PHYSICAL REVIEW A **110**, 012451 (2024)

Multiparty Spohn's theorem for a combination of local Markovian and non-Markovian quantum dynamics

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(Received 13 September 2022; revised 25 August 2023; accepted 26 June 2024; published 22 July 2024)

We obtain a Gorini-Kossakowski-Sudarshan-Lindblad-like master equation for two or more quantum systems connected locally to a combination of Markovian and non-Markovian heat baths. The master equation was originally formulated for multiparty systems with either exclusively Markovian or non-Markovian environments. We extend it to encompass the case of multiple quantum systems connected to a mixture of Markovian and non-Markovian heat baths. The coexistence of both non-Markovian and Markovian environments is a plausible scenario, particularly when studying hybrid physical systems such as atom-photon arrangements. We analyze the thermodynamic quantities for such a set of local environments, and derive a modified form of the Spohn's theorem for the setup. The modification of the theorem naturally leads to a witness as well as an easily computable quantifier of non-Markovianity. Expectedly, we find that for multiparty situations, where a combination of Markovian and non-Markovian heat baths are active, the response in thermodynamic system characteristics due to non-Markovian baths is prominent at times close to the initial time of evolution, whereas the long-time behavior is predominantly controlled by the Markovian ones.

DOI: 10.1103/PhysRevA.110.012451

I. INTRODUCTION

Quantum thermodynamics is an emerging field of research and the interconnections of it with quantum information theory have been studied from myriad perspectives. The study of quantum thermal devices [1–12] and that of their dynamics governed by open quantum systems [13–16] has significantly contributed to the understanding of thermodynamics in the quantum regime [17–29]. A significant body of work that analyze quantum devices deals with Markovian evolution, although non-Markovian dynamics has also been considered. Markovian environments are rare in nature and they exhibit rather specific behaviors [13,30]. The bosonic baths with infinite numbers of harmonic oscillators, within some restrictions, usually behave as Markovian environments, while certain thermal baths, such as spin baths [31–37], do not fit the Markovian framework easily and are categorized as non-Markovian reservoirs. Some non-Markovian baths may have Markovian limits, but for systems such as the spin star model, this limit can be elusive [32]. Detecting and characterizing non-Markovianity has been achieved through various measures [38–46], which are not all equivalent.

Heat current and entropy production rate (EPR) are two fundamental quantities that give an idea about the thermal properties of a system. The second law of thermodynamics leads to a balance equation, relating EPR (σ), the von Neumann entropy (S), and heat current (J) for a single system immersed in a heat bath, given by $\sigma = \frac{dS}{dt} + J$. Spohn's theorem [47,48] states that for a Markovian evolution, with bath initial states being thermal, EPR of the system is always positive. It is known that for non-Markovian evolutions, the EPR may take negative values [36,49–55]. For a deeper understanding of the entropy production rate, see, e.g., [21,56–98].

In physical systems, the presence of memory effects and strong system-bath correlations may lead to deviations from Markovian dynamics. In some cases, certain components or interactions within a system may exhibit Markovian behavior, while others display non-Markovian behavior. This can arise due to the complexity of the system's architecture or the interplay between different timescales involved in the dynamics. For example, this type of model holds significant relevance as a plausible approach for investigating hybrid systems, such as atom-photon arrangements. In atom-photon systems [99–101], the timescales of atomic and photonic interactions with their respective environments can vary. For instance, atomic transitions may occur on a different timescale compared to the relaxation processes involving emitted or absorbed photons. This can lead to a situation where certain aspects of the system-environment interaction exhibit Markovian characteristics, while others show non-Markovian features. Moreover, in atom-photon setups, the environment may not be homogeneous. Some components of the environment may induce memory effects and correlations, resulting in a non-Markovian influence, while other components may exhibit Markovian behavior. So, understanding and characterizing the interplay between Markovian and non-Markovian elements in atom-photon systems is essential for optimizing their performance in quantum technologies and information processing applications.

Here, we consider a situation where the local parts of the physical system under study are affected by local environments, which can be a few non-Markovian and the remaining not so. We derive a Gorini-Kossakowski-Sudarshan-Lindblad (GKSL)-like master equation of the system for this case and study the thermodynamic quantities such as heat current and EPR, for the composite system. Furthermore, we obtain a modified form of the well-known Spohn's theorem [47,48]

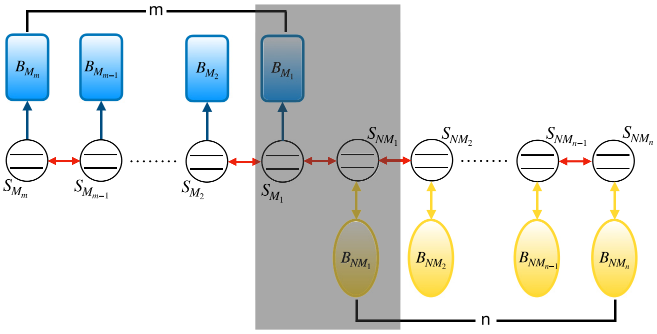

FIG. 1. Combination of local Markovian and non-Markovian dynamics of a multiparty system. Here we present a schematic diagram of $m+n$ subsystems, evolving under their system Hamiltonian as well as local environments, some of which are Markovian and the rest are not so. $B_{\mathrm{M}_1}\cdots B_{\mathrm{M}_m}$ are the baths which can be treated under the Born-Markov approximations, hence are Markovian baths, and $B_{N_1}\cdots B_{N_n}$ are the baths residing in a non-Markovian family. A simplified scenario with only two subsystems interacting with two baths locally, among which one is Markovian and the other is non-Markovian is presented in the gray box.

in connection to the second law of thermodynamics for this multiparty setup and propose an easily computable quantity that can be treated as a quantifier of non-Markovianity. For a four-qubit system, under the combined evolution of local Markovian and non-Markovian dynamics we observe that, the response in thermodynamic system characteristics is dominated by the effect of non-Markovian baths at short times. However, as expected, with the increase of time, non-Markovianity effects reduce, and the dynamics is more and more Markovian-like.

II. MULTIPARTY GKSL-LIKE EQUATION FOR LOCAL MARKOVIAN AND NON-MARKOVIAN BATHS

We consider $m+n$ subsystems locally coupled to m Markovian and n non-Markovian baths, respectively. The situation is depicted in Fig. 1. The composite system consisting of $m+n$ subsystems will evolve under the combined influence of the local Markovian and non-Markovian baths. Before considering the case of arbitrary m and n, we deal with the case of two subsystems (S_{M_1} and S_{NM_1}), locally coupled, respectively, to two heat baths, one of which is a Markovian bath (B_{M_1}) that can be treated under the Born-Markov approximation, while the other is a non-Markovian one (B_{NM_1}), whose frequency spectrum is discrete and goes beyond the Markovian regime. This two-party two-bath setup is illustrated in the gray box of Fig. 1. The Hamiltonian of the composite setup is given by $H=H_s+H_B+H_I$, where H_s describes the Hamiltonian of the composite system consisting of the two subsystems, H_B stands for the combined local Hamiltonian of the two baths and $H_I=\sum_X H_{I_{X_1}}$ for $X\in\{M,NM\}$. Here $H_{I_{\mathrm{M}_1}}$ represents the interaction between S_{M_1} and B_{M_1}, and $H_{I_{\mathrm{NM}_1}}$ presents the interaction between S_{NM_1} and B_{NM_1}. Note that the Hamiltonian H_s, describing the Hamiltonian of the composite system containing two subsystems, is a general Hamiltonian encompassing both the local and interacting part of the two subsystems. Precisely, this H_s can be written as $H_s=H_{\mathrm{loc}}+V$, where H_{loc} is the local Hamiltonian of the two subsystems and V represents the interaction between them. In the Schrödinger picture, let the density matrix of the composite two-party two-bath setup at time t be represented by $\rho(t)$. It is useful to perform the calculation in the interaction picture [13]. The von Neumann equation in that picture will be

$$\frac{d\rho_{\mathrm{int}}(t)}{dt}=-\frac{i}{\hbar}\left[H_{I_{\mathrm{int}}}(t),\,\rho_{\mathrm{int}}(t)\right],\tag{1}$$

where $H_{I_{\mathrm{int}}}(t)=e^{\frac{i(H_s+H_B)t}{\hbar}}H_I e^{-\frac{i(H_s+H_B)t}{\hbar}}$, $\rho_{\mathrm{int}}(t)=e^{\frac{i(H_s+H_B)t}{\hbar}}\rho(t)e^{-\frac{i(H_s+H_B)t}{\hbar}}$, without assuming a commutativity relation of H_I and $\rho(t)$ with H_s and H_B. Here we assume that initially the systems are uncorrelated to the baths, and that the baths themselves are also in a product state, so that at time $t=0$, $\rho(0)=\rho_s(0)\otimes\rho_{B_{\mathrm{M}_1}}(0)\otimes\rho_{B_{\mathrm{NM}_1}}(0)$, where B_{M_1} is initially in its stationary state, i.e., $[H_{B_{\mathrm{M}_1}},\rho_{B_{\mathrm{M}_1}}(0)]=0$, with $H_{B_{\mathrm{NM}_1}}$ being the free Hamiltonian of the non-Markovian bath B_{NM_1}. The derivation of the GKSL-like equation for this two-party two-bath setup is given in Appendix A. As we mentioned earlier, the bath B_{M_1} is Markovian and therefore in the derivation, while talking about B_{M_1}, we have imposed the Born-Markov approximations, which tells that the coupling between the subsystem S_{M_1} and B_{M_1} is weak, so that the state of B_{M_1} regains its initial state after every time step of interaction with S_{M_1}, and that any correlation created between B_{M_1} and S_{M_1} is also destroyed after the same time step. Moreover, B_{M_1} will also be assumed to remain uncorrelated with B_{NM_1} during the evolution. And also, the development of the reduced state of the system with respect to the bath B_{M_1}, at each time, is assumed memoryless. The reduced system dynamics in the Schrödinger picture turns out to be

