

Quadrature coherence scale of linear combinations of Gaussian functions in phase spaceAnaëlle Hertz *National Research Council of Canada, 100 Sussex Drive, Ottawa, Ontario, Canada K1N 5A2*Aaron Z. Goldberg *National Research Council of Canada, 100 Sussex Drive, Ottawa, Ontario, Canada K1N 5A2
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(Received 6 February 2024; revised 26 April 2024; accepted 10 June 2024; published 1 July 2024)

The quadrature coherence scale (QCS) is a recently introduced measure that was shown to be an efficient witness of nonclassicality. It takes a simple form for pure and Gaussian states, but a general expression for mixed states tends to be prohibitively unwieldy. In this paper we introduce a method for computing the quadrature coherence scale of quantum states characterized by Wigner functions expressible as linear combinations of Gaussian functions. Notable examples within this framework include cat states, Gottesman-Kitaev-Preskill states, and states resulting from Gaussian transformations, measurements, and breeding protocols. In particular, we show that the quadrature coherence scale serves as a valuable tool for examining the scalability of nonclassicality in the presence of loss. Our findings lead us to put forth a conjecture suggesting that, subject to 50% loss or more, all pure states lose any QCS-certifiable nonclassicality. We also consider the quadrature coherence scale as a measure of quality of the output state of the breeding protocol.

DOI: [10.1103/PhysRevA.110.012408](https://doi.org/10.1103/PhysRevA.110.012408)**I. INTRODUCTION**

Counterintuitive quantum phenomena including superposition and entanglement have transformed from problems [1] to curiosities [2–5] to features that confer advantages in multiple domains [6–13]. Superposition and entanglement are based off of coherence, which underlies all interference effects. This makes the generation [14–16], manipulation [17–20], quantification [21–27], and measurement [28–33] of coherence essential tasks, especially when viewed through the lens of coherence as a resource [21,22,34–36].

Coherence between macroscopically distinct states is a good indicator of quantumness or nonclassicality. Several measures and witnesses of nonclassicality have been introduced, with their strengths and weaknesses. For example, the distance to the set of classical set [37] is a precise measure of nonclassicality, but is typically difficult to evaluate because it requires an optimization over a set of states. The negative volume of the Wigner function [38] is also a very popular witness that can be hard to compute due to looking for small rapid interferences via numerical integration. The latter also fails to detect many nonclassical states, e.g., squeezed states. The quadrature coherence scale (QCS) was recently introduced as such an indicator, quantifying both the amount of coherence and the macroscopicity of the coherence in a quantum state; the QCS is a bona fide witness of optical nonclassicality [39,40] with the important advantage of bounding the distance to the set of classical states while being relatively

easy to compute. Let us note also that for pure states QCS is equivalent to many other nonclassicality witnesses [41]. In addition, an interferometric scheme with two identical copies of a state suffices for measuring the QCS of the state without requiring full state tomography [42], unlike the nonclassicality distance or Wigner negativity, which require full tomography and then full optimization or integration; the QCS measurement was recently demonstrated on a cloud quantum computer [41].

For states that are pure or Gaussian, the QCS is simple to compute. However, a general computation requires calculations involving multiple integrals over a state's entire Wigner function, which is essential to practical situations because non-Gaussian states are crucial to linear optical quantum computation [43–46] and pure states are rare.

In this work we show that computation of the QCS is significantly easier for states that can be written as superpositions and convex combinations of Gaussian states, which includes non-Gaussian states that are important to quantum metrology and computation. This takes advantage of a recently developed formalism for states whose Wigner functions are linear combinations of not-necessarily-real Gaussian functions [47] and for simulating quantum optics with finite-rank superpositions of coherent states [48].

This description of states might sound restrictive, but it actually represents a large family of relevant states. As an example, Schrödinger cat states are important non-Gaussian states used for demonstrating quantumness [49],

quantum-enhanced measurement sensitivity [50], and error-correcting codes [51] and belong to this class of superpositions of Gaussian states. They can be combined via linear optical networks and homodyne measurement can be performed on some branches while remaining in this class, allowing them to iteratively “breed” [15,52,53] into another important type of states known as Gottesman-Kitaev-Preskill (GKP) states [54]. Moreover, such states subject to the dominant noise source for photonics, viz., loss, still remain in this class, as do states subject to any Gaussian channels. In contrast, other nonclassicality measures such as quantum Fisher information that may be easy to compute for pure states must be computed from scratch with more challenging calculations when the state becomes mixed due to loss. These allow the QCS to measure the nonclassicality and thereby the quality of a state intended for fault-tolerant quantum computation as it improves via breeding and degrades via loss.

In addition, using the description of the P function [see Eq. (6)], one can see that any state can be written as a combination of Gaussian (coherent) states, at least formally, because the coefficients in this combination need not be positive. Moreover, it was recently proven that any pure state can be represented with arbitrary precision as a *finite* combination of coherent states, while all mixed states are just convex combinations of such [48]. Our framework is thus fully general and allows the computation of the QCS of any state belonging to the correct class subject to any Gaussian transformations and measurements. It was developed with GKP states in mind and may prove useful throughout multimode quantum optics with continuous variables.

The paper is divided as follows. In Secs. II and III we start by introducing the main concept of quantum optics in phase space and then introduce the quadrature coherence scale, respectively. In Sec. IV we define the formalism related to the expression of the Wigner function of a state as a linear combination of Gaussian functions and in Sec. V we show how to compute the QCS for such states. In Sec. VI we give concrete examples: We compute the QCS of cat states and GKP states, we study the scalability of the QCS through a loss channel, and we consider the QCS as a measure of fidelity for the output of a breeding protocol. In Sec. VII we discuss other nonclassicality measures. We summarize in Sec. VIII and discuss some future research problems.

II. PHASE-SPACE FORMALISM

In this section we provide a brief overview of the symplectic formalism employed for continuous-variable states in quantum optics. More details can be found, for example, in [55,56].

A continuous-variable system is represented by n modes. Associated with each of them are the annihilation and creation operators a_i and a_i^\dagger verifying the bosonic commutation relation $[a_i, a_i^\dagger] = 1$. We define the vector of quadratures $\hat{\mathbf{r}} = (\hat{x}_1, \hat{p}_1, \hat{x}_2, \hat{p}_2, \dots, \hat{x}_n, \hat{p}_n)$, where¹

$$\hat{x}_j = \frac{1}{\sqrt{2}}(a_j + a_j^\dagger), \quad \hat{p}_j = -\frac{i}{\sqrt{2}}(a_j - a_j^\dagger) \quad \forall j = 1, \dots, n.$$

¹Note that we employ units in which $\hbar = 1$ throughout this paper.

Each quantum state ρ can be described by a Wigner function

$$W(\boldsymbol{\alpha}) = \frac{1}{\pi^{2n}} \int \chi(\mathbf{z}) e^{\mathbf{z} \cdot \boldsymbol{\alpha} - \mathbf{z} \cdot \bar{\boldsymbol{\alpha}}} d^{2n} \mathbf{z}, \quad (1)$$

where $\chi(\mathbf{z}) = \text{Tr}[\rho D(\mathbf{z})]$ is the characteristic function and $D(\mathbf{z}) = e^{\mathbf{z} \cdot \hat{\mathbf{a}}^\dagger - \bar{\mathbf{z}} \cdot \hat{\mathbf{a}}}$ the displacement operator. The Wigner function is normalized to 1 but can take negative values, hence the qualification of *quasiprobability* distribution.

The first-order moments of a state ρ constitute the displacement vector, defined as $\boldsymbol{\mu} = \langle \hat{\mathbf{r}} \rangle = \text{Tr}(\hat{\mathbf{r}} \rho)$, while the second moments make up the symmetric covariance matrix $\boldsymbol{\gamma}$ whose elements are given by

$$\gamma_{ij} = \frac{1}{2} \langle \{\hat{r}_i, \hat{r}_j\} \rangle - \langle \hat{r}_i \rangle \langle \hat{r}_j \rangle, \quad (2)$$

where $\{\cdot, \cdot\}$ represents the anticommutator. To be the covariance matrix of a valid quantum state, $\boldsymbol{\gamma}$ must satisfy the uncertainty principle $\boldsymbol{\gamma} + \frac{i}{2} \boldsymbol{\Omega} \geq 0$, with

$$\boldsymbol{\Omega} = \bigoplus_{j=1}^n \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

the symplectic form.

