Einstein-Podolsky-Rosen steering criterion and monogamy relation via correlation matrices in tripartite systems

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Quantum steering is considered one of the most well-known nonlocal phenomena in quantum mechanics. Unlike entanglement and Bell nonlocality, the asymmetry of quantum steering makes it vital for one-sided device-independent quantum information processing. Although there has been much progress on steering detection for bipartite systems, the criterion for Einstein-Podolsky-Rosen steering in tripartite systems remains challenging and inadequate. In this paper we first derive a promising steering criterion for any three-qubit states via correlation matrix. Furthermore, we propose the monogamy relation between the tripartite steering of system and the bipartite steering of subsystems based on the derived criterion. Finally, as illustrations, we demonstrate the performance of the steering criterion and the monogamy relation by means of several representative examples. We believe that the results and methods presented in this work could be beneficial to capture genuine multipartite steering in the near future.

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I. INTRODUCTION

Einstein, Podolsky, and Rosen put forward the celebrated paradox in which they pointed out the incompleteness of quantum mechanics, known as the Einstein-Podolsky-Rosen (EPR) paradox [1]. To formalize this argument, Schrödinger [2] subsequently introduced the notion of quantum steering. Specifically, it was assumed that Alice and Bob share a maximally entangled state

$$|\psi_{AB}\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle), \tag{1}$$

where $|1\rangle$ and $|0\rangle$ denote the two eigenstates of the spin operator σ_z. Because of the perfect anticorrelations of the above state, if Alice measures her particle with observable σ_z and obtains the result of $+1$ or -1, the state of Bob's corresponding particle will collapse to $|1\rangle$ or $|0\rangle$, whereas if Alice's measurement choice is the observable σ_x, then the state of Bob's particle will be collapsed to either $|x_+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ or $|x_-\rangle = (|0\rangle - |1\rangle)/\sqrt{2}$. Herein it can be seen that Alice is capable of making another particle instantly collapse into a different state by performing local measurements on her own particle, which Schrödinger called quantum steering. In other words, quantum steering is a unique property of quantum systems, which describes the ability to instantaneously influence one subsystem in a two-party system by performing a local measurement of the other.

In order to attempt to interpret incompleteness of quantum mechanics, scientists put forward a local hidden-variable (LHV) model theory [3]. Notably, Bell [4] derived the famous Bell inequality by using the LHV model and found that Bell inequality can be violated in reality. Note that the violation of this inequality means that the predictions of quantum theory cannot be explained by the LHV model, revealing the nonlocality of quantum mechanics. At that time the concept of quantum steering had not yet been mathematically defined. Eventually, Wiseman and co-workers [5,6] formally introduced the definition of quantum steering. They described quantum steering as a quantum nonlocal phenomenon that cannot be explained by local hidden-state (LHS) models.

As a kind of quantum nonlocality, quantum steering is different from quantum entanglement [7–9] and Bell nonlocality. To be explicit, the characteristic of quantum steering is inherent asymmetry [10–15], and even one-way steering may occur. In some cases, party A can steer B, while B cannot steer A [16,17]. Therefore, quantum steering, as an effective quantum resource, plays a crucial role in various quantum information processing tasks, such as one-sided device-independent quantum key distribution [18–20], secure quantum teleportation [21,22], quantum randomness certification [23,24], and subchannel discrimination [25,26].

To judge whether a quantum state is steerable, several authors have significantly contributed to explaining this issue and brought up numerous different criteria [27–42] of quantum steering, for example, the steering criterion based on linear steering inequalities [27–29], the local uncertainty relation [30–35], the all-versus-nothing proof [36], and Clauser-Horne-Shimony-Holt–like inequalities [37–39]. Then detection of steerability can be achieved by steering robustness [25], steerable weight [43], and violating those various steering inequalities, etc. In experimental research, several criteria for quantum steering have been verified [28,44–50]. A groundbreaking experiment was proposed by Ou *et al.* [45] using Reid's criterion [42] to demonstrate the existence of quantum steering. To date, many criteria for bipartite steering detection have been proposed. However, there have been

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few investigations into tripartite steering detection, which still needs to be addressed.

Among those steering criteria of bipartite systems, Lai and Luo [51] employed the correlation matrix of the local observations and proposed the steerability criterion for bipartite systems of any dimension, and three classes of local measurements, including local orthogonal observables (LOOs) [52], mutually unbiased measurements [53], and general symmetric informationally complete measurements [54], were applied on attaining the proposed steering criteria. Inspired by Lai and Luo's work, we first derive the steering criterion for an arbitrary three-qubit quantum state via correlation matrices with LOOs. In addition, for a three-qubit state, we also propose a monogamy relation between three-party steering and subsystem two-party steering.

The remainder of the paper is arranged as follows. Section II introduces the notion of EPR steering and several well-known criteria. In Sec. III we present a new criterion of EPR steering in tripartite systems and also present its proof. Furthermore, we put forward the monogamy relation between three-party steering and subsystem two-party steering. As illustrations, we render several representative examples to demonstrate the detection ability of our criterion in Sec. IV. We summarize the paper in Sec. V.

II. EPR STEERING

Wiseman *et al.* [5] provided the definition of quantum steering. To be more specific, Alice prepares two entangled particles, sends one to Bob, and declares that she can steer the state of Bob's particle by measuring her remaining particle. For each measurement choice x and measurement result a of Alice, Bob will gain the corresponding unnormalized conditional state $\sigma_{a|x}$. These unnormalized conditional states satisfy $\sum_a \sigma_{a|x} = \rho_B$, which ensures that Bob's reduced state $\rho_B = \mathrm{tr}_A(\rho_{AB})$ does not depend on Alice's choice of measurements. Bob then verifies that the unnormalized conditional state can be described as the LHS model

$$\sigma_{a|x} = \int d\lambda\, p_\lambda p_C(a|x,\lambda) p_\lambda^B, \tag{2}$$

where λ represents the hidden-variable parameter, $p_C(a|x,\lambda)$ represents the local response function, and p_λ^B represents the hidden state. If the conditional state $\sigma_{a|x}$ can be described by the local hidden-state model, then the quantum state ρ_{AB} is not steerable; otherwise, it is steerable.

