Monogamy of k-entanglement

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Multipartite entanglement includes not only the genuine entanglement but also the k-entanglement ($k \geqslant 2$). It is known that the most bipartite entanglement measures have been shown to be monogamous, but the monogamy relation is involved in the bipartite entanglement measures rather than the multipartite ones. So how we can explore the monogamy relation for k-entanglement becomes a basic open problem. In this paper we establish an axiomatic definition of the monogamy relation for the k-entanglement measure based on the coarser relation of the system partition. We also present the axiomatic definition of the complete k-entanglement measure and the associated complete monogamy relation according to the framework of the complete multipartite entanglement measure we established in [Phys. Rev. A **101**, 032301 (2020)], which is shown to be an efficient tool for characterizing the multipartite quantum correlation as complementary to the monogamy relation associated with the bipartite measure. Consequently, the relation and the difference between monogamy, complete monogamy, and tightly complete monogamy are clearly depicted in light of the three types of coarser relation of the system partition. We then illustrate our approach with two classes of k-entanglement measures in detail. We find that all these k-entanglement measures are monogamous if the reduced function is strictly concave, they are not completely monogamous, and they are not complete generally.

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I. INTRODUCTION

With the rapid progress in the experimental study of quantum entanglement [1–3], efficient quantifying and characterizing of entanglement has been a problem of ubiquitous interest in quantum information science [4–36]. Although bipartite entanglement is well understood theoretically as it can be probed by the appropriate reduced states, multipartite entanglement remains challenging to understand undeniably since the complexity scales exponentially with the number of parties.

The empirical approach to classifying multipartite entanglement is the k-entanglement scenario, in which different types of entanglement are determined and the genuine entanglement is just the 2-entanglement as a special case. In the last two decades, a series of multipartite entanglement measures have been proposed along this line but mainly focus on genuine entanglement, such as the "residual tangle" [8], Q_m [11], genuinely multipartite concurrence [18], k-ME concurrence [19], parametrized k-ME concurrence [34], m-flip concurrence [13], generalization of negativity [17], SL-invariant multipartite measure of entanglement [9,10,12,14–16], α-entanglement entropy [20], concurrence triangle [23,31,32], concentratable entanglement [24], geometric mean of bipartite concurrence [27], a general way of constructing multipartite entanglement monotone [21,35], genuine multipartite entanglement monotone [28,29,35], etc. Only a few measures for k-entanglement have been discussed [19,34].

Among the characterizing multiparty quantum systems of importance, the distribution of entanglement over the subsystems is indispensable since it reveals fundamental insights into the nature of quantum correlations [2] and has profound applications in both quantum communication [37,38] and other area of physics [39–43]. The distribution of entanglement admits the monogamy relation [37,44], which means that the more entangled two parties are, the less correlated they can be with other parties. Quantitatively, the monogamy of entanglement is always described by an inequality [8,39,44–47], and later by equality [21,30,48,49], involving a bipartite entanglement measure. Recently, we developed a framework of a complete multipartite quantum correlation measure and the associated complete monogamy relation [21,29,35,50,51] which is shown to be complementary to the traditional monogamy relation [30,51]. By now, it is proved that most of the bipartite entanglement measures are monogamous [30,49], and most of the multipartite entanglement measures are completely monogamous [21,29,35].

Then the following problems arise naturally: How can we define the monogamy of k-entanglement and the complete k-entanglement measure? Is the k-entanglement monogamous or completely monogamous? The aim of this paper is to address such an issue. Note that, in general, we cannot discuss the monogamy of a multipartite entanglement measure, and similarly the complete monogamy of a bipartite entanglement measure, since they are incompatible with each other. However, k-entanglement corresponds to the entanglement of the split state over the subsystems with a fixed number of partition "k," which is similar to both the bipartite entanglement measure and the general multipartite ones to some extent. We thus need take into account both the monogamy and the complete monogamy for a given k-entanglement measure (k-EM).

The paper is organized as follows. In Sec. II we introduce some preliminaries. We propose the definition of the monogamy relation for the k-entanglement in Sec. III. Then, in Secs. IV and V, in the same spirit as that of the complete multipartite measure of quantum correlation and the associated complete monogamy relation we established before [21,29,30,35,50,51], we discuss when a k-entanglement measure can be said to be a complete k-entanglement measure and when it can be defined to be completely monogamous, respectively. In Sec. V we propose four different ways of constructing k-entanglement measures in terms of the sum of the reduced functions, while Sec. VI proposes three classes of k-entanglement measures in terms of the product of the reduced functions. For any one of these measures, we discuss whether it is monogamous, complete, and completely monogamous. We present a conclusion in Sec. VII.

II. NOTATIONS AND PRELIMINARIES

For the convenience of discussing the monogamy and the complete measure of the k-entanglement in the next sections, we need introduce the coarser relation of a multipartite partition, which is also introduced in Refs. [29,35], and then review some basic notations and terminologies related with the k-entanglement.