A Gaussian state ρ_G is fully characterized by its displacement vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\gamma}$. The name comes from the fact that its Wigner (and characteristic) function is a Gaussian function in the phase space:

$$W_G(\mathbf{r}) = \frac{\exp\left[-\frac{1}{2}(\mathbf{r} - \boldsymbol{\mu})^T \boldsymbol{\gamma}^{-1}(\mathbf{r} - \boldsymbol{\mu})\right]}{(2\pi)^n \sqrt{\det \boldsymbol{\gamma}}}. \quad (3)$$

A Gaussian transformation is a transformation that will map a Gaussian state onto a Gaussian state. In the phase space, this translates into updating the covariance matrix and mean value of the Gaussian function as follows:

$$\boldsymbol{\gamma} \rightarrow X \boldsymbol{\gamma} X^T + Y, \quad \boldsymbol{\mu} \rightarrow X \boldsymbol{\mu} + \mathbf{d}. \quad (4)$$

When the Gaussian transformation is unitary, X is a symplectic matrix and $Y = 0$. This includes displacement, rotation, and squeezing operations. Another example of a Gaussian transformation is the loss channel, where

$$X = \sqrt{\eta} \mathbb{1}, \quad Y = (1 - \eta) \mathbb{1} / 2, \quad \mathbf{d} = 0. \quad (5)$$

III. GENERAL COMPUTATION OF THE QCS

A state ρ is said to be (optically) classical [57] if and only if there exists a positive function $P(\alpha)$, called the Sudarshan-Glauber P function, such that

$$\rho = \int P(\alpha) |\alpha\rangle \langle \alpha| d\alpha, \quad (6)$$

where $|\alpha\rangle = D(\alpha)|0\rangle$ is a coherent state, $D(\alpha)$ is the displacement operator, and $|0\rangle$ is the vacuum state. Otherwise, the state is said to be nonclassical. In other words, a state is said to be nonclassical if it is not a probabilistic mixture of coherent states.

It is typically challenging to determine if such a P function exists and even more so to assess its positivity. Therefore, there is a need for defining measures and witnesses of nonclassicality. The most famous one is probably the negativity of the Wigner function: If the Wigner function of a state ρ

takes negative values at any point on the phase space, the state is nonclassical [38]. Nevertheless, many other measures and witnesses exist [37,58–79]. In this work we focus on the quadrature coherence scale, a recently introduced witness of nonclassicality [39–41,80,81].

Nonclassicality is linked to the coherences present in a state, but, while many nonclassicality witnesses and measures consider the size of the coherences, the QCS also measures where those coherences are located, showcasing a different aspect of the nonclassicality of the state. As an example, the nonclassicality of a squeezed state will be detected by the QCS while it is concealed from the Wigner negativity.

The QCS was calculated for several benchmark states including Fock states, squeezed thermal states, and cat states [39], as well as states with more complicated QCSs like photon-added and -subtracted Gaussian states [81]. For an n -mode mixed state ρ , the QCS (squared), \mathcal{C}^2 can be computed through the Wigner function as²

$$\mathcal{C}_\rho^2 = \frac{1}{2n} \frac{\int |\nabla W(\mathbf{r})|^2 d^2\mathbf{r}}{\int |W(\mathbf{r})|^2 d^2\mathbf{r}}. \quad (7)$$

For a pure state $|\psi\rangle$, it simplifies to the total noise (i.e., the sum of the variances) and many other indicators of nonclassicality as enumerated in Ref. [41]. It is thus the trace of the covariance matrix \mathbf{y} :

$$\mathcal{C}_{|\psi\rangle}^2 = \frac{1}{n} \text{Tr} \mathbf{y} = \frac{1}{n} \sum_{j=1}^n [(\Delta x_j)^2 + (\Delta p_j)^2]. \quad (8)$$

For a Gaussian state ρ_G , the QCS also takes a simpler form as it is proportional to the trace of the inverse of the covariance matrix:

$$\mathcal{C}_{\rho_G}^2 = \frac{1}{4n} \text{Tr} \mathbf{y}^{-1}. \quad (9)$$

The QCS is not a measure, but a witness of nonclassicality. All classical states have $\mathcal{C}_\rho \leq 1$. Therefore, a value of the QCS greater than 1 certifies that the state is nonclassical. In addition, despite not being a proper measure of nonclassicality, it was proven in [40] that the distance³ $D(\rho, \mathcal{E}_{\text{cl}})$ between the state ρ and the set of nonclassical states \mathcal{E}_{cl} is bounded by the QCS in the following way:

$$\mathcal{C}_\rho - 1 \leq D(\rho, \mathcal{E}_{\text{cl}}) \leq \mathcal{C}_\rho. \quad (10)$$

Hence, a state with $\mathcal{C}_\rho \leq 1$ will be either classical or so weakly nonclassical that it precludes certification, while $\mathcal{C}_\rho > \mathcal{C}_\sigma + 1$ implies that a state ρ is more nonclassical than σ .

IV. LINEAR COMBINATION OF GAUSSIAN FUNCTIONS IN PHASE SPACE

It is convenient to work only with Gaussian states, but this limits the pool of possibility. However, one can consider

²Here the vector \mathbf{r} contains all the quadratures x_j and p_j of each mode j and we use the correspondence $\alpha_j = \frac{x_j + ip_j}{\sqrt{2}}$.

³The distance is defined as $D(\rho, \mathcal{E}_{\text{cl}}) = \inf_{\sigma \in \mathcal{E}_{\text{cl}}} \|\tilde{\rho} - \tilde{\sigma}\|$ with $\|A\| = \sqrt{\langle A|A \rangle}$ and $\tilde{\rho} = \rho / \sqrt{\text{Tr} \rho^2}$.

linear combinations of Gaussian functions that describe non-Gaussian states. This then describes a much larger family of states, yet allows us to still use many of the properties of Gaussian states.

Let ρ be an n -mode state whose Wigner function can be written as a linear combination of Gaussian functions in phase space with mean values $\boldsymbol{\mu}_m$ and covariance matrix $\boldsymbol{\gamma}_m$:

$$W(\mathbf{r}) = \sum_m c_m G_{\boldsymbol{\mu}_m, \boldsymbol{\gamma}_m}(\mathbf{r}), \quad (11)$$

with

$$G_{\boldsymbol{\mu}_m, \boldsymbol{\gamma}_m}(\mathbf{r}) = \frac{\exp\left[-\frac{1}{2}(\mathbf{r} - \boldsymbol{\mu}_m)^T \boldsymbol{\gamma}_m^{-1}(\mathbf{r} - \boldsymbol{\mu}_m)\right]}{\sqrt{\det(2\pi \boldsymbol{\gamma}_m)}}. \quad (12)$$

Since a Wigner function is normalized to 1, we have $\sum_m c_m = 1$. Note that if $G_{\boldsymbol{\mu}_m, \boldsymbol{\gamma}_m}$ represents a valid quantum state, then it is equivalent to Eq. (3) and is real. However, $G_{\boldsymbol{\mu}_m, \boldsymbol{\gamma}_m}(x)$ can also be a complex (normalized) function, with Eq. (11) still describing a valid quantum state. More details can be found in Ref. [47], where this definition was introduced.

Writing the Wigner function in the form of Eq. (11) is useful to describe, for example, cat states, Fock states, and GKP states. We will develop this last example in the next section. Even more interesting is that this formalism is useful in describing the resultant state when a Gaussian transformation or measurement is applied to a state of the form of Eq. (11). Indeed, a Gaussian transformation maps a Gaussian state into another Gaussian state; hence by linearity, a state of the form of Eq. (11) will still be written in this specific form after the Gaussian transformation. In addition, a deterministic Gaussian map (such as a loss channel) will change neither the number nor the weights of the coefficients c_m in the sum, and each covariance matrix $\boldsymbol{\gamma}_m$ and mean value vector $\boldsymbol{\mu}_m$ will be updated according to Eq. (4). Once again, we refer the reader to Ref. [47] for more details regarding the application of Gaussian transformations to states written as a linear combination of Gaussian functions. We also direct the reader's attention to Ref. [82] for a more advanced work with linear combinations of Gaussian functions.

Note that, in theory, every state can be formatted in this way, but the description might involve arbitrary infinite sums or integrals, which makes this formalism impractical. If one is willing to tolerate any arbitrarily small error in a description of their state, then any state can be written in this form with only a finite number of terms [48].

V. COMPUTATION OF THE QCS FOR A STATE WHOSE WIGNER FUNCTION CAN BE WRITTEN AS A SUM OF GAUSSIAN FUNCTIONS

The crucial observation is that all of the integrals required for computing the QCS simplify to sums of integrals of products of Gaussians. Since products of Gaussians yield new Gaussians and integration of Gaussians is well understood, closed-form solutions for the QCS of this class of states can be established, as follows.