In the experiment, we use x and y to represent the measurement choices of Alice and Bob, respectively, and use a and b to represent the measurement results obtained by measuring x and y, respectively. For a quantum state ρ_{AB} that conforms to the LHS model, the joint probability distribution can be written as

$$p(a,b|x,y) = \int d\lambda\, p_\lambda p_C(a|x,\lambda) p_Q\big(b|y,p_\lambda^B\big), \tag{3}$$

where $p_C(a|x,\lambda)$ denotes classical probability and $p_Q(b|y,p_\lambda^B) = \mathrm{tr}(M_{b|y} p_\lambda^B)$ denotes quantum probability. If the probability distribution obtained by the experiment cannot obey this formula, then we say that a bipartite state is steerable from Alice to Bob.

As mentioned in the Introduction, many steering criteria have been proposed to judge whether a quantum state is steerable. Here we briefly introduce one typical criterion for arbitrary bipartite systems via correlation matrices [51]. Suppose that Alice and Bob share a bipartite state ρ on a Hilbert space. Then $\mathcal{A} = \{A_i : i = 1,2,\ldots,m\}$ and $\mathcal{B} = \{B_i : i = 1,2,\ldots,n\}$ are the local observables of the two sets of parties a and b, respectively. The corresponding correlation matrix can be written as

$$C(\mathcal{A},\mathcal{B}|\rho) = (c_{ij}), \tag{4}$$

with

$$c_{ij} = \mathrm{tr}[(\mathrm{A_i}\otimes \mathrm{B_j})(\rho - \rho_\mathrm{a}\otimes\rho_\mathrm{b})]. \tag{5}$$

Lai and Luo proposed and proved that if ρ is unsteerable from Alice to Bob, then

$$\|C(\mathcal{A},\mathcal{B}|\rho)\|_{\mathrm{tr}} \leqslant \sqrt{\Lambda_a\Lambda_b}, \tag{6}$$

where

$$\Lambda_a = \sum_{i=1}^m V(A_i,\rho_a),$$

$$\Lambda_b = \max_{\sigma_b}\left(\sum_{j=1}^n (\mathrm{tr}B_j\sigma_b)^2\right) - \sum_{j=1}^n (\mathrm{tr}B_j\rho_b)^2. \tag{7}$$

With respect to the matrix C, $\|C\|_{\mathrm{tr}}$ represents the trace norm, i.e., the sum of singular values. Additionally, the conventional variance of A_i in the state ρ_A is given by $V(A_i,\rho_a) = \mathrm{tr}(A_i^2\rho_a) - \mathrm{tr}(A_i\rho_a)^2$ and the maximum is over all states σ_b on Bob's side.

III. DETECTING EPR STEERING FOR TRIPARTITE SYSTEMS VIA CORRELATION MATRICES

Quantum steering describes the ability to instantaneously influence a subsystem in a two-body system by taking a measurement on the other subsystem. For a three-qubit system, if we would like to explore the system's steering, we have to divide the tripartite system into two parties. Here we divide it into two parties $1 \to 2$; then we can consider this system as the $\mathbb{C}^2\otimes\mathbb{C}^4$ state. Therefore, based on the steering criterion proposed by Lai and Luo, we extend the two-party criterion to a three-party version.

A complete set of LOOs $\{G_i : i = 1,2,\ldots,d^2\}$ form the orthonormal basis for all operators in the Hilbert space of a d-level system and satisfy the orthogonal relations $\mathrm{tr}(G_iG_j) = \delta_{ij}$. For a tripartite system, we divide it into two parties A and BC, and in this paper, two sets of LOOs G_m^A and G_n^{BC} are chosen for A and BC, respectively, to detect the steerability of ρ,

$$G_m^A = \frac{1}{\sqrt{2}}\sigma_m, \quad m\in\{0,1,2,3\}, \tag{8}$$

$$G_n^{BC} = \tfrac{1}{2}\sigma_{[n/4]}\otimes\sigma_{(n/4)}, \quad n\in\{0,1,2,\ldots,15\}, \tag{9}$$

where σ_m are the Pauli matrices and the signs $[n/4]$ and $(n/4)$ represent the integer function and remainder function, respectively.

Theorem 1. For an arbitrary tripartite state $\rho_{abc} = \frac{1}{8}\sum_{i,j,k=0}^{3}\Theta_{ijk}\sigma_i\otimes\sigma_j\otimes\sigma_k$, if

$$\|\mathbb{M}\|_{\mathrm{tr}} > \sqrt{\left[2-\mathrm{tr}\left(\rho_a^2\right)\right]\left[1-\mathrm{tr}\left(\rho_{bc}^2\right)\right]}, \tag{10}$$

then ρ is steerable from A to BC, where $\mathbb{M}$ is the correlation matrix constructed with two sets of LOOs G_m^A and G_n^{BC}, and $\Theta_{ijk} = \mathrm{tr}(\rho\sigma_i\otimes\sigma_j\otimes\sigma_k)$, the matrix elements, can be given by

$$M_{mn} = \Theta_{m[n/4](n/4)} - \Theta_{m00}\Theta_{0[n/4](n/4)}. \tag{11}$$

Proof. The elements of the correlation matrix $\mathbb{M}$ can be calculated by

$$M_{mn} = \mathrm{tr}\left[\left(G_m^A\otimes G_n^{BC}\right)(\rho_{abc}-\rho_a\otimes\rho_{bc})\right], \tag{12}$$

where the reduced states can be expressed as

$$\rho_a = \mathrm{tr}_{bc}(\rho_{abc}) = \frac{1}{2}\sum_{i=0}^{3}\Theta_{i00}\sigma_i, \tag{13}$$

$$\rho_{bc} = \mathrm{tr}_a(\rho_{abc}) = \frac{1}{4}\sum_{j,k=0}^{3}\Theta_{0jk}\sigma_j\otimes\sigma_k. \tag{14}$$

