Optimal unilocal virtual quantum broadcasting

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Quantum broadcasting is central to quantum information processing and characterizes the correlations within quantum states. Nonetheless, traditional quantum broadcasting encounters inherent limitations dictated by the principles of quantum mechanics. In a previous study, Parzygnat *et al.* [Phys. Rev. Lett. **132**, 110203 (2024)] introduced a canonical broadcasting quantum map that goes beyond the quantum no-broadcasting theorem through a virtual process. In this work, we generalize the concept of virtual broadcasting to unilocal broadcasting by incorporating a reference system and introduce protocols that can be approximated using physical operations with minimal cost. First, we propose a universal unilocal protocol enabling multiple parties to share the correlations of a target bipartite state, which is encoded in the expectation value for any observable. Second, we formalize the simulation cost of a virtual quantum broadcasting protocol into a semidefinite programming problem. Notably, we propose a specific protocol with optimal simulation cost for the two-broadcasting scenario, revealing an explicit relationship between simulation cost and the quantum system's dimension. Moreover, we establish upper and lower bounds on the simulation cost of the virtual n-broadcasting protocol and demonstrate the convergence of the lower bound to the upper bound as the quantum system's dimension increases.

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I. INTRODUCTION

In classical information processing, creating duplicates is a straightforward task. However, the quantum realm presents a challenge due to the no-cloning theorem [1,2], rendering direct copies impossible. Quantum broadcasting [3,4], a concept milder than quantum cloning, offers a distinct perspective on the classical-quantum interface. Unfortunately, there are also fundamental restrictions on quantum broadcasting [5]. The no-broadcasting theorem states that it is only possible to broadcast a set of quantum states if they commute with each other. In other words, if the quantum states have properties that can be simultaneously measured without disturbing each other, it is possible to broadcast them.

These no-go theorems can be further extended to the setting of local broadcasting for composite quantum systems [6–8]. Given a bipartite quantum state ρ_{AB} shared by Alice and Bob, the local broadcasting aims to perform local operations $\Lambda_{A\to A_1A_2}$ and $\Gamma_{B\to B_1B_2}$ to produce a state $\widehat{\rho}_{A_1A_2B_1B_2} := (\Lambda_{A\to A_1A_2}\otimes\Gamma_{B\to B_1B_2})\rho_{AB}$ such that $\mathrm{Tr}_{A_1B_1}[\widehat{\rho}_{A_1A_2B_1B_2}] = \mathrm{Tr}_{A_2B_2}[\widehat{\rho}_{A_1A_2B_1B_2}] = \rho_{AB}$. Furthermore, unilocal broadcasting is considered when the local operations are only allowed for one party, e.g., Bob. It is shown that the unilocal broadcasting can be done if and only if ρ_{AB} is classical on B [6–9]. More generally, a unilocal n-broadcasting performs the local operation $\Gamma_{B\to B_1\cdots B_n}$ to produce the state $\widehat{\rho}_{AB_1\ldots B_n} := \Gamma_{B\to B_1\cdots B_n}(\rho_{AB})$ such that $\mathrm{Tr}_{\backslash AB_1}[\widehat{\rho}_{AB_1\ldots B_n}] = \cdots = \mathrm{Tr}_{\backslash AB_n}[\widehat{\rho}_{AB_1\ldots B_n}] = \rho_{AB}$ [4], which is shown in Fig. 1.

Although any physical process cannot achieve quantum broadcasting due to these no-go theorems, Parzygnat *et al.* [10] presented a canonical broadcasting quantum map going beyond the quantum no-broadcasting theorem via a virtual process, which focuses on broadcasting measurement statistics of a target state rather than the state itself. They presented three natural conditions that virtual broadcasting maps should satisfy and provided several physical interpretations, such as that the universal quantum cloner is the optimal physical approximation to their canonical broadcasting map. However, when considering using physical operators to approximate the nonphysical process with minimal sampling cost, the optimal protocols for virtual broadcasting and the more general unilocal broadcasting are unknown. To overcome the above challenges as well as the limitations of the quantum no-local-broadcasting theorem, we investigate *unilocal virtual quantum broadcasting*, which aims to broadcast the correlation of a bipartite quantum system encoded in the expectation values of any possible observables.

Virtual processes concern the classical information discerned after measurement, referred to as *shadow information* [11,12], that we mainly focus on in the majority of quantum information and quantum computing tasks, rather than the whole information of a state. Therefore, for a bipartite quantum state ρ_{AB}, we concentrate on a specific broadcasting task that the local operations employed by Bob enable n parties $B_1, \ldots, B_n$ to access the same shadow information $\mathrm{Tr}[O\rho_{AB}]$ with respect to any observable O. It is worth noting that we are not focusing on distributing the expectation value as classical bits to different parties. In fact, Alice and Bob are considered

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geographically separated laboratories where a global expectation value cannot be obtained in the first place. Instead, our framework works by supplying Alice and Bob with multiple identical copies of the states, which ensures that each bipartite party AB_j can access the correlation of AB by sharing the same expectation value.

Technically, we extend the traditional quantum broadcasting by employing Hermitian-preserving and trace-preserving (HPTP) maps, which can be physically implemented by quasiprobability decomposition (QPD) [13–16] and measurement-controlled postprocessing [17]. Such physical simulation of unphysical maps plays a crucial role in applications such as entanglement detection [18–21], error mitigation [14–16,22,23], and two-point correlator [13]. In specific, we may construct an HPTP map $\Gamma_{B\to B^n}$ and decompose it into a linear combination of local channels $\mathcal{N}_j$ for Bob, i.e., $\Gamma_{B\to B^n}=\sum_{j=1}c_j\mathcal{N}_j$, where c_j are certain real numbers. One can estimate the shadow information by sampling quantum channels $\mathcal{N}_j$ and postprocessing the measurement outcomes [22] for an observable O and quantum state ρ_{AB} (see Proposition 2 for a precise statement). Then, it is essential to understand the power and limitations of such virtual quantum broadcasting as the following two questions arise:

(1) *Is there a universal virtual quantum broadcasting protocol?*

(2) *What is the optimal protocol with minimum sampling cost?*

In this paper, we fully address these two questions. In Sec. II, we demonstrate the existence of a *universal* unilocal virtual n-broadcasting protocol, for any bipartite quantum state ρ_{AB} and observable O. In Sec. III, we formalize the simulation cost of a unilocal virtual n-broadcasting into a semidefinite programming (SDP) [24]. Notably, we provide an analytical universal unilocal virtual two-broadcasting protocol to elucidate the optimal simulation cost. In addition, we investigate the upper and lower bounds on the simulation cost of the unilocal virtual n-broadcasting protocol.

