Entanglement degradation under local dissipative Landau-Zener noise

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(Received 23 February 2024; revised 17 June 2024; accepted 24 June 2024; published 24 July 2024)

We study entanglement degradation when one share of an entangled pair experiences noise. We consider the dissipative Landau-Zener model for describing local noise. The dissipative Landau-Zener model provides a convenient setting for modeling entanglement degradation with applications in quantum network developments and communications. Here, we study the problem in the fast- and slow-driving regimes. In the slow-driving regime, our results are analytical, while in the fast-driving regime, we use numerical techniques to obtain the results. Our study addresses the role of two main properties of the dynamics in entanglement degradation, namely, spin-coupling direction and adiabaticity of the dynamics. We derive an analytical expression for entanglement survival time versus bath temperature in the slow-driven regime. In both regimes, when the bath temperature is zero, we show that transversal spin-coupling has a less destructive effect on entanglement than longitudinal spin-coupling. We also show that entanglement degradation is weaker in the nonadiabatic regime compared to the adiabatic regime. Therefore, our results determine the proper choice of parameters for having less entanglement degradation in the presence of local noise described by the dissipative Landau-Zener model.

DOI: 10.1103/PhysRevA.110.012456

I. INTRODUCTION

The essential role of entanglement in quantum information science and technology [1–5] urges analyzing its generation [6–9], distribution [10,11], and protection against noise [12–14]. In particular, analyzing entanglement degradation when one share of an entangled pair experiences noise is essential due to its application in the development of quantum networks [15–17], quantum sensing [18], quantum illumination [19], and quantum communication protocols [20]. Isolating one share of an entanglement pair and using the other pair for entanglement distribution or remote sensing and communication has been demonstrated experimentally [21–24].

Regarding the vital role of entanglement, its dynamics and degradation have been studied from different aspects, theoretically and experimentally [25–39]. In Ref. [32] the role of local noise in entanglement degradation is studied where noise is parametrized by local unital channels. A more realistic model is discussed in Ref. [33] where one share of an entangled pair is protected against noise and the other interacts with a thermal bath. The dynamic of this qubit is modeled by a Markovian master equation and the role of coherent and incoherent parts of dynamics in entanglement degradation is discussed. Here we consider a setting as depicted in Fig. 1, where one share of an entangled pair $|\Psi\rangle_{SR}$, namely, reference qubit R is protected against noise and its local unitary evolution does not affect the entanglement properties of the pair. System qubit S interacts with a thermal bath and its coherent dynamics is generated by the time-dependent Landau-Zener Hamiltonian [40,41]. The Landau-Zener model is the simplest time-dependent model for addressing adiabatic/nonadiabatic transitions which have significant applications in adiabatic quantum computation [42,43]. Also, the Landau-Zener model serves as a successful model for describing the lowest-energy levels employed in quantum annealing [44].

We consider the noise on the system qubit to be described by the dissipative Landau-Zener model. In this model, system qubit with Landau-Zener Hamiltonian interacts with a bosonic bath. The interaction is described by the coupling of different spin directions to harmonic oscillators of the bath. To be more specific, in this model which is also discussed in Refs. [45,46], by varying the parameter θ (see Fig. 1) we allow the coupling to change from the σ_z direction to the σ_x direction or from longitudinal to transversal. The intermediate values of this parameter define different spin-coupling directions.

Here, we discuss the role of the bath temperature in entanglement survival time. Also, we analyze the role of the spin-coupling direction in entanglement degradation. The role of the spin-coupling direction in thermally assisted quantum annealing and the benefit of having transverse spin-coupling for the ground-state probability have been discussed in Ref. [46]. Here, for entanglement degradation under the dissipative Landau-Zener model, we show that a larger noise coupling angle is beneficial for preserving entanglement in the slow-driving regime with zero bath temperature, and for a transverse coupling, the dynamic is not entanglement breaking at all. In the fast-driving regime with zero bath temperature, with transversal noise, entanglement decay is less than entanglement degradation with longitudinal noise. We also address the role of adiabaticity of the dynamics in entanglement degradation. We show that by going from a fast-driving regime to a slow-driving regime, entanglement breakdown is more serious.

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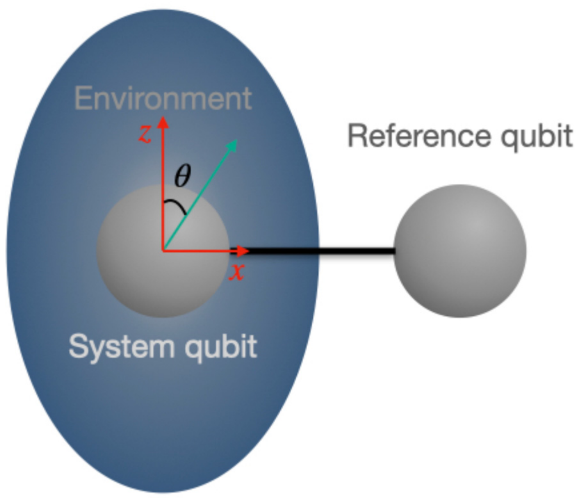

FIG. 1. Schematic representation of the two-qubit system, where the system qubit with the Landau-Zener Hamiltonian interacts with the environment in a thermal state and the reference qubit does not experience noise. Parameter θ indicates the spin-coupling direction in the interaction between the system qubit and the environment.

For our analysis, we work in the weak-coupling regime to have Markovian dynamics. Deriving the master equation for the Landau-Zener Hamiltonian (like any other time-dependent Hamiltonian) requires taking care of further subtleties. We use the framework introduced in Ref. [45] which is based on using instantaneous eigenstates of the system's Hamiltonian. We derive the time-dependent generator of the dissipative Landau-Zener model for a bath with a finite temperature. Considering a bath with a finite temperature allows us to discuss the role of bath temperature in entanglement degradation in the slow-driving regime. The derived Markovian master equation is valid in adiabatic and nonadiabatic regimes for a particular range of parameters [45]. We derive the range of validity of the derived master equation and in this range discuss the role of adiabaticity of the dynamics in entanglement degradation. The other challenging issue is solving the master equation with a time-dependent generator. The analytical solution does not exist unless in some particular range of parameters. Hence, besides providing an analytical solution in the slow-driving regime, we use numerical techniques to explore a wider range of parameters and extend our analysis to the fast-driving regime.

The structure of the paper is as follows. In Sec. II after describing a single-qubit dissipative Landau-Zener model, we derive the master equation and discuss the proper range of the parameters for having a valid Markov master equation. We discuss the evolution of a general two-qubit system under local dissipative Landau-Zener noise in Sec. III. Then we proceed to analyze entanglement dynamics in Sec. IV and report our results in slow- and fast-driving regimes. We discuss the results and conclude in Sec. V with an outlook for potential future work.