A. Coarser relation of multipartite partition

We denote by $A_1A_2\cdots A_n$ an n-partite quantum system. Let $X_1|X_2|\cdots|X_k$ and $Y_1|Y_2|\cdots|Y_l$ be two partitions of $A_1A_2\cdots A_n$ or subsystem of $A_1A_2\cdots A_n$ (for instance, partition $AB|C|DE$ is a 3-partition of the five-particle system $ABCDE$ with $X_1 = AB$, $X_2 = C$ and $X_3 = DE$). We denote by [29]

$$X_1|X_2|\cdots|X_k \succ^a Y_1|Y_2|\cdots|Y_l, \tag{1}$$

$$X_1|X_2|\cdots|X_k \succ^b Y_1|Y_2|\cdots|Y_l, \tag{2}$$

$$X_1|X_2|\cdots|X_k \succ^c Y_1|Y_2|\cdots|Y_l \tag{3}$$

if $Y_1|Y_2|\cdots|Y_l$ can be obtained from $X_1|X_2|\cdots|X_k$ by

(a) Discarding some subsystem(s) of $X_1|X_2|\cdots|X_k$,

(b) Combining some subsystems of $X_1|X_2|\cdots|X_k$,

(c) Discarding some subsystem(s) of some subsystem(s) X_t provided that $X_t = A_{t(1)}A_{t(2)}\cdots A_{t(f(t))}$ with $f(t)\geqslant 2$, $1\leqslant t\leqslant k$,

respectively. For example,

$$A|B|C|D \succ^a A|B|D \succ^a B|D,$$
$$A|B|C|D \succ^b AC|B|D \succ^b AC|BD,$$
$$A|BC \succ^c A|B.$$

We call $Y_1|Y_2|\cdots|Y_l$ coarser than $X_1|X_2|\cdots|X_k$ if $Y_1|Y_2|\cdots|Y_l$ can be obtained from $X_1|X_2|\cdots|X_k$ by one or some of the ways in item (a) to item (c), and we denote it by $X_1|X_2|\cdots|X_k \succ Y_1|Y_2|\cdots|Y_l$ uniformly.

Furthermore, if $X_1|X_2|\cdots|X_k \succ Y_1|Y_2|\cdots|Y_l$, we denote by

$$\Xi(X_1|X_2|\cdots|X_k - Y_1|Y_2|\cdots|Y_l) \tag{4}$$

the set of all the partitions that are coarser than $X_1|X_2|\cdots|X_k$ but (i) neither coarser than $Y_1|Y_2|\cdots|Y_l$ nor the one from which one can derive $Y_1|Y_2|\cdots|Y_l$ by the coarsening means, and (ii) if it includes some or all subsystems of $Y_1|Y_2|\cdots|Y_l$, then all the subsystems Y_js included are regarded as one subsystem, and (iii) if $Y_1|Y_2|\cdots|Y_l = X_1|X_2|\cdots|X_{l-1}|X_l\cdots X_k$, $\Xi(X_1|X_2|\cdots|X_k - Y_1|Y_2|\cdots|Y_l)$ contains only $X_l|\cdots|X_k$ and the one coarser than it. We call $\Xi(X_1|X_2|\cdots|X_k - Y_1|Y_2|\cdots|Y_l)$ the complementarity of $Y_1|Y_2|\cdots|Y_l$ up to $X_1|X_2|\cdots|X_k$. For example,

$$\begin{aligned}\Xi(A|B|CD|E - A|B) = \{&CD|E, A|CD|E, B|CD|E, A|CD, B|CD, B|C|E, B|D|E, A|D|E, A|C|E, A|E,\\ &B|E, A|C, A|D, B|C, B|D, C|E, D|E, AB|CDE, AB|CD|E, AB|CD, AB|E\}.\\ \Xi(A|B|C|D|E - A|B|C) = \{&D|E, A|D|E, A|D, A|E, B|D|E, B|D, B|E, C|D|E, C|D, C|E, AB|D|E,\\ &AB|D, AB|E, AC|D|E, AC|D, AC|E, BC|D|E, BC|D, BC|E, ABC|DE,\\ &ABC|D|E, ABC|D, ABC|E, AB|DE, AC|DE, BC|DE\}.\\ \Xi(A|B|C|D - A|BCD) = \{&B|C|D, B|CD, BC|D, C|BD, B|C, C|D, B|D\}.\end{aligned}$$

B. k-entanglement

A pure state $|\psi\rangle$ of an n-partite system $A_1A_2\cdots A_n$ with state space $\mathcal{H}^{A_1A_2\cdots A_n}$ is said to be k-separable if $|\psi\rangle = |\psi\rangle^{X_1}|\psi\rangle^{X_2}\cdots|\psi\rangle^{X_k}$ for some k-partition of $A_1A_2\cdots A_n$. An n-partite mixed state ρ is k-separable if it can be written as a convex combination of k-separable pure states $\rho = \sum_i p_i|\psi_i\rangle\langle\psi_i|$, wherein the contained $\{|\psi_i\rangle\}$ can be k-separable with respect to different k-partitions (i.e., a mixed k-separable state does not need to be separable with respect to any particular k-partition). If ρ is not 2-separable (2-separable is also called biseparable), then it is called genuinely entangled.