With regard to the Pauli matrices, we have $\mathrm{tr}(\sigma_m\sigma_n) = 2\delta_{mn}$. We substitute these formulas into Eq. (12) and obtain

$$\begin{aligned} M_{mn} &= \mathrm{tr}\left[\left(G_m^A\otimes G_n^{BC}\right)(\rho_{abc}-\rho_a\otimes\rho_{bc})\right] \\ &= \frac{1}{8}\sum_{i,j,k=0}^{3}(\Theta_{ijk}-\Theta_{i00}\Theta_{0jk})\mathrm{tr} \\ &\quad\times(\sigma_m\sigma_i)\mathrm{tr}(\sigma_j\sigma_{[n/4]})\mathrm{tr}(\sigma_k\sigma_{(n/4)}) \\ &= \Theta_{m[n/4](n/4)} - \Theta_{m00}\Theta_{0[n/4](n/4)}. \end{aligned} \tag{15}$$

As for the right-hand side of Eq. (10), Ref. [55] has pointed out that a d-dimensional single-particle state ρ' meets

$$\sum_i \mathrm{tr}\left(G_i^2\rho'\right) = d, \tag{16}$$

$$\sum_i \mathrm{tr}(G_i\rho')^2 = \mathrm{tr}(\rho'^2). \tag{17}$$

Thus, combining Eqs. (7), (16), and (17), we get

$$\Lambda_a = \sum_{i=1}^{m} V(G_i,\rho_a) = 2-\mathrm{tr}\left(\rho_a^2\right), \tag{18}$$

which belongs to the right-hand side of Eq. (10). One can find the maximum $\max_{\sigma_{bc}}[\sum_{j=1}^{n}(\mathrm{tr}G_n^{BC}\sigma_{bc})^2]$ for all possible quantum state σ_{bc}, namely,

$$\sum_{j=1}^{n}\left(\mathrm{tr}G_n^{BC}\sigma_{bc}\right)^2 = \mathrm{tr}\left(\sigma_{bc}^2\right), \tag{19}$$

where $\sigma_{bc} = \frac{1}{4}\sum_{i,j=0}^{3}T_{ij}\sigma_i\otimes\sigma_j$ and $\mathrm{tr}(\sigma_{bc}^2)$ is related to the purity of any two-particle states. Incidentally, the maximum of the purity can reach 1. As a result, we have

$$\begin{aligned} \Lambda_{bc} &= \max_{\sigma_{bc}}\left(\sum_{j=1}^{n}\left(\mathrm{tr}G_n^{BC}\sigma_{bc}\right)^2\right) - \sum_{j=1}^{n}\left(\mathrm{tr}G_n^{BC}\rho_{bc}\right)^2 \\ &= 1-\mathrm{tr}\left(\rho_{bc}^2\right). \end{aligned} \tag{20}$$

Based on Eqs. (15), (18) and (20), Eq. (10) has been proved.

In addition, considering the intrinsic asymmetry of quantum steering, we can judge whether BC can steer A relying on the following criterion. First, we can use a commutative operator that can change ρ_{abc} to ρ_{bca}. In this case, $\rho_{bca} = \frac{1}{8}\sum_{i,j,k=0}^{3}\Theta_{ijk}\sigma_j\otimes\sigma_k\otimes\sigma_i$. If

$$\|\mathbb{M}'\|_{\mathrm{tr}} > \sqrt{\left[4-\mathrm{tr}\left(\rho_{bc}^2\right)\right]\left[1-\mathrm{tr}\left(\rho_a^2\right)\right]}, \tag{21}$$

then ρ is steerable from BC to A, where $\mathbb{M}'$ is the correlation matrix, and the matrix elements are

$$\begin{aligned} M'_{nm} &= \mathrm{tr}\left[\left(G_n^{BC}\otimes G_m^A\right)(\rho_{bca}-\rho_{bc}\otimes\rho_a)\right] \\ &= \Theta_{[n/4](n/4)m} - \Theta_{[n/4](n/4)0}\Theta_{00m}. \end{aligned} \tag{22}$$

Here Theorem 1 presents a steering criterion for evaluating the steerability of a tripartite system. Then we define the difference of the left- and right-hand sides of the inequality as

$$H_{A\to BC} = \|\mathbb{M}\|_{\mathrm{tr}} - \sqrt{\left[2-\mathrm{tr}\left(\rho_a^2\right)\right]\left[1-\mathrm{tr}\left(\rho_{bc}^2\right)\right]}. \tag{23}$$

Physically, as long as $H_{A\to BC}$ is greater than 0, it means that ρ is steerable from A to BC. Therefore, the quantification of the steering can be expressed as

$$S_{A\to BC} = \max[H_{A\to BC}, 0]. \tag{24}$$

Canonically, one can get ρ_{ac} or ρ_{ab} when tracing out B or C. As a result, the corresponding steering of the states ρ_{ab}, ρ_{ac}, and ρ_{bc} can be written as

$$\begin{aligned} S_{A\to B} &= \max[H_{A\to B}, 0], \\ S_{A\to C} &= \max[H_{A\to C}, 0], \\ S_{B\to C} &= \max[H_{B\to C}, 0], \end{aligned} \tag{25}$$

where

$$\begin{aligned} H_{A\to B} &= \|C(G,G|\rho_{ab})\|_{\mathrm{tr}} - \sqrt{\Lambda_a\Lambda_b}, \\ H_{A\to C} &= \|C(G,G|\rho_{ac})\|_{\mathrm{tr}} - \sqrt{\Lambda_a\Lambda_c}, \\ H_{B\to C} &= \|C(G,G|\rho_{bc})\|_{\mathrm{tr}} - \sqrt{\Lambda_b\Lambda_c}. \end{aligned} \tag{26}$$

■

Theorem 2. Based on the criterion proposed above (Theorem 1), for any three-qubit pure state, the monogamy relation can be obtained as

$$S_{A\to BC} \geqslant H_{A\to B} + H_{A\to C} + H_{B\to C}. \tag{27}$$

A detailed proof of Theorem 2 is provided in the Appendix.