II. UNIVERSAL VIRTUAL BROADCASTING PROTOCOL

We consider a finite-dimensional Hilbert space $\mathcal{H}$ and denote A and B as two parties, each possessing their respective Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$. We denote the dimension of $\mathcal{H}_B$ as d. Let $\{|j\rangle\}_{j=0,\dots,d-1}$ be a standard computational basis. Denote $\mathcal{L}(\mathcal{H}_A)$ as the set of linear operators that map from $\mathcal{H}_A$ to itself. A linear operator in $\mathcal{L}(\mathcal{H}_A)$ is called a density operator if it is positive semidefinite with trace one, and denotes $\mathcal{D}(\mathcal{H}_A)$ as the set of all density operators on $\mathcal{H}_A$. We denote $F_{B_1B_2}:=\sum_{i,j=0}^{d-1}|ij\rangle\langle ji|$ as the swap operator between subsystems B_1 and B_2, and denote $\Phi_{BB_1}:=\sum_{i,j=0}^{d-1}|ii\rangle\langle jj|_{BB_1}$ as the unnormalized $d\otimes d$ maximally entangled state. In the absence of ambiguity, subsystems may be omitted, i.e., Φ_d. A quantum channel $\mathcal{N}_{A\to B}$ is a linear map from $\mathcal{L}(\mathcal{H}_A)$ to $\mathcal{L}(\mathcal{H}_B)$ that is completely positive and trace preserving (CPTP). Its associated Choi-Jamiołkowski operator is expressed as $J_{AB}^{\mathcal{N}}:=\sum_{i,j=0}^{d-1}|i\rangle\langle j|\otimes\mathcal{N}_{A\to B}(|i\rangle\langle j|)$.

Formally, a unilocal virtual n-broadcasting protocol for a bipartite quantum state ρ_{AB} is defined as follows.

Definition 1 (Unilocal virtual n-broadcasting protocol). For a bipartite state $\rho_{AB}\in\mathcal{D}(\mathcal{H}_A\otimes\mathcal{H}_B)$, an HPTP map $\Gamma_{B\to B^n}$ is called a unilocal virtual n-broadcasting protocol for ρ_{AB} if

$$\rho_{AB}=\mathrm{Tr}_{\backslash AB_j}[\Gamma_{B\to B^n}(\rho_{AB})],\quad\forall j=1,2,\dots,n,\tag{1}$$

where the identity map is omitted, $\mathrm{Tr}_{\backslash AB_j}$ denotes taking partial trace on the subsystems excluding AB_j, and B^n is the abbreviation of the subsystems $B_1B_2\dots B_n$.

We note that if there is a unilocal virtual n-broadcasting protocol $\Gamma_{B\to B^n}$ for all quantum states $\rho_{AB}\in\mathcal{D}(\mathcal{H}_A\otimes\mathcal{H}_B)$, we call it *a universal unilocal virtual n-broadcasting protocol*. Equivalently, a universal unilocal virtual n-broadcasting protocol $\Gamma_{B\to B_1\cdots B_n}$ can be characterized by its Choi operator $J_{BB^n}^{\Gamma}$ as the following Lemma.

Lemma 1. An HPTP map $\Gamma_{B\to B^n}$ is a universal unilocal virtual n-broadcasting protocol if and only if

$$J_{BB_j}^{\Gamma}=\Phi_{BB_j},\quad j=1,\dots,n,\tag{2}$$

where Φ_{BB_j} denotes the unnormalized $d\otimes d$ maximally entangled state on system BB_j, $J_{BB_j}^{\Gamma}:=\mathrm{Tr}_{\backslash BB_j}[J_{BB^n}^{\Gamma}]$, and $J_{BB^n}^{\Gamma}$ is the Choi operator of $\Gamma_{B\to B^n}$.

Lemma 1 states that a universal unilocal virtual n-broadcasting protocol can be described by its Choi operator, which means we can check constraints on Choi operators instead of constraints involving input and output states. The proof can be found in the Appendix. One of the remarkable and valuable findings in this paper is that there indeed exists a universal virtual n-broadcasting protocol. As a warm-up example, we present a universal unilocal virtual two-broadcasting protocol as follows:

$$\Gamma_{B\to B_1B_2}(\rho_{AB}):=\rho_{AB_1}\otimes\frac{I_{B_2}}{d}+\mathcal{S}_{B_1B_2}\left(\rho_{AB_1}\otimes\frac{I_{B_2}}{d}\right)-\mathcal{R}_{B\to B_1B_2}(\rho_{AB}),$$

where $\mathcal{S}_{B_1B_2}(\cdot)$ denotes the swap operation between the subsystem B_1 and B_2, $\mathcal{R}_{B\to B_1B_2}(\cdot)$ denotes the replacement channel yielding the normalized $d\otimes d$ maximally entangled state between subsystem B_1 and B_2 for any input state. Its Choi operator can be written as

$$J_{BB_1B_2}^{\Gamma_{B\to B_1B_2}}:=\frac{1}{d}\Phi_{BB_1}\otimes I_{B_2}+\frac{1}{d}\Phi_{BB_2}\otimes I_{B_1}-\frac{1}{d}\Phi_{B_1B_2}\otimes I_B.$$

It is straightforward to check that $J_{BB_1}^{\Gamma_{BB_1B_2}}=J_{BB_2}^{\Gamma_{BB_1B_2}}=\Phi_{BB_1}=\Phi_{BB_2}$. Consequently, $\Gamma_{B\to B_1B_2}$ is a universal unilocal virtual two-broadcasting protocol by Lemma 1. Furthermore, we extend our investigation to encompass the realm of n-broadcasting, where we demonstrate the existence of a universal unilocal virtual n-broadcasting protocol as follows.

Proposition 2. For any bipartite quantum system AB, there exists a universal unilocal virtual n-broadcasting protocol.