II. SINGLE-QUBIT DISSIPATATIVE LANDAU-ZENER MODEL

In this section, we describe the dynamics of a single qubit with Landau-Zener Hamiltonian that interacts with a thermal bath. In Sec. II A we use instantaneous eigenstates of the Hamiltonian and follow the approach of Ref. [45] to derive the master equation in the weak-coupling limit. To have a compact representation for Lindblad operators, we need to work in a rotated basis which we explain in Sec. II B. As entanglement is invariant under a change of basis, this time-dependent rotated frame enables us to obtain analytical results in a slow-driving regime and also have a numerical analysis of entanglement dynamics in the fast-driving regime.

The Landau-Zener Hamiltonian which describes the coherent dynamics of the system is given by

$$H_S(t) = \hbar(\Delta\sigma_x + vt\sigma_z), \tag{1}$$

where $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ are Pauli operators and Δ and v are positive real parameters of the Hamiltonian. From here on, we use natural units where the Planck constant $\hbar = 1$. The energy levels of system and the corresponding eigenvectors are given by

$$\epsilon_\pm(t) = \pm\Omega(t), \quad |\epsilon_\pm(t)\rangle = \frac{1}{N_\pm(t)}\begin{pmatrix} \Delta \\ \pm\Omega(t) - vt \end{pmatrix}, \tag{2}$$

with $N_\pm(t) = \sqrt{2\Omega(t)[\Omega(t) \mp vt]}$ and

$$\Omega(t) = \sqrt{v^2t^2 + \Delta^2}. \tag{3}$$

To model noise on the system of study, we assume that the system qubit interacts with a thermal environment described by the following Hamiltonian:

$$H_E = \int_0^{\omega_{\max}} d\nu\, b^\dagger(\nu) b(\nu), \tag{4}$$

where $b(\nu)$ and $b(\nu)^\dagger$ are bosonic operators of the environment $[b(\nu), b^\dagger(\nu')] = \delta(\nu - \nu')$ and $\omega_{\max}$ is the cutoff frequency of the harmonic oscillators in the bath. The total Hamiltonian of the system and the environment is given by

$$H_{\text{total}}(t) = H_S(t) + H_E + \lambda H_{SE}, \tag{5}$$

where $\lambda > 0$ is a constant denoting the strength of the interaction between the system and the environment and H_{SE} describes the interaction between the system and the environment:

$$H_{SE} = \mathbf{A} \otimes \mathbf{B}, \tag{6}$$

where

$$\mathbf{A} = \frac{1}{2}[\cos(\theta)\sigma_z + \sin(\theta)\sigma_x],$$
$$\mathbf{B} = \int_0^{\omega_{\max}} d\nu\, g(\nu)[b(\nu) + b^\dagger(\nu)]. \tag{7}$$

Here $g(\nu)$ is the coupling function and $\theta \in [0, 2\pi)$ determines the spin-coupling direction. By varying θ from 0 to $\frac{\pi}{2}$, the spin-coupling direction changes from longitudinal to transversal.

A. Master equation in the weak-coupling limit

In this subsection, we derive the master equation describing the dynamics of the system qubit, with the Landau-Zener Hamiltonian in Eq. (1), that interacts with a thermal bath, with the interaction Hamiltonian given in Eq. (6). For rigorous construction of the Markov quantum master equation we follow the formalism in Ref. [45] and discuss the range of parameters that guarantees the validity of the derived Markov master equation for describing the dynamics even in a nonadiabatic regime [45].

Following the approach described in Ref. [45] we derive the following quantum master equation:

$$\begin{aligned}
\dot{\rho}_S(t) &= \mathcal{L}_t[\rho_S(t)],\\
\mathcal{L}_t[\bullet] &= -i[H_S(t) + \lambda^2 H_{\mathrm{LS}}(t), \bullet] + \lambda^2 \mathcal{D}_t[\bullet],\\
\mathcal{D}_t[\bullet] &= \sum_m \gamma(\omega_m(t))\bigg\{A(\omega_m(t)) \bullet A^\dagger(\omega_m(t))\\
&\quad - \frac{1}{2}[A^\dagger(\omega_m(t))A(\omega_m(t)), \bullet]\bigg\},
\end{aligned} \tag{8}$$

where $\bullet$ stands for any linear operator acting on $\mathbb{C}^2$ (two-dimensional complex Hilbert space). For deriving Lindblad operators $A(\omega_m(t))$ instantaneous eigenstates of Hamiltonian $H_S(t)$ are used:

$$\begin{aligned}
A(\omega_1(t)) &= \Pi_{\epsilon_-}(t)\mathbf{A}\Pi_{\epsilon_+}(t),\\
A(\omega_2(t)) &= \Pi_{\epsilon_+}(t)\mathbf{A}\Pi_{\epsilon_-}(t),\\
A(\omega_3(t)) &= \Pi_{\epsilon_-}(t)\mathbf{A}\Pi_{\epsilon_-}(t) + \Pi_{\epsilon_+}(t)\mathbf{A}\Pi_{\epsilon_+}(t).
\end{aligned} \tag{9}$$

Here $\Pi_{\epsilon_\pm} = |\epsilon_\pm(t)\rangle\langle\epsilon_\pm(t)|$ are respectively projectors to eigenstates $|\epsilon_\pm(t)\rangle$ in Eq. (2) and $\omega_m(t)$ denotes transition frequencies:

$$\omega_1(t) = -\omega_2(t) = 2\Omega(t), \quad \omega_3(t) = 0. \tag{10}$$

Following the description in the Appendix, we have

$$\begin{aligned}
\gamma(\omega_m(t)) = & 2\pi J(|\omega_m(t)|)\Theta(\omega_m(t))[\bar{n}(|\omega_m(t)|) + 1]\\
& + 2\pi J(|\omega_m(t)|)\Theta(-\omega_m(t))[\bar{n}(|\omega_m(t)|)]\\
& + 2\pi T\delta(\omega_m(t)),
\end{aligned} \tag{11}$$

where $m = 1$, 2, and 3 addresses different transition frequencies in Eq. (10), $\Theta(x)$ is the step function, and

$$\bar{n}(\omega) = \frac{1}{e^{\frac{\omega}{T}} - 1} \tag{12}$$

is the environment's mean photon number with frequency ω when its temperature is T. We recall that in the natural units the Boltzmann constant $\kappa_B = 1$. In Eq. (11), $J(\omega_m(t)) = g^2(\omega_m(t))$ is the spectral density of the environment which is assumed to be ohmic:

$$J(\omega_m(t)) = \omega_m(t)e^{-\frac{\omega_m(t)}{\omega_c}}, \tag{13}$$

where ω_c is the cutoff frequency. The Lamb-shift Hamiltonian in Eq. (8) is the correction to Hamiltonian arising from the weak-coupling approximation and is given in terms of Lindblad operators:

$$H_{\mathrm{LS}}(t) = \sum_{m=1}^{3} \mathcal{S}(\omega_m(t))A^\dagger(\omega_m(t))A(\omega_m(t)). \tag{14}$$

For $\mathcal{S}(\omega_m(t))$, see the Appendix.