Corollary 1. By virtue of Theorem 2, if $H_{A\to B}$, $H_{A\to C}$, and $H_{B\to C}$ are greater than or equal to 0, the monogamy relation corresponding to the steering of tripartite system can be further generalized as

$$S_{A\to BC} \geqslant S_{A\to B} + S_{A\to C} + S_{B\to C}. \tag{28}$$

Proof. Logically speaking, if $H_{A\to B}$, $H_{A\to C}$, and $H_{B\to C}$ are all greater than 0, the following relations hold:

$$\begin{aligned} H_{A\to B} &= S_{A\to B}, \\ H_{A\to C} &= S_{A\to C}, \\ H_{B\to C} &= S_{B\to C}. \end{aligned} \tag{29}$$

Due to the above equivalence relations, Eq. (27) can be further rewritten as Eq. (28). ■

Corollary 2. On the basis of Theorem 2, if $H_{A\to B}$, $H_{A\to C}$, and $H_{B\to C}$ are less than 0, the monogamy relation corresponding to the steering of tripartite system can be further generalized as

$$S_{A\to BC} \geqslant S_{A\to B} + S_{A\to C} + S_{B\to C}. \qquad (30)$$

Proof. If $H_{A\to B}$, $H_{A\to C}$, and $H_{B\to C}$ are less than 0, the steering of ρ_{ab} and ρ_{ac} can be expressed as

$$\begin{aligned} S_{A\to B} &= \max[H_{A\to B}, 0] = 0,\\ S_{A\to C} &= \max[H_{A\to C}, 0] = 0, \qquad (31)\\ S_{B\to C} &= \max[H_{B\to C}, 0] = 0 \end{aligned}$$

and $S_{A\to BC} \geqslant 0$ holds; we thus have that Eq. (27) can be rewritten as Eq. (30). ■

IV. ILLUSTRATIONS

In what follows, several representative examples will be offered to illustrate the performance of our steering criteria and the monogamy relation, by employing the randomly generated states, the generalized Greenberger-Horne-Zeilinger (GHZ) state, and the generalized W state.

As a matter of fact, there are various effective methods to generate random states [56–58]. Here we proceed by introducing the method used for constructing random three-qubit states. It is well established that an arbitrary three-qubit state can be represented by its eigenvalues and normalized eigenvectors as

$$\rho = \sum_{n=1}^{8} \lambda_n |\Psi_n\rangle\langle\Psi_n|, \quad n \in \{1, 2, 3, 4, 5, 6, 7, 8\}. \qquad (32)$$

Herein λ_n can be interpreted as the probability that ρ is in the pure state $|\Psi_n\rangle$ and the normalized eigenvector of state can establish arbitrary unitary operations $E = \{\Psi_1, \Psi_2, \Psi_3, \Psi_4, \Psi_5, \Psi_6, \Psi_7, \Psi_8\}$. Thus, an arbitrary three-qubit state can be composed of an arbitrary probability set λ_n and an arbitrary E. The random number function $\Gamma(a_1, a_2)$ generates a random real number within a closed interval $[a_1, a_2]$. At first, we can generate eight random numbers in this way:

$$\begin{aligned} &\mathcal{N}_1 = \Gamma(0, 1), \quad \mathcal{N}_2 = \mathcal{N}_1\Gamma(0, 1),\\ &\mathcal{N}_3 = \mathcal{N}_2\Gamma(0, 1), \quad \mathcal{N}_4 = \mathcal{N}_3\Gamma(0, 1),\\ &\mathcal{N}_5 = \mathcal{N}_4\Gamma(0, 1), \quad \mathcal{N}_6 = \mathcal{N}_5\Gamma(0, 1),\\ &\mathcal{N}_7 = \mathcal{N}_5\Gamma(0, 1), \quad \mathcal{N}_8 = \mathcal{N}_7\Gamma(0, 1). \qquad (33) \end{aligned}$$

The random probability is the set of λ_n ($n \in \{1, 2, 3, 4, 5, 6, 7, 8\}$) controlled by random numbers $\mathcal{N}_m$, which is expressed as

$$\lambda_n = \frac{\mathcal{N}_m}{\sum_{m=1}^{8} \mathcal{N}_m}. \qquad (34)$$

In this way, we get a set of random probabilities in descending order. For the random generation of unitary operation, we first randomly give an eighth-order real matrix K by the random number function $f(-1, 1)$ with the closed interval $[-1, 1]$.

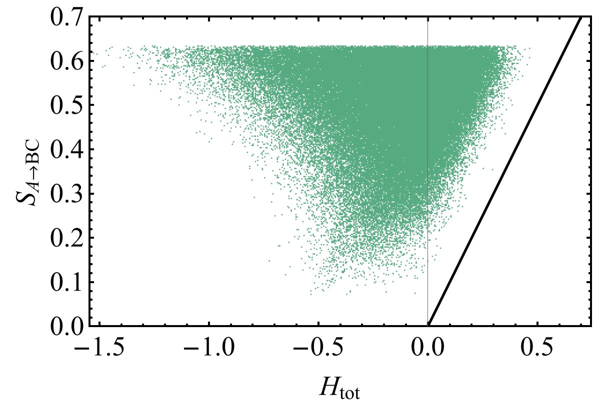

FIG. 1. Plot of $S_{A\to BC}$ vs H_{tot} for 10^5 randomly generated three-qubit pure states. Each green dot represents a random pure state. Here $H_{tot} = H_{A\to B} + H_{A\to C} + H_{B\to C}$ is set. The black line denotes the proportional function with a slope of unity.