$$\begin{aligned}\frac{d\tilde{\rho}_s(t)}{dt}=\mathcal{L}[\tilde{\rho}_s(t)]\equiv-\frac{i}{\hbar}[H_s,\tilde{\rho}_s(t)]+\mathcal{D}_{\mathrm{M}_1}[\tilde{\rho}_s(t)]\\ +\mathcal{D}_{\mathrm{NM}_1}[\rho_{sB_2}(t)],\end{aligned}\tag{2}$$

where

$$\begin{aligned}\mathcal{D}_{\mathrm{M}_1}[\tilde{\rho}_s(t)]=\frac{1}{\hbar^2}\sum_{\omega}\sum_{k,l}\gamma_{k,l}(\omega)\Big(A_{\mathrm{M}_{1_l}}(\omega)\tilde{\rho}_s(t)A^{\dagger}_{\mathrm{M}_{1_k}}(\omega)\\ -\frac{1}{2}\big\{A^{\dagger}_{\mathrm{M}_{1_k}}(\omega)A_{\mathrm{M}_{1_l}}(\omega),\,\tilde{\rho}_s(t)\big\}\Big),\end{aligned}$$

$$\mathcal{D}_{\mathrm{NM}_1}\big[\rho_{sB_{\mathrm{NM}_1}}(t)\big]=-\frac{i}{\hbar}\mathrm{tr}_{B_{\mathrm{NM}_1}}\big[H_{I_{\mathrm{NM}_1}}(t),\,\rho_{sB_{\mathrm{NM}_1}}(t)\big].\tag{3}$$

We denote $\tilde{\rho}_s(t)=\mathrm{tr}_{B_{\mathrm{NM}_1}}[e^{\frac{iH_{B_{\mathrm{NM}_1}}t}{\hbar}}\rho_{sB_{\mathrm{NM}_1}}(t)e^{\frac{-iH_{B_{\mathrm{NM}_1}}t}{\hbar}}]=\mathrm{tr}_{B_{\mathrm{NM}_1}}[\rho_{sB_{\mathrm{NM}_1}}(t)]$. Here ω is the transition energy, $A_{\mathrm{M}_{1_k}}(\omega)$ are the Lindblad operators defined after Eq. (A1), and $\gamma_{k,l}(\omega)$ are the transition rates defined after Eq. (A8) in Appendix A. Here we have neglected the possible effects of the Lamb-shift Hamiltonian in the dynamics [55,102]. Effectively, $\mathcal{D}_{\mathrm{M}_1}[\tilde{\rho}_s(t)]$ contains the dissipative part of the GKSL-like equation due to the influence of the Markovian bath and $\mathcal{D}_{\mathrm{NM}_1}[\rho_{sB_{\mathrm{NM}_1}}(t)]$ contains the non-Markovian contribution. Interestingly, although the bath B_{M_1} is treated as Markovian, the dissipator $\mathcal{D}_{\mathrm{M}_1}[\tilde{\rho}_s(t)]$ of Eq. (3) reveals a crucial distinction between our approach and the traditional Markovian cases [13,15]. In the Markovian scenario, the dissipative term would involve the system's state uncorrelated with any environment. However, in our formulation, this term incorporates $\tilde{\rho}_s(t)$, which represents the system's state correlated with the non-Markovian bath. The correlation between the system and non-Markovian baths implies that the composite state of the system and the non-Markovian bath at time t cannot be expressed as a product state, unlike in the case of Markovian baths. Therefore, a crucial distinction arises due to the presence of the non-Markovian effect of B_{NM_1}, which persists within the contribution of B_{M_1}. Therefore, both $\mathcal{D}_{\mathrm{M}_1}[\tilde{\rho}_s(t)]$ and $\mathcal{D}_{\mathrm{NM}_1}[\rho_{sB_{\mathrm{NM}_1}}(t)]$ serve as non-Markovian dissipators for this combined evolution.

The two-party dynamical equation given in Eq. (2) can be extended to the situation where $m+n$ subsystems are connected to m Markovian and n non-Markovian baths locally (see Fig. 1). For that general case, the dynamical equation of the system takes the form,

$$\begin{aligned}\mathcal{L}[\tilde{\rho}_s(t)] = {} & -\frac{i}{\hbar}[H_s, \tilde{\rho}_s(t)] + \sum_{j=1}^{m} \mathcal{D}_{\mathrm{M}_j}[\tilde{\rho}_s(t)] \\ & + \sum_{j=1}^{n} \mathcal{D}_{\mathrm{NM}_j}\big[\rho_{sB_{\mathrm{NM}_{1\ldots n}}}(t)\big]. \end{aligned} \tag{4}$$

Here $\tilde{\rho}_s(t) = \mathrm{tr}_{B_{\mathrm{NM}_{1\ldots n}}}[\rho_{sB_{\mathrm{NM},1\ldots n}}(t)]$. The $\mathcal{D}_{\mathrm{M}_j}[\tilde{\rho}_s(t)]$ presents the contribution of the jth Markovian bath. Similarly, $\mathcal{D}_{\mathrm{NM}_j}[\rho_{sB_{\mathrm{NM}_{1\ldots n}}}(t)]$ represents the contribution of the jth non-Markovian bath. With the increase in the number of non-Markovian baths, the system in general will tend to become more and more correlated with the non-Markovian baths, but the effect of the Markovian baths will also in general become significantly altered in comparison to the situation where non-Markovian environments are absent. This in turn may affect the general properties and interrelations between thermodynamic quantities that are typically considered in either Markovian or non-Markovian situations before. Below, we study the response of the heat current and EPR, and their interrelation via the Spohn's theorem, for the combination of local Markovian and non-Markovian dynamics.

III. THERMODYNAMIC QUANTITIES AND THEIR PROPERTIES FOR COMBINATION OF LOCAL MARKOVIAN AND NON-MARKOVIAN ENVIRONS

Heat current and EPR are two fundamental thermodynamic properties of a system, which provide information about heat flow from the system to its environment or vice versa, and further aspects of equilibrium and nonequilibrium physics of the system. It is known that for a non-Markovian evolution, EPR can take a negative value [36,49–55] and as a consequence, it can be treated as a witness of non-Markovianity. The definitions of the thermodynamic quantities can strongly depend on the character of the environments under which the system is being evolved. In general, entropy flux or heat current can be defined as the amount of entropy exchanged per unit time between the open system and its environment [13]. Entropy flux for the composite two-party system depicted in the grey box of Fig. 1 can be defined as $J_{\{\mathrm{M}_1,\mathrm{NM}_1\}} = \frac{d}{dt}|_{\mathrm{diss}}E$, which indicates the change in internal energy resulting from the dissipative effects. Here $E = \mathrm{tr}[H_s\tilde{\rho}_s(t)]$. So,

$$\begin{aligned} J_{\{\mathrm{M}_1,\mathrm{NM}_1\}} &= \mathrm{tr}\big[H_s\big(\mathcal{D}_{\mathrm{M}_1}\big[\tilde{\rho}_s(t)\big] + \mathcal{D}_{\mathrm{NM}_1}\big[\rho_{sB_{\mathrm{NM}_1}}(t)\big]\big)\big] \\ &= \mathrm{tr}\big[H_s\mathcal{L}[\tilde{\rho}_s(t)]\big]. \end{aligned} \tag{5}$$

We define the local heat currents of each subsystem as $J_{\mathrm{M}_1} = \mathrm{tr}[H_s\mathcal{D}_{\mathrm{M}_1}[\tilde{\rho}_s(t)]]$ and $J_{\mathrm{NM}_1} = \mathrm{tr}[H_s\mathcal{D}_{\mathrm{NM}_1}[\rho_{sB_{\mathrm{NM}_1}}(t)]]$. The definition of J_{M_1} is quite similar to that in the Markovian approach, but the effect of non-Markovianity resides in the state $\tilde{\rho}_s(t)$, as the system's state is correlated with the non-Markovian bath. The formulation of J_{NM_1} is inherently non-Markovian.