The denominator of Eq. (7) is proportional to the purity $\text{Tr} \rho^2$ of the state, which can be computed as follows, using

the definition (11) of the Wigner function:

$$\begin{aligned}
\text{Tr}\rho^2 &= (2\pi)^n \int |W(\mathbf{r})|^2 d^2\mathbf{r} \\
&= (2\pi)^n \int \left| \sum_m c_m G_{\boldsymbol{\mu}_m, \boldsymbol{\gamma}_m}(\mathbf{r}) \right|^2 d^2\mathbf{r} \\
&= (2\pi)^n \int \sum_m \sum_n c_m \bar{c}_n G_{\boldsymbol{\mu}_m, \boldsymbol{\gamma}_m}(\mathbf{r}) G_{\bar{\boldsymbol{\mu}}_n, \bar{\boldsymbol{\gamma}}_n}(\mathbf{r}) d^2\mathbf{r} \\
&= (2\pi)^n \sum_m \sum_n c_m \bar{c}_n \\
&\quad \times \int G_{d, \Gamma}(\mathbf{r}) e^A \sqrt{\frac{\det(2\pi \Gamma)}{\det(2\pi \boldsymbol{\gamma}_m) \det(2\pi \bar{\boldsymbol{\gamma}}_n)}} d^2\mathbf{r}.
\end{aligned}$$

Here we used the fact that the product of two Gaussian functions gives a new Gaussian function [see, for example, Eq. (337) in Ref. [83]], with

$$\Gamma^{-1} = \boldsymbol{\gamma}_m^{-1} + \bar{\boldsymbol{\gamma}}_n^{-1}, \quad (13)$$

$$\mathbf{d} = \Gamma(\boldsymbol{\gamma}_m^{-1} \boldsymbol{\mu}_m + \bar{\boldsymbol{\gamma}}_n^{-1} \bar{\boldsymbol{\mu}}_n), \quad (14)$$

$$\begin{aligned}
A &= \frac{1}{2}(\boldsymbol{\mu}_m^T \boldsymbol{\gamma}_m^{-1} + \bar{\boldsymbol{\mu}}_n^T \bar{\boldsymbol{\gamma}}_n^{-1}) \Gamma(\boldsymbol{\gamma}_m^{-1} \boldsymbol{\mu}_m + \bar{\boldsymbol{\gamma}}_n^{-1} \bar{\boldsymbol{\mu}}_n) \\
&\quad - \frac{1}{2}(\boldsymbol{\mu}_m^T \boldsymbol{\gamma}_m^{-1} \boldsymbol{\mu}_m + \bar{\boldsymbol{\mu}}_n^T \bar{\boldsymbol{\gamma}}_n^{-1} \bar{\boldsymbol{\mu}}_n). \quad (15)
\end{aligned}$$

Since the Gaussian function is normalized, $\int G_{d, \Gamma}(\mathbf{r}) d^2\mathbf{r} = 1$. In addition, we can write $\Gamma = \boldsymbol{\gamma}_m(\boldsymbol{\gamma}_m + \bar{\boldsymbol{\gamma}}_n)^{-1} \bar{\boldsymbol{\gamma}}_n$ [83], which implies $\det \Gamma = \det(\boldsymbol{\gamma}_m) \det(\bar{\boldsymbol{\gamma}}_n) / \det(\boldsymbol{\gamma}_m + \bar{\boldsymbol{\gamma}}_n)$. Thus,

$$\text{Tr}\rho^2 = \sum_m \sum_n \frac{c_m \bar{c}_n e^A}{\sqrt{\det(\boldsymbol{\gamma}_m + \bar{\boldsymbol{\gamma}}_n)}}. \quad (16)$$

In a similar way, noting that the gradient of a Gaussian is given by $\nabla G_{\boldsymbol{\mu}_m, \boldsymbol{\gamma}_m}(\mathbf{r}) = \boldsymbol{\gamma}_m^{-1}(\mathbf{r} - \boldsymbol{\mu}_m) G_{\boldsymbol{\mu}_m, \boldsymbol{\gamma}_m}(\mathbf{r})$, the numerator of Eq. (7) can be computed as follows:

$$\begin{aligned}
&\int |\nabla W(\mathbf{r})|^2 d^2\mathbf{r} \\
&= \int \left| \sum_m c_m \nabla G_{\boldsymbol{\mu}_m, \boldsymbol{\gamma}_m}(\mathbf{r}) \right|^2 d^2\mathbf{r} \\
&= \int \left| \sum_m c_m \boldsymbol{\gamma}_m^{-1}(\mathbf{r} - \boldsymbol{\mu}_m) G_{\boldsymbol{\mu}_m, \boldsymbol{\gamma}_m}(\mathbf{r}) \right|^2 d^2\mathbf{r} \\
&= \int \sum_{m,n} c_m \bar{c}_n G_{\boldsymbol{\mu}_m, \boldsymbol{\gamma}_m}(\mathbf{r}) G_{\bar{\boldsymbol{\mu}}_n, \bar{\boldsymbol{\gamma}}_n}(\mathbf{r}) \\
&\quad \times (\mathbf{r} - \bar{\boldsymbol{\mu}}_n)^T \bar{\boldsymbol{\gamma}}_n^{-1} \boldsymbol{\gamma}_m^{-1}(\mathbf{r} - \boldsymbol{\mu}_m) d^2\mathbf{r} \\
&= \sum_{m,n} c_m \bar{c}_n \int G_{d, \Gamma}(\mathbf{r}) e^A \sqrt{\frac{\det(2\pi \Gamma)}{\det(2\pi \boldsymbol{\gamma}_m) \det(2\pi \bar{\boldsymbol{\gamma}}_n)}} \\
&\quad \times (\mathbf{r} - \bar{\boldsymbol{\mu}}_n)^T \bar{\boldsymbol{\gamma}}_n^{-1} \boldsymbol{\gamma}_m^{-1}(\mathbf{r} - \boldsymbol{\mu}_m) d^2\mathbf{r} \\
&= \sum_{m,n} \frac{c_m \bar{c}_n e^A}{2\pi \sqrt{\det(\boldsymbol{\gamma}_m + \bar{\boldsymbol{\gamma}}_n)}} E[(\mathbf{r} - \bar{\boldsymbol{\mu}}_n)^T \bar{\boldsymbol{\gamma}}_n^{-1} \boldsymbol{\gamma}_m^{-1}(\mathbf{r} - \boldsymbol{\mu}_m)]. \quad (17)
\end{aligned}$$

Here $E[x]$ is the mean value of x with respect to the probability distribution $G_{d, \boldsymbol{\gamma}}$. With Eq. (357) in Ref. [83] and a bit of algebra, we have

$$\begin{aligned}
&E[(\mathbf{r} - \bar{\boldsymbol{\mu}}_n)^T \bar{\boldsymbol{\gamma}}_n^{-1} \boldsymbol{\gamma}_m^{-1}(\mathbf{r} - \boldsymbol{\mu}_m)] \\
&= (d - \bar{\boldsymbol{\mu}}_n)^T \bar{\boldsymbol{\gamma}}_n^{-1} \boldsymbol{\gamma}_m^{-1}(d - \boldsymbol{\mu}_m) + \text{Tr}[(\boldsymbol{\gamma}_m + \bar{\boldsymbol{\gamma}}_n)^{-1}]. \quad (18)
\end{aligned}$$

Hence,

$$\begin{aligned}
\int |\nabla W(\mathbf{r})|^2 d^2\mathbf{r} &= \sum_{m,n} \frac{c_m \bar{c}_n e^A}{2\pi \sqrt{\det(\boldsymbol{\gamma}_m + \bar{\boldsymbol{\gamma}}_n)}} \{ \text{Tr}[(\boldsymbol{\gamma}_m + \bar{\boldsymbol{\gamma}}_n)^{-1}] \\
&\quad + (d - \bar{\boldsymbol{\mu}}_n)^T \bar{\boldsymbol{\gamma}}_n^{-1} \boldsymbol{\gamma}_m^{-1}(d - \boldsymbol{\mu}_m) \}. \quad (19)
\end{aligned}$$

With Eqs. (16) and (19), we can now compute the QCS of Eq. (7) without the need for any integration.