We demonstrate Proposition 2 by explicitly constructing an HPTP map $\Gamma'_{B\to B^n}$ as follows:

$$\Gamma'_{B\to B^n}(\rho_{AB}):=\sum_{j=1}^{n}\mathcal{S}_{B_1B_j}\left(\rho_{AB_1}\otimes\frac{I_{B_2\dots B_n}}{d^{n-1}}\right)-(n-1)\mathcal{R}_{B\to B^n}(\rho_{AB}),\tag{3}$$

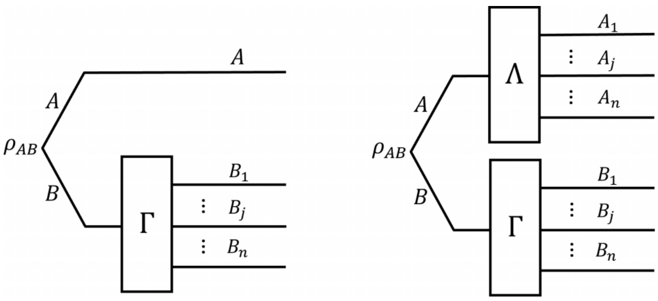

FIG. 1. Unilocal (left) and bilocal (right) n-broadcasting for bipartite state ρ_{AB}. The goal is for the map Γ to minimize the dissimilarity between the states on ρ_{AB_j} and ρ_{AB} in a certain measure. Conventionally, Γ is a CPTP map, i.e., quantum channel. This paper focuses on the scenario where Γ is an HPTP map.

where $\mathcal{S}_{B_1B_j}(\cdot)$ denotes the swap operation between the subsystems B_1 and B_j, and $\mathcal{R}_{B\to B^n}(\cdot)$ denotes the replacement channel yielding $\Phi_{B_1B_2}\otimes\frac{I_{B_3\ldots B_n}}{d^{n-3}}$ for any input state. By checking its Choi operator and applying Lemma 1, we know it is a universal unilocal n-broadcasting protocol.

Such a universal unilocal virtual n-broadcasting protocol can be implemented via a quasiprobability decomposition strategy [15,22,23] as shown in Fig. 2. Given an observable O and M copies of a bipartite state ρ_{AB} shared between Alice and Bob, a unilocal virtual n-broadcasting protocol can be decomposed as $\Gamma_{B\to B^n}=p_1\mathcal{N}_1-p_2\mathcal{N}_2$, where $\mathcal{N}_1$ and $\mathcal{N}_2$ are quantum channels [15]. In the mth round of sampling, Bob samples a quantum channel $\mathcal{N}^{(m)}\in\{\mathcal{N}_1,\mathcal{N}_2\}$ with probability $p^{(m)}\in\{p_1/\gamma,p_2/\gamma\}$ where $\gamma=p_1+p_2$. Then apply the channel to ρ_{AB} obtaining $\rho_{AB^n}^{(m)}$. Repeat this process M times to obtain M copies of state $\{\rho_{AB^n}^{(1)},\rho_{AB^n}^{(2)},\ldots,\rho_{AB^n}^{(M)}\}$. Without loss of generality, if a global measurement is performed on a computational basis on AB_j, i.e., $O=\sum_k\lambda_k|k\rangle\langle k|$, $\lambda_k\in[-1,1]$, we construct

$$\xi:=\frac{\eta}{M}\sum_{m=1}^{M}\operatorname{sgn}(p^{(m)})\lambda^{(m)} \tag{4}$$

as an estimator of $\operatorname{Tr}[O\rho_{AB}]$. Subsequently, each bipartite system AB_j, where $j=1,2,\ldots,n$, acquires the information of $\operatorname{Tr}[O\rho_{AB}]$ by measuring their subsystems in the eigenbasis of O and postprocessing [22].

Note that for any bipartite system AB_j, there are M copies of quantum states $\rho_{AB_j}^{(m)}$, $m=1,2,\ldots,M$, each of which is labeled by a classical bit $\operatorname{sgn}(p^{(m)})$ where $p^{(m)}=p_1$ or p_2. Instead of directly transmitting the expectation value, AB_j can further apply any further quantum operations to these samples and access the same correlation as AB through measurement statistics. This universal unilocal virtual n-broadcasting protocol can apply to any state ρ_{AB} and any observable O. However, it is impossible when one considers using one channel to deal with this task. Consequently, we may extend the no-go theorem for local broadcasting by involving HPTP maps in the broadcasting procedure.

III. OPTIMAL VIRTUAL BROADCASTING

In this section, we explore the universal unilocal virtual n-broadcasting protocol, which can be simulated by physical operations with minimum costs. Treating the unilocal virtual n-broadcasting protocol as a general HPTP map, its simulation or sampling cost can be characterized via the following physical implementability, which plays the key role in quantifying the number of rounds required to reach the desired estimating precision [15].

Definition 2 (Simulation cost of an HPTP map [15]). The simulation cost (or physical implementability) of an HPTP map Γ is defined as

$$\nu(\Gamma):=\log_2\min\{p_1+p_2\,|\,\Gamma=p_1\mathcal{N}_1-p_2\mathcal{N}_2,\; p_1,p_2\geqslant 0,\mathcal{N}_1,\mathcal{N}_2\in\text{CPTP}\}. \tag{5}$$

By Hoeffding's inequality, denoting $\gamma=p_1+p_2$, it requires at least $\mathcal{O}(\frac{\gamma^2}{\delta^2}\ln\frac{2}{\epsilon})$ samples of ρ_{AB} to achieve the estimation error δ with a probability $1-\epsilon$, for estimating $\operatorname{Tr}(O\rho_{AB})$ by estimator ξ in Eq. (4). Based on the above, we define the optimal simulation cost of a universal unilocal virtual n-broadcasting protocol as follows.