A multipartite entanglement measure is always defined via the bipartite entanglement measure, which is associated with the corresponding bipartite partition. We need to review the bipartite entanglement measure first. Let $\mathcal{S}^X$ be the set of all density operators acting on the state space $\mathcal{H}^X$. Recall that a function $E : \mathcal{S}^{AB} \to \mathbb{R}_+$ is called a measure of entanglement [5,6] if (1) $E(\sigma^{AB}) = 0$ for any separable state $\sigma^{AB} \in \mathcal{S}^{AB}$, and (2) E behaves monotonically decreasing under local operations and classical communication (LOCC). Moreover, a convex measure of bipartite entanglement that does not increase *on average* under LOCC is called an entanglement monotone [6,7]. By replacing $\mathcal{S}^{AB}$ with $\mathcal{S}^{A_1A_2\cdots A_n}$, it is just the multipartite entanglement measure/monotone. Any bipartite entanglement monotone E corresponds to a concave function h on the reduced state when it is evaluated for the pure states [7],

$$h(\rho^A) = E(|\psi\rangle\langle\psi|^{AB}) \tag{5}$$

with h satisfying $h[\lambda\rho_1+(1-\lambda)\rho_2]\geqslant\lambda h(\rho_1)+(1-\lambda)h(\rho_2)$ for any states ρ_1, ρ_2, and any $0\leqslant\lambda\leqslant 1$. If E is bipartite entanglement measure and h is a positive function that satisfies Eq. (5), we call h the *reduced function* of E [52]. Let E be an entanglement measure, then $E(\rho^{AB})\equiv\min\sum_{j=1}^{n}p_jE(|\psi_j\rangle\langle\psi_j|^{AB})$ is called the convex roof extension of E, where the minimum is taken over all pure state decompositions of $\rho^{AB}=\sum_{j=1}^{n}p_j|\psi_j\rangle\langle\psi_j|^{AB}$.

A function $E_k:\mathcal{S}^{A_1A_2\cdots A_n}\to\mathbb{R}_+$ is called a k-entanglement measure if it admits the following conditions: (k-E1) $E_k(\rho)=0$ for any k-separable $\rho\in\mathcal{S}^{A_1A_2\cdots A_n}$; ($k$-E2) $E_k(\rho)\geqslant 0$ for any k-entangled state $\rho\in\mathcal{S}^{A_1A_2\cdots A_n}$; and ($k$-E3) $E_k(\rho)\geqslant E_k(\rho')$ for any n-partite LOCC ε, $\varepsilon(\rho)=\rho'$. A convex function E_k is said to be a k-entanglement monotone (abbreviated k-EMo) if it satisfies items (k-E1)–(k -E2) and does not increase on average under n-partite stochastic LOCC additionally. Obviously, a k-EMo must be a k-EM. Item (k-E3) implies that k-EM is locally unitary invariant, i.e., $E_k(\rho)=E_k(U_1\otimes U_2\otimes\cdots\otimes U_n\rho U_1^\dagger\otimes U_2^\dagger\otimes\cdots\otimes U_n^\dagger)$ for any local unitary operator U_i acting on $\mathcal{H}^{A_i}$, $i=1,2,\ldots,n$. If $E_k(\rho)>0$ for any k-entangled state $\rho\in\mathcal{S}^{A_1A_2\cdots A_n}$, we say E_k is faithful.

III. MONOGAMY OF k-EM

We recall the monogamy of the bipartite entanglement measure from the point of view of the coarser relation of the multipartite partition first. A bipartite entanglement measure E is said to be monogamous, if for any state $\rho^{ABC}\in\mathcal{S}^{ABC}$ that satisfies [21,35]

$$E(A|BC)=E(AB) \tag{6}$$

we have that $E(AC)=0$. Hereafter, $E(X)$ denotes $E(\rho^X)$; the vertical bar indicates the split across which the entanglement is measured. Note here that this definition is equivalent to the traditional definition of the monogamy relation in terms of the inequality [8,53]

$$E(A|BC)\geqslant E(AB)+E(AC) \tag{7}$$

whenever E is continuous (see Ref. [48], Theorem 1): a continuous measure E is monogamous according to the equality-based definition by Eq. (6) if and only if there exists $0<\alpha<\infty$ such that

$$E^\alpha(\rho^{A|BC})\geqslant E^\alpha(\rho^{AB})+E^\alpha(\rho^{AC}) \tag{8}$$

for all ρ acting on the state space $\mathcal{H}^{ABC}$ with fixed dim $\mathcal{H}^{ABC}$. Moreover, the equality-based definition by Eq. (5) offers significant advantages for checking whether a measure is monogamous compared with the traditional one [30,48,49]. So we follow the equality-based approach throughout this paper. Clearly, in these monogamy relations, the corresponding coarser relations are actually $A|BC\succ^c A|B$ and $A|C\in\Xi(A|BC-A|B)$. If we consider $\rho^{ABCD}\in\mathcal{S}^{ABCD}$, the monogamy of E can be stated as

$$\begin{aligned}&E(AB|CD)=E(BC)\\&\Rightarrow E(AB|D)=E(A|CD)=E(AD)=0.\end{aligned}$$

The associated coarser relations are $AB|CD\succ^c B|C$, and the corresponding complementary relations $AB|D$, $A|CD$, and $A|D$ in $\Xi(AB|CD-B|C)$. Namely, the monogamy can be determined by the relation between states under the coarser relation of type (c), i.e., discarding the subsystem of the subsystems, together with the associated complementary relations. With this principle in mind, we can establish the monogamy of k-EM.