VI. EXAMPLES

A. Cat states

Let us start with the simple example of a cat state defined as $|\text{cat}\rangle = \sqrt{\mathcal{N}}(|\alpha\rangle + |-\alpha\rangle)$, where $|\alpha\rangle$ is a coherent state and $\mathcal{N} = (2 + 2e^{-2|\alpha|^2})^{-1}$ a normalization constant. The larger the value of α is, the more macroscopically distinguishable the two terms in the coherent superposition become and thus the larger we expect the QCS to be. Since it is a pure state, the QCS of a cat state can be computed as the sum of its variances [see Eq. (8)] and we find [39]

$$C_{\text{cat}}^2 = 1 + 2\alpha^2 \tanh |\alpha|^2. \quad (20)$$

To use our method, we write the cat state as the sum of four Gaussian functions with the following parameters and coefficients [47]:

$$\begin{aligned}
c_1 &= c_2 = \mathcal{N}, \quad c_3 = c_4 = e^{-2|\alpha|^2} \mathcal{N}, \\
\boldsymbol{\gamma}_1 &= \boldsymbol{\gamma}_2 = \boldsymbol{\gamma}_3 = \boldsymbol{\gamma}_4 = \frac{1}{2} \mathbb{1}, \\
\boldsymbol{\mu}_1 &= -\boldsymbol{\mu}_2 = \sqrt{2}(\mathcal{R}(\alpha), \mathcal{I}(\alpha)), \\
\boldsymbol{\mu}_3 &= \boldsymbol{\mu}_4^* = \sqrt{2}(i\mathcal{I}(\alpha), -i\mathcal{R}(\alpha)). \quad (21)
\end{aligned}$$

As a test, one can use the equations introduced in the preceding section and compute

$$\begin{aligned}
\boldsymbol{\gamma} &= \frac{1}{4} \mathbb{1}, \quad \mathbf{d} = \frac{1}{2}(\boldsymbol{\mu}_m + \bar{\boldsymbol{\mu}}_n), \\
A &= \frac{1}{2}(\boldsymbol{\mu}_m + \bar{\boldsymbol{\mu}}_n)^T (\boldsymbol{\mu}_m + \bar{\boldsymbol{\mu}}_n) - \boldsymbol{\mu}_m^T \boldsymbol{\mu}_m - \bar{\boldsymbol{\mu}}_n^T \bar{\boldsymbol{\mu}}_n, \quad (22)
\end{aligned}$$

which inserted into Eqs. (16), (19), and (7) confirms that the same value of QCS is obtained. This method will be useful later when we study the effect of the loss channel on the QCS of a cat state.

B. GKP states

Let us now consider a more interesting example, which is the grid state introduced by Gottesman, Kitaev, and Preskill [54]. An ideal GKP state is represented by an infinite number of Dirac delta functions equally spaced by $2\sqrt{\pi}$ in the

phase space

$$\begin{aligned} |0\rangle_{\text{GKP}} &\propto \sum_{s=-\infty}^{\infty} |\sqrt{\pi\hbar}(2s)\rangle_q, \\ |1\rangle_{\text{GKP}} &\propto \sum_{s=-\infty}^{\infty} |\sqrt{\pi\hbar}(2s+1)\rangle_q, \end{aligned} \quad (23)$$

where $|\cdot\rangle_q$ denotes an eigenstate of the position quadrature. Due to this gridlike property, GKP states can be used to correct displacement errors, making them essential resources for continuous-variable quantum computation [84–86].

The ideal GKP state has infinite energy and cannot be normalizable. One thus needs to define finite-energy GKP states in order to deal with them. One option is to apply a Fock damping operator $E(\epsilon) = e^{-\epsilon\hat{n}}$ with $\epsilon > 0$ [47,87]. It can then be shown that the Wigner function of the GKP states can be written as a linear combination of Gaussian functions in phase space, as defined in Eq. (11). Let us introduce

$$|\psi\rangle_{\text{GKP}} = a_0|0\rangle_{\text{GKP}} + a_1|1\rangle_{\text{GKP}}. \quad (24)$$

Then the density matrix of the GKP state can be expressed as a sum of Gaussian functions $G_{\mu_m, \gamma_m}(\mathbf{r})$ with covariance matrix γ_m , mean values μ_m , and coefficients c_m given by

$$\begin{aligned} \gamma_m &= \frac{1}{2} \begin{pmatrix} \tanh(\epsilon) & 0 \\ 0 & -\coth(\epsilon) \end{pmatrix}, \\ \mu_m &= \frac{\sqrt{\pi}}{2} \begin{pmatrix} \text{sech}(\epsilon)(2k+2l+s+t) \\ i \text{csch}(\epsilon)(-2k+2l+s-t) \end{pmatrix}, \\ c_m &= \frac{a_s a_t^*}{\mathcal{N}} \exp\left(\frac{\pi}{2} \text{csch}(2\epsilon)(2k+2l+s+t)^2\right. \\ &\quad \left. - \frac{\pi}{2} \coth(\epsilon)((2k+t)^2 + (2l+s)^2)\right), \end{aligned} \quad (25)$$

where \mathcal{N} is such that $\sum c_m = 1$, $m = (k, l, s, t) | s, t \in \{0, 1\}$, and $k, l \in \mathbb{Z}$.

Using Eqs. (16) and (19), we can compute the QCS of the GKP state defined above. As an example, we focus here on the state $|0\rangle_{\text{GKP}}$ ($s = t = 0$, $a_0 = 1$, and $a_1 = 0$). The result is shown in Fig. 1, where we plotted the QCS as a function of the damping parameter ϵ . As expected, for an ideal GKP state, that is, when $\epsilon \rightarrow 0$, the QCS tends to infinity. Indeed, because they are superpositions of many different positions, GKP states have many coherences located infinitely far away from each other. The damping operator gradually erases the most distant superpositions, which in turn reduces the QCS. When ϵ is large enough, the QCS tends to 1, which means we lose all nonclassicality.

The analytical formulation of the QCS is complicated, but as shown by the green curve in Fig. 1 we can see that it evolves like

$$\mathcal{C}_{\text{GKP}}^2 \approx e^{-\epsilon} (\tanh \epsilon + \coth \epsilon). \quad (26)$$

C. Evolution of the QCS through a loss channel

It is essential to consider loss in any actual experiment. In the domain of state generation, if the goal is to generate a nonclassical state, it is important to understand how the

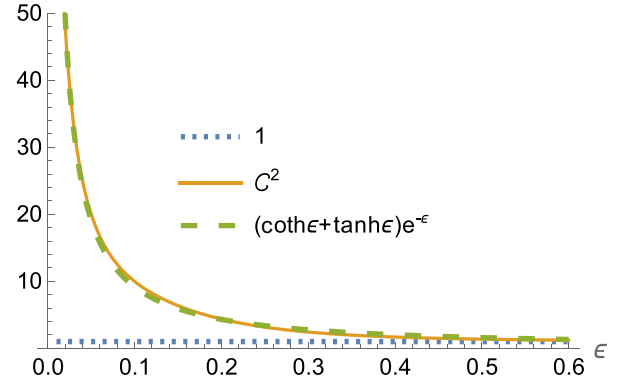


FIG. 1. Plot of the QCS (squared) of a GKP state as a function of the damping operator ϵ . The green dashed curve shows the general behavior of the QCS. All units here and in all subsequent figures are dimensionless.

nonclassicality, e.g., the QCS, evolves with losses. A simple model of loss is the loss channel whose action in the phase space is described by Eq. (5). As mentioned in Sec. IV, since a loss channel is described by a Gaussian transformation, when the Wigner function of the input state can be written as a sum of Gaussian functions, so will the Wigner function of the output state. The coefficient c_m in Eq. (11) remains the same and the covariance matrices and mean values are updated as follows:

$$\gamma_m(\eta) = \eta \gamma_m + (1 - \eta)\mathbb{1}/2, \quad \mu_m(\eta) = \sqrt{\eta} \mu_m. \quad (27)$$

Hence, Eqs. (16) and (19) can be used to compute the QCS of the output state in terms of the loss parameter η , which allows us to analyze how the nonclassicality scales with losses.

Loss is the dominant source of errors in photon-based protocols and can arise from multiple physical mechanisms including imperfect coupling and detector inefficiencies. It naturally arose in the reported measurements of the QCS on a cloud quantum computer [41] and similarly arises in many real-life experiments with quantum states [88–91]. Fortunately, loss is additive, such that the effects of disparate loss processes can be aggregated into a single parameter η . This provides a microscopic description that, as expected, reduces the average energy in a beam by a multiplicative factor of η . Without loss, one might expect fault-tolerant quantum computation with continuous variables to already be prevalent.

Let us study in more detail the evolution of the QCS through a loss channel for two specific input states: the cat states and the GKP states.