By using the formal definition of Lindblad operators in Eq. (9), the explicit expressions of Lindblad operators are given by matrices with time-dependent elements. Such a representation is not suitable for the aim of analyzing entanglement degradation. In the next subsection, we explain how to derive a compact explicit form of Lindblad operators for a qubit with the Landau-Zener Hamiltonian interacting with a thermal bath according to the interaction Hamiltonian H_{SE} in Eq. (6).

It is worth mentioning that the Markovian master equation in Eq. (8) is valid in both adiabatic and nonadiabatic regimes if the following inequalities are satisfied [45]:

$$\begin{aligned}
\tau_S(t) &\ll \tau_R(t),\\
\tau_S(t) &\ll \tau_A(t).
\end{aligned} \tag{15}$$

Here, $\tau_A(t)$, $\tau_S(t)$, and $\tau_R(t)$ are respectively the temporal change timescale of $H_S(t)$, the intrinsic beat timescale, and the relaxation timescale. As defined in Ref. [45], $\tau_A(t)$ reflects the temporal change of the system's Hamiltonian. This timescale is defined in terms of the temporal change of eigenstates and eigenvalues of the system's Hamiltonian as follows:

$$\tau_A(t) = \min\{\tau_{AE}(t), \tau_{AS}(t)\}, \tag{16}$$

with

$$\begin{aligned}
\tau_{AE}^{-1}(t) &= \max_n \left|\frac{1}{\epsilon_n(t)}\frac{d}{dt}\epsilon_n(t)\right|,\\
\tau_{AS}^{-1}(t) &= \max_{m\neq n} \left|\langle m(t)|\frac{d}{dt}|n(t)\rangle\right|,
\end{aligned} \tag{17}$$

where eigenstates and eigenenergies of $H_S(t)$ are denoted by $\epsilon_n(t)$ and $|n(t)\rangle$, respectively. According to this definition for the Landau-Zener Hamiltonian (1), we have

$$\tau_A(t) = \begin{cases} \frac{2\Omega^2(t)}{v\Delta}, & t \leqslant \frac{\Delta}{v},\\ \frac{\Omega^2(t)}{v^2 t}, & t > \frac{\Delta}{v}. \end{cases} \tag{18}$$

The other two timescales are defined as follows [45,47–50]:

$$\begin{aligned}
\tau_S^{-1} &= \min_{\omega_m(t)\neq\omega_n(t)} |\omega_m(t) - \omega_n(t)| = 2\Omega(t),\\
\tau_R^{-1}(t) &= \max_m \gamma(\omega_m) = 2\pi J(\omega_1(t))[\bar{n}(\omega_1(t)) + 1].
\end{aligned} \tag{19}$$

In the proceeding sections, when we provide relevant figures for discussing the role of different parameters in entanglement degradation, we choose parameters such that the inequalities in Eq. (15) hold.

B. Time-dependent master equation in a rotated basis

In this subsection, we derive the explicit compact form for the generator of dynamic $\mathcal{L}_t$ in Eq. (8) in a rotated basis [51]. This generator is used in the following sections for describing local Landau-Zener dissipative dynamics.

To derive a compact form for Lindblad operators, we use instantaneous eigenstates of $H_S(t)$ in Eq. (1). Unitary time-dependent transformation that diagonalizes $H_S(t)$ is given by

$$\mathcal{R}(t) = e^{i\phi(t)\sigma_y}, \tag{20}$$

where $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ is the second Pauli matrix and

$$\phi(t) = \frac{1}{2}\arctan\left(\frac{\Delta}{vt}\right). \tag{21}$$

We denote any operator O in the rotated basis by $\tilde{O}(t)$:

$$\tilde{O}(t) = \mathcal{R}(t) O \mathcal{R}^\dagger(t). \tag{22}$$

Therefore, the system Hamiltonian in the rotated basis is given by

$$\tilde{H}_S(t) = \mathcal{R}(t) H_S(t) \mathcal{R}^\dagger(t) = \Omega(t)\sigma_z, \tag{23}$$

where $\Omega(t)$ is defined in Eq. (3). From Eq. (23) it is clear that $\tilde{H}_S(t)$ is diagonal for all time t. The master equation in Eq. (8) in the rotated basis is given by

$$\begin{aligned}
\dot{\tilde{\rho}}(t) &= \tilde{\mathcal{L}}_t[\tilde{\rho}(t)],\\
\tilde{\mathcal{L}}_t[\bullet] &= -i[\tilde{H}_S(t) + \lambda^2 \tilde{H}_{\mathrm{LS}}(t) + \dot{\phi}(t)\sigma_y, \bullet] + \lambda^2 \tilde{\mathcal{D}}_t[\bullet],\\
\tilde{\mathcal{D}}_t[\bullet] &= \sum_m \gamma(\omega_m(t))(\tilde{A}(\omega_m(t)) \bullet \tilde{A}^\dagger(\omega_m(t))\\
&\quad - \frac{1}{2}\{\tilde{A}^\dagger(\omega_m(t))\tilde{A}(\omega_m(t)), \bullet\},
\end{aligned} \tag{24}$$

where Lindblad operators in the rotated basis are given by

$$\begin{aligned}
\tilde{A}(\omega_1(t)) &= \tilde{\Pi}_{\epsilon_-}\tilde{\mathbf{A}}(t)\tilde{\Pi}_{\epsilon_+},\\
\tilde{A}(\omega_2(t)) &= \tilde{\Pi}_{\epsilon_+}\tilde{\mathbf{A}}(t)\tilde{\Pi}_{\epsilon_-},\\
\tilde{A}(\omega_3(t)) &= \tilde{\Pi}_{\epsilon_-}\tilde{\mathbf{A}}(t)\tilde{\Pi}_{\epsilon_-} + \tilde{\Pi}_{\epsilon_+}\tilde{\mathbf{A}}(t)\tilde{\Pi}_{\epsilon_+}.
\end{aligned} \tag{25}$$

Projector operators $\tilde{\Pi}_{\epsilon_\pm}$ in the rotated basis have a simple form: $\tilde{\Pi}_{\epsilon_-} = |0\rangle\langle 0|$ and $\tilde{\Pi}_{\epsilon_+} = |1\rangle\langle 1|$. In this basis, each Lindblad operator is derived as a multiplication of a time-dependent function with a time-independent operator: $\sigma_\pm = \frac{\sigma_x \pm i\sigma_y}{2}$ and σ_z:

$$\tilde{A}(\omega_1(t)) = \tfrac{1}{2}\sin[\theta - 2\phi(t)]\sigma_-, \tag{26}$$

$$\tilde{A}(\omega_2(t)) = \tfrac{1}{2}\sin[\theta - 2\phi(t)]\sigma_+, \tag{27}$$

$$\tilde{A}(\omega_3(t)) = \tfrac{1}{2}\cos[\theta - 2\phi(t)]\sigma_z. \tag{28}$$