Let E_k be a k-EM. Hereafter, $E_k(X_1|X_2|\cdots|X_p)$ for $\rho\in\mathcal{S}^{A_1A_2\cdots A_n}$ means that $E_k(X_1|X_2|\cdots|X_p)=E_k(\rho^{X_1|X_2|\cdots|X_p})$ with $\rho^{X_1|X_2|\cdots|X_p}$ being the state with respect to the k-partition $X_1|X_2|\cdots|X_p$ of $A_1A_2\cdots A_n$ or some subsystem of $A_1A_2\cdots A_n$. For example, $E_3(A|CD|E|F)$ for ρ^{ABCDEF} means $E_3(\rho^{A|CD|E|F})$ with $\rho^{ACDEF}=\mathrm{Tr}_B(\rho^{ABCDEF})$. E_k is said to be *monogamous* if for any $\rho\in\mathcal{S}^{A_1A_2\cdots A_n}$ that satisfies

$$E_k(X_1|X_2|\cdots|X_p)=E_k(X_1|X_2|\cdots|X_{s-1}|X_s'|X_{s+1}'|\cdots|X_p') \tag{9}$$

with $X_s|X_{s+1}|\cdots|X_p\succ^c X_s'|X_{s+1}'|\cdots|X_p'$, $1\leqslant s\leqslant p$, $k\leqslant p$, we have that

$$\left.\begin{aligned}E_{p-s+2}(X_1\cdots X_{s-1}|Z_s|Z_{s+1}|\cdots|Z_p)=0,&\quad\text{if } s\geqslant 2,\\E_p(Z_1|Z_2|\cdots|Z_p)=0,&\quad\text{if } s=1,\end{aligned}\right\} \tag{10}$$

and

$$E_2(W_m|Z_m)=0 \tag{11}$$

whenever $Z_m\in X_m-X_m'$, $W_m\in X_1X_2\cdots X_p-X_m$, $s\leqslant m\leqslant p$. Here X_m-X_m' denotes the subsystem(s) of X_m complementary to X_m'. For example, $ABCD-AB=CD$, $C\in ABCD-AB$, and $D\in ABCD-AB$. In such a sense, e.g., if E_3 is monogamous, then

$$\begin{aligned}E_3(A|B|CD)&=E_3(A|B|C)\Rightarrow E_2(AB|D)=0,\\E_3(A|B|CD|EF)&=E_3(A|B|C|E)\\\Rightarrow E_3(AB|D|F)&=E_2(ABEF|D)\\&=E_2(ABCD|F)=0,\\E_3(AB|CD|EF)&=E_3(A|C|E)\\\Rightarrow E_3(B|D|F)&=E_2(ABCD|F)\\&=E_2(ABEF|D)\\&=E_2(B|CDEF)=0,\end{aligned}$$

which is similar to that of $E(A|BC)=E(AB)$ leading to $E(AC)=0$ for the monogamy of the bipartite entanglement measure E. We call E_k *weakly monogamous* if for any $\rho\in\mathcal{S}^{A_1A_2\cdots A_n}$ that satisfies Eq. (9) with $p=k$ we have that Eqs. (10) and (11) hold. Note here that in Eq. (9) only the case of $X_1|X_2|\cdots|X_p\succ^c X_1|X_2|\cdots|X_{s-1}|X_s'|X_{s+1}'|\cdots|X_p'$ is presented with no loss of generality since all other cases [i.e., any partitions with coarser relation of type (c)] can be followed easily due to the symmetry of E_k.

IV. COMPLETENESS OF k-EM

We now discuss the completeness of k-EM. The key principle of a complete measure for multipartite quantum correlation is that there is a unified criterion for quantifying different subsystems or systems under different partition, which means the amount of the quantum correlations contained in different particles can be compared with each other consistently and compatibly [21,29,35,50,51]. So the first thing we need to figure out is that, for any given n-partite system, a complete multipartite measure $E^{(n)}$ on it indeed refers to a series of measures $\{E^{(k)} : 2 \leqslant k \leqslant n\}$ that are defined in a unified way. As we discussed in Refs. [21,29], when we deal with the multipartite entanglement, there are two steps to reveal such a completeness of a given measure. The first step is the unification condition, and the second one is the hierarchy condition. The unification condition is mainly related to the coarsened relation of type (a), while the hierarchy condition is corresponding to the coarsened relation of type (b). In general, the hierarchy condition is more restrictive, e.g., some multipartite entanglement measures satisfy the unification condition but violate the hierarchy condition [21,35]. If $E^{(n)}$ obeys the unification condition, it is called unified, and if it also admits the hierarchy condition, it is called complete [22,29,35].

Keeping the same spirit in mind, we give the definitions of the unified k-EM and the complete k-EM. A k-EM E_k is called *unified* if it satisfies the unification condition: (i) (symmetry) $E_k(A_1A_2\cdots A_n) = E_k(A_{\pi(1)}A_{\pi(2)}\cdots A_{\pi(n)})$ for all $\rho^{A_1A_2\cdots A_n} \in \mathcal{S}^{A_1A_2\cdots A_n}$ and any permutation π of $\{1, 2, \cdots, n\}$; (ii) (k-monotone)

$$E_k(A_1A_2\cdots A_n) \geqslant E_{k-1}(A_1A_2\cdots A_n) \tag{12}$$

holds for all $\rho^{A_1A_2\cdots A_n} \in \mathcal{S}^{A_1A_2\cdots A_n}$, $k \geqslant 3$; and (iii) (coarsening monotone)