1. Cat states in a loss channel

Using Eqs. (27), (7), (16), and (19) and a little bit of algebra, it is straightforward to compute the QCS of the output of a loss channel when the input state is a cat state as defined in Eq. (21). We obtain

$$\mathcal{C}_{\text{cat}}^2(\eta) = 1 + \frac{4\eta|\alpha|^2 \sinh(4\alpha^2\eta - 2|\alpha|^2)}{\cosh(4\alpha^2\eta - 2|\alpha|^2) + \cosh(2|\alpha|^2) + 2}. \quad (28)$$

The result is plotted in Fig. 2. Near $\eta = 1$, we can approximate the rate of losing nonclassicality as the derivative of the

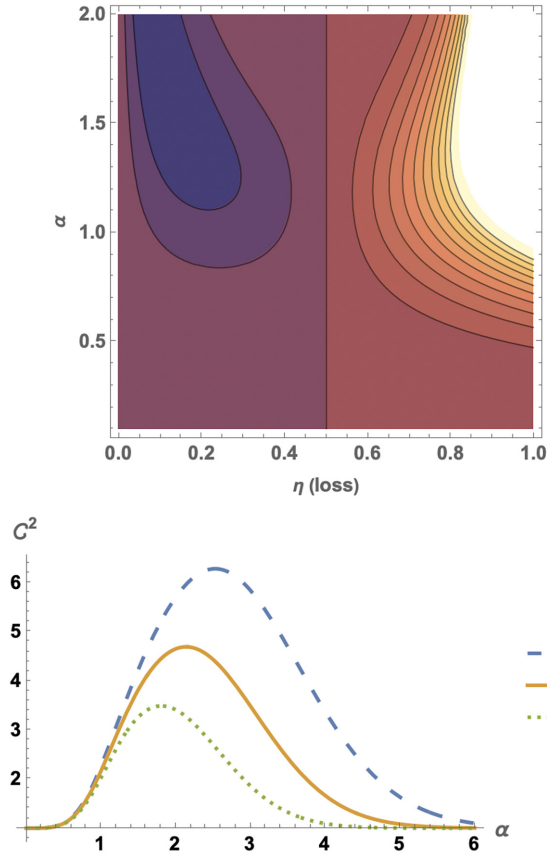


FIG. 2. Plot of the QCS (squared) of a cat state going through a loss channel as a function of α and η . No loss is represented by $\eta = 1$.

QCS (assuming $\alpha \in \mathbb{R}$):

$$\left. \frac{\partial \mathcal{C}_{\text{cat}}^2(\eta)}{\partial \eta} \right|_{\eta=1} = 2\alpha^2(2\alpha^2 + \tanh(\alpha^2)) = 4\alpha^4 - 1 + \mathcal{C}_{\text{cat}}^2. \quad (29)$$

As already shown in [39], we observe that the decoherence rate grows quadratically with the QCS (and quadratically with the energy) of the cat state: The larger the nonclassicality is, the quicker we lose it.

It is interesting to note that, no matter the value of α , the QCS of the states reaches 1 (which is the nonclassicality threshold) for $\eta = \frac{1}{2}$. This phenomenon was already observed in Ref. [41].

We also remark that, in the regime of small losses (η close to 1), the largest nonclassicality, as measured by the QCS, not necessarily obtained with the largest value of α . An example is given at the bottom of Fig. 2. When there is no loss ($\eta = 1$), the QCS tends to infinity when α tends to infinity; however, as soon as $\eta < 1$, the QCS will tend to 1 when $\alpha \rightarrow \infty$ (and the smaller the value of η is, the smallest α needs to be in order to reach 1). This means that, if one knows the loss of a setup available in a laboratory and the aim is to create a state as nonclassical as possible (for some computing task), then α needs to be chosen wisely. Nevertheless, let us keep in mind that the QCS is not a proper measure of nonclassicality, but rather provides a bound on the distance $D(\rho, \mathcal{E}_{\text{cl}})$ between the

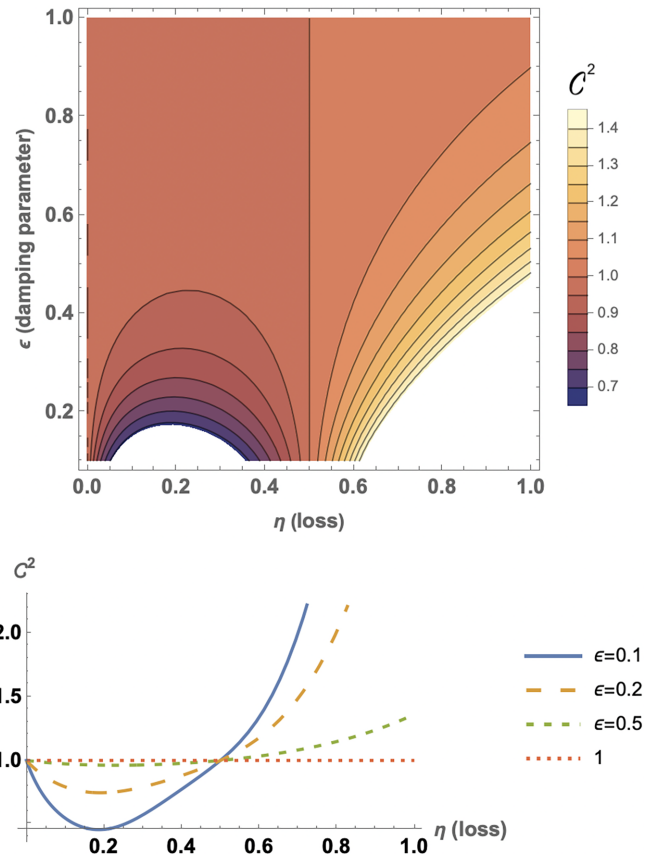


FIG. 3. Plot of the QCS (squared) of the GKP state as a function of the damping parameter ϵ and loss parameter η . No loss is represented by $\eta = 1$. An ideal GKP state has $\epsilon \rightarrow 0$.

state ρ and the set of classical states \mathcal{E}_{cl} [see Eq. (10)]. Hence, to ensure that a state is more nonclassical in the sense that it is more distant from the set of classical states, we need a variation of QCS greater than 1. When $\eta \leq \frac{1}{2}$, we observe that $\mathcal{C}_{\text{cat}}^2(\eta) \leq 1$, which implies that the state is always classical or weakly nonclassical. Intriguingly, per Fig. 2, a cat state with $\alpha = 3$ subject to 5% loss is much further away from the set of classical states than an initially nonclassical state with $\alpha \gg 3$ subject to the same 5% loss. More emphatically, a less quantum state with $\alpha = 2$ subject to more loss $\eta = 0.93$ is further from any classical state than a more quantum state with $\alpha \gg 3$ subject to more loss $\eta = 0.95$.

2. GKP states in a loss channel

Once again, using Eqs. (27), (7), (16), and (19), one can compute the QCS $\mathcal{C}_{\text{GKP}}^2(\eta, \epsilon)$ of the output of a loss channel when the input state is a GKP state as defined in Eq. (25). The result is plotted in Fig. 3. No loss is represented by $\eta = 1$. Hence, Fig. 1 represents the vertical slice of the top of Fig. 3 when $\eta = 1$.

As for the cat states, GKP states with better approximations (i.e., smaller value of ϵ) lose quantumness more quickly with loss. The rate of losing QCS could be evaluated by the derivative of $\mathcal{C}_{\text{GKP}}^2(\eta)$, but since the equation is more involved, so is its derivative. Nevertheless, one can see via Fig. 3 that here too

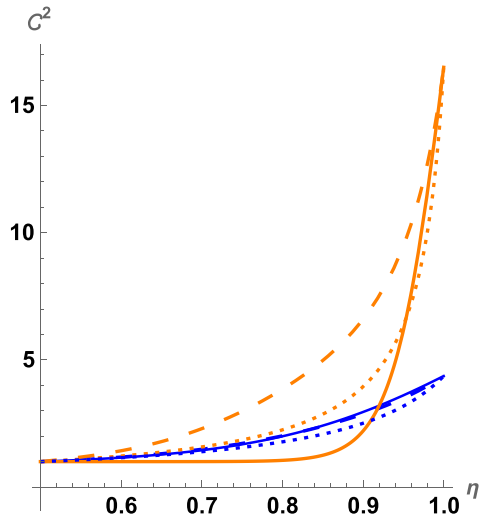


FIG. 4. Comparison of the loss (parametrized by η) of nonclassicality, as measured by the QCS for a cat state (solid line), a GKP state (dashed line), and a squeezed state (dotted line). The orange curves (highest three at far right) represent states starting with $C^2 \approx 16.5$ (i.e., $\alpha \approx 2.8$, $\epsilon \approx 0.05$, and $r = 1.7$) and blue curves (lowest three at far right) states starting with $C^2 \approx 4.4$ (i.e., $\alpha \approx 1.3$, $\epsilon \approx 0.2$, and $r = 1.1$).

the initially more nonclassical states lose their nonclassicality more rapidly. In addition, in this case as well, the QCS of the GKP state reaches 1 when there is 50% loss, regardless of the initial value. However, unlike cat states, no matter the value of $\eta > \frac{1}{2}$, a smaller ϵ implies a larger value of the QCS. When $\eta \leq \frac{1}{2}$, although it varies slightly, the QCS is always smaller than or equal to 1, which implies that the state is classical or weakly nonclassical. We can consider that above 50% loss (i.e., $\eta \leq \frac{1}{2}$), all nonclassicality is dissipated.