We find the Lamb-shift Hamiltonian in Eq. (14) in the rotated basis by using the explicit forms of Lindblad operators in Eqs. (26)–(28):

$$\begin{aligned}
\tilde{H}_{\mathrm{LS}}(t) &= \tfrac{1}{2}\sin^2[\theta - 2\phi(t)][\mathcal{S}(\omega_1(t)) - \mathcal{S}(\omega_2(t))](\mathrm{id} + \sigma_z)\\
&\quad + \tfrac{1}{4}\cos^2[\theta - 2\phi(t)]\,\mathrm{id}\,.
\end{aligned} \tag{29}$$

The use of the explicit form of the system Hamiltonian in Eq. (23), the Lindblad operators in Eqs. (26)–(28), and the Lamb-shift term in Eq. (29) yields the explicit form of the dynamics generator in the rotated basis:

$$\begin{aligned}
\tilde{\mathcal{L}}_t[\bullet] =& -i[k(t)\sigma_z - \dot{\phi}(t)\sigma_y, \bullet]\\
&+ f(t)\big(\sigma_- \bullet \sigma_+ - \tfrac{1}{2}\{\sigma_+\sigma_-, \bullet\}\big)\\
&+ g(t)\big(\sigma_+ \bullet \sigma_- - \tfrac{1}{2}\{\sigma_-\sigma_+, \bullet\}\big)\\
&+ l(t)(\sigma_z \bullet \sigma_z - \bullet),
\end{aligned} \tag{30}$$

where $k(t)$, $f(t)$, $g(t)$, and $l(t)$ are given by

$$k(t) = \Omega(t) + \frac{\lambda^2}{2}\sin^2[\theta - 2\phi(t)][\mathcal{S}(\omega_1(t)) - \mathcal{S}(\omega_2(t))], \tag{31}$$

$$f(t) = \frac{\lambda^2}{4}\sin^2[\theta - 2\phi(t)]\gamma(\omega_1(t)), \tag{32}$$

$$g(t) = \frac{\lambda^2}{4}\sin^2[\theta - 2\phi(t)]\gamma(\omega_2(t)), \tag{33}$$

$$l(t) = \frac{\lambda^2}{4}\cos^2[\theta - 2\phi(t)]\gamma(\omega_3(t)). \tag{34}$$

In the next section, we use this explicit form of the dynamics generator in Eq. (30) for deriving the local Markovian master equation in a bipartite system.

III. LOCAL DISSIPATIVE LANDAU-ZENER MODEL

In this section, we describe the local dissipative Landau-Zener model. We derive the master equation governing the dynamics of a bipartite system where the system qubit experiences noise and the reference qubit remains invariant. We derive a set of coupled first-order differential equations for coefficients of a general bipartite input state. For a specific range of parameters, we solve this set of coupled first-order differential equations analytically for a one-parameter initial pure state. By varying this parameter, the initial state's entanglement interpolates between 0 and 1.

We consider a pair of entangled qubits. As depicted in Fig. 1 in the setting under study, the reference qubit is protected against noise; therefore, the dissipative dynamics of the system qubit is the only source of noise that affects the entanglement between reference and system qubits. We denote the system-reference bipartite density matrix at time t with $\rho_{SR}(t)$. Following the discussion of Sec. II, when the system undergoes a dissipative Landau-Zener dynamics with system-environment interaction as described in Eq. (6) and the reference qubit does not evolve, the dynamics of the pair is generated by $\mathcal{L}_t \otimes \mathrm{id}$ with $\mathcal{L}_t$ defined in Eq. (8). Following the arguments in Sec. II B, to derive the explicit forms of Lindblad operators as in Eqs. (26)–(28), it is required to rotate the basis by the time-dependant unitary operator $\mathcal{R}(t)$ given in Eq. (20). Because we aim to analyze entanglement dynamics of a bipartite system described by the density matrix $\rho_{SR}(t)$ and entanglement is invariant under local unitary operations, we work in the rotated basis and analyze the entanglement dynamics of

$$\tilde{\rho}_{SR}(t) = [\mathcal{R}(t) \otimes \mathrm{id}]\rho_{SR}(t)[\mathcal{R}^\dagger(t) \otimes \mathrm{id}]. \tag{35}$$

Therefore, in what follows, we focus on solving the following master equation in the rotated basis:

$$\dot{\tilde{\rho}}_{SR}(t) = (\tilde{\mathcal{L}}_t \otimes \mathrm{id})\tilde{\rho}_{SR}(t), \tag{36}$$

where $\tilde{\mathcal{L}}_t$ is given in Eq. (30). To solve Eq. (36) for the density matrix $\tilde{\rho}_{SR}(t)$, we take it into account that any bipartite density matrix of $\tilde{\rho}_{SR}(t)$ can be written as

$$\tilde{\rho}_{SR}(t) = \frac{1}{4}\Bigg(\mathrm{id}\otimes\mathrm{id} + \vec{s}(t)\cdot\sigma\otimes\mathrm{id} + \mathrm{id}\otimes\vec{r}(t)\cdot\sigma + \sum_{i,j=1}^{3}\chi_{ij}(t)\sigma_i\otimes\sigma_j\Bigg), \tag{37}$$

where the scalar product is defined as $\vec{r}\cdot\sigma = \sum_{i=1}^{3} r_i\sigma_i$. Here the three elements of vectors $\vec{r}(t)$ and $\vec{s}(t)$ are given by

$$\begin{aligned} s_i(t) &= \mathrm{Tr}[(\sigma_i\otimes\mathrm{id})\tilde{\rho}_{SR}(t)],\\ r_i(t) &= \mathrm{Tr}[(\mathrm{id}\otimes\sigma_i)\tilde{\rho}_{SR}(t)],\\ \chi_{ij} &= \mathrm{Tr}[(\sigma_i\otimes\sigma_j)\tilde{\rho}_{SR}], \end{aligned} \tag{38}$$

where σ_i's are Pauli operators. Furthermore, by using $\tilde{\mathcal{L}}_t$ in Eq. (30) we have

$$\begin{aligned} \tilde{\mathcal{L}}_t[\mathrm{id}] &= -2a_-(t)\sigma_z,\\ \tilde{\mathcal{L}}_t[\sigma_x] &= 2k(t)\sigma_y - b(t)\sigma_x + 2\dot{\phi}(t)\sigma_z,\\ \tilde{\mathcal{L}}_t[\sigma_y] &= -2k(t)\sigma_x - b(t)\sigma_y,\\ \tilde{\mathcal{L}}_t[\sigma_z] &= -2\dot{\phi}(t)\sigma_x - 2a_+(t)\sigma_z, \end{aligned} \tag{39}$$

with

$$\begin{aligned} a_\pm(t) &= \tfrac{1}{2}[f(t)\pm g(t)],\\ b(t) &= a_+(t) + 2l(t). \end{aligned} \tag{40}$$