Thus, we can construct a random Hermitian matrix by using the matrix K,

$$H = D + (U^{T} + U) + i(L^{T} - L), \qquad (35)$$

where D denotes the diagonal part of the real matrix K and L (U) represents the strictly lower (upper) triangular part of the real matrix K. The superscript T represents the transpose of the corresponding matrix.

By means of this method, we can obtain normalized eigenvectors $|\Psi_n\rangle$ of the Hermitian matrix H that forms the random unitary operation E. We thereby attain the random three-qubit state $\rho = \sum_{n=1}^{8} \lambda_n |\Psi_n\rangle\langle\Psi_n|$. Here $\mathcal{N}_1 = 1$ corresponds to the case of generating a three-qubit pure random state.

Example 1. By utilizing the above method, we prepare 10^5 random three-qubit pure states and plot $S_{A\to BC}$ versus $H_{tot} = H_{A\to B} + H_{A\to C} + H_{B\to C}$ in Fig. 1. Following the figure, the green dots corresponding to the 10^5 random states are always above the black diagonal line with the slope of unity, that is to say, the inequality (27) is held for all the generated random states.

Example 2. On the basis of Example 1, we extract those random states that satisfy the conditions of Corollaries 1 and 2 and draw the steering distribution in Figs. 2(a) and 2(b), respectively. One can easily see that $S_{A\to BC} \geqslant S_{tot} =$

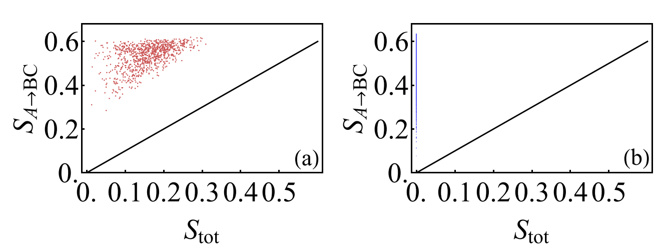

FIG. 2. Plots of (a) $S_{A\to BC}$ vs $S_{tot} = S_{A\to B} + S_{A\to C} + S_{B\to C}$ for the selected random states (the number of these selected random states, which satisfy $H_{A\to B} \geqslant 0$, $H_{A\to C} \geqslant 0$, and $H_{B\to C} \geqslant 0$, is 990) and (b) $S_{A\to BC}$ vs S_{tot} for the selected random states (the number of these selected random states, which satisfy $H_{A\to B} < 0$, $H_{A\to C} < 0$, and $H_{B\to C} < 0$, is 13 366). The black line denotes the proportional function with a slope of unity.

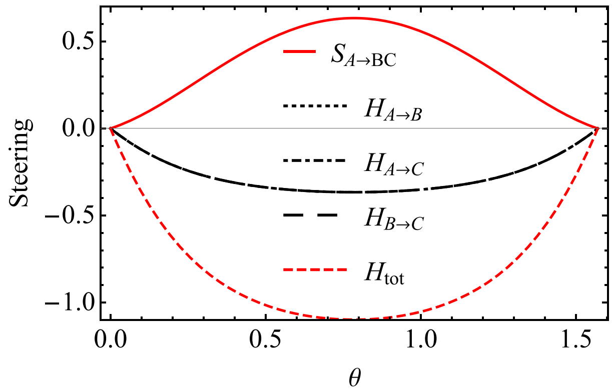

FIG. 3. Quantum steering vs the state's parameter θ in the case of the generalized GHZ state. The black dotted line denotes $H_{A\to B}$, the black dash-dotted line denotes $H_{A\to B}$, and the black dashed line denotes $H_{B\to C}$. Specifically, the three black lines are overlapped due to the same expressions (38). The red solid line denotes $S_{A\to BC}$ and the red dashed line represents $H_{\rm tot}$.

$S_{A\to B}+S_{A\to C}+S_{B\to C}$ is maintained, which essentially displays Corollaries 1 and 2.

Example 3. Let us first consider a type of three-qubit state, the generalized GHZ state, which can be described as

$$|\psi\rangle_{\rm GHZ}=\sin\theta|000\rangle+\cos\theta|111\rangle, \tag{36}$$

where $0\leqslant\theta\leqslant\pi/2$. Incidentally, the quantum state will become separable without any quantum correlation, when $\theta=0$ or $\pi/2$. To determine whether the state is steerable, we can judge by whether it conforms to the inequality (10). If the inequality is satisfied, it means that $\rho_1=|\psi_{\rm GHZ}\rangle\langle\psi_{\rm GHZ}|$ is steerable from A to BC. According to Eq. (10), we have

$$\begin{aligned}\|\mathbb{M}_1\|_{\rm tr}&=2|\cos\theta\sin\theta|+2\cos^2\theta\sin^2\theta,\\ 2-{\rm tr}\big(\rho_a^2\big)&=\tfrac{1}{4}(5-\cos4\theta),\\ 1-{\rm tr}\big(\rho_{bc}^2\big)&=2\cos^2\theta\sin^2\theta.\end{aligned} \tag{37}$$

For clarity, the variation trend of the corresponding steering $S_{A\to BC}$ with the coefficient θ is plotted in Fig. 3. As can be seen from Fig. 3, in the range of $\theta\in[0,2\pi]$, $S_{A\to BC}$ is always greater than 0, which illustrates the relative tightness of our EPR steering criterion (10).