Definition 3 (Optimal simulation cost). The optimal simulation cost of all universal unilocal virtual n-broadcasting

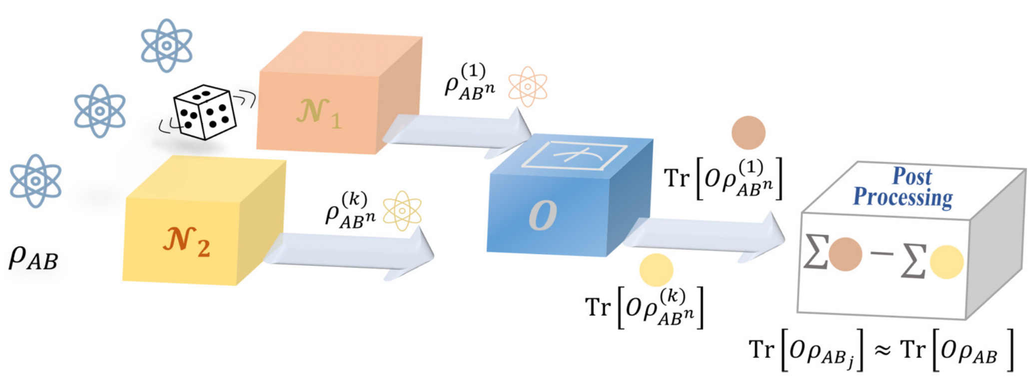

FIG. 2. Illustration of using a universal virtual n-broadcasting $\Gamma_{B\to B^n}=p_1\mathcal{N}_1-p_2\mathcal{N}_2$ to share shadow information between different parties. For a given observable O and many copies of a bipartite state ρ_{AB}, we sample local quantum channels $\mathcal{N}_1$ and $\mathcal{N}_2$ with probability $p_1/(p_1+p_2)$ and $p_2/(p_1+p_2)$, respectively. Iterating this procedure m times, we can obtain $\rho_{AB^n}^{(k)}$ for $k=1,2,\ldots,m$. Afterward, each party AB_j, where $j=1,2,\ldots,n$, obtains $\operatorname{Tr}[O\rho_{AB}]$ since $\operatorname{Tr}[O\rho_{AB}]=\operatorname{Tr}[O\operatorname{Tr}_{\backslash AB_j}[\Gamma_{B\to B^n}(\rho_{AB})]]$.

protocols is defined as

$$\gamma_n^* := \min\{\nu(\Gamma_{B\to B^n}) : \Gamma_{B\to B^n} \in \mathcal{T}_n\}, \tag{6}$$

where $\mathcal{T}_n$ denotes the set of all universal unilocal virtual n-broadcasting protocols. The corresponding protocol $\Gamma^*_{B\to B^n} := \mathrm{argmin}\{\nu(\Gamma_{B\to B^n}) : \Gamma_{B\to B^n} \in \mathcal{T}_n\}$ is the optimal universal n-broadcasting protocols.

Combined with the properties that a universal virtual broadcasting should satisfy as stated in Lemma 1, the optimal simulation cost can be formalized as follows.

Proposition 3. The optimal simulation cost of all universal unilocal virtual n-broadcasting protocols can be characterized as the following SDP:

$$\begin{aligned} 2^{\gamma_n^*} = \min\ & p_1 + p_2 \\ \text{subject to } & \mathrm{Tr}_{\backslash BB_j}\left[J^{\mathcal{N}_1}_{BB^n} - J^{\mathcal{N}_2}_{BB^n}\right] = \Phi_{BB_j},\ j = 1, \dots, n \\ & \mathrm{Tr}_{B^n}\left[J^{\mathcal{N}_1}_{BB^n}\right] = p_1 I_B, \\ & \mathrm{Tr}_{B^n}\left[J^{\mathcal{N}_2}_{BB^n}\right] = p_2 I_B, \\ & J^{\mathcal{N}_1}_{BB^n} \geqslant 0, J^{\mathcal{N}_2}_{BB^n} \geqslant 0, \end{aligned} \tag{7}$$

with variables $J^{\mathcal{N}_1}_{BB^n}$, $J^{\mathcal{N}_2}_{BB^n}$, p_1, and p_2. Φ_{BB_j} is the unnormalized $d \otimes d$ maximally entangled state on system BB_j.

Proof. Let $\mathcal{T}_n$ be the set of all universal unilocal virtual n-broadcasting protocols, and $J^{\Gamma}_{BB^n}$ be the Choi operator of $\Gamma_{B\to B^n} \in \mathcal{T}_n$. By definition of the simulation cost given in Definition 2, there exist $p_1, p_2 \geqslant 0$ and $\mathcal{N}_1, \mathcal{N}_2 \in$ CPTP such that

$$\nu(\Gamma_{BB^n}) = \log_2(p_1 + p_2). \tag{8}$$

The Choi operator of Γ_{BB^n} satisfies $J^{\Gamma}_{BB^n} = p_1 \hat{J}^{\mathcal{N}_1}_{BB^n} - p_2 \hat{J}^{\mathcal{N}_2}_{BB^n}$, where $\hat{J}^{\mathcal{N}_1}_{BB^n}$ and $\hat{J}^{\mathcal{N}_2}_{BB^n}$ denote the Choi operators of $\mathcal{N}_1$ and $\mathcal{N}_2$, respectively. We further rewrite $J^{\mathcal{N}_1}_{BB^n} := p_1 \hat{J}^{\mathcal{N}_1}_{BB^n}$ and $J^{\mathcal{N}_2}_{BB^n} := p_2 \hat{J}^{\mathcal{N}_2}_{BB^n}$ for simplifying this optimization problem. According to Lemma 1 and Definition 3, one can obtain the SDP in Eq. (7), which completes this proof. ■

We further present its dual SDP as follows:

$$\begin{aligned} \max\ & \sum_{j=1}^{n} \mathrm{Tr}\left[X_{BB_j}\Phi_{BB_j}\right] \\ \text{subject to } & \mathrm{Tr}[Z_B] \leqslant 1, \mathrm{Tr}[K_B] \leqslant 1, \\ & Z_B \otimes I_{B^n} - \sum_{j=1}^{n} \mathcal{S}_{B_1B_j}\left(X_{BB_1} \otimes I_{B_2\cdots B_n}\right) \geqslant 0, \\ & K_B \otimes I_{B^n} + \sum_{j=1}^{n} \mathcal{S}_{B_1B_j}\left(X_{BB_1} \otimes I_{B_2\cdots B_n}\right) \geqslant 0, \\ & (j = 1, \dots, n), \end{aligned} \tag{9}$$

where X_{BB_j}, Z_B, and K_B are optimization variables, and $\mathcal{S}_{B_1B_j}$ denotes the swap operator between system B and B_j. We retain the derivation in the Appendix.

The above SDPs allow us to explore the optimal universal virtual broadcasting protocols that can achieve the optimal simulation cost. Specifically, we give the analytical optimal simulation cost for a universal unilocal virtual two-broadcasting and obtain the optimal universal protocol.