Hence, by comparing the left- and right-hand sides of Eq. (36) for $\tilde{\rho}_{SR}(t)$ as in Eq. (37) and considering the action of $\tilde{\mathcal{L}}_t$ on identity and Pauli operators in Eq. (39), we find a set of coupled first-order differential equations for $\vec{s}(t)$, $\vec{r}(t)$, and $\chi(t)$. For vector $\vec{r}(t)$ it results that

$$\forall t > t_{\mathrm{int}},\qquad \vec{r}(t) = \vec{r}(t_{\mathrm{int}}), \tag{41}$$

where t_{int} denotes the initial time of the dynamics. Furthermore, for $\vec{s}(t)$ and $\chi_{ij}(t)$ the following differential equations hold:

$$\begin{aligned} \frac{d}{dt}\vec{s}(t) &= Q(t)s(\vec{t}) + \vec{q}(t),\\ \frac{d}{dt}\vec{\chi}_j(t) &= Q(t)\vec{\chi}_j(t),\qquad j = 1,2,3, \end{aligned} \tag{42}$$

where

$$\vec{\chi}_j(t) = \begin{pmatrix}\chi_{1j}(t)\\ \chi_{2j}(t)\\ \chi_{3j}(t)\end{pmatrix},\qquad \vec{q}(t) = \begin{pmatrix}0\\ 0\\ -2a_-(t)\end{pmatrix}, \tag{43}$$

and

$$Q(t) = \begin{pmatrix}-b(t) & -2k(t) & -2\dot{\phi}(t)\\ 2k(t) & -b(t) & 0\\ 2\dot{\phi}(t) & 0 & -2a_+(t)\end{pmatrix}. \tag{44}$$

The differential equations in Eq. (42) have time-dependent coefficients. Therefore solving Eq. (42) analytically and expressing the solution in a compact form is not possible. Hence, we analytically solve Eq. (42) in a particular range of parameters where $v\to 0$ and Δ is finite for $t \ll \frac{\Delta}{v}$. For the rest of the range of parameters, we solve the master equation in Eq. (36) numerically.

For analytical solution, first, we discuss the limit of elements of matrix $Q(t)$ in Eq. (44) in the limit of $v\to 0$ and $t\ll\frac{\Delta}{v}$ with finite Δ. From Eq. (3) we conclude that, for $t\ll\frac{\Delta}{v}$, $\Omega(t)$ approaches Δ. On the other hand, from Eq. (21) it is easy to see that

$$\dot{\phi}(t) = -\frac{\Delta v}{2\Omega^2(t)}. \tag{45}$$

Therefore, for $v\to 0$ and $t\ll\frac{\Delta}{v}$, $\dot{\phi}(t)$ approaches zero. Also it is clear from Eq. (21) that in this regime $\phi(t)\approx\frac{\pi}{4}$. Therefore, from Eqs. (31)–(34) we conclude that $k(t)$, $f(t)$, $g(t)$, and $l(t)$ in this regime are respectively given by

$$\begin{aligned} k &= \Delta + \frac{\lambda^2}{2}\cos^2(\theta)[\mathcal{S}(2\Delta) - \mathcal{S}(-2\Delta)],\\ f &= \frac{\lambda^2}{4}\cos^2(\theta)\gamma(2\Delta),\\ g &= \frac{\lambda^2}{4}\cos^2(\theta)\gamma(-2\Delta),\\ l &= \frac{\lambda^2}{4}\sin^2(\theta)\gamma(0). \end{aligned} \tag{46}$$

Therefore, in this regime for coefficients $a_\pm(t)$ and $b(t)$ in Eq. (40) we have

$$\begin{aligned} a_\pm &= \frac{\lambda^2}{4}\cos^2(\theta)[\gamma(2\Delta)\pm\gamma(-2\Delta)],\\ b &= a_+ + 2l. \end{aligned} \tag{47}$$

Hence, in this regime $Q(t)$ in Eq. (44) is time independent. Furthermore, as $\dot{\phi}(t)\approx 0$, matrix $Q(t)$ is block-diagonal in this regime. Therefore, it is possible to solve the coupled differential equations in Eq. (42) analytically and the solution of differential equations in Eq. (42) in the regime of $v\to 0$ and finite Δ for $t\ll\frac{\Delta}{v}$ is summarized as follows:

$$\vec{s}(t) = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-b(t-t_{\mathrm{int}})}\big(A_0^+e^{2ik(t-t_{\mathrm{int}})} + A_0^-e^{-2ik(t-t_{\mathrm{int}})}\big)\\ ie^{-b(t-t_{\mathrm{int}})}\big(A_0^+e^{2ik(t-t_{\mathrm{int}})} - A_0^-e^{-2ik(t-t_{\mathrm{int}})}\big)\\ \sqrt{2}\big(B_0e^{-2a_+(t-t_{\mathrm{int}})} + \frac{a_-}{a_+}\big)\end{pmatrix}, \tag{48}$$

and

$$\vec{\chi}_j(t) = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-b(t-t_{\mathrm{int}})}\big(A_j^+e^{2ik(t-t_{\mathrm{int}})} + A_j^-e^{-2ik(t-t_{\mathrm{int}})}\big)\\ ie^{-b(t-t_{\mathrm{int}})}\big(A_j^+e^{2ik(t-t_{\mathrm{int}})} - A_j^-e^{-2ik(t-t_{\mathrm{int}})}\big)\\ \sqrt{2}B_je^{-2a_+(t-t_{\mathrm{int}})}\end{pmatrix}, \tag{49}$$

where $A_j^\pm$ and B_j, with $j = 0, 1, 2, 3$, are constant coefficients which are determined by the initial state:

$$\begin{aligned} A_0^\pm &= \frac{1}{\sqrt{2}}[s_1(t_{\mathrm{int}})\mp is_2(t_{\mathrm{int}})],\qquad B_0 = \Big(s_3(t_{\mathrm{int}}) - \frac{a_-}{a_+}\Big),\\ A_j^\pm &= \frac{1}{\sqrt{2}}[\chi_{1j}(t_{\mathrm{int}})\mp i\chi_{2j}(t_{\mathrm{int}})],\quad B_j = \chi_{j3}(t_{\mathrm{int}}). \end{aligned} \tag{50}$$

The analytical solutions for a regime of $v\to 0$ and $t\ll\frac{\Delta}{v}$ as given in Eqs. (48) and (49) are used in the next section for analytical analysis of entanglement degradation under local dissipative Landau-Zener evolution.

IV. ENTANGLEMENT DECAY

In this section, we analyze entanglement degradation in a bipartite qubit system when one share of an entangled pair undergoes dissipative Landau-Zener evolution. After a general review of the measure of the entanglement we use, we analyze slow- and fast-driving regimes separately. Starting with a maximally entangled state, in the slow-driving regime we provide an analytical expression for entanglement in time. We discuss the dependence of entanglement survival time on bath temperature and also spin-coupling direction to the environment when the bath temperature is zero. In the fast-driving regime when the bath temperature is zero we discuss entanglement behavior in time for different ranges of parameters. Furthermore, we address the role of adiabaticity in entanglement degradation.

