,n\in\mathbb{N}}, a_{mn} \in \mathbb{C}$, be an infinite Hermitian matrix. If $\sup_{n\in\mathbb{N}}\sum_{m=0}^{\infty}|a_{mn}|$ is finite, then A is bounded.

Lemma 47. Let $A = (a_{mn})_{m,n\in\mathbb{N}}, a_{mn} \in \mathbb{C}$, be a bounded Hermitian infinite matrix. Then A is positive semidefinite as an operator on $\ell^2(\mathbb{N})$ if and only if $A^{(d)} := (a_{mn})_{m,n=0,1,\ldots,d-1}$ is positive semidefinite for all $d \in \mathbb{N}$, where $A^{(d)}$ is the $d \times d$ top left corner of A.

Proof. Assume that $A^{(d)}$ is positive semidefinite for all $d \in \mathbb{N}$. Let us pick an arbitrary $x \in \ell^2(\mathbb{N})$. It is known that for any $\varepsilon > 0$, there exists $d \in \mathbb{N}$ and $y^{(d)} := (y_n^{(d)})_{n\in\mathbb{N}}, y_n^{(d)} \in \mathbb{C}$, with $y_n^{(d)} = 0$ for all $n > d$, such that $\|x - y^{(d)}\| < \varepsilon$. Note that

$$\begin{aligned} x^\dagger Ax &= (x - y^{(d)})^\dagger A\,x + (y^{(d)})^\dagger A\,(x - y^{(d)}) + (y^{(d)})^\dagger Ay^{(d)} \\ &\overset{\text{(i)}}{\geqslant} -\|x - y^{(d)}\|\,\|A\|_\infty\,(\|x\| + \|y^{(d)}\|) + (y^{(d)})^\dagger A^{(d)}\,y^{(d)} \\ &\overset{\text{(ii)}}{\geqslant} -\varepsilon\,\|A\|_\infty\,(2\|x\| + \varepsilon), \end{aligned}$$

where in (i) we applied Cauchy-Schwarz inequality twice and the definition of infinity norm as follows:

$$\begin{aligned} |(x - y^{(d)})^\dagger A\,x| &\leqslant \|x - y^{(d)}\|\|A\,x\| \leqslant \|x - y^{(d)}\|\|A\|_\infty\|x\|, \\ |(y^{(d)})^\dagger A\,(x - y^{(d)})| &\leqslant \|y^{(d)}\|\|A\,(x - y^{(d)})\| \\ &\leqslant \|y^{(d)}\|\|A\|_\infty\|x - y^{(d)}\|, \end{aligned}$$

and in (ii) we exploited triangular inequality to derive

$$\|y^{(d)}\| \leqslant \|y^{(d)} - x\| + \|x\| \leqslant \varepsilon + \|x\|, \tag{E2}$$

together with the fact that $A^{(d)}$ is positive semidefinite. Hence, since $\varepsilon > 0$ is arbitrary, we conclude that $x^\dagger A x \geqslant 0$, meaning that A is positive semidefinite as an operator on $\ell^2(\mathbb{N})$. ■

A square matrix is said to be diagonally dominant if

$$\sum_{m\neq n} |a_{mn}| \leqslant |a_{nn}|, \quad \forall n.$$

In words, a square matrix is said to be diagonally dominant if for every row of the matrix, the absolute value of the diagonal entry in a row is larger than or equal to the sum of the absolute values of all the other (nondiagonal) entries in that row. Note that for Hermitian matrices one can exchange row with column in this definition. We proceed to state a sufficient condition for a matrix A to be positive semidefinite as an operator on $\ell^2(\mathbb{N})$. While the following sufficient condition is usually stated for finite matrices, it can be generalized to infinite matrices as per Lemma 47.

Lemma 48. [[43], Chapter 6] For any $d \in \mathbb{N}$, let $A = (a_{mn})_{m,n=0,1,\dots,d-1}$, $a_{mn} \in \mathbb{C}$, be a $d \times d$ Hermitian matrix with $a_{nn} \in \mathbb{R}_{\geqslant 0}$ for all $n \in \{0,\dots,d-1\}$. A is positive semidefinite if it is diagonally dominant.