Theorem 4 (Optimal simulation cost of virtual two-broadcasting). The optimal simulation cost of all universal unilocal virtual two-broadcasting protocols which broadcast system B to B_1B_2 is given by

$$\gamma_2^* = \log_2\left(3 - \frac{4}{d+1}\right), \tag{10}$$

where d denotes the dimension of Hilbert spaces $\mathcal{H}_B$, $\mathcal{H}_{B_1}$, and $\mathcal{H}_{B_2}$.

Proof. First, we are going to prove $2^{\gamma_2^*} \leqslant \frac{3d-1}{d+1}$ using the primal SDP in Eq. (7). Denoting $M = \Phi_{BB_1} \otimes I_{B_2}$ and $N = I_B \otimes F_{B_1B_2}$, we shall show that $\{p_1, p_2, J^{\mathcal{N}_1}, J^{\mathcal{N}_2}\}$ is a feasible solution, where $p_1 = \frac{2d}{d+1}$, $p_2 = \frac{d-1}{d+1}$, and

$$\begin{aligned} J^{\mathcal{N}_1} &:= \frac{M + NMN + MN + NM}{2(d+1)}, \\ J^{\mathcal{N}_2} &:= \frac{1}{d^2-2}\left(I - \frac{d(M + NMN) - (MN + NM)}{d^2-1}\right), \end{aligned} \tag{11}$$

respectively. It is straightforward to check that the equality constraints in Eq. (7) hold. For the inequality constraints, we find that $(J^{\mathcal{N}_1})^2 = J^{\mathcal{N}_1}$ and $(J^{\mathcal{N}_2})^2 = \frac{1}{d^2-2} J^{\mathcal{N}_2}$. Thus, 1 and $\frac{1}{d^2-2}$ are unique non-negative eigenvalues of $J^{\mathcal{N}_1}$ and $J^{\mathcal{N}_2}$, respectively, which means $J^{\mathcal{N}_1} \geqslant 0$ and $J^{\mathcal{N}_2} \geqslant 0$. Therefore, $\{p_1 J^{\mathcal{N}_1}, p_2 J^{\mathcal{N}_2}\}$ is a feasible solution with the cost of $\frac{3d-1}{d+1}$, which implies $2^{\gamma_2^*} \leqslant \frac{3d-1}{d+1}$.

Second, we use dual SDP in Eq. (9) to show that $2^{\gamma_2^*} \geqslant \frac{3d-1}{d+1}$. We show that $\{X_{BB_1}, Y_{BB_2}, Z_B, K_B\}$ is a feasible solution, where

$$Z_B = K_B = \frac{1}{d} I \text{ and } X_{BB_1} = Y_{BB_2} = \frac{2}{d(d+1)}\Phi_d - \frac{1}{2d} I. \tag{12}$$

Still, we can check that $\{X_{BB_1}, Y_{BB_2}, Z_B, K_B\}$ satisfies the constraints SDP in Eq. (9). Specifically, we have $I - \frac{M+NMN}{d+1} \geqslant 0$ since $\frac{M+NMN}{d+1}$ is a Hermitian matrix with a maximal eigenvalue of one [25]. Therefore, $\{X_{BB_1}, Y_{BB_2}, Z_B, K_B\}$ is a feasible solution. Finally, we further check the objective function,

$$\mathrm{Tr}\left[X_{BB_1}\Phi_{BB_1}\right] + \mathrm{Tr}\left[Y_{BB_2}\Phi_{BB_2}\right] = 3 - \frac{4}{d+1}, \tag{13}$$

which yields $2^{\gamma_2^*} \geqslant \frac{3d-1}{d+1}$. Combining the primal part and the dual part, we conclude that

$$\gamma_2^* = \log_2\left(3 - \frac{4}{d+1}\right), \tag{14}$$

which completes this proof. ■

Proposition 5 (Optimal universal two-broadcasting protocol). The optimal universal two-broadcasting protocol is given by $\Gamma^*_{B\to B_1B_2} = p_1\mathcal{N}_1 - p_2\mathcal{N}_2$, where $p_1 = \frac{2d}{d+1}$, $p_2 =$

$\frac{d-1}{d+1}$, and

$$\mathcal{N}_1(\rho_{AB}) := \frac{d}{d+1}\mathcal{P}(\rho_{AB}) + \frac{1}{d+1}\mathcal{Q}(\rho_{AB}),$$

$$\mathcal{N}_2(\rho_{AB}) := \frac{d^2}{d^2-2}\mathcal{I}(\rho_{AB}) - \frac{2d^2}{(d^2-2)(d^2-1)}\mathcal{P}(\rho_{AB}) + \frac{2}{(d^2-2)(d^2-1)}\mathcal{Q}(\rho_{AB}), \tag{15}$$

where d denotes the dimension of Hilbert spaces $\mathcal{H}_B$, $\mathcal{H}_{B_1}$, and $\mathcal{H}_{B_2}$, $\mathcal{Q}(\rho_{AB}) := \frac{1}{2}[F_{B_1B_2}(\rho_{AB_1}\otimes I_{B_2}) + (\rho_{AB_1}\otimes I_{B_2})F_{B_1B_2}]$, $\mathcal{P}(\rho_{AB}) := \frac{1}{2d}[\rho_{AB_1}\otimes I_{B_2} + \mathcal{S}_{B_1B_2}(\rho_{AB_1}\otimes I_{B_2})]$, $\mathcal{S}_{B_1B_2}(\cdot)$ is the swap operation between B_1 and B_2 corresponding to the swap operator $F_{B_1B_2} := \sum_{i,j=1}^{d-1}|ij\rangle\langle ji|$, and $\mathcal{I}(\cdot)$ denotes the replacement channel yielding $\frac{1}{d^2}I_{B_1B_2}$.