On the basis of Eq. (25), the steering of subsystems ρ_{ab}, ρ_{ac}, and ρ_{bc} can be calculated as

$$\begin{aligned}H_{A\to B}=H_{A\to C}=H_{B\to C}&=2\cos^2\theta\sin^2\theta\\ &-\sqrt{[2-(\cos^4\theta+\sin^4\theta)][1-(\cos^4\theta+\sin^4\theta)]}.\end{aligned} \tag{38}$$

Figure 3 also has plotted $H_{A\to B}$, $H_{A\to C}$, $H_{B\to C}$, and $H_{\rm tot}$ as a function of the state's parameter θ. It is interesting to see that $H_{A\to B}$, $H_{A\to C}$, and $H_{B\to C}$ coincide perfectly and $H_{A\to B},H_{A\to C},H_{B\to C}\leqslant0$ and $S_{A\to BC}\geqslant H_{\rm tot}$ are satisfied all the time, which show the performance of Theorem 2 and Corollary 2, respectively.

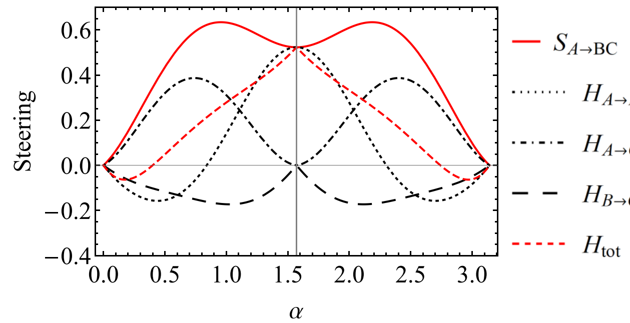

FIG. 4. Steering of the generalized W state vs the state's parameter α. The red solid line represents steering $S_{A\to BC}$; the black dotted line denotes $H_{A\to B}$; the black dash-dotted line denotes $H_{A\to B}$; the black dashed line denotes $H_{B\to C}$; the red dashed line represents $H_{\rm tot}$, which here means $H_{A\to B}+H_{A\to C}+H_{B\to C}$; and the gray vertical line represents $\alpha=\pi/2$.

Example 4. Let us consider another three-qubit state, the generalized W state, which can be expressed as

$$|\psi\rangle_W=\sin\theta\sin\alpha|100\rangle+\sin\alpha\cos\theta|010\rangle+\cos\alpha|001\rangle, \tag{39}$$

where $\theta\in[0,\pi]$ and $\alpha\in[0,\pi]$. Without loss of generality, we choose here $\theta=\frac{\pi}{3}$; hence the two sides of Eq. (10) can be expressed as

$$\begin{aligned}\|\mathbb{M}_2\|_{\rm tr}=&\sqrt{\tfrac{3}{8}(5+3\cos2\alpha)\sin^2\alpha}\\ &+\big|\tfrac{3}{32}(5+3\cos2\alpha)\sin^2\alpha\big|,\end{aligned} \tag{40}$$

$$2-{\rm tr}\big(\rho_a^2\big)=\tfrac{1}{64}(85-12\cos2\alpha-9\cos4\alpha), \tag{41}$$

$$2-{\rm tr}\big(\rho_{bc}^2\big)=\tfrac{3}{16}(5+3\cos2\alpha)\sin^2\alpha. \tag{42}$$

Consequently, $S_{A\to BC}$ can be drawn as a function of the state's parameter α in Fig. 4. It is straightforward to see that $S_{A\to BC}\geqslant0$, demonstrating the effectiveness of our criterion in detecting the steering for the generalized W state.

In addition, we have the trace norms of the correlation matrix and purities as

$$\begin{aligned}\big\|C\big(G,G|\rho_2^{AB}\big)\big\|_{\rm tr}&=\frac{\sqrt{3}}{2}\sin^2\alpha+\frac{3}{8}\sin^4\alpha,\\ \big\|C\big(G,G|\rho_2^{AC}\big)\big\|_{\rm tr}&=\frac{\sqrt{3}}{2}\sin2\alpha+\frac{3}{8}\sin^22\alpha,\\ \big\|C\big(G,G|\rho_2^{BC}\big)\big\|_{\rm tr}&=\frac{1}{2}\sin2\alpha+\frac{1}{2}\sin^2\alpha\cos^2\alpha,\\ {\rm tr}\big(\rho_a^2\big)&=\frac{1}{64}(43+12\cos2\alpha+9\cos4\alpha),\\ {\rm tr}\big(\rho_b^2\big)&=\frac{1}{64}(51+12\cos2\alpha+\cos4\alpha),\\ {\rm tr}\big(\rho_c^2\big)&=\cos^4\alpha+\sin^4\alpha.\end{aligned} \tag{43}$$

By combining Eqs. (26) and (43), $H_{A\to B}$, $H_{A\to C}$, and $H_{B\to C}$ can be worked out exactly. All the above quantities, $S_{A\to BC}$, and $H_{\rm tot}$ with respect to the state's parameter α are depicted in Fig. 4. It is apparent that $S_{A\to BC}$ (the red solid line) consistently exceeds or equals $H_{\rm tot}$ (the red dashed line), and

$H_{A\to B}$, $H_{A\to C}$, and $H_{B\to C}$ are greater than or equal to 0, for $\alpha = \pi/2$ (the gray vertical line in the figure). With these in mind, we say that Theorem 2 and Corollary 1 are illustrated in the current architecture.

V. CONCLUSION

Multipartite quantum steering is considered as a promising and significant resource for implementing various quantum communication tasks in quantum networks, which consist of multiple observers sharing multipartite quantum states. In this paper we have derived the steering criterion for tripartite systems based on the correlation matrix, which might be of fundamental importance in prospective quantum networks. In particular, we utilize LOOs as local measurements to provide operational criteria of quantum steering.