Lemma 49. Let $A = (a_{mn})_{m,n\in\mathbb{N}}, a_{mn} \in \mathbb{C}$ be an infinite Hermitian matrix with $a_{nn} \in \mathbb{R}_{\geqslant 0}, \forall n \in \mathbb{N}$. Assume that $\sup_{n\in\mathbb{N}} a_{nn}$ is finite and that A is diagonally dominant. Then A is bounded and positive semidefinite when seen as an operator on $\ell^2(\mathbb{N})$.

Proof. Since

$$\sup_{n\in\mathbb{N}} \sum_{m=0}^{\infty} |a_{mn}| \leqslant 2 \sup_{n\in\mathbb{N}} a_{n,n} < \infty,$$

Corollary 46 implies that A is bounded. The fact that A is positive semidefinite as an operator on $\ell^2(\mathbb{N})$ follows from Lemma 48 together with Lemma 47. ■

We now present a lemma from the literature that establishes necessary and sufficient conditions for an infinite matrix A to ensure that the associated Hadamard map H is a quantum channel. We then use it to derive an explicit sufficient condition for an infinite matrix A to give rise to a CPTP Hadamard map.

Lemma 50. [[28], Lemma S4] Let $A := (a_{mn})_{m,n\in\mathbb{N}}, a_{mn} \in \mathbb{C}$ be a bounded infinite matrix. The following requirements establish the necessary and sufficient conditions for the associated Hadamard map H to qualify as a quantum channel:

(1) $a_{nn} = 1, \forall n \in \mathbb{N}$

(2) A is positive semidefinite as on operator on $\ell^2(\mathbb{N})$.

As a consequence of Lemma 50 and Lemma 49, we obtain the following lemma.

Lemma 51. Let $A := (a_{mn})_{m,n\in\mathbb{N}}$ be an infinite Hermitian matrix that is diagonally dominant with $a_{n,n} = 1$ for all $n \in \mathbb{N}$. In this case, its associated Hadamard map H is a quantum channel.

2. Miscellaneous lemmas

Lemma 52. Let $\mathcal{H}_A$ and $\mathcal{H}_B$ be two Hilbert spaces and let $\mathcal{N} : \mathcal{T}(\mathcal{H}_A) \to \mathcal{T}(\mathcal{H}_B)$ be a quantum channel. For all normalized states $|\psi_1\rangle, |\psi_2\rangle \in \mathcal{H}_A$ and $|\phi_1\rangle, |\phi_2\rangle \in \mathcal{H}_B$ it holds that

$$|\langle\phi_1|\mathcal{N}(|\psi_1\rangle\langle\psi_2|)|\phi_2\rangle| \leqslant 1. \tag{E3}$$

Proof. It holds that

$$|\langle\phi_1|\mathcal{N}(|\psi_1\rangle\langle\psi_2|)|\phi_2\rangle| \overset{\text{(i)}}{\leqslant} \|\mathcal{N}(|\psi_1\rangle\langle\psi_2|)\|_\infty \overset{\text{(ii)}}{\leqslant} \|\mathcal{N}(|\psi_1\rangle\langle\psi_2|)\|_1 \overset{\text{(iii)}}{\leqslant} \||\psi_1\rangle\langle\psi_2|\|_1 = 1. \tag{E4}$$

Here in (i) we exploited one of the definition of the operator norm in (A2). In (ii) we exploited that the trace norm is always an upper bound on the operator norm. Finally, in (iii) we leveraged the monotonicity of the trace norm under quantum channels [57]. ■

Lemma 53 ([59]). Let $\mathcal{H}_A, \mathcal{H}_{A'}$ be isomorphic Hilbert spaces, possibly infinite dimensional. Let $|\psi\rangle_{A'A}$ be a pure state that satisfies $\mathrm{Tr}_{A'}[|\psi\rangle\langle\psi|_{AA'}] > 0$. The generalized Choi-Jamiołkowski matrix defines an isomorphism between the set of quantum channels from $\mathcal{H}_{A'}$ to $\mathcal{H}_B$ and the set of bipartite states $\sigma_{AB} \in \mathcal{P}(\mathcal{H}_{AB})$ such that $\mathrm{Tr}_B\sigma_{AB} = \mathrm{Tr}_{A'}[|\psi\rangle\langle\psi|_{AA'}]$. Specifically, for any quantum channel $\mathcal{N}_{A'\to B} : \mathcal{T}(\mathcal{H}_{A'}) \to \mathcal{T}(\mathcal{H}_B)$, it holds that