EPR is a thermodynamic quantity of a system, which is defined as a source term in the balance equation involving the rate of change of entropy with time and heat current [47,48,103]. For a two-party two-bath composite setup, this balance equation can be considered as

$$\sigma_{\{\mathrm{M}_1,\mathrm{NM}_1\}} = \frac{dS(\tilde{\rho}_s(t))}{dt} - \frac{1}{T_{\mathrm{M}_1}}J_{\mathrm{M}_1} - \frac{1}{T_{\mathrm{NM}_1}}J_{\mathrm{NM}_1}, \tag{6}$$

where $S(\cdot)$ is the von Neumann entropy of its argument and defined as $S(\rho) = -k_B\mathrm{tr}[\rho\ln(\rho)] = -k_B\sum_i\lambda_i\ln(\lambda_i)$, with λ_i's being the eigenvalues of the density matrix ρ. T_{M_1} and T_{NM_1} are the temperatures of the Markovian and non-Markovian baths, respectively. k_B is the Boltzmann constant. In this formulation, the balance equation is the definition of the EPR, denoted by $\sigma_{\{\mathrm{M}_1,\mathrm{NM}_1\}}$.

We now move over to the case of $m+n$ subsystems (see Fig. 1). For $m+n$ parties, the global heat current and the global EPR take the following forms:

$$\begin{aligned} J_{\{\mathrm{M}_{1\cdots m},\mathrm{NM}_{1\cdots n}\}} &= \sum_{j=1}^{m} J_{\mathrm{M}_j} + \sum_{j=1}^{n} J_{\mathrm{NM}_j}, \\ \sigma_{\{\mathrm{M}_{1\cdots m},\mathrm{NM}_{1\cdots n}\}} &= \frac{dS(\tilde{\rho}_s(t))}{dt} - \sum_{j=1}^{m}\frac{1}{T_{\mathrm{M}_j}}J_{\mathrm{M}_j} - \sum_{j=1}^{n}\frac{1}{T_{\mathrm{NM}_j}}J_{\mathrm{NM}_j}. \end{aligned} \tag{7}$$

Here the local heat currents, $J_{\mathrm{M}_j} = \mathrm{tr}[H_s\mathcal{D}_{\mathrm{M}_j}[\tilde{\rho}_s(t)]]$, presents the heat current flowing to or from the jth Markovian bath for $j = 1,\ldots,m$, and the second term, $J_{\mathrm{NM}_j} = \mathrm{tr}[H_s\mathcal{D}_{\mathrm{NM}_j}[\rho_{sB_{\mathrm{NM}_{1\ldots n}}}(t)]]$, signifies the local heat currents flowing towards or outwards from the jth non-Markovian bath for $j = 1,\ldots,n$.

IV. MODIFICATION TO SPOHN'S THEOREM

For the evolution of a system under a Markovian reservoir, which is initially kept in its canonical equilibrium state, Spohn's theorem assures the positivity of EPR, as under this circumstance, the canonical equilibrium state of the system is a stationary state with respect to the Markovian dynamics. For non-Markovian evolutions, the positivity of the same quantity may hold, but is not guaranteed even if the bath is initially in its thermal state. So, there may exist a modified form of Spohn's theorem for non-Markovian evolutions, which can describe the thermodynamics of non-Markovian scenarios. Our aim is to obtain a thermodynamic condition similar to that in the Spohn's theorem, which can describe a multiparty situation with a combination of local Markovian and non-Markovian environs.

For the ease of notation and calculations, we take the simple two-party two-bath situation depicted in the gray box of Fig. 1. The dynamical equation of the system for this setup is given in Eq. (2). We define the partial superoperators [104],

$$\mathcal{L}_{\text{M}_1}[\tilde{\rho}_s(t)] = -\frac{ip}{\hbar}[H_s, \tilde{\rho}_s(t)] + \mathcal{D}_{\text{M}_1}[\tilde{\rho}_s(t)],$$
$$\mathcal{L}_{\text{NM}_1}\big[\rho_{sB_{\text{NM}_1}}(t)\big] = -\frac{i(1-p)}{\hbar}[H_s, \tilde{\rho}_s(t)] + \mathcal{D}_{\text{NM}_1}\big[\rho_{sB_{\text{NM}_1}}(t)\big], \tag{8}$$

where $\mathcal{L}[\tilde{\rho}_s(t)] = \mathcal{L}_{\text{M}_1}[\tilde{\rho}_s(t)] + \mathcal{L}_{\text{NM}_1}[\rho_{sB_{\text{NM}_1}}(t)]$ and p is a weight factor. The two parts $\mathcal{L}_{\text{M}_1}[\tilde{\rho}_s(t)]$ and $\mathcal{L}_{\text{NM}_1}[\rho_{sB_{\text{NM}_1}}(t)]$ act as GKSL-like equation operators individually, with $\text{tr}[\mathcal{L}_{\text{M}_1}[\tilde{\rho}_s(t)]] = \text{tr}[\mathcal{L}_{\text{NM}_1}[\rho_{sB_{\text{NM}_1}}(t)]] = 0$. Hence, the local heat currents can be described in terms of the local superoperators as $J_{\text{M}_1} = \text{tr}[H_s\mathcal{L}_{\text{M}_1}[\tilde{\rho}_s(t)]]$, $J_{\text{NM}_1} = \text{tr}[H_s\mathcal{L}_{\text{NM}_1}[\rho_{sB_{\text{NM}_1}}(t)]]$. Now we introduce two local canonical equilibrium states of the composite system at temperatures T_{M_1} and T_{NM_1} [104] having the form, $\tilde{\rho}_{\text{th}_{X_1}} = \frac{e^{-\beta_{X_1}H_s}}{Z_{X_1}}$, for $X \in \{M, NM\}$. Note that H_s is a Hamiltonian of two parties. Z_{X_1}'s stand for the corresponding partition functions and defined as $Z_{X_1} = \text{tr}[e^{-\beta_{X_1}H_s}]$. Here $\beta_{X_1} = \frac{1}{k_BT_{X_1}}$. Thus, using $\tilde{\rho}_{\text{th}_{X_1}}$ and the partial superoperators, we get

$$\sigma_{\{\text{M}_1,\text{NM}_1\}} = -\frac{d}{dt}\Big|_{\text{M}} S\big(\tilde{\rho}_s(t)||\tilde{\rho}_{\text{th}_{\text{M}_1}}\big) - k_B\text{tr}\Big[\mathcal{L}_{\text{NM}_1}\big[\rho_{sB_{\text{NM}_1}}(t)\big]\big(\ln(\tilde{\rho}_s(t)) - \ln(\tilde{\rho}_{\text{th}_{\text{NM}_1}})\big)\Big], \tag{9}$$

where the relative entropy distance, $S(\rho||\sigma) = k_B\text{tr}(\rho\ln\rho - \rho\ln\sigma)$, is used to quantify the distance between the evolved state and the local canonical equilibrium state at temperature T_{M_1}, and $\frac{d}{dt}|_{\text{M}}S(\tilde{\rho}_s(t)||\tilde{\rho}_{\text{th}_{\text{M}_1}}) = k_B\text{tr}\{\mathcal{L}_{\text{M}_1}[\tilde{\rho}_s(t)](\ln(\tilde{\rho}_s(t)) - \ln(\tilde{\rho}_{\text{th}_{\text{M}_1}}))\}$. In the Markovian limit of the setup under consideration, i.e., when both the baths are Markovian, the first term of Eq. (9) will be duplicated for the other bath, and the second term will be nonexistent. Hence, Eq. (9) can be presented as a general expression of EPR for a two-party system evolving under a combination of local Markovian and non-Markovian dynamics.

We now try to establish the Spohn's theorem with the altered definition of EPR. From Eq. (9), we can write

$$\sigma_{\{\text{M}_1,\text{NM}_1\}} + k_B\text{tr}\big\{\mathcal{L}_{\text{NM}_1}\big[\rho_{sB_{\text{NM}_1}}(t)\big]\big(\ln(\tilde{\rho}_s(t)) - \ln(\tilde{\rho}_{\text{th}_{N\tilde{M}_1}})\big)\big\} = -k_B\text{tr}\big\{\mathcal{L}_{\text{M}_1}\big[\tilde{\rho}_s(t)\big]\big(\ln(\tilde{\rho}_s(t)) - \ln(\tilde{\rho}_{\text{th}_{\text{M}_1}})\big)\big\}. \tag{10}$$