$$E_k(X_1|X_2|\cdots|X_p) \geqslant E_l(Y_1|Y_2|\cdots|Y_q) \tag{13}$$

holds for all k-entangled states $\rho \in \mathcal{S}^{X_1X_2\cdots X_p}$ whenever $X_1|X_2|\cdots|X_p \succ^a Y_1|Y_2|\cdots|Y_q$ with $l \leqslant q \leqslant p$ and $l \leqslant k \leqslant p$. Item (i) is clear, i.e., the symmetry is an inherent feature of any entanglement measure indeed. If a state is k-separable, it must be $(k-1)$-separable, but not vice versa, so we require condition (ii). For the generalized n-qudit GHZ state $\frac{1}{\sqrt{d}}(|00\cdots 0\rangle + |11\cdots 1\rangle + \cdots |d-1\rangle|d-1\rangle\cdots|d-1\rangle)$, Eq. (13) is always true for any k-EM. Hereafter, if a k-EM E_k obeys Eq. (12) and Eq. (13), we call it is *k-monotonic* and *coarsening monotonic*, respectively. We remark here that, Eq. (13) does not hold for k-separable state $\rho^{A_1A_2\cdots A_n} \in \mathcal{S}^{A_1A_2\cdots A_n}$ in general. For example, $E_3(|\psi\rangle^{AB}|\psi\rangle^C|\psi\rangle^D) = 0$, but $E_3(|\psi\rangle^{AB}|\psi\rangle^C) > 0$ if $|\psi\rangle^{AB}$ is entangled generally.

A unified k-EM E_k is called *complete* if it satisfies the hierarchy condition additionally: (iv) (tight coarsening monotone)

$$E_k(X_1|X_2|\cdots|X_p) \geqslant E_l(Y_1|Y_2|\cdots|Y_q) \tag{14}$$

holds for all k-entangled state $\rho \in \mathcal{S}^{X_1X_2\cdots X_p}$ whenever $X_1|X_2|\cdots|X_p \succ^b Y_1|Y_2|\cdots|Y_q$ with $l \leqslant q \leqslant p$ and $l \leqslant k \leqslant p$. For k-separable state $\rho \in \mathcal{S}^{A_1A_2\cdots A_n}$, Eq. (14) fails in general. If a k-EM E_k satisfies Eq. (14), we call it is *tightly coarsening monotonic*. One need note here that, for any given k-EM E_k,

$$E_k(X_1|X_2|\cdots|X_p) \geqslant E_k(X_1'|X_2'|\cdots|X_p') \tag{15}$$

holds for any $\rho \in \mathcal{S}^{A_1A_2\cdots A_n}$ whenever $X_1|X_2|\cdots|X_p \succ^c X_1'|X_2'|\cdots|X_p'$ since $\rho^{X_1'|X_2'|\cdots|X_p'}$ is obtained from $\rho^{X_1|X_2|\cdots|X_p}$ by a partial trace and such a partial trace is indeed a p-partite LOCC, $2 \leqslant k \leqslant p < n$.

We take the 4-partite system $ABCD$, for example. E_4 is k-monotonic means

$$E_4(ABCD) \geqslant E_3(ABCD) \geqslant E_2(ABCD)$$

for any $\rho^{ABCD} \in \mathcal{S}^{ABCD}$, E_3 is coarsening monotonic refers to

$$E_3(A|B|CD) \geqslant E_2(A|CD),$$
$$E_3(A|B|CD) \geqslant E_2(B|CD),$$
$$E_3(A|B|CD) \geqslant E_2(A|B)$$

for any 3-entangled state $\rho^{ABCD} \in \mathcal{S}^{ABCD}$, and E_4 is coarsening monotonic refers to

$$E_4(ABCD) \geqslant E_3(ABC),$$
$$E_4(ABCD) \geqslant E_3(ABD),$$
$$E_4(ABCD) \geqslant E_3(ACD),$$
$$E_4(ABCD) \geqslant E_3(BCD),$$
$$E_4(ABCD) \geqslant E_2(AB),$$
$$E_4(ABCD) \geqslant E_2(AC),$$
$$E_4(ABCD) \geqslant E_2(BC),$$
$$E_4(ABCD) \geqslant E_2(AD),$$
$$E_4(ABCD) \geqslant E_2(BD),$$
$$E_4(ABCD) \geqslant E_2(CD)$$

for any 4-entangled state $\rho^{ABCD} \in \mathcal{S}^{ABCD}$. E_3 is tightly coarsening monotonic means

$$E_3(A|B|CD) \geqslant E_2(A|BCD),$$
$$E_3(A|B|CD) \geqslant E_2(AB|CD),$$
$$E_3(A|B|CD) \geqslant E_2(ACD|B)$$

for any 3-entangled state $\rho^{ABCD} \in \mathcal{S}^{ABCD}$, while E_4 is tightly coarsening monotonic means

$$E_4(ABCD) \geqslant E_3(A|B|CD),$$
$$E_4(ABCD) \geqslant E_3(A|BC|D),$$
$$E_4(ABCD) \geqslant E_3(AB|C|D),$$
$$E_4(ABCD) \geqslant E_3(AC|B|D),$$
$$E_4(ABCD) \geqslant E_3(AD|B|C),$$
$$E_4(ABCD) \geqslant E_3(A|C|BD),$$
$$E_4(ABCD) \geqslant E_2(AB|CD),$$
$$E_4(ABCD) \geqslant E_2(AC|BD),$$
$$E_4(ABCD) \geqslant E_2(AD|BC)$$

for any 4-entangled state $\rho^{ABCD} \in \mathcal{S}^{ABCD}$.