3. Squeezed state in a loss channel

As another example, it is also possible to compute the QCS of a lossy squeezed vacuum state $S(r)|0\rangle$, where $S(r) = e^{\frac{1}{2}(\bar{r}a^2 - ra^{12})}$ is the squeezing operator. We obtain (this result was already derived in Ref. [41])

$$C_{\text{sq}}^2(\eta) = \frac{1}{\frac{(1-2\eta)(\eta \cosh(2r) - \eta)}{-\eta + \eta \cosh(2r) + 1} + 1}. \quad (30)$$

In Fig. 4, starting with different values of the QCS, we compare the loss of nonclassicality, as measured by the QCS, for GKP states (dashed line), cat states (solid line), and squeezed states (dotted line). We first observe that, similar to GKP states, lines representing the QCS loss of different squeezed states will never cross. This means that even though higher initial squeezing (so high initial QCS value) will lose QCS quickly, a smaller squeezing value (thus smaller initial QCS) will always lead to lower QCS for a fixed amount of loss, even if the decay is slower.

Comparing the different families of states, we observe that there is no particular order for which state loses QCS faster. For large initial QCS values, GKP states are the most resilient and squeezed states lose QCS most quickly, but this

conclusion does not hold anymore when initial QCS values get smaller.

Again, we see that, for all those example states, the QCS reaches 1 at exactly 50% loss. This phenomenon can be proven for any pure one-mode Gaussian state or Fock state (see the Appendix). This leads us to believe that all pure states become classical (according to the QCS) after this threshold. It was shown in Ref. [92] that the Wigner function of the output of a loss channel is always positive as soon as there is at least 50% loss. Indeed, the s -ordered quasiprobability function of the output of the loss channel $P_{\text{out}}(\alpha, s)$ can be written as

$$P_{\text{out}}(\alpha, s) = \frac{1}{\eta} P_{\text{in}}\left(\frac{\alpha}{\sqrt{\eta}}, \frac{s + \eta - 1}{\eta}\right), \quad (31)$$

where $P_{\text{in}}(\alpha, s)$ is the s -ordered quasiprobability function of the input. The Wigner function corresponds to $s = 0$, which implies that $s' = (s + \eta - 1)/\eta \leq -1$ as soon as $\eta \leq \frac{1}{2}$. All s -ordered quasiprobability distributions with $s \leq -1$ are positive, which certifies the positivity of the Wigner function of the output when losses are greater than $\frac{1}{2}$. This does not prove that the state becomes classical according to Sudarshan-Glauber condition, but gives a hint that some quantum signature is lost at this specific point. Note that this conjecture does not extend to mixed states. The simplest example is to take a lossy squeezed state as the initial state: After 50% loss, the QCS of this state will be below 1. A direct corollary of our conjecture is that all states become classical according to the QCS for loss equal to or greater than 50%.

D. Breeding

Several schemes exist that strive to efficiently generate GKP states in optical setups [85,87,93–102]. The breeding protocol [14,15,52,103,104] is a procedure that gradually generates (or breeds) GKP states by impinging squeezed cat states $[1 + D(\alpha)]S(r)|0\rangle$ on a 50:50 beam splitter and measuring the p quadrature of one output mode. By postselecting on specific values of the p quadrature, one can obtain a GKP state in the other output mode of the beam splitter. This procedure can be repeated several times in order to increase the fidelity of the output state. In one recent protocol, the vast majority of p -measurement values suffice for generating GKP states, singling out GKP states as nonclassical fixed points of linear optics. The result is an approximation of a GKP state with $\sqrt{2}\alpha$ spacing and e^r squeezing.

In what is called a slow breeding protocol, the output of the first round of breeding is inserted again with another squeezed cat state of displacement $\alpha/\sqrt{2}$ at the second input, and the p quadrature of one of the outputs is once again measured. This continues, where in the M th round the input squeezed cat state is of the form $(1 + D(\alpha/\sqrt{2}^{M-1}))S(r)|0\rangle$. More details can be found in Ref. [15]. Note that, at each round, one had to do a postselection on the value of the p quadrature. As in Refs. [15,52,103], we choose here to postselect on $p = 0$. Nevertheless, keep in mind that the variance of the probability distribution of measuring $p = 0$ scales as e^{2r} .

One important question is to understand how many rounds of breeding one needs to get a GKP state with high enough fidelity. Further, how can this fidelity be measured? We

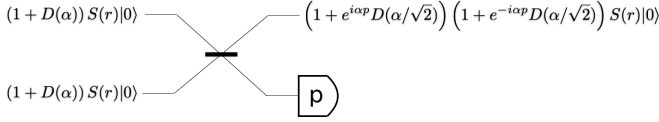


FIG. 5. Scheme of the first round of a breeding protocol.

suggest here to look at the QCS. Indeed, we know that the QCS of a cat state grows with its amplitude α and we thus expect the QCS to increase through the breeding protocol. Since the output can be written as a sum of Gaussian functions and the breeding circuit is a Gaussian transformation, the QCS of the output can be easily computed with the equations presented in Sec. V.

The first round of the breeding protocol is depicted in Fig. 5 and the output state is given by

$$|\psi\rangle = \left[1 + e^{i\alpha p}D\left(\frac{\alpha}{\sqrt{2}}\right)\right] \left[1 + e^{-i\alpha p}D\left(\frac{\alpha}{\sqrt{2}}\right)\right] S(r)|0\rangle.$$

To describe the output state after M rounds of the slow breeding protocol, let us define the measurement operator

$$\mathcal{M}(\phi, \alpha) = 1 + e^{i\phi}D(\alpha). \quad (32)$$

The output state will then have the form [15]

$$\begin{aligned} |\text{out}\rangle &= \prod_{j=1}^M \mathcal{M}(\tilde{\phi}_j, \tilde{\alpha}_j) \mathcal{M}(\tilde{\psi}_M, \tilde{\beta}_M) S(r)|0\rangle \\ &= \prod_{j=1}^M \mathcal{M}\left(\phi_j + \alpha_j p, \frac{\alpha_j}{\sqrt{2}}\right) \mathcal{M}\left(\psi_M - \beta_M p, \frac{\beta_M}{\sqrt{2}}\right) S(r)|0\rangle \\ &= \prod_{j=1}^M \mathcal{M}\left(\theta_j, \frac{\alpha}{\sqrt{2^M}}\right) \mathcal{M}\left(-\frac{\alpha p}{\sqrt{2^{M-1}}}, \frac{\alpha}{\sqrt{2^M}}\right) S(r)|0\rangle \\ &= \prod_{j=1}^M (1 + e^{i\theta_j p} D_M) (1 + e^{-i\alpha p M / \sqrt{2^{M-1}}} D_M) S(r)|0\rangle, \end{aligned} \quad (33)$$

where

$$\begin{aligned} \theta_j &= \sum_{k=1}^M \frac{\alpha}{\sqrt{2^{k-1}}} (-1)^{(j+1)k}, \\ D_M &= D\left(\frac{\alpha}{\sqrt{2^M}}\right). \end{aligned} \quad (34)$$

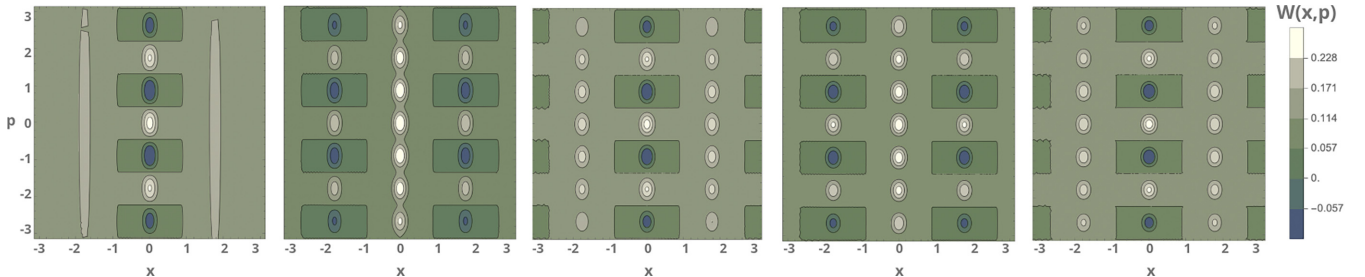


FIG. 6. Wigner functions of a GKP state generated after $M = 0, 1, 2, 3, 4$ rounds of the slow breeding protocol with final spacing of $2\sqrt{\pi}$ and squeezing $r = -\ln(0.2)$. Note that at each step, we applied a final displacement of $(M + 1)\sqrt{\pi}/2$ in the x direction to center the state on 0. The color bar indicates the magnitude and sign of the Wigner function.