If the initial state of the Markovian bath is the canonical equilibrium state, then the state $\tilde{\rho}_{\text{th}_{\text{M}_1}}$ will be the stationary state with respect to $\mathcal{L}_{\text{M}_1}[\cdot]$, i.e., $\mathcal{L}_{\text{M}_1}[\tilde{\rho}_{\text{th}_{\text{M}_1}}] = 0$. Spohn's inequality [47] tells us that for any superoperator of Lindblad form, say $\mathcal{L}_{\text{M}_1}[\cdot]$, with a stationary state, say $\tilde{\rho}_{\text{th}_{\text{M}_1}}$, the right-hand side (RHS) of Eq. (10) is always $\geqslant 0$. On the contrary, when one must go beyond the Markovian approximations while considering the dynamics of a system, the existence of a steady state is not guaranteed. Moreover, in the second term of the left-hand side (LHS) of (10), $\tilde{\rho}_{\text{th}_{\text{NM}_1}}$ will in general not be the steady state corresponding to $\mathcal{L}_{\text{NM}_1}[\cdot]$ irrespective of the initial state of the non-Markovian bath. So, we cannot infer the sign of that term, as can, e.g., be seen for the case involving four qubits undergoing a combined local Markovian and non-Markovian evolution presented in Appendix D, where we show that this term can take both positive and negative values. Thus, for the case of a two-party system with the subsystems being attached separately to two baths, one of which is Markovian and the other not, we get the altered form of Spohn's theorem as

$$\sigma_{\{\text{M}_1,\text{NM}_1\}} + k_B\text{tr}\big\{\mathcal{L}_{\text{NM}_1}\big[\rho_{sB_{\text{NM}_1}}(t)\big]\big(\ln(\tilde{\rho}_s(t)) - \ln(\tilde{\rho}_{\text{th}_{\text{NM}_1}})\big)\big\} \geqslant 0, \tag{11}$$

provided that the Markovian bath is initially in its canonical equilibrium state. The second term in the LHS of the inequality is an extra term that has got appended due to the presence of the non-Markovian bath in the set of local environments. Hence, a modified version of Spohn's theorem arises, which states that not the EPR, but the conjunction of EPR and $M^1_{\text{NM}}[\rho_{sB_{\text{NM}_1}}(t)] = k_B\text{tr}\{\mathcal{D}_{\text{NM}_1}[\rho_{sB_{\text{NM}_1}}(t)](\ln(\tilde{\rho}_s(t)) - \ln(\tilde{\rho}_{\text{th}_{\text{NM}_1}}))\}$ is assured to be positive for a combination of local Markovian and non-Markovian environments. The $\mathcal{L}_{\text{NM}_1}[\rho_{sB_{\text{NM}_1}}(t)]$ in (11) can be replaced by $\mathcal{D}_{\text{NM}_1}[\rho_{sB_{\text{NM}_1}}(t)]$, as the first term of the local superoperator $\mathcal{L}_{\text{NM}_1}[\rho_{sB_{\text{NM}_1}}(t)]$ has no contribution in $M^1_{\text{NM}}[\rho_{sB_{\text{NM}_1}}(t)]$. The presence of $M^1_{\text{NM}}[\rho_{sB_{\text{NM}_1}}(t)]$ in the inequality (11), therefore, indicates a deviation from the Markovian regime.

For a general $(m+n)$ subsystem, schematically depicted in Fig. 1, the GKSL-like equation takes the form, $\frac{d}{dt}\tilde{\rho}_s(t) = \mathcal{L}[\tilde{\rho}_s(t)] \equiv \sum_{j=1}^{m}\mathcal{L}_{\text{M}_j}[\tilde{\rho}_s(t)] + \sum_{j=1}^{n}\mathcal{L}_{\text{NM}_j}[\rho_{sB_{\text{NM}_{1\cdots n}}}(t)]$. The modified Spohn's theorem in this general case of $m+n$ parties turns out to be

$$\sigma_{\{\text{M}_{1\cdots m},\text{NM}_{1\cdots n}\}} + \sum_{j=1}^{n} M^j_{\text{NM}}\big[\rho_{sB_{\text{NM}_{1\cdots n}}}(t)\big] \geqslant 0, \tag{12}$$

for all Markovian baths being kept in their canonical equilibrium states at $t = 0$. Here, the number of non-Markovian baths, n, can be interpreted as the count of partial superoperators for which the corresponding states $\tilde{\rho}_{\text{th}_{\text{NM}_j}}$ are not the stationary states. Conversely, if all the baths are Markovian, then $\tilde{\rho}_{\text{th}_{\text{M}_j}}$ will represent the stationary states of their respective partial superoperators for all j, resulting in $n = 0$. Hence, Eq. (12) reverts to, $\sigma_{\{\text{M}_{1\cdots m},\text{NM}_{1\cdots n}\}} \geqslant 0$, which is the original form of the Spohn's theorem. On the other hand, in case all baths are non-Markovian, we obtain an altered form of the Spohn's theorem that follows directly from the balance equation and the concept of EPR. Appendix B contains a detailed discussion on this matter.

In the context of inequality (12), we can introduce a witness for detecting non-Markovian behavior, as well as a measure for the same. Let us consider a situation where we have q parties, each connected to q environments locally, and the initial states of these environments are the respective canonical equilibrium states. Now, we evaluate the quantity $M^k[\rho'(t)] = k_B\text{tr}\{\mathcal{D}_k[\rho'(t)](\ln(\tilde{\rho}_s(t)) - \ln(\tilde{\rho}_{\text{th}_k}))\}$ associated with the dissipators $\mathcal{D}_k[\rho'(t)]$ coming from the kth environment where k runs from 1 to q. The form of $\rho'(t)$ depends on whether the environment associated with the dissipator is Markovian or non-Markovian. If the environment is Markovian, then $\rho'(t)$ equals $\tilde{\rho}_s(t) = \text{tr}_{B_{1\cdots q}}[\rho_{sB_{1\cdots q}}(t)]$. If the environment is non-Markovian, then $\rho'(t)$ equals $\rho_{sB_{1\cdots q}}(t)$, representing the composite state of the systems and the baths. This construction of $\rho'(t)$ is possible as we can replace the trace taken over the non-Markovian baths, denoted as $\mathrm{tr}_{B_{\mathrm{NM}_{1\cdots n}}}$ in Eq. (4), with the trace taken over all the baths, denoted as $\mathrm{tr}_{B_{1\cdots q}}$ while constructing the dissipators, because tracing out the Markovian baths has no impact on the dissipators, as they are product states with the remaining part of the system-bath setup. Therefore, in this q-party scenario, we can use $\rho_{sB_{1\cdots q}}$ instead of $\rho_{sB_{\underline{1\cdots n}}}$. With the above definitions, we can now define a quantity $\overline{M}[t, \rho_s(0)] = \sum_{k=1}^{q} \max\{0, M^k[\rho'(t)]\}$. If all the baths are Markovian, $\overline{M}[t, \rho_s(0)]$ will be zero. However, if at least one bath is non-Markovian, $\overline{M}[t, \rho_s(0)]$ can take values greater than zero. Hence, this quantity $\overline{M}[t, \rho_s(0)]$ serves as a witness of non-Markovianity as it detects the deviation of the altered Spohn's theorem for the combined local Markovian and non-Markovian dynamics from the original version of Spohn's theorem. Note that it is crucial to start with environments initially in their canonical equilibrium states. If the environments do not begin in these states, then $\overline{M}[t, \rho_s(0)]$ can yield positive values even for Markovian environments. We can therefore define a quantifier of non-Markovianity as

$$\mathcal{M}_{\mathrm{NM}} = \max_{\rho_s(0)} \int_0^\infty \overline{M}[t, \rho_s(0)]dt. \tag{13}$$

For a Markovian dynamics, we get $\mathcal{M}_{\mathrm{NM}} = 0$. In case there is at least one non-Markovian bath, the quantifier $\mathcal{M}_{\mathrm{NM}}$ may give a positive (nonzero) value. For a single system, $\mathcal{M}_{\mathrm{NM}}$ reduces to the well-known BLP measure of non-Markovianity proposed in Ref. [38] within some restrictions, while for higher number of parties this equality does not hold. See Appendix C in this regard. Note that the quantifiers of non-Markovianity described in the literature are typically not easily computable. The quantifier $\mathcal{M}_{\mathrm{NM}}$ is, however, easily computable, and therein lies its potential utility, viz. in providing a computable strength of non-Markovianity in the dynamics of a system. In Appendix D, we have explored how introducing non-Markovian baths or substituting Markovian baths with non-Markovian ones impacts the dynamics of the system. We find that, initially non-Markovian baths have a strong effect, but over time, Markovian baths dominate, suppressing the amplitude of oscillations of the witness of non-Markovianity to approximately zero. This is expected because for a long time, the impact of memory effects, arising from the presence of non-Markovianity, diminishes. Also, more the number of Markovian baths, the quicker is the suppression. For a complete non-Markovian situation, where all the baths are from the non-Markovian family, the periodic oscillatory pattern of the witness of non-Markovianity gets disrupted.

V. CONCLUSION

To summarize, we have derived the GKSL-like equation for a situation containing more than one system, each interacting locally with a separate heat bath, some of which are Markovian, while others are non-Markovian. We present the dynamics of a multipartite system evolving under a mixture of Markovian and non-Markovian local environments. Our work provides a significant broadening of the area of investigation of open quantum dynamics, as a combination of non-Markovian and Markovian environments is a reasonable possibility in a realistic situation, especially when considering hybrid physical systems such as atom-photon arrangements. Our setup leads to a modification of the Spohn's theorem, taken to the multiparty case with a set of local Markovian and non-Markovian environments. As a consequence of the modification, we obtained a computable quantifier of non-Markovianity, which can detect the deviation from a Markovian situation. Most of the known quantifiers of non-Markovianity available in the literature are not easily computable. The computability of our measure can potentially be a useful tool to detect non-Markovianity. Analysis of the time dynamics of the quantifier expectedly showed that for an evolution affected by a combination of local Markovian and non-Markovian baths, non-Markovian effects are prominent for times close to initial time, but with the increase of time, non-Markovianity of the dynamics decreases and the evolution tends to be more and more Markovian-like.