Proof. We further denote $M := \Phi_{BB_1}\otimes I_{B_2}$ and $N := I_B\otimes F_{B_1B_2}$ for short. Based on the proof of Theorem 4, one can find that there exists a virtual two-broadcasting protocol $\Gamma^*_{B\to B_1B_2} = p_1\mathcal{N}_1 - p_2\mathcal{N}_2$ with the optimal simulation cost, where $p_1 = \frac{2d}{d+1}$, $p_2 = \frac{d-1}{d+1}$, and $\mathcal{N}_1$ and $\mathcal{N}_2$ are quantum channels with Choi operators $J^{\mathcal{N}_1} := \frac{M+NMN+MN+NM}{2(d+1)}$ and $J^{\mathcal{N}_2} := \frac{1}{d^2-2}(I - \frac{d(M+NMN)-(MN+NM)}{d^2-1})$, respectively. According to the statement of Definition 3, we can refer to $\Gamma^*_{B\to B_1B_2}$ as the optimal universal two-broadcasting protocol, which completes this proof. ∎

Theorem 4 proposes the optimal universal virtual two-broadcasting protocol, taking into account the sampling cost required to broadcast the correlation inherent in the classical information $\mathrm{Tr}[O\rho_{AB}]$ with a desired estimating precision. Note that what we obtained here is the minimum cost protocol among all possible universal unilocal virtual two-broadcasting protocols. We first find the HPTP protocol for the desired simulation cost and then utilize the dual SDP in Eq. (7) to establish the optimality of this protocol. Moreover, Theorem 4 reveals an intriguing relationship between the sampling cost and the system's dimension. As the dimension of the quantum system grows, the simulation cost for universal virtual two-broadcasting converges to a constant value of $\log_3$, which means that even in high-dimensional quantum systems, the simulation cost is still within a controllable range.

We further extend our investigation to the context of unilocal virtual n-broadcasting, to analyze the change in simulation cost in relation to the number of parties involved, i.e., from system B to $B_1,\ldots,B_n$. In particular, we derive an upper bound and a lower bound for the simulation cost of universal virtual n-broadcasting.

Theorem 6 (Upper and lower bounds). The optimal simulation cost of all universal unilocal virtual n-broadcasting protocols which broadcast system B to $B_1\ldots B_n$ satisfies

$$\log_2\left(\frac{2nd}{n+d-1}-1\right) \leqslant \gamma_n^* \leqslant \log_2(2n-1), \tag{16}$$

where d denotes the dimension of Hilbert spaces $\mathcal{H}_B$ and $\mathcal{H}_{B_j}$ for $j = 1,\ldots,n$.

Proof. We first show the upper bound on the minimum simulation cost. According to Proposition 2, one can find that the simulation cost of the universal protocol $\Gamma'_{B\to B^n}$ can be an upper bound of γ_n^*, i.e., $\gamma_n^* \leqslant \nu(\Gamma'_{B\to B^n})$. Then, rewrite the universal virtual n-broadcasting protocol $\Gamma'_{B\to B^n}$ into the linear combination of two quantum channels $\mathcal{M}_1$ and $\mathcal{M}_2$ as

$$\Gamma'_{B\to B^n} = n\mathcal{M}_1 - (n-1)\mathcal{M}_2, \tag{17}$$

where the Choi operators of $\mathcal{M}_1$ and $\mathcal{M}_2$ can be written as $J^{\mathcal{M}_1} := \frac{1}{nd^{n-1}}\sum_{j=1}^{n}\mathcal{S}_{B_1B_j}(\Phi_{BB_1}\otimes I_{B_2\ldots B_n})$ and $J^{\mathcal{M}_2} := \frac{1}{d^{n-1}}\Phi_{B_1B_2}\otimes I_{BB_3\ldots B_n}$, respectively. Then, by definition, we have $\nu(\Gamma'_{B\to B^n}) \leqslant \log_2(2n-1)$, which directly gives $\gamma_n^* \leqslant \log_2(2n-1)$.

Second, we are going to derive the lower bound by showing that $\{X_{BB_1},\ldots,X_{BB_n},Z_B,K_B\}$ is a feasible solution of the dual SDP in Eq. (7), where $Z_B = K_B = \frac{I_B}{d}$, and $X_{BB_1} = \cdots = X_{BB_n} = \frac{2}{d(n+d-1)}\Phi_d - \frac{1}{nd}I$. It is straightforward to check that $\{X_{BB_1},\ldots,X_{BB_n},Z_B,K_B\}$ satisfies the constrains of SDP in Eq. (7). We further check the objective function

$$\sum_{j=1}^{n}\mathrm{Tr}[X_{BB_j}\Phi_{BB_j}] = \frac{2nd}{n+d-1} - 1. \tag{18}$$

According to the fact that the optimal solution of dual SDP in Eq. (7) is a lower bound of the optimal solution of primal SDP in Eq. (7), we have the following inequality:

$$\log_2\left(\frac{2nd}{n+d-1}-1\right) \leqslant \gamma_n^*, \tag{19}$$

which completes the proof. ∎

Remarkably, in Fig. 3, one can find that the lower bound matches the optimal simulation cost γ_n^* in numerical experiments. Furthermore, according to Theorem 6, it is straightforward to find that the lower bound converges to the upper bound as the dimension of the quantum system grows. These mean the upper bound is significantly valuable at a high system level. The simulation cost will not exhibit exponential growth with the dimension of the system, which suggests our capability to effectively tackle the unilocal

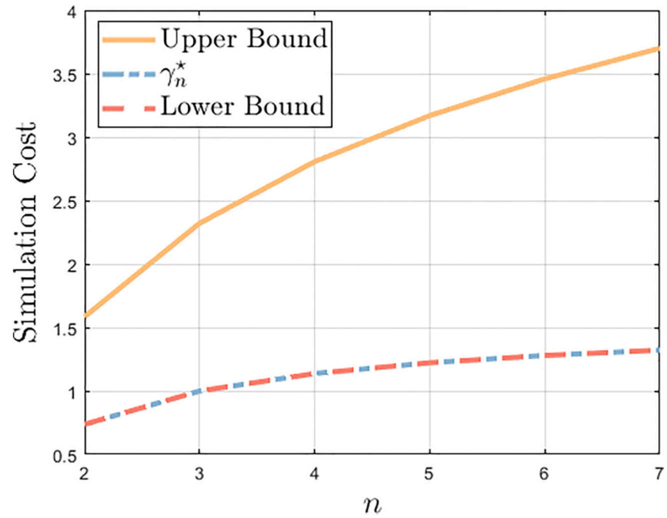

FIG. 3. Simulation cost of universal unilocal virtual n-broadcasting. Here, the dimensions of the Hilbert spaces $\mathcal{H}_B$ and $\mathcal{H}_{B_j}$ ($j = 1,\ldots,n$) are all equal to two. The x axis corresponds to the number of parties on Bob's side involved in the broadcasting. The y axis corresponds to the simulation cost of the protocol.

virtual n-broadcasting task, even for a bipartite system with a high dimension. In summary, Theorems 4 and 6 reveal that engaging in virtual n-broadcasting not only enables the acquisition of correlation of a bipartite quantum system encoded in the expectation values but also grants control over the associated costs. These remind us that it is feasible to employ nonphysical operations to overcome the limitations of quantum mechanics at an acceptable cost.