Furthermore, we have put forward the monogamy relation between tripartite steering $S_{A\to BC}$ and H_{tot} of the subsystems, based on our derived criterion. We proved that $S_{A\to BC} \geqslant H_{A\to B} + H_{A\to C} + H_{B\to C}$ is always satisfied for arbitrary pure tripartite states. In addition, we presented two corollaries in terms of the proposed theorem. At the same time, we employed various types of states, including the randomly generated three-qubit pure states, generalized GHZ states, and generalized W states, as illustrations for our findings. These examples also show the detection ability of our steering criterion. We believe our criterion provides a valuable methodology for detecting the steerability of any three-qubit quantum states, which may be constructive to generalize into steering criteria for multipartite states in the future.

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APPENDIX

Basically, the Schmidt decomposition for an arbitrary three-particle pure state can be written in the form [59]

$$|\Psi\rangle = x|000\rangle + ye^{i\phi}|100\rangle + z|101\rangle + h|110\rangle + \lambda|111\rangle, \tag{A1}$$

where $x^2 + y^2 + z^2 + h^2 + \lambda^2 = 1$ and $x, y, z, h, \lambda \geqslant 0$ are maintained. For simplicity, we can set $\phi = 0$ and $\lambda = 0$. To prove Theorem 2, we first need to prove the inequality

$$H_{A\to BC} \geqslant H_{A\to B} + H_{A\to C} + H_{B\to C}. \tag{A2}$$

In accordance with Eqs. (23) and (26), we can obtain $H_{A\to BC}$, $H_{A\to B}$, $H_{A\to C}$, and $H_{B\to C}$ for any pure three-qubit state. If we want to prove that the inequality (A2) is valid, we only need to prove that

$$H_{A\to BC} - (H_{A\to B} + H_{A\to C} + H_{B\to C}) \geqslant 0. \tag{A3}$$

As a result, the left item of the resulting formula can be reexpressed as

$$\begin{aligned} f(x,y,z,h) &= H_{A\to BC} - (H_{A\to B} + H_{A\to C} + H_{B\to C}) \\ &= \sqrt{2}z\sqrt{[1+2x^2(z^2+h^2)](x^2+h^2)} - xh - z(x+h) + \sqrt{2}h\sqrt{[1+2x^2(z^2+h^2)](x^2+z^2)} \\ &\quad + 2x\sqrt{z^2+h^2} + 2x^2(z^2+h^2) - zh\sqrt{8zh + (-1+2y^2+2zh)^2} + \sqrt{2}z\sqrt{[1+2h^2(z^2+x^2)](x^2+h^2)} \\ &\quad - \sqrt{2}x\sqrt{[1+2x^2(z^2+h^2)](z^2+h^2)} - \tfrac{1}{2}(|f+g| + |f-g| + |w+v| + |w-v|), \end{aligned} \tag{A4}$$

with

$$\begin{aligned} f &= x(h - 2y^2h + 2x^2h), \quad g = x\sqrt{h^2\{1+4y^4-4y^2(1+2xh+h^2) - 4h[x+(-1+z^2)h+h^3]\}}, \\ w &= x(z - 2y^2z + 2x^2z), \quad v = x\sqrt{z^2\{1+4y^4-4y^2(1+2xz+z^2) - 4z[x+(-1+h^2)z+z^3]\}}. \end{aligned} \tag{A5}$$

More specifically, if we can prove that $f(x,y,z,h) \geqslant 0$ in the region of $x^2+y^2+z^2+h^2 = 1$ with $x \geqslant 0, y \geqslant 0, z \geqslant 0$, and $h \geqslant 0$, then the inequality (A2) is proved. There are four absolute values in the above formula. In order to facilitate calculation, we can divide the region for removing the absolute value symbols. As a matter of fact, the region can be divided into 16 subregions and the internal and external boundaries.

In addition, we make use here of the Lagrange multiplier method to prove the inequity (A2). If the local minima of Eq. (A4) are greater than 0, it indicates the desired inequality is true, as the function $f(x,y,z,h)$ is continuous. According to the Lagrange multiplier method, to solve the local minima of $f(x,y,z,h)$ under the condition $x^2+y^2+z^2+h^2-1=0$, we require constructing the Lagrange function

$$f(x,y,z,h,k) = f(x,y,z,h) - k(-1+x^2+y^2+z^2+h^2), \tag{A6}$$

where k denotes the Lagrange multiplier. Then we take the derivatives of x, y, z, h, and k as

$$\frac{\partial f(x,y,z,h,k)}{\partial x} = 0,$$
$$\frac{\partial f(x,y,z,h,k)}{\partial y} = 0,$$
$$\frac{\partial f(x,y,z,h,k)}{\partial z} = 0,$$

TABLE I. Specific situation of 16 regions and the number of the critical points of the corresponding regions.

$f+g$	$f-g$	$w+v$	$w-v$	No. of critical points
+	+	+	+	2
+	+	+	−	0
+	+	−	+	0
+	−	+	+	0
−	+	+	+	0
−	−	+	+	0
−	+	−	+	0
+	−	−	+	0
+	+	−	−	0
−	+	+	−	0
+	−	+	−	0
−	−	+	−	0
−	+	−	−	0
+	−	−	−	0
−	−	−	+	0
−	−	−	−	0

$$\frac{\partial f(x,y,z,h,k)}{\partial h} = 0,$$
$$\frac{\partial f(x,y,z,h,k)}{\partial k} = 0, \tag{A7}$$

respectively. The solutions satisfying these equations are called the critical points. Finally, we choose the critical point satisfying conditions $x \geqslant 0$, $y \geqslant 0$, $z \geqslant 0$, and $h \geqslant 0$ to get the local minimum of $f(x, y, z, h)$.