ACKNOWLEDGMENTS

We acknowledge computations performed using ARMADILLO [105,106] and QICLIB [107] on the cluster computing facility of the Harish-Chandra Research Institute, India. This research was supported in part by the INFOSYS scholarship for senior students. We also acknowledge partial support from the Department of Science and Technology, Government of India, through QuEST with Grant No. DST/ICPS/QUST/Theme-3/2019/120.

APPENDIX A: DERIVATION OF THE TWO-PARTY GKSL-LIKE EQUATION FOR LOCAL MARKOVIAN AND NON-MARKOVIAN BATHS

A schematic diagram of the two-party two-bath setup is depicted in the right panel of Fig. 2. (The left panel depicts the more general case of an arbitrary number of parties with some connected to Markovian baths and the rest to non-Markovian ones.) The Hamiltonian of the composite setup is given by $H = H_s + H_B + H_I$, where H_s describes the Hamiltonian of the composite system consisting of the two parties, H_B stands for the combined local Hamiltonian of the two baths and $H_I = \sum_X H_{I_{X_1}}$ for $X \in \{M, NM\}$. Here $H_{I_{\mathrm{M}_1}}$ represents the interaction between S_{M_1} and B_{M_1}, and $H_{I_{\mathrm{NM}_1}}$ presents the interaction between S_{NM_1} and B_{NM_1}. Note that the Hamiltonian H_s, describing the Hamiltonian of the composite system containing two subsystems, is a general Hamiltonian encompassing both the local and interacting part of the two subsystems. Precisely, this H_s can be written as $H_s = H_{\mathrm{loc}} + V$, where H_{loc} is the local Hamiltonian of the two subsystems and V represents the interaction between them. In the Schrödinger picture, let the density matrix of the composite two-party two-bath setup at time t be represented by $\rho(t)$. It is useful to perform the calculation in the interaction picture [13]. The von Neumann equation in interaction picture will be

$$\frac{d\rho_{\mathrm{int}}(t)}{dt} = -\frac{i}{\hbar}[H_{I_{\mathrm{int}}}(t), \rho_{\mathrm{int}}(t)], \tag{A1}$$

where $H_{I_{\mathrm{int}}}(t) = e^{\frac{i(H_s+H_B)t}{\hbar}} H_I e^{-\frac{i(H_s+H_B)t}{\hbar}}$, $\rho_{\mathrm{int}}(t) = e^{\frac{i(H_s+H_B)t}{\hbar}} \rho(t) e^{-\frac{i(H_s+H_B)t}{\hbar}}$, without assuming a commutativity

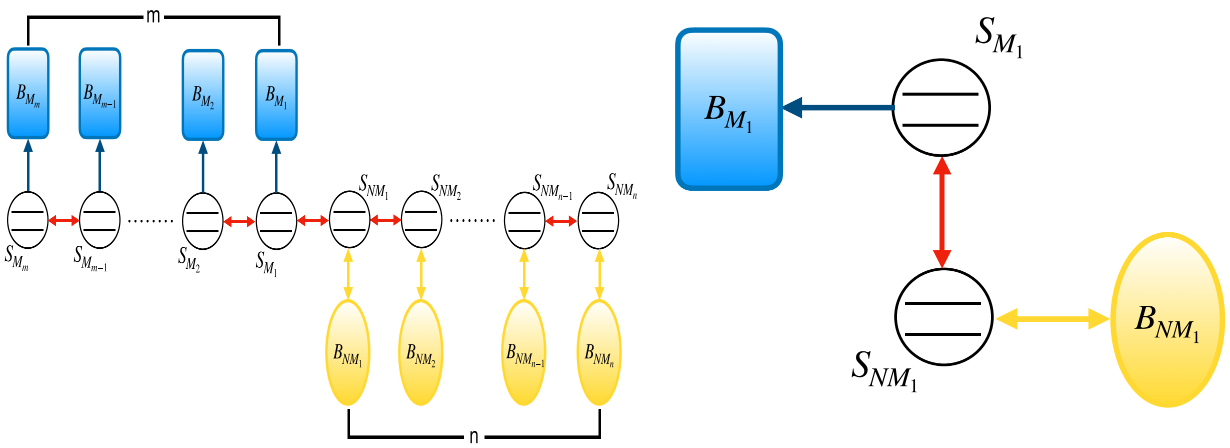

FIG. 2. A combination of local environments. In the left panel, we present a schematic diagram of $m+n$ TLSs, evolving under their system Hamiltonian as well as local environments, which are some Markovian and the rest not so. $B_{\mathrm{M}_1}\cdots B_{\mathrm{M}_m}$ are the baths, which can be treated under the Born-Markov approximations, hence are Markovian baths, and $B_{N_1}\cdots B_{N_n}$ are the baths residing in a non-Markovian family. A special case of the left panel is presented in the right one where only two TLSs are interacting with two baths locally, among which one is Markovian and the other is non-Markovian. S_{M_1} and S_{NM_1} are the two TLSs. B_{M_1} is the Markovian bath, while B_{NM_1} is non-Markovian one.

relation of H_I and $\rho(t)$ with H_s and H_B. The interaction Hamiltonian in the Schrödinger picture can be decomposed in the form [13,15], $H_{I_{X_1}}=\sum_k A_{X_{1k}}\otimes B_{X_{1k}}$, with $A_{X_{1k}}$ and $B_{X_{1k}}$ being the system and bath operators, respectively. Here $X\in\{M,NM\}$ and k runs from 1 to the maximum number of terms required for the decomposition for each j. Reverting to the interaction picture we get, $H_{I_{X_1,\mathrm{int}}}(t)=\sum_k A_{X_{1k}}(t)\otimes B_{X_{1k}}(t)$, where $A_{X_{1k}}(t)=e^{\frac{iH_st}{\hbar}}A_{X_{1k}}e^{-\frac{iH_st}{\hbar}}$, $B_{X_{1k}}(t)=e^{\frac{iH_Bt}{\hbar}}B_{X_{1k}}e^{-\frac{iH_Bt}{\hbar}}$. The system operators $A_{X_{1k}}$ can be decomposed in the eigenspace of the system Hamiltonian as $A_{X_{1k}}(\omega)=\sum_{\epsilon'-\epsilon=\omega}\Pi(\epsilon)A_{X_{1k}}\Pi(\epsilon')$, where $\Pi(\epsilon)$'s are the projectors onto the eigenspaces of the system Hamiltonian H_s corresponding to the eigenfrequency ϵ. Therefore, we have the properties, $[H_s,A_{X_{1k}}(\omega)]=-\hbar\omega A_{X_{1k}}(\omega)$ and $[H_s,A^\dagger_{X_{1k}}(\omega)]=\hbar\omega A^\dagger_{X_{1k}}(\omega)$, indicating the fact that $A_{X_{1k}}(t)$ and $A_{X_{1k}}(\omega)$ are related by a Fourier transformation from the t space to the ω space, as $A_{X_{1k}}(t)=\sum_\omega e^{-i\omega t}A_{X_{1k}}(\omega)$ and $A^\dagger_{X_{1k}}(t)=\sum_\omega e^{i\omega t}A^\dagger_{X_{1k}}(\omega)$. Now the interaction Hamiltonian becomes

$$H_{I_{\mathrm{int}}}(t)=\sum_X H_{I_{X_1,\mathrm{int}}}=\sum_{X,k,\omega}e^{-i\omega t}A_{X_{1k}}(\omega)\otimes B_{X_{1k}}(t). \quad \text{(A2)}$$

The state of the entire setup $\rho_{\mathrm{int}}(t)$, is given by

$$\rho_{\mathrm{int}}(t)=\rho(0)-\frac{i}{\hbar}\int_0^t ds\big[H_{I_{\mathrm{int}}}(s),\,\rho_{\mathrm{int}}(s)\big]. \quad \text{(A3)}$$

We assume that initially the systems are uncorrelated to the baths, and that the baths themselves are also in a product state, so that at time $t=0$, $\rho(0)=\rho_s(0)\otimes\rho_{B_{\mathrm{M}_1}}(0)\otimes\rho_{B_{\mathrm{NM}_1}}(0)$. As we mentioned earlier, that the bath B_{M_1} is Markovian and therefore in the further calculations, while talking about B_{M_1}, we will impose the Born-Markov approximations, which tells that the coupling between the subsystem S_{M_1} and B_{M_1} is weak, so that the state of B_{M_1} regains its initial state after every time step of interaction with S_{M_1}, and that any correlation created between B_{M_1} and S_{M_1} is also destroyed after the same time step. Moreover, B_{M_1} will also be assumed to remain uncorrelated with B_{NM_1} during the evolution. So, at time t, the state of the entire setup will take the form, $\rho(t)\approx\rho_{sB_{\mathrm{NM}_1}}(t)\otimes\rho_{B_{\mathrm{M}_1}}$, where $\rho_{sB_{\mathrm{NM}_1}}(t)$ is the density matrix of the systems S_{M_1} and S_{NM_1} combined along the bath B_{NM_1} at time t, and $\rho_{B_{\mathrm{M}_1}}=\rho_{B_{\mathrm{M}_1}}(0)$. Now we make a further assumption for the Markovian bath B_{M_1} [13], viz.