To analyze entanglement between system and reference qubits we use negativity as a measure of entanglement. Like all entanglement measures, negativity is invariant under local unitary operations. Therefore, by taking into account Eq. (35), we have

$$\mathcal{N}(\tilde{\rho}_{SR}(t)) = \mathcal{N}(\rho_{SR}(t)). \tag{51}$$

Hence, to address entanglement between system and reference qubits, we focus on entanglement in state $\tilde{\rho}_{SR}(t)$. By definition, negativity of the bipartite qubit density matrix $\tilde{\rho}_{SR}(t)$ is given by [52]

$$N(\tilde{\rho}_{SR}(t)) := \frac{1}{2}\sum_{i=1}^{4}[|\mu_i(t)| - \mu_i(t)], \tag{52}$$

where $\mu_i(t)$'s are eigenvalues of $\tilde{\rho}_{SR}^{\top_R}(t)$, which is the partial transposed of $\tilde{\rho}_{SR}(t)$ with respect to the reference qubit R. For maximally entangled states, like the Bell state, negativity reaches its maximum value $\frac{1}{2}$, and for separable states, it is zero.

We consider a general pure bipartite state in its Schmidt decomposition [53] for the initial state:

$$|\phi\rangle = \cos\eta|00\rangle + \sin\eta|11\rangle. \tag{53}$$

By varying $\eta \in [0, \frac{\pi}{2}]$, the initial state $|\phi\rangle$ in Eq. (53) interpolates between a separable and a maximally entangled state. For this initial state, we have

$$\begin{aligned} r_3(t_{\text{int}}) &= s_3(t_{\text{int}}) = \cos(2\eta),\\ \chi_{11}(t_{\text{int}}) &= -\chi_{22}(t_{\text{int}}) = \sin(2\eta),\\ \chi_{33}(t_{\text{int}}) &= 1, \end{aligned} \tag{54}$$

and the rest of $r_i(t_{\text{int}})$, $s_i(t_{\text{int}})$, and $\chi_{ij}(t_{\text{int}})$ are zero. Hence, according to Eq. (41) for arbitrary time $t > t_{\text{int}}$, we have

$$\vec{r}(t) = \begin{pmatrix} 0\\ 0\\ \cos(2\eta)\end{pmatrix}. \tag{55}$$

For the rest of the coefficients, we work in two different regimes in the following two subsections. For the regime of $v \to 0$ and $vt \ll \Delta$ we use the analytical results in Sec. III, and for other ranges of parameters we solve the master equation using the QUTIP library [54,55].

A. Slow-driving regime

In this subsection, we work in the regime of $v \to 0$ and $vt \ll \Delta$. We derive the explicit expression for entanglement when the initial state is maximally entangled. We discuss entanglement survival time, and for zero temperature bath, we discuss the behavior of entanglement in terms of the spin-coupling direction to the environment.

For the initial state in Eq. (53), by using Eq. (48) for the slow-driving regime, we have

$$\vec{s}(t) = \begin{pmatrix} 0\\ 0\\ \left[\cos(2\eta) - \frac{a_-}{a_+}\right]e^{-2a_+(t-t_{\text{int}})} + \frac{a_-}{a_+}\end{pmatrix}. \tag{56}$$

Also, regarding Eq. (49) for the initial state in Eq. (53), the nonvanishing elements of matrix $\chi(t)$ have the following simple form:

$$\begin{aligned} \chi_{11}(t) &= -\chi_{22}(t) = e^{-b(t-t_{\text{int}})}\cos 2k(t-t_{\text{int}})\sin 2\eta,\\ \chi_{12}(t) &= \chi_{21}(t) = -e^{-b(t-t_{\text{int}})}\sin 2k(t-t_{\text{int}})\sin 2\eta,\\ \chi_{33}(t) &= e^{-2a_+(t-t_{\text{int}})}\sin 2\eta. \end{aligned} \tag{57}$$

From $\vec{r}(t)$ and $\vec{s}(t)$ in Eqs. (55) and (56) and $\chi(t)$ in Eq. (57), we derive the density matrix $\tilde{\rho}_{SR}(t)$ which has the X form. Hence, its partial transpose $\tilde{\rho}_{SR}^{\top_R}(t)$ has the X form and is given by

$$\tilde{\rho}_{SR}^{\top_R}(t) = \frac{1}{4}\Bigg[\text{id}\otimes\text{id} + s_3(t)\sigma_z\otimes\text{id} + r_3(t)\,\text{id}\otimes\sigma_z + \sum_{i,j=1}^{3}(-1)^{j+1}\chi_{ij}(t)\sigma_i\otimes\sigma_j\Bigg]. \tag{58}$$

This block-diagonal structure enables us to derive the eigenvalues of $\tilde{\rho}_{SR}^{\top_B}(t)$ analytically:

$$\begin{aligned} \mu_{1,2}(t) &= \tfrac{1}{4}\Big\{1 - \chi_{33}(t) \pm \sqrt{4\left[\chi_{11}^2(t) + \chi_{12}^2(t)\right] + [s_3(t) - r_3(t)]^2}\Big\},\\ \mu_{3,4}(t) &= \tfrac{1}{4}[1 + \chi_{33}(t) \pm |s_3(t) + r_3(t)|], \end{aligned} \tag{59}$$

with $r_3(t)$, $s_3(t)$, and $\chi_{ij}(t)$ given in Eqs. (55)–(57). To see when the dynamics become entanglement breaking [56], it is sufficient to consider a maximally entangled state as the initial state and investigate at what time the entanglement between two qubits vanishes [56]. Hence, in Eq. (53) and subsequent equations, we set $\eta = \frac{\pi}{4}$ for the initial maximally entangled state. By examining eigenvalues of $\tilde{\rho}_{SR}^{\top_B}(t)$ in Eq. (59) we see that, for $\eta = \frac{\pi}{4}$, the only eigenvalue that might get negative values is $\mu_2(t)$. Hence, for a maximally entangled initial state and in the limit of $v \to 0$ and $t \ll \frac{\Delta}{v}$, negativity is given by

$$\mathcal{N}(\rho_{SR}(t)) = |\mu_2(t)|. \tag{60}$$

When one share of an entangled pair experiences noise, the maximum entanglement survival time τ_{ent} is defined as the smallest evolution time after which the dynamics become

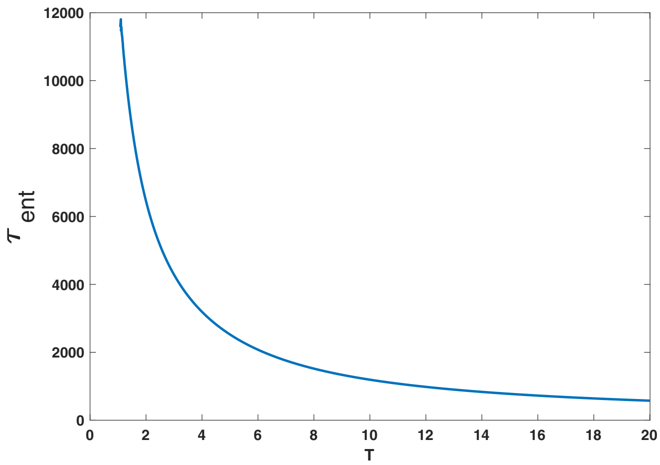

FIG. 2. Entanglement survival time [Eq. (61)] with dimension $\frac{1}{E}$ (E stands for energy) versus temperature T, with dimension E, in the slow-driving regime for $\theta = 0$, $\lambda = 0.1$, $\Delta = 10$, and $\omega_c = \frac{\Delta}{3}$.