V. COMPLETE MONOGAMY OF k-EM

By reviewing the complete monogamy and the tightly complete monogamy discussed in Refs. [21,29,50,51], we can conclude that the complete monogamy is related to the unification condition while the tightly complete monogamy is to the hierarchy condition. Accordingly, we suggest the

TABLE I. Comparing of monogamy, complete monogamy, and tightly complete monogamy. Abbreviations are as follow: bipartite (B), unified (U), complete (C), monogamy (M), complete monogamy (CM), tightly complete monogamy (TCM), entanglement measure (EM), multipartite entanglement measure (MEM), and genuine multipartite entanglement measure (GMEM).

	Coarser relation	Compatible EM
M	$\succ^c$	k-EM, BEM
CM	$\succ^a$	Uk-EM, UMEM, UGMEM
TCM	$\succ^b$	Ck-EM, CMEM, CGMEM

following two concepts. A unified k-EM E_k is *completely monogamous* if for any k-entangled state $\rho \in \mathcal{S}^{A_1A_2\cdots A_n}$ that satisfies

$$E_k(X_1|X_2|\cdots|X_p) = E_l(Y_1|Y_2|\cdots|Y_q) \tag{16}$$

with $X_1|X_2|\cdots|X_p \succ^a Y_1|Y_2|\cdots|Y_q$ we have that

$$E_*(\Gamma) = 0 \tag{17}$$

holds for all $\Gamma \in \Xi(X_1|X_2|\cdots|X_p - Y_1|Y_2|\cdots|Y_q)$, $k \leqslant p$, $l \leqslant q$, $l \leqslant k$, and hereafter the asterisk (*) subscript is associated with the partition Γ, e.g., if Γ is a t-partite partition, then $* = t$. A complete E_k is defined to be *tightly complete monogamous* if Eqs. (16) and (17) hold by replacing $X_1|X_2|\cdots|X_p \succ^a Y_1|Y_2|\cdots|Y_q$ with $X_1|X_2|\cdots|X_p \succ^b Y_1|Y_2|\cdots|Y_q$. For more clarity, we compare these three kinds of monogamy relations in Table I.

As a illustrated example, we consider the 4-partite case $ABCD$. If E_4 is completely monogamous, then

$$E_4(ABCD) = E_3(ABC) \Rightarrow E_2(ABC|D) = 0,$$
$$E_4(ABCD) = E_2(AC) \Rightarrow E_3(AC|B|D) = E_2(BD) = 0$$

for any 4-entangled state $\rho^{ABCD} \in \mathcal{S}^{ABCD}$ (the other cases can be easily followed); if E_3 is completely monogamous, then

$$E_3(A|B|CD) = E_2(A|CD) \Rightarrow E_2(ACD|B) = 0$$

for any 3-entangled state $\rho^{ABCD} \in \mathcal{S}^{ABCD}$ (the other cases can be easily followed). If E_4 is tightly complete monogamous, then

$$E_4(ABCD) = E_3(A|B|CD) \Rightarrow E_2(CD) = 0,$$
$$E_4(ABCD) = E_2(AB|CD) \Rightarrow E_2(AB) = E_2(CD) = 0$$

for any 4-entangled state $\rho^{ABCD} \in \mathcal{S}^{ABCD}$ (the other cases can be easily followed); if E_3 is tightly completely monogamous, then

$$E_3(A|B|CD) = E_2(AB|CD) \Rightarrow E_2(AB) = 0$$

for any 3-entangled state $\rho^{ABCD} \in \mathcal{S}^{ABCD}$ (the other cases can be easily followed).

VI. k-EM FROM SUM OF THE REDUCED FUNCTIONS

A. The minimal sum

We denote the set of all the k-partitions of $A_1|A_2|\cdots|A_n$ by Γ_k, $2 \leqslant k < n$, and write $\Gamma_k = \{\gamma_i\}$, where $\gamma_i = X_{1(i)}|X_{2(i)}|\cdots|X_{k(i)}$. For example $\gamma_i = A_1|A_2A_3|A_4A_5$ is a 3-partition of $A_1|A_2|A_3|A_4|A_5$ with $X_{1(i)} = A_1$, $X_{2(i)} = A_2A_3$, and $X_{3(i)} = A_4A_5$. Let $|\psi\rangle = |\psi\rangle^{A_1A_2\cdots A_n}$ be a pure state in $\mathcal{H}^{A_1A_2\cdots A_n}$ and h be a non-negative concave function on $\mathcal{S}^X$ with some abuse of notations. For any $\gamma_i \in \Gamma_k$, we write

$$\mathcal{P}_k^{\gamma_i}(|\psi\rangle) \equiv \frac{1}{2}\sum_{t=1}^{k} h(\rho^{X_{t(i)}}), \quad 2 \leqslant k < n, \tag{18}$$

where $\rho^X = \mathrm{Tr}_{\overline{X}}|\psi\rangle\langle\psi|$, and $\overline{X}$ denotes the subsystems complementary to those of X. The coefficient "1/2" is fixed by the unification condition when the measures defined via $\mathcal{P}_k^{\gamma_i}$ are regarded as unified k-EMs (see below). We define

$$E_k(|\psi\rangle) = \min_{\Gamma_k} \mathcal{P}_k^{\gamma_i}(|\psi\rangle), \tag{19}$$