$$\mathrm{tr}_B\big[H_{I_{\mathrm{M}_1,\mathrm{int}}}(t),\,\rho_{\mathrm{int}}(0)\big]=0. \quad \text{(A4)}$$

As a consequence of this assumption, the Markovian bath B_{M_1} possesses the property,

$$\mathrm{tr}_{B_{\mathrm{M}_1}}\left(B_{\mathrm{M}_{1k}}\rho_{B_{\mathrm{M}_1}}\right)=0. \quad \text{(A5)}$$

Here we assume B_{M_1} is initially in its stationary state, i.e., $[H_{B_{\mathrm{M}_1}},\rho_{B_{\mathrm{M}_1}}(0)]=0$. This Eq. (A5) is a very important relation for the succeeding calculations of this paper. Now, using the integral form of $\rho_{\mathrm{int}}(t)$, given in Eq. (A3), and then using Eq. (A4) in the von Neumann equation, we get

$$\frac{d\rho_{\mathrm{int}}(t)}{dt}=\sum_X-\frac{1}{\hbar^2}\int_0^t ds\big[H_{I_{X_1,\mathrm{int}}}(t),\,\big[H_{I_{X_1,\mathrm{int}}}(s),\,\rho_{sB_{\mathrm{NM}_1},\mathrm{int}}(s)\otimes\rho_{B_{\mathrm{M}_1}}\big]\big]-\frac{i}{\hbar}\big[H_{I_{\mathrm{NM}_1,\mathrm{int}}}(t),\,\rho_{sB_{\mathrm{NM}_1},\mathrm{int}}(t)\otimes\rho_{B_{\mathrm{M}_1}}\big], \quad \text{(A6)}$$

for $X\in\{M,NM\}$. The first term for $X=NM$ vanishes by using the relation given in Eq. (A5). Thus the reduced system dynamics comes out to be

$$\frac{d\tilde{\rho}_{s,\mathrm{int}}(t)}{dt}=-\frac{1}{\hbar^2}\int_0^t ds\,\mathrm{tr}_{B_{\mathrm{M}_1}}\big[H_{I_{\mathrm{M}_1,\mathrm{int}}}(t),\,\big[H_{I_{\mathrm{M}_1,\mathrm{int}}}(s),\,\tilde{\rho}_{s,\mathrm{int}}(s)\otimes\rho_{B_{\mathrm{M}_1}}\big]\big]-\frac{i}{\hbar}\mathrm{tr}_B\big[H_{I_{\mathrm{NM}_1,\mathrm{int}}}(t),\,\rho_{sB_{\mathrm{NM}_1},\mathrm{int}}(t)\otimes\rho_{B_{\mathrm{M}_1}}\big]. \quad \text{(A7)}$$

Here $\tilde{\rho}_{s,\mathrm{int}}(t)=\mathrm{tr}_{B_{\mathrm{NM}_1}}\{\rho_{sB_{\mathrm{NM}_1},\mathrm{int}}(t)\}$. Next we use the Markov approximation, i.e., replace the integrand $\tilde{\rho}_{s,\mathrm{int}}(s)$ in the first term by $\tilde{\rho}_{s,\mathrm{int}}(t)$, so that the development of the reduced state of the system at time t only depends on the present state

$\tilde{\rho}_{s,\mathrm{int}}(t)$. Furthermore, we substitute s by $t-s$ and let the upper limit of the integral go to infinity so that we can avoid the dependency of the reduced density matrix on the explicit choice of the initial preparation at time $t=0$. See Ref. [13] for more details about the Markovian approximations. Hence, after applying the rotating wave approximation, the right-hand side of the above equation comes out to be

$$\begin{aligned}
&= \frac{1}{\hbar^2}\sum_{\omega}\sum_{k,l}\gamma_{k,l}(\omega)\big[A_{\mathrm{M}_{1_l}}(\omega)\tilde{\rho}_{s,\mathrm{int}}(t)A^{\dagger}_{\mathrm{M}_{1_k}}(\omega)\\
&\quad - A^{\dagger}_{\mathrm{M}_{1_k}}(\omega)A_{\mathrm{M}_{1_l}}(\omega)\tilde{\rho}_{s,\mathrm{int}}(t)\big] + \mathrm{H.c.}\\
&\quad - \frac{i}{\hbar}\mathrm{tr}_{B_{\mathrm{NM}_1}}\big[H_{I_{\mathrm{NM}_1,\mathrm{int}}}(t),\, \rho_{sB_{\mathrm{NM}_1},\mathrm{int}}(t)\big]. \qquad \text{(A8)}
\end{aligned}$$

$\gamma_{k,l}(\omega)$ is given by $\gamma_{k,l}(\omega) = \int_0^{\infty} ds e^{i\omega s}\mathrm{tr}_{B_{\mathrm{M}_1}}\{B^{\dagger}_{\mathrm{M}_{1_k}}(t)B_{\mathrm{M}_{1_l}}(t-s)\rho_{B_{\mathrm{M}_1}}\}$. Now we go to the Schrödinger picture. $\rho_{B_{\mathrm{M}_1}}$ will be the same in both the pictures as $\rho_{B_{\mathrm{M}_1}}$ commutes with $H_{B_{\mathrm{M}_1}}$. So, $\tilde{\rho}_{s,\mathrm{int}}(t) = \mathrm{tr}_{B_{\mathrm{NM}_1}}[e^{\frac{i(H_s+H_{B_{\mathrm{NM}_1}})t}{\hbar}}\rho_{sB_{\mathrm{NM}_1}}(t)e^{-\frac{i(H_s+H_{B_{\mathrm{NM}_1}})t}{\hbar}}]$. Using $\tilde{\rho}_s(t) = \mathrm{tr}_{B_{\mathrm{NM}_1}}\{e^{\frac{iH_{B_{\mathrm{NM}_1}}t}{\hbar}}\rho_{sB_{\mathrm{NM}_1}}(t)e^{\frac{-iH_{B_{\mathrm{NM}_1}}t}{\hbar}}\} = \mathrm{tr}_{B_{\mathrm{NM}_1}}(\rho_{sB_{\mathrm{NM}_1}})$, the reduced system dynamics in the Schrödinger picture turns out to be

$$\begin{aligned}
\frac{d\tilde{\rho}_s(t)}{dt} &= \mathcal{L}\big[\tilde{\rho}_s(t)\big] \equiv -\frac{i}{\hbar}\big[H_s,\tilde{\rho}_s(t)\big] + \mathcal{D}_{\mathrm{M}_1}\big[\tilde{\rho}_s(t)\big]\\
&\quad + \mathcal{D}_{\mathrm{NM}_1}\big[\rho_{sB_{\mathrm{NM}_1}}(t)\big], \qquad \text{(A9)}
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{D}_{\mathrm{M}_1}\big[\tilde{\rho}_s(t)\big] &= \frac{1}{\hbar^2}\sum_{\omega}\sum_{k,l}\gamma_{k,l}(\omega)\big(A_{\mathrm{M}_{1_l}}(\omega)\tilde{\rho}_s(t)A^{\dagger}_{\mathrm{M}_{1_k}}(\omega)\\
&\quad - \frac{1}{2}\big[A^{\dagger}_{\mathrm{M}_{1_k}}(\omega)A_{\mathrm{M}_{1_l}}(\omega),\,\tilde{\rho}_s(t)\big]\big),\\
\mathcal{D}_{\mathrm{NM}_1}\big[\rho_{sB_{\mathrm{NM}_1}}(t)\big] &= -\frac{i}{\hbar}\mathrm{tr}_{B_{\mathrm{NM}_1}}\big[H_{I_{\mathrm{NM}_1}}(t),\,\rho_{sB_{\mathrm{NM}_1}}(t)\big]. \qquad \text{(A10)}
\end{aligned}$$

APPENDIX B: ALTERATION OF SPOHN'S THEOREM IN CASE OF ALL NON-MARKOVIAN ENVIRONMENTS

The modified form of the Spohn's theorem for the combined local evolution under Markovian and non-Markovian environments is presented in Eq. (11) for a two-party setup and the generalization of this two-party scenario to a multiparty situation is given in Eq. (12). This altered version of the Spohn's theorem has been derived by imposing strict restrictions on the Markovian environments. If the restrictions corresponding to the Markovian baths are relaxed for the Markovian environments, i.e., if we consider all the environments to be non-Markovian, i.e., $m=0$, then there will only be the non-Markovian dissipator $\mathcal{D}_{\mathrm{NM}_j}[\rho_{sB_{\mathrm{NM}_{1\cdots n}}}(t)]$ for $j=1$ to n in the GKSL-like master equation given in Eq. (4) of the main text. Let us first consider the simplest situation of the two-party two-bath setup. If both the baths are non-Markovian, then we will not get the term $-k_B\mathrm{tr}\{\mathcal{L}_{\mathrm{M}_1}[\tilde{\rho}_s(t)](\ln(\tilde{\rho}_s(t)) - \ln(\tilde{\rho}_{\mathrm{th}_{\mathrm{M}_1}}))\}$ in the RHS of Eq. (10) of the main text. Instead, we will get the equation,

$$\begin{aligned}
\sigma_{\{\mathrm{NM}_1,\mathrm{NM}_2\}} + \sum_{j=1}^{2}k_B\mathrm{tr}\big\{\mathcal{L}_{\mathrm{NM}_j}\big[\rho_{sB_{\mathrm{NM}_{12}}}(t)\big]\big(\ln(\tilde{\rho}_s(t))\\
- \ln(\tilde{\rho}_{\mathrm{th}_{\mathrm{NM}_j}})\big)\big\} = 0. \qquad \text{(B1)}
\end{aligned}$$