When postselecting on $p = 0$ at each step, this simplifies to

$$\begin{aligned} |\text{out}\rangle &= \sum_{k=0}^{M+1} \binom{M+1}{k} D(\alpha/\sqrt{2^M})^k S(r)|0\rangle \\ &= \sum_{k=0}^{M+1} \binom{M+1}{k} D(k\alpha/\sqrt{2^M}) S(r)|0\rangle. \end{aligned} \quad (35)$$

The output grid state has a final spacing of $\alpha/\sqrt{2^{M-1}}$. Hence, for a GKP state with spacing $2\sqrt{\pi}$, the initial cat state must have a coherent amplitude of $\sqrt{2^{M+1}}\pi$. Figure 6 shows the evolution of the Wigner function after $M = 0, 1, 2, 3, 4$ rounds of slow breeding; $M = 0$ corresponds to a squeezed cat state.

The density matrix of the output state has the form $\rho_{\text{breed}} = |\text{out}\rangle\langle\text{out}|$, which is indeed a sum of Gaussian functions since a state of the form $D(\beta)S(r)|0\rangle\langle 0|S^\dagger(r)D^\dagger(\delta)$ has the Wigner function of a normalized Gaussian of covariance matrix $\boldsymbol{\gamma}$ and mean value $\boldsymbol{\mu}$ given by (see Appendix A of Ref. [47])

$$\boldsymbol{\gamma} = \frac{1}{2} \begin{pmatrix} e^{-2r} & 0 \\ 0 & e^{2r} \end{pmatrix}, \quad \boldsymbol{\mu} = \sqrt{\frac{1}{2}} \begin{pmatrix} \beta + \delta \\ ie^{2r}(\delta - \beta) \end{pmatrix}, \quad (36)$$

multiplied by the prefactor $e^{-e^{2r}(\beta - \delta)^2/2}$. Note that, for simplicity, we assume that the displacement is real.

Using the equations of Sec. V, we can compute the QCS at each round. Note that $M = 0$ corresponds to the input squeezed cat state and the QCS is given by

$$C_{\text{in}}^2 = \frac{1}{2} \alpha^2 \left(1 - \frac{e^{4r} + 1}{e^{\alpha^2 e^{2r}/2} + 1}\right) + \cosh(2r). \quad (37)$$

In Fig. 7 we compare the QCS of a GKP state as defined in Sec. VIB with the QCS of the output state of M rounds of the (slow) breeding protocol. We compare two cases: $r = -\ln(0.2)$ (blue line) as in Ref. [15] and $r = -\ln(0.3)$ (orange line). We observe that the QCS increases rapidly and reaches the target value (i.e., the GKP state as described in Sec. VIB with $\tanh \epsilon = e^{-2r}$) after only a few rounds. However, more importantly, we see that the QCS increases with each round of breeding and thus tends to infinity, as we would expect from a perfect GKP state. This is in contrast with the GKP state as described by the damping operator, but can be explained by looking at the Wigner function.

In Fig. 8 we plotted a slice of the Wigner function $W(x, p = 0)$ of a GKP state ($\epsilon \approx 0.09$) and the output state of five rounds of the breeding protocol [$r = -\ln(0.3) \approx 1.2$].

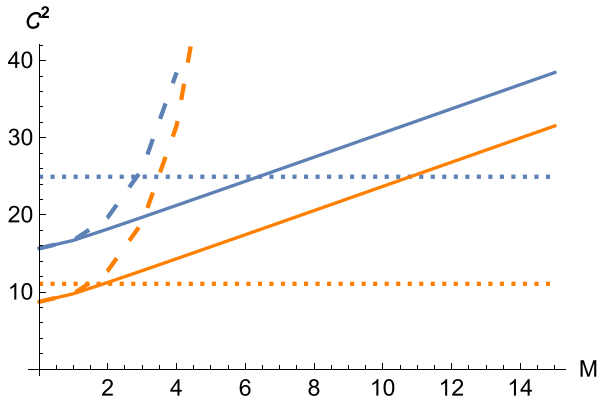


FIG. 7. Evolution of the QCS (squared) after M rounds of the slow (solid line) and efficient (dashed line) breeding protocol. The dotted reference line corresponds to the value of the QCS of a GKP state with spacing $2\sqrt{\pi}$ and a Fock damping operator $\tanh \epsilon = e^{-2r}$. Here we chose $r = -\ln(0.2) \approx 1.6$ (blue line; upper of each pair) and $r = -\ln(0.3) \approx 1.2$ (orange line; lower of each pair).

As we can see, the spacing of each Gaussian peak is the same, but the Gaussian envelope is not. The variance of the Gaussian envelope of the GKP state as described by the damping operator is smaller, so the total energy of the state as well as the QCS are smaller. On the other hand, after each round of the breeding protocol, the Wigner function has more and more nonzero peaks further from the origin so that both energy and QCS tend to infinity when the number of rounds M grows. This explains why the QCS of the breeding protocol does not tend to the value of the QCS of the targeted GKP state as described by the damping operator in Sec. VIB.

A more efficient scheme of breeding is realized by using a parallelized procedure. The first round is the same as the slow protocol, but then, instead of inserting a squeezed cat state in the second input of the beam splitter, we insert the same bred

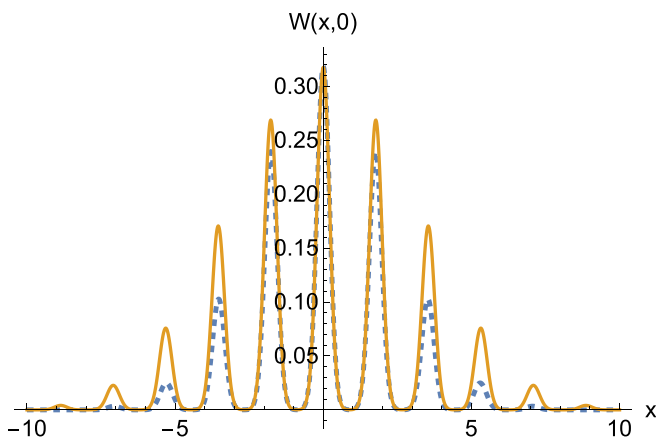


FIG. 8. Comparison of the Wigner function $W(x, p=0)$ of a GKP state (blue dashed line) and the output state of five rounds of the breeding protocol (orange solid line).

state to both inputs. The output state is then described by

$$|\text{out}\rangle = \sum_{k=0}^{2^M} \binom{2^M}{k} D(k\alpha/\sqrt{2^M}) S(r)|0\rangle. \quad (38)$$

We now need 2^M input squeezed cat states for M rounds of breeding (as opposed to $M+1$ inputs states for the slow protocol), but, as can be seen in Fig. 7, this scheme is much more efficient and we reach an acceptable value of QCS after only two to three rounds.

VII. OTHER MEASURES FOR SUMS OF GAUSSIAN FUNCTIONS

Suppose we restrict our states to those whose Wigner functions can be written as sums of Gaussian functions. Does that simplify the computation of other measures of quantumness?