IV. CONCLUDING REMARKS

In this work, we have proposed a framework known as *unilocal virtual quantum broadcasting*, employing HPTP maps. We have demonstrated the existence of a universal unilocal virtual n-broadcasting protocol capable of distributing information from any bipartite quantum state to multiple parties via local operations. Furthermore, we have formalized the simulation cost of this broadcasting protocol as a semidefinite programming problem. Notably, we have provided an analytical universal unilocal virtual two-broadcasting protocol to clarify the optimal simulation cost. By accurately characterizing simulation cost, we found that virtual two-broadcasting remains applicable in high-dimensional quantum systems, as the corresponding simulation cost converges to a constant $\log_3$ with increasing dimensions. Furthermore, we have provided upper and lower bounds on the simulation cost of the virtual n-broadcasting protocol and demonstrated that the lower bound converges to the upper bound $\log_2(2n-1)$ that is independent of the system dimension. The findings above demonstrate the practical potential of our virtual broadcasting protocol, as the simulation costs are always controllable. It is worth noting that Parzygnat *et al.* [10] have explored broadcasting tasks via a virtual process. They focused on the conditions that virtual quantum broadcasting maps should fulfill and provided physical interpretations for their canonical quantum broadcasting map from multiple perspectives. Our work generalizes the virtual broadcasting [10] to virtual unilocal broadcasting by allowing a reference system and shows that unilocal virtual broadcasting maps can efficiently accomplish the broadcasting task through simulated physical operations with minimal cost. It is noteworthy that if system A is trivial, the canonical virtual broadcasting map presented in Ref. [10] is also a feasible two-broadcasting protocol but not the one with the minimum simulation cost. Specifically, one can check that the simulation cost of the canonical virtual broadcasting map is $\log_2 d$.

Our results open new avenues for understanding and harnessing the unique properties of quantum mechanics. This demonstrates the possibility of overcoming the limitations of quantum mechanics using controllable nonphysical operations. The exploration of virtual broadcasting not only broadens our comprehension of quantum information distribution [26–28] but also provides a valuable tool for advancing quantum communication and computing technologies [29]. Future work will focus on the implementation of quantum circuits of our proposed virtual broadcasting protocol and its further practical applications in the areas of quantum communication and computing.

Note added. Recently, we became aware of the arXiv preprint of a closely related work [10] that independently proposed the idea of virtual broadcasting. The main distinction is discussed in the above concluding remarks.

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APPENDIX: OPTIMAL UNILOCAL VIRTUAL QUANTUM BROADCASTING

1. The proof of Lemma 1 and Proposition 2

Lemma 1. An HPTP map $\Gamma_{B\to B^n}$ is a universal unilocal virtual n-broadcasting protocol if and only if

$$J^{\Gamma}_{BB_j} = \Phi_{BB_j}, \quad j = 1, \dots, n, \tag{A1}$$

where Φ_{BB_j} denotes the unnormalized $d \otimes d$ maximally entangled state on system BB_j, $J^{\Gamma}_{BB_j} := \mathrm{Tr}_{\backslash BB_j}[J^{\Gamma}_{BB^n}]$, and $J^{\Gamma}_{BB^n}$ is the Choi operator of $\Gamma_{B\to B^n}$.

Proof. Considering the "if" part, for $\forall j \in \{1, \dots, n\}$, $\mathrm{Tr}_{\backslash BB_j}[J^{\Gamma}_{BB^n}] = \Phi_{BB_j}$ implies that

$$\mathrm{Tr}_B\left[\rho_{AB}^{T_B}\,\mathrm{Tr}_{\backslash BB_j}\left[J^{\Gamma}_{BB^n}\right]\right] = \rho_{AB}, \tag{A2}$$

for all states $\rho_{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$. For the "only if" part, we assume $\Gamma_{B\to B^n}$ is a universal unilocal virtual n-broadcasting protocol. Then, for any input state $\rho_{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$, we have

$$\rho_{AB} = \mathrm{Tr}_{\backslash AB_j}\left[\rho_{AB}^{T_B} J^{\Gamma}_{BB^n}\right] = \mathrm{Tr}_B\left[\rho_{AB}^{T_B}\,\mathrm{Tr}_{\backslash BB_j}\left[J^{\Gamma}_{BB^n}\right]\right], \tag{A3}$$

which means $\mathrm{Tr}_{\backslash BB_j}[J^{\Gamma}_{BB^n}]$ is a Choi operator of the identity operator from B to B_j, i.e., $\mathrm{Tr}_{\backslash BB_j}[J^{\Gamma}_{BB^n}] = \Phi_{BB_j}$. Thus, we complete the proof. ■

Proposition 2. For any bipartite quantum system AB, there exists a universal unilocal virtual n-broadcasting protocol.

Proof. The universal unilocal virtual n-broadcasting protocol $\Gamma'_{B\to B^n}$ can be written as

$$\Gamma'_{B\to B^n}(\rho_{AB}) := \sum_{j=1}^{n} \mathcal{S}_{B_1B_j}\left(\rho_{AB_1} \otimes \frac{I_{B_2\cdots B_n}}{d^{n-1}}\right) - (n-1)\mathcal{R}_{B\to B^n}(\rho_{AB}), \tag{A4}$$

where $\mathcal{S}_{B_1B_j}(\cdot)$ denotes the swap operation between the subsystems B_1 and B_j, and $\mathcal{R}_{B\to B^n}(\cdot)$ denotes the replacement channel yielding $\Phi_{B_1B_2} \otimes \frac{I_{B_3\dots B_n}}{d^{n-3}}$ for any input state.