For n parties, j will run from 1 to n in this equation, $\sigma_{\{\mathrm{NM}_1,\mathrm{NM}_2\}}$ is to be replaced by $\sigma_{\{\mathrm{NM}_1,\cdots,\mathrm{NM}_n\}}$, and $\rho_{sB_{\mathrm{NM}_{12}}}(t)$ will be replaced by $\rho_{sB_{\mathrm{NM}_{1\ldots n}}}(t)$. Therefore, this equation will now become the altered version of the Spohn's theorem, valid for the situation where all the environments exhibit non-Markovian behavior, and is a direct consequence of the balance equation and the concept of EPR. This version of the Spohn's result does not incorporate any assumption about weak coupling, or correlation destruction (system-bath as well as bath-bath), or about an evolution being memoryless.

APPENDIX C: RELATION OF $\mathcal{M}_{\mathrm{NM}}$ WITH THE BLP MEASURE OF NON-MARKOVIANITY

In this section, we aim to establish a relationship between the proposed quantifier of non-Markovianity in the main text, denoted as $\mathcal{M}_{\mathrm{NM}}$, and the well-known BLP measure [38]. First, let us introduce the BLP measure of non-Markovianity. For any two quantum states ρ_1 and ρ_2, we define the rate of change of the distance between these states over time as

$$\Lambda(t,\rho_{1,2}(0)) = \frac{d}{dt}D(\rho_1(t),\rho_2(t)), \qquad \text{(C1)}$$

where $\rho_i(t) = \Phi(t)(\rho_i(0))$ for $i=1$ and 2, and $\Phi(t)(\cdot)$ represents any quantum channel. The function D denotes a distance measure in the space of quantum states. For the purposes of this discussion, we will choose the relative entropy as the distance measure, which leads to

$$\Lambda(t,\rho_{1,2}(0)) = \frac{d}{dt}S(\rho_1(t),\rho_2(t)). \qquad \text{(C2)}$$

When $\Lambda \leqslant 0$, the dynamical process exhibits the divisibility property of a dynamical semigroup, indicating a Markovian dynamics. However, if divisibility breaks during the dynamical evolution of the system, Λ may take a positive (nonzero) value. As a result, Λ serves as a witness of non-Markovianity. To quantify the non-Markovian behavior, we can therefore define the corresponding measure of non-Markovianity as

$$\mathcal{N} = \max_{\rho_{1,2}(0)}\int_{\Lambda>0}\Lambda(t,\rho_{1,2}(0))\,dt, \qquad \text{(C3)}$$

where $\mathcal{N}=0$ for all Markovian processes, i.e., the ones that satisfy the divisibility property of a quantum dynamical semigroup. If $\mathcal{N}>0$ the evolution must be non-Markovian. This is the quantifier of non-Markovianity proposed by Breuer-Laine-Piilo in Ref. [38].

To facilitate a comparison between the non-Markovianity quantifier of BLP and the quantifier $\mathcal{M}_{\mathrm{NM}}$ [proposed in Eq. (13) of the main text], we will impose an additional restriction on the BLP quantifier. Consider a scenario where a single system, described by the Hamiltonian H_s, is immersed in a heat bath, initially kept in its canonical equilibrium state $\tilde{\rho}_{\mathrm{th}_{B_1}} = \frac{e^{-\beta_1 H_{B_1}}}{\mathrm{tr}(e^{-\beta_1 H_{B_1}})}$ with the inverse temperature $k_B\beta_1$ and H_{B_1}

being the free Hamiltonian of the bath. Consequently, the canonical equilibrium state of the system can be expressed as $\tilde{\rho}_{\text{th}_1} = \frac{e^{-\beta_1 H_s}}{\text{tr}(e^{-\beta_1 H_s})}$. Now, we set $\rho_2(t) = \tilde{\rho}_{\text{th}_1}$ for all time. Hence the BLP witness of non-Markovianity Λ reduces to

$$\Lambda(t, \rho_1(0), \tilde{\rho}_{\text{th}_1}) = \frac{d}{dt} S(\rho_1(t), \tilde{\rho}_{\text{th}_1}). \tag{C4}$$

Accordingly, the corresponding quantifier of non-Markovianity can be expressed as

$$\tilde{\mathcal{N}} = \max_{\rho_1(0)} \int_{\Lambda>0} \Lambda(t, \rho_1(0), \tilde{\rho}_{\text{th}_1})\, dt. \tag{C5}$$

In contrast, for the case of a single system, the witness of non-Markovianity, denoted as $\overline{M}[t, \rho_s(0)]$ in the main text, is given by

$$\begin{aligned}\overline{M}[t, \rho_s(0)] &= \max\{0, M^1[\rho'(t)]\} \\ &= \max\left\{0, \frac{d}{dt} S(\tilde{\rho}_s(t)||\tilde{\rho}_{\text{th}_1})\right\},\end{aligned} \tag{C6}$$

where $S(\tilde{\rho}_s(t)||\tilde{\rho}_{\text{th}_1}) = k_B \text{tr}\{\mathcal{L}_1[\rho'(t)](\ln(\tilde{\rho}_s(t)) - \ln(\tilde{\rho}_{\text{th}_1}))\}$. Therefore, the quantifier of non-Markovianity becomes

$$\begin{aligned}\mathcal{M}_{\text{NM}} &= \max_{\rho_s(0)} \int_0^\infty \overline{M}[t, \rho_s(0)] dt \\ &= \tilde{\mathcal{N}}.\end{aligned} \tag{C7}$$

Hence, we observe that the quantifier of non-Markovianity $\mathcal{M}_{\text{NM}}$ reduces to the BLP measure when the state $\rho_2(t)$ in BLP quantifier is fixed at $\tilde{\rho}_{\text{th}_1}$. However, it is important to note that for systems comprising two or more subsystems, this type of relationship between $\mathcal{M}_{\text{NM}}$ and $\tilde{\mathcal{N}}$ is not attainable.

APPENDIX D: FOUR QUBITS COUPLED TO A COMBINATION OF FOUR MARKOVIAN AND NON-MARKOVIAN HEAT BATHS LOCALLY

We consider here the case of four noninteracting two-level systems (TLSs), each locally immersed in a bath that is either Markovian or not so. To begin, let us consider four noninteracting single-qubit subsystems, each locally immersed in a Markovian bosonic heat bath, kept in their canonical equilibrium states at temperatures T_1, T_2, T_3, and T_4, respectively, so that under the time evolution of the system in this circumstance, the local canonical equilibrium states are the steady states of the partial superoperators. Here we use the dimensionless temperatures of the baths, defined as $T_j = \frac{\hbar\tilde{\eta}}{k_B \tilde{T}_j}$, where $\tilde{T}_j$ are the real temperatures for $j = 1$–4 and $\tilde{\eta}$ is a constant having the dimension of frequency. So, as we discussed in the main text, the quantifier, $\mathcal{M}_{\text{NM}} = 0$ in this case. We now replace the Markovian baths one by one with non-Markovian ones, and study its response in the non-Markovian witness $\overline{M}[t, \rho_s(0)]$ with time.