$$\mathcal{N}_{A'\to B}(|e_i\rangle\langle e_j|) = \frac{1}{\sqrt{\lambda_i\lambda_j}} \mathrm{Tr}_A[(|e_j\rangle\langle e_i|_A \otimes \mathbb{1}_B)\,\sigma_{AB}], \quad \forall\, i, j \in \mathbb{N}, \tag{E5}$$

where $(|e_i\rangle)_{i\in\mathbb{N}}$ and $(\lambda_i)_{i\in\mathbb{N}}$ form a spectral decomposition of $\mathrm{Tr}_{A'}[|\psi\rangle\langle\psi|_{AA'}]$, i.e., $\mathrm{Tr}_{A'}[|\psi\rangle\langle\psi|_{AA'}] = \sum_i \lambda_i |e_i\rangle\langle e_i|_A$, and where the state $\sigma_{AB} := \mathrm{id}_A \otimes \mathcal{N}_{A'\to B}(|\psi\rangle\langle\psi|_{AA'})$ is called the generalized Choi state of $\mathcal{N}$. Equation (E5) is enough to specify the channel $\mathcal{N}_{A'\to B}$ completely, as the linear span of the operators $(|e_i\rangle\langle e_j|)_{i,j\in\mathbb{N}}$ (i.e., the set of finite-rank operators) is dense in $\mathcal{T}(\mathcal{H}_{A'})$.

Lemma 54 ([41]). Let $\mathcal{H}_A, \mathcal{H}_{A'}, \mathcal{H}_B$ be isomorphic Hilbert spaces, possibly infinite dimensional. Let $\mathcal{N}_{A'\to B} : \mathcal{T}(\mathcal{H}_{A'}) \to \mathcal{T}(\mathcal{H}_B)$ be a quantum channel. Let $|\psi\rangle_{A'A} \in \mathcal{H}_A \otimes \mathcal{H}_{A'}$ be a pure state such that the reduced state $\mathrm{Tr}_{A'}[|\psi\rangle\langle\psi|_{AA'}]$ is positive definite. Then $\mathcal{N}_{A'\to B}$ is antidegradable if and only if the state $\mathrm{id}_A \otimes \mathcal{N}_{A'\to B}(|\psi\rangle\langle\psi|_{AA'})$ is two-extendible on B, meaning that there exists a state $\rho_{AB_1B_2} \in \mathcal{P}(\mathcal{H}_A \otimes \mathcal{H}_{B_1} \otimes \mathcal{H}_{B_2})$ such that

$$\begin{aligned}\mathrm{Tr}_{B_2}[\rho_{AB_1B_2}] &= \mathrm{id}_A \otimes \mathcal{N}_{A'\to B_1}(|\psi\rangle\langle\psi|_{AA'}),\\ \mathrm{Tr}_{B_1}[\rho_{AB_1B_2}] &= \mathrm{id}_A \otimes \mathcal{N}_{A'\to B_2}(|\psi\rangle\langle\psi|_{AA'}),\end{aligned} \tag{E6}$$

where $\mathcal{H}_{B_1}$ and $\mathcal{H}_{B_2}$ are Hilbert spaces that are isomorphic to $\mathcal{H}_B$.

Proof. Let $U_{A'E\to BE}$ be a Stinespring dilation of the channel $\mathcal{N}_{A'\to B}$. Further assume that $\mathcal{N}_{A'\to B}$ is antidegradable. By definition, there exists a quantum channel $\mathcal{A}_{E\to B}$ such that $\mathcal{A}_{E\to B} \circ \mathcal{N}^{\mathrm{c}}_{A'\to B} = \mathcal{N}_{A'\to B}$. Let us consider the tripartite state $\rho_{AB_1B_2}$, with B_1, B_2 being copies of B, defined as

$$\rho_{AB_1B_2} = \mathrm{id}_A \otimes \mathrm{id}_{B_1} \otimes \mathcal{A}_{E\to B_2}\big(U_{A'E\to B_1E}\,(|\psi\rangle\langle\psi|_{AA'} \otimes |0\rangle\langle 0|_E)\,U^\dagger_{A'E\to B_1E}\big). \tag{E7}$$