entanglement breaking [33]. Here in the regime of $v \rightarrow 0$ and $t \ll \frac{\Delta}{v}$, we can find τ_{ent} by equating Eq. (60) to zero for $\theta = 0$:

$$\tau_{\text{ent}} = -\frac{\ln(\xi)}{2b}, \tag{61}$$

where b is defined in Eq. (47) and

$$\xi = \frac{3 - \ell^2 - 2\sqrt{2 - \ell^2}}{1 - \ell^2}, \quad \ell = \frac{1}{2\bar{n}(2\Delta) + 1}. \tag{62}$$

Figure 2 shows τ_{ent} in Eq. (61) for $\theta = 0$ versus temperature T in the slow-driving regime when $\Delta = 10$, $\omega_c = \Delta/3$, and $\lambda = 0.1$. This choice of parameters assures that the inequalities in Eq. (15) are satisfied. As expected, temperature has a destructive effect on τ_{ent}. What is reflected in Eq. (61) and Fig. 2 is for $\theta = 0$. For further investigation on the role of noise-coupling direction in entanglement behavior, we focus on the zero temperature environment. By using Eq. (60) in the regime of $v \rightarrow 0$ and $t \ll \frac{\Delta}{v}$ at $T = 0$, we have

$$\mathcal{N}(\rho_{SR}(t)) = \tfrac{1}{2} e^{-(t - t_{\text{int}})\lambda^2 \pi J(2\Delta)\cos^2(\theta)}. \tag{63}$$

It exhibits the exponential decay of entanglement in time and also provides an explicit relation between entanglement and spin-coupling direction to the bath at $T = 0$. In Fig. 3 we see entanglement between two qubits versus θ at $T = 0$, for $t_{\text{int}} = -100$, $t = 100$, $\Delta = 10$, $\omega_c = \Delta/3$, and $\lambda = 0.1$ as given in Eq. (63). As entanglement is nonincreasing in time under local noise, from Fig. 3 we conclude that, at any instant of the evolution, for larger values of θ, the noisy environment performs less destructively and at $\theta = \frac{\pi}{2}$ the initial entanglement is preserved. This is expected as in the regime of $vt \ll \Delta$ and $v \rightarrow 0$, the system Hamiltonian in the rotated basis $\tilde{H}_S(t)$ is proportional to σ_z [see Eq. (23)]. On the other hand, by using Eqs. (30) and (46) in this regime we have

$$\tilde{\mathcal{L}}_t[\bullet] = -ik[\sigma_z, \bullet] + \frac{\lambda^2}{4}\cos\theta^2\left(\sigma_- \bullet \sigma_+ - \frac{1}{2}\{\sigma_+\sigma_-, \bullet\}\right). \tag{64}$$

This is a representation of an amplitude damping channel when coupling to the bath is proportional to $|\lambda \cos(\theta)|$. The larger the value of θ is, the weaker the role of decoherence is, and the coherent terms play the dominant role in the generator of the dynamics. At $\theta = \frac{\pi}{2}$, there is no dissipation term in the generator and the coherent part leads to a unitary evolution which does not change entanglement.

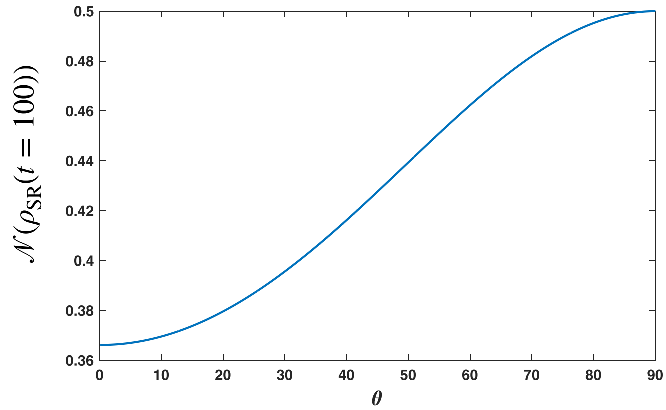

FIG. 3. Entanglement (in ebits) versus parameter θ (in degrees) for $T = 0$, $\Delta = 10$, $t_0 = -100$, and $t = 100$ as given in Eq. (63).

B. Fast-driving regime

In this subsection, we analyze the behavior of entanglement in the fast-driving regime. We address its decay in time for different values of Δ and also discuss how it behaves in terms of the ratio $\frac{\Delta^2}{v}$, which characterizes adiabatic and nonadiabatic regimes in the absence of interaction with any environment.

To investigate the role of parameters affecting entanglement in the fast-driving regime, one should solve the differential equations with time-dependent coefficients in Eq. (42).

Unlike the limit of $v \rightarrow 0$ and $vt \ll \Delta$, analytic results cannot be obtained in the fast-driving regime. Hence, we use the QUTIP library [54,55] to solve the master equation and analyze entanglement behavior numerically. The outcome of our code is in agreement with our analytical result in the range of $v \rightarrow 0$ and $vt \ll \Delta$. In Fig. 4, the behavior of negativity versus time is shown for the initial state $|\phi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$,

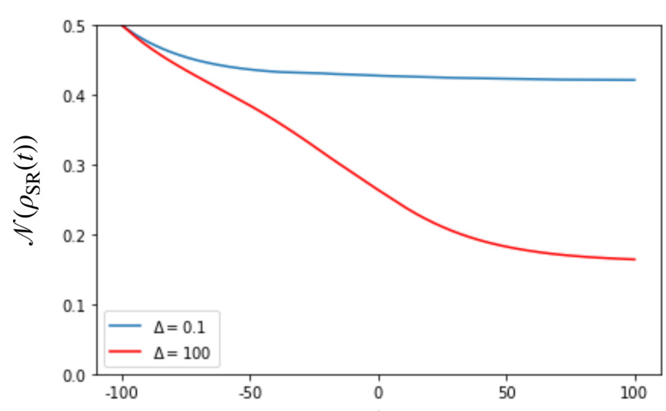

FIG. 4. Negativity (in ebits) versus time (in $\frac{1}{E}$) for the initial state $|\phi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, $\lambda = 0.1$, $T = 0$, $\omega_c = \frac{\Delta}{3}$, $\theta = 0$, and $v = 1$. From top to bottom, $\Delta = 0.1$ and 100.