The general answer is negative. Consider computation of the quantum Fisher information for a state's sensitivity to displacements, averaged over all displacement directions. More quantumness means higher average quantum Fisher information, which is a metric specifically tied to the usefulness of the state for metrology. For pure states, this is simply proportional to the average number of photons in the state, the same as the QCS [105]. For mixed states ρ , such as a cat state subject to loss, the quantum Fisher information is given by [106]

$$\mathcal{Q}_\rho = - \int_0^\infty ds \text{Tr}\{([\hat{x}, \rho]e^{-\rho s})^2 + ([\hat{p}, \rho]e^{-\rho s})^2\}. \quad (39)$$

If not for the factors $e^{-\rho s}$ and the integral over s , this would look very similar to the QCS (hence their agreement for pure states); that ρ has a Wigner function that is a linear combination of Gaussians does not simplify this formula in any known way.

One metric that is easy to compute for states that are linear combinations of Gaussians is the total noise, i.e., the trace of the state's covariance matrix. Unfortunately, this quantity tends to be most useful for pure states, where it is equal to the QCS, but it becomes less insightful for mixed states, where expressions like the quantum Fisher information \mathcal{Q}_ρ are more useful and yet do not inherit the ease of computation with linear combinations of Gaussians.

Next consider computation of a state's negative Wigner volume

$$\mathcal{N}_\rho = \int d^2\mathbf{r} \frac{|W(\mathbf{r})| - W(\mathbf{r})}{2}. \quad (40)$$

This formula is straightforward to manipulate when the state is subject to loss, as the Wigner function's transformation is known [see Eq. (31)]. However, computing the absolute value $|W(\mathbf{r})|$ for a linear combination of Gaussian functions is challenging because each coefficient in the linear combination is, in general, complex. Moreover, the Gaussian functions themselves may be complex, and adding loss changes those complex functions, even if the complex coefficients c_m are unchanged by loss.

As for distance to classical states, it is not known whether there exists a faster optimization procedure for states whose Wigner functions are linear combinations of Gaussians. It is indeed the case that the classical states belong to this

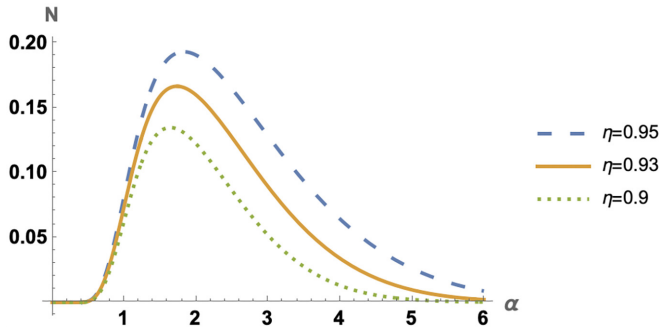


FIG. 9. Plot of N , the negative volume of the Wigner function of a cat state going through a loss channel, as a function of α and η . The peak in N versus α shifts with η in the same way as the QCS in Fig. 2.

category, but that does not imply that the most direct path from a given state to a given classical state is through the set of states whose Wigner functions are combinations of Gaussians. New optimizations are required for each state and each set of parameters such as loss, so this special class of states does not ameliorate the prohibitiveness of distance measures for straightforward computations.

With that all in mind, we compare the behavior of N_ρ to \mathcal{C}^2 for the particular class of lossy cat states in Fig. 9. The surprising features that we learned from the QCS again feature in the Wigner negativity, showcasing a correlation in their ability to find physical properties. Nonetheless, N is not known to bound the distance to classical states and thus alone could not be interpreted in any other way than witnessing nonclassicality. In addition, the computation of the QCS was rapid with our formalism, while the computation of N_ρ required a costly integration over phase space for each point on the curve, as would be required for computing the negativity of any filtered quasiprobability distribution [79]. That similar information can be gleaned from each confirms the usefulness of our results.

VIII. CONCLUSION

This paper presented a general formulation of the QCS of mixed quantum states whose Wigner functions can be expressed as a linear combination of Gaussian functions. This family of states encompasses important non-Gaussian states like Schrödinger cat states and GKP states, but also the output of a breeding protocol or, more generally, all resulting states of Gaussian operations applied to any states of such a form. Note that, in this work, we only considered transformations that are deterministic Gaussian completely positive and trace-preserving maps, but the class of Gaussian operations falling into this formalism is much larger and one could consider conditional dynamics (when the measurement of some modes updates the remaining modes), which shows the versatility and applicability of the proposed framework. With such a transformation, the number as well as the weight of each coefficient c_m in Eq. (11) would change. More details can be found in Ref. [47].

The methods presented here could be applied to states whose Wigner functions are more complicated than linear

combinations of Gaussian functions, with some modification. If some of the functions in the linear combination are given by $f(\mathbf{r})G(\mathbf{r})$ for some polynomial $f(\mathbf{r})$, there are closed-form solutions for each Gaussian integral for each monomial in the polynomials. This happens, for example, when a Gaussian state is acted on by a polynomial function of creation and annihilation operators. The expressions become more complicated as the degrees of the polynomials increase, especially when faced with linear combinations of different polynomials, but the crux of the methods here that rely on Gaussian integration can be broadly applied.

Loss as described by a loss channel falls into this framework, which allowed us to assess the scalability of nonclassicality, as measured by the QCS, with loss. As expected, for all states, the QCS decreases when loss increases, as long as the loss is less than 50%. At this exact point, the QCS of all our examples of pure states reached a value of exactly 1, which led us to conjecture that this is true for all pure states. We extended the conjecture by suggesting that all states lose their nonclassicality (as measured by the QCS) at the latest after 50% loss, a statement which is supported by all Wigner functions becoming strictly non-negative at the 50%-loss mark.

Because GKP states are so important, in particular for their error-correcting properties in quantum computing codes, we studied the breeding protocol, which is one way to experimentally create GKP states and can be expressed using the formalism described in this paper. We used the QCS as a way to measure the quality of the resulting GKP state and showed that a parallelized procedure is more efficient. In a realistic scenario, one should include loss in the protocol and understand the threshold limit of the parameters allowing one to keep a sufficient amount of nonclassicality. In particular, it would be interesting to study how to optimize the squeezing parameter for state generation under realistic conditions. One could ask the following question: Is it better to start with highly squeezed states (so very sensitive to loss of nonclassicality) and directly generate cat or GKP states, or is it better to input less squeezed states in a breeding protocol, which we have seen has the power to increase the nonclassicality?

ACKNOWLEDGMENTS

The authors thank Eli Bourassa, Ilan Tzitrin, and Rafael Alexander from Xanadu for fruitful discussions. The authors acknowledge that the NRC headquarters is located on the traditional unceded territory of the Algonquin, Anishinaabe, Haudenosaunee, and Mohawk people. This work was supported by the NRC's Quantum Sensors Challenge program. A.Z.G. acknowledges support from an NSERC postdoctoral fellowship. K.H. acknowledges support from NSERC's Discovery Grant program.

APPENDIX

1. QCS of a pure Gaussian state after 50% loss

Since the QCS is invariant under displacement, we can assume that the state is centered at the origin. A pure one-mode

Gaussian state has the covariance matrix

$$\boldsymbol{\gamma} = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix} \quad (\text{A1})$$

such that $\det(\boldsymbol{\gamma}) = \frac{1}{4}$. After 50% loss ($\eta = \frac{1}{2}$), the covariance matrix becomes $\frac{1}{2}(\boldsymbol{\gamma} + \mathbb{1}/2)$. We can compute the QCS with Eq. (9),

$$\begin{aligned} c^2 &= \frac{1}{2} \text{Tr} \left(\frac{\boldsymbol{\gamma} + \mathbb{1}/2}{2} \right)^{-1} \\ &= \frac{2(\sigma_{11} + \sigma_{22} + 1)}{4 \det(\boldsymbol{\gamma}) + 2\sigma_{11} + 2\sigma_{22} + 1} = 1, \end{aligned} \quad (\text{A2})$$

and see that the result is 1 for all one-mode pure Gaussian states.

2. QCS of a Fock state after 50% loss

We know from Ref. [92] that the Wigner function of the output state of the loss channel is proportional to the Q

function of the input state [see Eq. (31)]. The Q function of a Fock state $|k\rangle$ is given by [107]

$$Q_k(x, p) = \frac{1}{\pi} \frac{(x^2 + p^2)^k}{2^k k!} e^{-(x^2 + p^2)/2}. \quad (\text{A3})$$

Therefore, after 50% loss ($\eta = 0.5$), the Wigner function is given by

$$W_k(x, p; \eta = \frac{1}{2}) = \frac{1}{\pi} \frac{(x^2 + p^2)^k}{k!} e^{-(x^2 + p^2)}, \quad (\text{A4})$$

where we used the correspondence $\alpha = (x + ip)/\sqrt{2}$. Using Eq. (7), we can compute the QCS of the lossy Fock state and confirm that we obtain 1 for all k .

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