Then, the Choi operator of $\Gamma'_{B\to B^n}$ is

$$J_{BB^n}^{\Gamma'} := \frac{1}{d^{n-1}}\mathcal{S}_{B_1B_j}\left(\sum_{j=1}^{n}\Phi_{BB_1}\otimes I_{B_2\ldots B_n}\right) - \frac{n-1}{d^{n-1}}\Phi_{B_1B_2}\otimes I_{BB_3\ldots B_n}, \tag{A5}$$

Then, it is straightforward to check that

$$\mathrm{Tr}_{\backslash BB_j}\left[J_{BB^n}^{\Gamma'}\right] = \Phi_{BB_j}, \quad j=1,\ldots,n \tag{A6}$$

Hence, we conclude that the HPTP map with Choi operator $J_{BB^n}^{\Gamma'}$ achieves n-broadcasting for all states ρ_{AB} by Lemma 1. ■

2. SDP for unilocal virtual broadcasting protocol

Proposition 3. The optimal simulation cost of all universal unilocal virtual n-broadcasting protocols can be characterized as the following SDP:

$$\begin{aligned} 2^{\gamma_n^*} = \min \quad & p_1+p_2 \\ \text{subject to } & \mathrm{Tr}_{\backslash BB_j}\left[J_{BB^n}^{\mathcal{N}_1}-J_{BB^n}^{\mathcal{N}_2}\right] = \Phi_{BB_j},\ j=1,\ldots,n \\ & \mathrm{Tr}_{B^n}\left[J_{BB^n}^{\mathcal{N}_1}\right] = p_1 I_B, \\ & \mathrm{Tr}_{B^n}\left[J_{BB^n}^{\mathcal{N}_2}\right] = p_2 I_B, \\ & J_{BB^n}^{\mathcal{N}_1}\geqslant 0, J_{BB^n}^{\mathcal{N}_2}\geqslant 0, \end{aligned} \tag{A7}$$

with variables $J_{BB^n}^{\mathcal{N}_1}$, $J_{BB^n}^{\mathcal{N}_2}$, p_1, and p_2. Φ_{BB_j} is the unnormalized $d\otimes d$ maximally entangled state on system BB_j.

Now, we derive its dual SDP for the case of two-broadcasting. Based on the primal SDP, the Lagrange function can be written as

$$\begin{aligned} &\mathcal{L}(X_{BB_1}, Y_{BB_2}, Z_B, K_B, J^{\mathcal{N}_1}, J^{\mathcal{N}_2}, p_1, p_2) \qquad \text{(A8)} \\ &:= p_1+p_2+\langle X_{BB_1},\ \Phi_{BB_1}-\mathrm{Tr}_{B_2}[J^{\mathcal{N}_1}-J^{\mathcal{N}_2}]\rangle \\ &\quad+\langle Y_{BB_2},\ \Phi_{BB_2}-\mathrm{Tr}_{B_1}[J^{\mathcal{N}_1}-J^{\mathcal{N}_2}]\rangle \\ &\quad+\langle Z_B,\ p_1 I_B-\mathrm{Tr}_{B_1B_2}[J^{\mathcal{N}_1}]\rangle \\ &\quad+\langle K_B,\ p_2 I_B-\mathrm{Tr}_{B_1B_2}[J^{\mathcal{N}_2}]\rangle \qquad \text{(A9)} \\ &= \mathrm{Tr}[X_{BB_1}\Phi_{BB_1}]+\mathrm{Tr}[Y_{BB_2}\Phi_{BB_2}] \\ &\quad+p_1(\mathrm{Tr}[Z_B]+1)+p_2(\mathrm{Tr}[K_B]+1) \\ &\quad+\langle -Z_B\otimes I_{B_1B_2},\ J^{\mathcal{N}_1}\rangle+\langle -K_B\otimes I_{B_1B_2},\ J^{\mathcal{N}_2}\rangle \\ &\quad+\langle -X_{BB_1}\otimes I_{B_2},\ J^{\mathcal{N}_1}-J^{\mathcal{N}_2}\rangle \\ &\quad+\langle -Y_{BB_2}\otimes I_{B_1},\ J^{\mathcal{N}_1}-J^{\mathcal{N}_2}\rangle, \qquad \text{(A10)} \end{aligned}$$

where X_{BB_1}, Y_{BB_2}, Z_B, and K_B are Lagrange multipliers. Then, the Lagrange dual function can be written as

$$\begin{aligned} &\mathcal{G}(X_{BB_1}, Y_{BB_2}, Z_B, K_B) \\ &:= \inf_{J^{\mathcal{N}_1}\geqslant 0, J^{\mathcal{N}_2}\geqslant 0, p_1, p_2} \mathcal{L}(p_1, p_2, X_{BB_1}, Y_{BB_2}, \\ &\qquad\times Z_B, K_B, J^{\mathcal{N}_1}, J^{\mathcal{N}_2}). \end{aligned} \tag{A11}$$

Since $J^{\mathcal{N}_1}\geqslant 0$ and $J^{\mathcal{N}_2}\geqslant 0$, it holds that $\mathrm{Tr}[Z_B]\geqslant -1$, $\mathrm{Tr}[K_B]\geqslant -1$,

$$\begin{aligned} -Z_B\otimes I_{B_1B_2}-(X_{BB_1}\otimes I_{B_2}+Y_{BB_2}\otimes I_{B_1})\geqslant 0, \\ -K_B\otimes I_{B_1B_2}+(X_{BB_1}\otimes I_{B_2}+Y_{BB_2}\otimes I_{BB_1})\geqslant 0, \end{aligned}$$

otherwise, the inner norm is unbounded. Redefine Z_B as $-Z_B$ and K_B as $-K_B$. Then, we obtain the following dual SDP:

$$\begin{aligned} \max \quad & \mathrm{Tr}[X_{BB_1}\Phi_{BB_1}]+\mathrm{Tr}[Y_{BB_2}\Phi_{BB_2}] \\ \text{subject to} \quad & \mathrm{Tr}[Z_B]\leqslant 1, \\ & \mathrm{Tr}[K_B]\leqslant 1, \\ & Z_B\otimes I_{B_1B_2}-X_{BB_1}\otimes I_{B_2}-Y_{BB_2}\otimes I_{B_1}\geqslant 0, \\ & K_B\otimes I_{B_1B_2}+X_{BB_1}\otimes I_{B_2}+Y_{BB_2}\otimes I_{B_1}\geqslant 0. \end{aligned} \tag{A12}$$

It is worth noting that the strong duality is held by Slater's condition. Similarly, it is straightforward to generalize it to the case of n-broadcasting shown in Eq. (9).

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