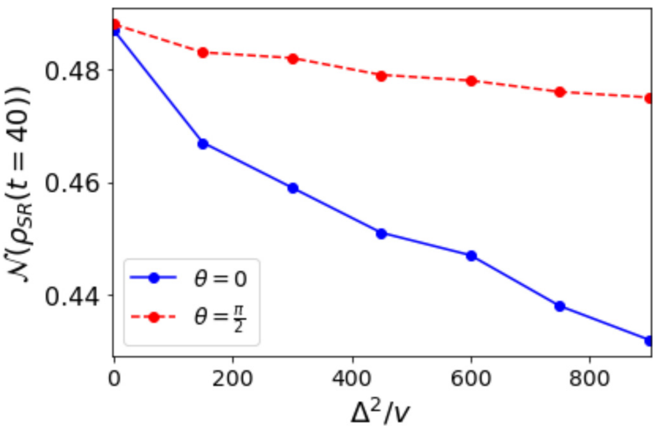

FIG. 5. Negativity (in ebits) versus $\frac{\Delta^2}{v}$ (dimensionless parameter) for $\theta = 0$ (blue curve) and $\theta = \frac{\pi}{2}$ (dashed red curve) at $t = 40$ when the initial state is maximally entangled, $t_0 = -40$, $T = 0$, $\omega_c = \frac{\Delta}{3}$, $\lambda = 0.1$, and $v = 1$.

$v = 1$, $\theta = 0$, $\omega_c = \frac{\Delta}{3}$, and $T = 0$. From top to bottom, $\Delta = 0.1$ and 100. As it is seen in this figure, for smaller values of Δ, entanglement degradation is slower and after a shorter time it achieves its steady value. In other words, when $\frac{\Delta^2}{v} \ll 1$, which indicates a nonadiabatic regime, entanglement remains less intact by the environment compared to the adiabatic regime. The same behavior is seen for $\theta = \frac{\pi}{2}$. To better illustrate the effect of adiabaticity of the dynamics on entanglement degradation, in Fig. 5 we present the behavior of entanglement versus the ratio $\frac{\Delta^2}{v}$ for $\theta = 0$ (blue solid curve) and $\theta = \frac{\pi}{2}$ (red dashed curve) at $t = 40$. Like previous cases, the initial state is maximally entangled. As it is seen in this figure, the larger the value of $\frac{\Delta^2}{v}$ is, the stronger the entanglement decay is. Therefore, the nonadiabatic regime is more favorable for preserving entanglement.

V. CONCLUSION

We have studied entanglement degradation when one share of an entangled pair undergoes a dissipative Landau-Zener evolution. Regarding the role of the Landau-Zener model in adiabatic quantum computation [42,43] and its success in describing quantum annealing [44,46], the dissipative Landau-Zener model serves as a realistic model for modeling local noise and analyzing entanglement degradation. This study has many applications in different areas such as communication and quantum network developments [15–17].

The important problem of entanglement degradation in the presence of local dissipative Landau-Zener noise is a challenging problem because derivation and solving of the master equation require considering many subtleties [45] After careful derivation of the master equation using the approach in Ref. [45], to overcome the complications for solving coupled differential equations with time-dependent coefficients, first we concentrate on the slow-driving regime with $v \to 0$ and $vt \ll \Delta$. In this regime, when noise is longitudinal we drive an analytical expression for entanglement survival time which confirms the destructive role of bath temperature on entanglement. Also, we present an analytic expression for entanglement when the bath temperature is zero. Our analytic results in the slow-driving regime show the important role of spin-coupling direction in the environment in entanglement preservation. In Ref. [46], the role of spin-coupling direction in the ground-state probability is discussed and the beneficial effect of the bath temperature on the ground-state probability is seen only for transversal noise ($\theta = \frac{\pi}{2}$). Here, we have another aspect of the importance of the spin-coupling direction. In the slow-driving regime with a zero temperature bath, by increasing θ the destructive effect of the environment on entanglement decreases, and for transversal coupling, entanglement is preserved. The role of spin-coupling direction is also evident in the fast-driving regime where we solve the master equation numerically. Figure 5 shows that for all values of $\frac{\Delta^2}{v}$ entanglement between two qubits is larger for transversal noise compared to longitudinal noise.

Another important question is about the role of adiabaticity of the dynamics in entanglement degradation. Despite the primary naive expectation, our results indicate that, in the nonadiabatic regime, entanglement degradation is less compared to that in the adiabatic evolution. In other words, when the minimum gap has a fixed value Δ, for larger values of v the entanglement degradation is smaller.

While the case of a two-level system has its own importance and impact, a promising direction to follow would be to extend this study to systems with higher dimensions. In systems with higher dimensions, in addition to the adiabatic property of the dynamics and, the role of spin-coupling direction, one can study different classes of time-dependent Landau-Zener-type Hamiltonians [57]. This would give us more knowledge about the factors affecting entanglement degradation. Another direction for future exploration is the study of entanglement degradation when both qubits experience local independent identical dissipative Landau-Zener noise. A concrete example of such an event is when each qubit of an entangled pair is sent to distant network nodes. Although the initial qualitative impression is that local noise in both qubits is more destructive, precise analysis is essential to investigate the role of spin coupling and adiabaticity of the dynamics, which involves solving a more complicated set of differential equations.

ACKNOWLEDGMENTS

M.B. and L.M. acknowledge financial support from the Iran National Science Foundation (INSF) under Project No. 4022322. L.M. acknowledges support from the ICTP through the Associates Programme (2019-2024).

APPENDIX: KOSSAKOWSKI COEFFICIENTS

In this Appendix we explain how to compute the coefficients $\gamma(\omega_m(t))$ in Eq. (11) and $\mathcal{S}(\omega_m(t))$ in Eq. (14). These coefficients are given in terms of $\Gamma(\omega_m(t))$:

$$\Gamma(\omega_m(t)) = \int_0^\infty d\tau \int_{-a}^{a} d\nu e^{i(\omega_m(t)-\nu)\tau} \operatorname{Tr}\left(\mathbf{B}(\nu)\mathbf{B}\rho_E^{\text{th}}\right), \quad \text{(A1)}$$

where a is determined such that all transition frequencies belong to the interval $(-a, a)$ and

$$\mathbf{B}(\nu) := \int_0^{\omega_{\max}} d\mu |\mu\rangle\langle\mu|\mathbf{B}|\mu+\nu\rangle\langle\mu+\nu|, \quad \text{(A2)}$$

with $|\mu\rangle$ representing the eigenstate of $b^\dagger(\mu)b(\mu)$ with eigenvalue μ. The coefficient $\gamma(\omega_m(t))$ is the real part of $\Gamma(\omega_m(t))$. With straightforward calculations [45], we have

$$\gamma(\omega_m(t)) = 2\pi \operatorname{Tr}\left[\mathbf{B}(\omega_m(t))\mathbf{B}\rho_E^{\text{th}}\right]. \quad \text{(A3)}$$

By using the definition in Eq. (A2) and the definition of the thermal state in Eq. (A2), the more explicit form of $\gamma(\omega_m(t))$ in Eq. (11) is derived. The coefficients $\mathcal{S}(\omega_m(t))$ in Eq. (14) are given by the imaginary part of $\Gamma(\omega_m(t))$ in Eq. (A1) [45].

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