where the minimum is taken over all feasible k-partitions in Γ_k. For a mixed state, we define it by the convex-roof structure. In what follows, we give only the measures for pure states; for the case of mixed states they are all defined by the convex-roof extension with no further statement. Obviously, any entanglement measure that is defined in this way is convex straightforwardly. By definition, for any $\rho \in \mathcal{S}^{A_1A_2\cdots A_n}$, $E_k(\rho) > 0$ if and only if ρ is k-entangled. If we take $k = n$, then E_k defined in Eq. (19) is reduced to the MEM $E^{(n)}$ in Ref. [35]. When $k = 2$ with h is the reduced function of the concurrence which is defined as $C(|\psi\rangle^{AB}) = \sqrt{2(1 - \mathrm{Tr}\rho_A^2)}$ [54,55], it reduces to the GMEM C_{GME} in Ref. [18]. Throughout this paper, for any k-EM we defined, it refers to that the involved global system is an n-partite system with $n > k$ unless otherwise specified.

Theorem 1. E_k is a k-EMo, and it is k-monotonic whenever the reduced function is subaddtive.

Proof. In order to show E_k is an entanglement monotone we need to prove only that it does not increase on average under n-partite stochastic LOCC for pure states since the case of a mixed state can be easily followed according to the argument in Ref. [7]. For any given $|\psi\rangle^{A_1A_2\cdots A_n} \in \mathcal{H}^{A_1A_2\cdots A_n}$, we assume $E_k(|\psi\rangle^{A_1A_2\cdots A_n}) = \frac{1}{2}[h(\rho^{X_1}) + h(\rho^{X_2}) + \cdots + h(\rho^{X_k})]$ for some k-partition $X_1|X_2|\cdots|X_k$ of $A_1A_2\cdots A_n$. For arbitrarily given n-partite stochastic LOCC ε, we let the output state under ε is $\{p_i, |\phi_i\rangle^{A_1'A_2'\cdots A_n'}\}$ and denote the X' part marginal state of $|\phi_i\rangle^{A_1'A_2'\cdots A_n'}$ by $\sigma_i^{X'}$. For any subsystem X of $A_1A_2\cdots A_n$, we have $h(\rho^X) \geqslant \sum_i p_i h(\rho_i^{X'})$ since $h(\rho^X)$ is indeed a bipartite entanglement monotone of $|\psi\rangle^{X|\overline{X}}$. It follows that

$$\begin{aligned} E_k(|\psi\rangle^{A_1A_2\cdots A_n}) &= \frac{1}{2}[h(\rho^{X_1}) + h(\rho^{X_2}) + \cdots + h(\rho^{X_k})] \\ &\geqslant \sum_i p_i E_k(|\phi_i\rangle^{X_1'X_2'\cdots X_k'}) \\ &= \frac{1}{2}\sum_i p_i\big[h(\sigma_i^{X_1'}) + h(\sigma_i^{X_2'}) + \cdots + h(\sigma_i^{X_k'})\big] \\ &\geqslant \sum_i p_i E_k(|\phi_i\rangle^{A_1'A_2'\cdots A_n'}) \\ &\geqslant E_k(\sigma^{A_1'A_2'\cdots A_n'}), \end{aligned}$$

where $\sigma^{A_1'A_2'\cdots A_n'} = \sum_i p_i |\phi_i\rangle\langle\phi_i|^{A_1'A_2'\cdots A_n'}$. Therefore, E_k is an entanglement monotone.

We claim that if a k-EM is k-monotonic for pure states, then it is also true for mixed states. If $\rho^{A_1A_2\cdots A_n}$ is a mixed state, we let $E_k(\rho^{A_1A_2\cdots A_n})=\sum_i p_iE_3(|\psi_i\rangle^{A_1A_2\cdots A_n})$ for some decomposition $\rho^{A_1A_2\cdots A_n}=\sum_i p_i|\psi_i\rangle\langle\psi_i|^{A_1A_2\cdots A_n}$. Then

$$\begin{aligned}E_k(\rho^{A_1A_2\cdots A_n})&=\sum_i p_iE_k(|\psi_i\rangle^{A_1A_2\cdots A_n})\\&\geqslant\sum_i p_iE_{k-1}(|\psi_i\rangle^{A_1A_2\cdots A_n})\\&\geqslant E_{k-1}(\rho^{A_1A_2\cdots A_n}).\end{aligned}$$

If h is subadditive, then $E_k(|\psi\rangle^{A_1A_2\cdots A_n})\geqslant E_{k-1}(|\psi\rangle^{A_1A_2\cdots A_n})$ since for any k-partition $X_1|X_2|\cdots|X_k$ of $A_1A_2\cdots A_n$ and $(k-1)$-partition $Y_1|Y_2|\cdots|Y_{k-1}$ of $A_1A_2\cdots A_n$ that satisfies $X_1|X_2|\cdots|X_k\succ^b Y_1|Y_2|\cdots|Y_{k-1}$ we have

$$E_k(|\psi\rangle^{X_1|X_2|\cdots|X_k})\geqslant E_{k-1}(|\psi\rangle^{Y_1|Y_2|\cdots|Y_{k-1}}).$$

Namely, E_k is k-monotonic for a pure state. We thus conclude that E_k is k-monotonic. ■