The Hamiltonian of the composite four-qubit four-bath setup is

$$H_{\text{tot}} = \sum_{j=1}^{4} \left(H_{s_j} + H_{B_{X_j}} + H_{I_{X_j}}\right), \tag{D1}$$

with X in the suffixes of the second and the third terms indicating whether the contribution is coming from a Markovian (M) or a non-Markovian (NM) bath. The local Hamiltonian of each TLS is given by

$$H_{s_j} = \frac{\hbar\omega_j}{2}\sigma_j^z, \tag{D2}$$

having the ground-state eigenvalue $-\frac{\hbar\omega_j}{2}$ corresponding to the state $|1\rangle$ and the excited state eigenvalue $\frac{\hbar\omega_j}{2}$ corresponding to the state $|0\rangle$. For the Markovian harmonic oscillator baths, the local Hamiltonian of each bath can be expressed as

$$H_{B_{\text{M}_j}} = \int_0^{\eta_{\max_j}} \hbar\tilde{\eta}\, d\eta_j a_{\eta_j}^\dagger a_{\eta_j}, \tag{D3}$$

and the same for each non-Markovian bath is taken as

$$H_{B_{\text{NM}_j}} = \hbar\nu_j J_j^+ J_j^-. \tag{D4}$$

For the harmonic oscillator baths, $a_{\eta_j}^\dagger(a_{\eta_j})$ represents the bosonic creation (annihilation) operator of the harmonic oscillator of the jth mode of the bath, having the unit of $\frac{1}{\sqrt{\eta_j}}$ and connected by the commutation relation $[a_{\eta_i}, a_{\eta_i'}^\dagger] = \delta(\eta_i - \eta_i')$, $\eta_{\max_j}$ is the cutoff frequency of the jth Markovian bath. On the contrary, each of the non-Markovian baths, with frequency ν_j for each j, is taken as one described by the spin-star model [32,33] consisting of N localized quantum spin-$\frac{1}{2}$ particles centering the single-qubit system at equal distances on a sphere, with $J_j^\pm = \sum_{l=1}^N \sigma_{j,(l)}^\pm$. Here $\sigma_{j,(l)}^\pm = \frac{1}{2}(\sigma_{j,(l)}^x \pm i\sigma_{j,(l)}^y)$, with $\vec{\sigma}(\sigma^x, \sigma^y, \sigma^z)$ representing the Pauli matrices. The interaction between the systems and the local bosonic baths is considered as

$$H_{I_{\text{M}_j}} = \int_0^{\eta_{\max_j}} \hbar\sqrt{\tilde{\eta}}\, d\eta_j h(\eta_j)(a_{\eta_j}^\dagger \sigma_j^- + a_{\eta_j}\sigma_j^+), \tag{D5}$$

where $h(\eta_j)$ is a dimensionless function of η_j, and represents the system-bath coupling strength. For a harmonic oscillator bath, $\tilde{\eta}h^2(\eta_j) = \mathcal{J}(\eta_j)$, where $\mathcal{J}(\eta_j)$ is the ohmic spectral density function. As we consider the spectral density function as ohmic, $\mathcal{J}(\eta_j) \propto \eta_j$. Thus $\mathcal{J}(\eta_j) = \kappa_j\eta_j$, where κ_j are unitless constants. For the qubits connected to non-Markovian spin baths, the interaction is through a Heisenberg XY coupling [108], given by

$$H_{I_{\text{NM}_j}} = \hbar\alpha_j(\sigma_j^+ J_j^- + \sigma_j^- J_j^+). \tag{D6}$$

Here α_j quantifies the coupling strength between the jth system and the jth non-Markovian bath, having the unit of frequency. The dynamical equation for the $[(m+n) = 4]$-qubit system, locally connected to a combination of Markovian and non-Markovian heat baths, directly follows from Eq. (9), and is given by

$$\begin{aligned}\frac{d\rho}{d(\tilde{\eta}t)} &= \frac{1}{\tilde{\eta}}\mathcal{L}[\tilde{\rho}_s(t)] = -\frac{i}{\hbar\tilde{\eta}}[H_s, \tilde{\rho}_s(t)] \\ &\quad + \frac{1}{\tilde{\eta}}\sum_{j=1}^{m}\mathcal{D}_{\text{M}_j}[\tilde{\rho}_s(t)] + \frac{1}{\tilde{\eta}}\sum_{j=1}^{n}\mathcal{D}_{\text{NM}_j}\left[\rho_{sB_{\text{NM},1\ldots n}}(t)\right].\end{aligned} \tag{D7}$$

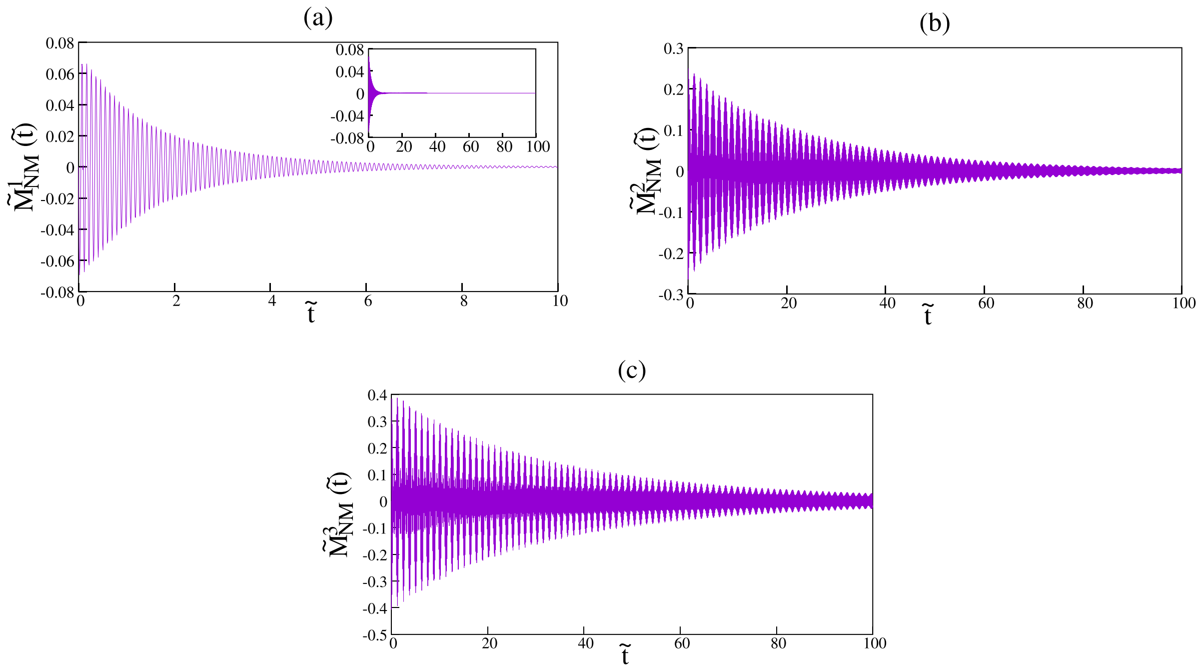

FIG. 3. Time dynamics of $\tilde{M}^n_{\rm NM}(\tilde{t})$. Here we plot the behavior of $\tilde{M}^n_{\rm NM}(\tilde{t})$ with time for (a) $n = 1$, where only the fourth bath is non-Markovian and the other three are Markovian, (b) $n = 2$, where the third and fourth baths are non-Markovian and the other two are Markovian, and (c) $n = 3$, where only the first bath is Markovian and the other three are non-Markovian, for the initial state of the system taken as $|\psi_s(0)\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle)$. For the demonstration, we have chosen $\omega_1 = 50.0$, $\omega_2 = 55.0$, $\omega_3 = 60.0$, $\omega_4 = 65.0$, and $\nu_1 = \nu_2 = \nu_3 = \nu_4 = 1.0$ and the temperatures of the heat baths are set to be $T_1 = 127.33$, $T_2 = 105.57$, $T_3 = 95.8$, and $T_4 = 68.6$. The coupling constants are taken to be $\kappa_1 = \kappa_2 = \kappa_3 = 10^{-3}$ and $\alpha_2 = \alpha_3 = \alpha_4 = 0.5$. The quantities plotted along the horizontal axes are dimensionless and the same along the vertical axes have the unit of $k_B\tilde{\eta}$.

Both sides of the equation are made dimensionless. For the purpose of our demonstration, we will take the variable $\tilde{t} = \tilde{\eta}t$ as the dimensionless time.

In Fig. 3, we have depicted the time dynamics of the quantity $\tilde{M}^n_{\rm NM}(\tilde{t}) = \sum_{k=1}^{n} M^k_{\rm NM}[\rho_{sB_{{\rm NM}_{1\ldots n}}}(\tilde{t})]$ for $n = 1$, 2, and 3 in Figs. 3(a)–3(c), respectively, n being the number of non-Markovian environments. In all the three panels, $\tilde{M}^n_{\rm NM}(\tilde{t})$ exhibits oscillating profiles. The envelope of $\tilde{M}^n_{\rm NM}(\tilde{t})$, having positive and negative values, is significant in magnitude for a short time near zero, but decreases monotonically, tending to zero for large time. The existence of the positive part of this envelope indicates that the witness of non-Markovianity, $\overline{M}[\tilde{t}, \rho_s(0)] > 0$. We find that the non-Markovian baths have a significant contribution for times near the initial time, while for larger time, the effect of Markovian baths dominate, suppressing the amplitude of oscillations of $\tilde{M}^n_{\rm NM}(\tilde{t})$ to approximately zero. This is expected because for a long time, the impact of memory effects, arising from the presence of non-Markovianity, diminishes. Although both positive as well as negative oscillations of $\tilde{M}^n_{\rm NM}(t)$ are suppressed in this specific scenario, it is the suppression of the positive oscillations that imply the dominance of the Markovian baths in the evolution. The greater the number of Markovian baths, the quicker is the suppression. [Compare Figs. 3(a)–3(c)]. For a complete non-Markovian situation, where all the baths are from the non-Markovian family, the periodic oscillatory pattern of $\tilde{M}^n_{\rm NM}(\tilde{t})$ gets disrupted. The amplitude of oscillation does not diminish with time, as there is no Markovian bath to suppress the oscillation like in case of the combination of local Markovian and non-Markovian dynamics.

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