All the subregions and the number of corresponding critical points are shown in Table I. After careful computation, two critical points can be found as

$$(x,y,z,h) = \begin{cases} (0.390\,368\,239\,272\,184\,67,\ 0, \\ 0.788\,617\,685\,785\,144\,8,\ 0.475\,073 \\ 450\,567\,846\,94) \\ (0.390\,368\,239\,272\,127\,77,\ 0, \\ 0.788\,617\,685\,785\,211\,7,\ 0.475\,073 \\ 450\,567\,758\,7). \end{cases}$$

Then we can substitute the critical points into Eq. (A4); the same local minimum $f_{\min}(x, y, z, h) = 0.361\,084$ is obtained, which is obviously greater than 0. In other words, the inequality (A2) is held in the subregions.

In addition to the critical points within these 16 subregions, there may also exist critical points on the boundaries including internal ones and external ones. Next let us turn to consider the cases of the region's boundaries. The internal boundaries refer to those with $f = g$ and $w = v$, while the external ones refer to those with $x = 0$, $y = 0$, $z = 0$, or $h = 0$. Likewise, we take advantage of the Lagrange multiplier method to judge whether the inequality is valid on the boundaries. In what follows, we will discuss the cases of the internal and external boundaries, respectively.

With respect to the internal boundary, there exist three cases, i.e.,

$$\begin{aligned} |f+g| + |f-g| + |w+v| + |w-v| &= 2f + 2w \quad \text{for } f = g > 0,\ w = v > 0, \\ |f+g| + |f-g| + |w+v| + |w-v| &= 2f - 2w \quad \text{for } w = -v < 0,\ f = g > 0, \\ |f+g| + |f-g| + |w+v| + |w-v| &= 2w - 2f \quad \text{for } w = v > 0,\ f = -g < 0. \end{aligned} \tag{A8}$$

By the Lagrange multiplier method, we obtain the critical points shown in Table II.

For the external boundaries, the critical points can be divided into the following cases. On the boundary with $x = 0$, the absolute value items of Eq. (A4) will disappear and the function consequently can be simplified into

$$\begin{aligned} f(y,z,h) = {}& \sqrt{2}(2zh + zh\sqrt{1+2z^2h^2}) \\ & - hz[1 + \sqrt{4y^4 + (1+2hz)^2 + y^2(-4+8hz)}]. \end{aligned} \tag{A9}$$

On the boundary of $y = 0$, the corresponding function can be reexpressed as

$$\begin{aligned} f(x,z,h) = {}& -4x^2 + 4x^4 + \sqrt{2}z[\sqrt{(-1-2h^2+2h^4)(-1+z^2)} + \sqrt{(-1-2x^2+2x^4)(-1+z^2)}] \\ & + x(2\sqrt{1-x^2} - \sqrt{2}\sqrt{1+x^2-4x^4+2x^6} - 2z) \\ & + h\{-2x + \sqrt{2}\sqrt{(-1+h^2)(-1-2x^2+2x^4)} - z[1 + \sqrt{(1+2hz)^2}]\}. \end{aligned} \tag{A10}$$

To be explicit, we have listed all the critical points of the five cases mentioned above in Table II. One critical point is $(x, y, z, h) = (0, 0, 0.707\,107, 0.707\,107)$ and we obtain the local minimum $f_{\min}(x, y, z, h) = 0.780\,239$; other critical point is $(x, y, z, h) = (0, 1, 0, 0)$ and $f_{\min}(x, y, z, h) = 0$ is obtained. Apparently, all the local minima are greater than or equal to 0, showing that the inequality (A2) is held on these boundaries.

TABLE II. Number of critical points of boundaries.

$2f+2w$	$2f-2w$	$2w-2f$	$x=0$	$y=0$
0	0	0	2	0

TABLE III. Number of critical points in different cases on the boundary $z = 0$.

$f_1 + g_1$	$f_1 - g_1$	No. of critical points
+	+	0
+	−	0
−	+	0
−	−	0

On the boundary of $z = 0$ and $h = 0$, the function f can be written as

$$f(x, y, h) = hx + 2h^2x^2 - \tfrac{1}{2}(|f_1 + g_1| + |f_1 - g_1|), \quad \text{(A11)}$$

$$f(x, y, z) = (\sqrt{2} + 1)zx + 2z^2x^2 - \tfrac{1}{2}(|w_1 + v_1| + |w_1 - v_1|), \quad \text{(A12)}$$

with $f_1 = x(h + 2h^2x - 2hy^2)$, $w_1 = 2x^2z^2 + x(z - 2y^2z)$, $g_1 = x\sqrt{h^2[1 - 4h(-h+h^3+x) - 4(1+h^2 + 2hx)y^2 + 4y^4]}$, and $v_1 = x\sqrt{z^2[1+4y^4 - 4y^2(1+2xz + z^2) - 4z(x - z+z^3)]}$. The numbers of critical points on the boundary of $z = 0$ and $h = 0$ are listed in Tables III and IV, respectively. The only critical point is found as $(x, y, z, h) = (0, 1, 0, 0)$, and we compute that the corresponding local minimum is equal to 0.

TABLE IV. Number of critical points in different cases on the boundary $h = 0$.

$w_1 + v_1$	$w_1 - v_1$	No. of critical points
+	+	1
+	−	0
−	+	0
−	−	0

Thus, the inequality (A2) is satisfied on the boundaries with $z = 0$ and $h = 0$ as well.

In summary, the local minima of the function $f_{\min}(x, y, z, h) \geqslant 0$ are satisfied all the time in the whole region, consisting of the above 16 subregions and all boundaries. Therefore, the inequality

$$H_{A\to BC} \geqslant H_{A\to B} + H_{A\to C} + H_{B\to C} \quad \text{(A13)}$$

holds for all three-qubit pure states. Owing to Eq. (24), we have $S_{A\to BC} \geqslant H_{A\to BC}$. In combination with the above inequality (A13), the monogamy relation (27) can be obtained as

$$S_{A\to BC} \geqslant H_{A\to B} + H_{A\to C} + H_{B\to C}. \quad \text{(A14)}$$

As a consequence, Theorem 2 has been proved.

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