It holds that

$$\begin{aligned}\mathrm{Tr}_{B_2}[\rho_{AB_1B_2}] &= \mathrm{Tr}_{B_2}\big[\mathrm{id}_A \otimes \mathrm{id}_{B_1} \otimes \mathcal{A}_{E\to B_2}\big(U_{A'E\to B_1E}(|\psi\rangle\langle\psi|_{AA'} \otimes |0\rangle\langle 0|_E)\,U^\dagger_{A'E\to B_1E}\big)\big]\\ &= \mathrm{id}_A \otimes \mathrm{Tr}_E\big[U_{A'E\to B_1E}\big(|\psi\rangle\langle\psi|_{AA'} \otimes |0\rangle\langle 0|_E\big)U^\dagger_{A'E\to B_1E}\big]\\ &= \mathrm{id}_A \otimes \mathcal{N}_{A'\to B_1}(|\psi\rangle\langle\psi|_{AA'}),\end{aligned}$$

and

$$\begin{aligned}\mathrm{Tr}_{B_1}[\rho_{AB_1B_2}] &= \mathrm{Tr}_{B_1}\big[\mathrm{id}_A \otimes \mathrm{id}_{B_1} \otimes \mathcal{A}_{E\to B_2}\big(U_{A'E\to B_1E}\,\big(|\psi\rangle\langle\psi|_{AA'} \otimes |0\rangle\langle 0|_E\big)\,U^\dagger_{A'E\to B_1E}\big)\big]\\ &= \mathrm{id}_A \otimes \mathcal{A}_{E\to B_2}\big(\mathrm{Tr}_{B_1}\big[U_{A'E\to B_1E}\big(|\psi\rangle\langle\psi|_{AA'} \otimes |0\rangle\langle 0|_E\big)U^\dagger_{A'E\to B_1E}\big]\big)\\ &= \mathrm{id}_A \otimes \mathcal{A}_{E\to B_2} \circ \mathcal{N}^{\mathrm{c}}_{A'\to B_2}(|\psi\rangle\langle\psi|_{AA'})\\ &= \mathrm{id}_A \otimes \mathcal{N}_{A'\to B_2}(|\psi\rangle\langle\psi|_{AA'}).\end{aligned}$$

Now let us establish the converse. Assume that there exists $\rho_{AB_1B_2}$ which satisfies (E6). Let $|\Psi\rangle_{AB_1B_2P} \in \mathcal{H}_A \otimes \mathcal{H}_{B_1} \otimes \mathcal{H}_{B_2} \otimes \mathcal{H}_P$ be a purification of $\rho_{AB_1B_2}$, with $\mathcal{H}_P$ being the purifying Hilbert space. Note that both $|\Psi\rangle_{AB_1B_2P}$ and $U_{A'E\to B_1E}\,(|\psi\rangle_{AA'} \otimes |0\rangle_E)$ are purifications of $\mathrm{id}_A \otimes \mathcal{N}_{A'\to B_1}(|\psi\rangle\langle\psi|_{AA'})$, with $\mathcal{H}_{B_2} \otimes \mathcal{H}_P$ and $\mathcal{H}_E$ being their purifying Hilbert spaces, respectively. It follows that [57] there exists an isometry $V_{E\to B_2P} : \mathcal{H}_E \to \mathcal{H}_{B_2} \otimes \mathcal{H}_P$ such that $V_{E\to B_2P}\,U_{A'E\to B_1E}\,(|\psi\rangle_{AA'} \otimes |0\rangle_E) = |\Psi\rangle_{AB_1B_2P}$. Hence, the quantum channel $\mathcal{A}_{E\to B_2} : \mathcal{T}(\mathcal{H}_E) \to \mathcal{T}(\mathcal{H}_{B_2})$, defined by $\mathcal{A}_{E\to B_2}(\cdot) = \mathrm{Tr}_P[V_{E\to B_2P}(\cdot)V^\dagger_{E\to B_2P}]$, satisfies that

$$\begin{aligned}\mathrm{id}_A \otimes \mathcal{A}_{E\to B_2} \circ \mathcal{N}^{\mathrm{c}}_{A'\to B_2}(|\psi\rangle\langle\psi|_{AA'}) &= \mathrm{id}_A \otimes \mathcal{A}_{E\to B_2}\big(\mathrm{Tr}_{B_1}\big[U_{A'E\to B_1E}(|\psi\rangle\langle\psi|_{AA'} \otimes |0\rangle\langle 0|_E)U^\dagger_{A'E\to B_1E}\big]\big)\\ &= \mathrm{Tr}_{B_1P}[|\Psi\rangle\langle\Psi|_{AB_1B_2P}]\\ &= \mathrm{Tr}_{B_1}[\rho_{AB_1B_2}]\\ &= \mathrm{id}_A \otimes \mathcal{N}_{A'\to B_2}(|\psi\rangle\langle\psi|_{AA'}).\end{aligned}$$

Consequently, since the pure state $|\psi\rangle_{A'A}$ satisfies $\mathrm{Tr}_{A'}[|\psi\rangle\langle\psi|_{AA'}] > 0$, Lemma 53 implies that $\mathcal{A}_{E\to B_2} \circ \mathcal{N}^{\mathrm{c}}_{A'\to B_2} = \mathcal{N}_{A'\to B_2}$, meaning that $\mathcal{N}_{A'\to B_2}$ is antidegradable. ■

Lemma 55. Let $\mathcal{N}, \mathcal{M} : \mathcal{T}(\mathcal{H}_S) \to \mathcal{T}(\mathcal{H}_S)$ be quantum channels. If either $\mathcal{M}$ or $\mathcal{N}$ is antidegradable, then the composition $\mathcal{M} \circ \mathcal{N}$ is antidegradable. Specifically, let E_1 and E_2 be the Stinespring environments of $\mathcal{N}$ and $\mathcal{M}$, respectively. If $\mathcal{N}$ is antidegradable with antidegrading map $\mathcal{A}_{E_1\to S}$, then $(\mathcal{M} \circ \mathcal{A}_{E_1\to S}) \otimes \mathrm{Tr}_{E_2}$ is an antidegrading map of $\mathcal{M} \circ \mathcal{N}$. Analogously, if $\mathcal{M}$ is antidegradable with antidegrading map $\mathcal{A}_{E_2\to S}$, then $\mathrm{Tr}_{E_1} \otimes \mathcal{A}_{E_2\to S}$ is an antidegrading map of $\mathcal{M} \circ \mathcal{N}$.

Proof. Let $V^{S\to SE_1}$ and $W^{S\to SE_2}$ be Stinespring isometries associated with $\mathcal{N}$ and $\mathcal{M}$, respectively. By considering the complementary channel of $\mathcal{M} \circ \mathcal{N}$,

$$(\mathcal{M} \circ \mathcal{N})^{\mathrm{c}}(\rho) = \mathrm{Tr}_S[W^{S\to SE_2}V^{S\to SE_1}\,\rho(V^{S\to SE_1})^\dagger(W^{S\to SE_2})^\dagger],\quad \forall\,\rho \in \mathcal{T}(\mathcal{H}_S),$$

one can easily check that if $\mathcal{N}$ is antidegradable with antidegrading map $\mathcal{A}_{E_1\to S}$, then

$$\big[(\mathcal{M} \circ \mathcal{A}_{E_1\to S}) \otimes \mathrm{Tr}_{E_2}\big] \circ (\mathcal{M} \circ \mathcal{N})^{\mathrm{c}} = \mathcal{M} \circ \mathcal{N}. \tag{E8}$$

Analogously, one can easily verify that if $\mathcal{M}$ is antidegradable with antidegrading map $\mathcal{A}_{E_2\to S}$, then

$$\big[\mathrm{Tr}_{E_1} \otimes \mathcal{A}_{E_2\to S}\big] \circ (\mathcal{M} \circ \mathcal{N})^{\mathrm{c}} = \mathcal{M} \circ \mathcal{N}. \tag{E9}$$

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