We consider $|\psi\rangle^{AB}|\psi\rangle^{CD}$ with both $|\psi\rangle^{AB}$ and $|\psi\rangle^{CD}$ are entangled, and $h(\rho^A)>h(\rho^C)$. Then it is clear that

$$E_3(|\psi\rangle^{AB}|\psi\rangle^{CD})=h(\rho^C)<h(\rho^A)=E_3(\rho^{ABC})$$

since $E_3(\rho^{ABC})=\min\sum_i p_iE_3(|\psi\rangle^{AB}|\psi_i\rangle^C)=\min\sum_i p_ih_A=h_A$, where the minimum is taken over all ensembles $\{p_i,|\psi\rangle^{AB}|\psi_i\rangle^C\}$ of ρ^{ABC}. Let

$$\begin{aligned}|\psi\rangle^{ABC_1}&=\sqrt{0.5}|000\rangle+0.1|101\rangle+0.7|110\rangle,\\|\psi\rangle^{C_2D}&=\sqrt{0.99}|20\rangle+\sqrt{0.01}|31\rangle,\end{aligned}\qquad(20)$$

then $|\Psi\rangle=|\psi\rangle^{ABC_1}|\psi\rangle^{C_2D}$ is a genuinely entangled pure state in $\mathcal{H}^{ABCD}$,

$$\rho^A=\begin{pmatrix}0.5&0\\0&0.5\end{pmatrix},\quad\rho^B=\begin{pmatrix}0.51&0\\0&0.49\end{pmatrix},$$
$$\rho^{C_1}=\begin{pmatrix}0.99&0\\0&0.01\end{pmatrix},$$

and

$$\rho^{AB}=\begin{pmatrix}0.5&0&0&0.245\\0&0&0&0\\0&0&0.01&0\\0.245&0&0&0.49\end{pmatrix}.$$

We take $h(\rho)=\sqrt{2(1-\mathrm{Tr}\rho^2)}$, i.e., E_2 is concurrence C. From the formula $C(\rho)=\lambda_1-\lambda_2-\lambda_3-\lambda_4$ [56], where λ_i's are the eigenvalues of $\sqrt{\rho^{1/2}\tilde{\rho}\rho^{1/2}}$, $\tilde{\rho}=\sigma_y\otimes\sigma_y\rho^*\sigma_y\otimes\sigma_y$, $\lambda_1\geqslant\lambda_2\geqslant\lambda_3\geqslant\lambda_4$, we get $E_2(\rho^{AB})=C(\rho^{AB})\approx0.9899>0.3388\approx E_3(|\Psi\rangle)$. That is, E_k is not coarsening monotonic (even for the genuine entangled states), namely, there exists $|\psi\rangle^{ABCD}$ which violates Eq. (13), so E_k is not coarsening monotonic.

It is straightforward that E_k is not tightly coarsening monotonic. In fact, we have that

$$E_k(X_1|X_2|\cdots|X_p)\leqslant E_k(Y_1|Y_2|\cdots|Y_q)\qquad(21)$$

holds for all k-entangled state $\rho\in\mathcal{S}^{A_1A_2\cdots A_n}$ whenever $X_1|X_2|\cdots|X_p\succ^b Y_1|Y_2|\cdots|Y_q$ with $k\leqslant q\leqslant p$.

Theorem 2. If h is strictly concave, E_k is monogamous.

Proof. We check the case of $n=5$ and $2\leqslant k\leqslant3$ with no loss of generality first. If $E_3(|\psi\rangle^{A|BC|DE})=E_3(\rho^{A|BC|D})$, we have

$$\begin{aligned}&h(\rho^A)+h(\rho^{BC})+h(\rho^{DE})\\&=h(\rho^A)+h(\rho^{BC})+h(\rho^{ABC})\\&=\sum_i p_i\big[h(\rho_i^A)+h(\rho_i^{BC})+h(\rho_i^D)\big]\\&=\sum_i p_i\big[h(\rho_i^A)+h(\rho_i^{BC})+h(\rho_i^{ABC})\big]\end{aligned}$$

for any decomposition $\rho^{ABCD}=\sum_i p_i|\psi_i\rangle\langle\psi_i|^{ABCD}$, where $h(\rho_i^X)=\mathrm{Tr}_{\overline{X}}|\psi_i\rangle\langle\psi_i|^{ABCD}$. Since h is strictly concave, we get $h(\rho^A)=\sum_i p_ih(\rho_i^A)$, $h(\rho^{BC})=\sum_i p_ih(\rho_i^{BC})$, $h(\rho^{ABC})=\sum_i p_ih(\rho_i^{ABC})$, and $h(\rho^{DE})=\sum_i p_ih(\rho_i^D)$. By Theorem 1 in [49], $\mathcal{H}^D$ has a subspace isomorphic to $\mathcal{H}^{D_1}\otimes\mathcal{H}^{D_2}$ such that up to local unitary operation on D_1D_2,

$$|\psi\rangle^{ABCDE}=|\psi\rangle^{ABCD_1}|\psi\rangle^{D_2E},$$

which reveals $E_2(\rho^{A|BC|E})=0$.

Moreover, if $E_3(|\psi\rangle^{A|BC|DE})=E_3(\rho^{ABD})$, then $E_3(|\psi\rangle^{A|BC|DE})=E_3(\rho^{A|BC|D})$ at first. Therefore $\rho^{ABCD}=|\psi\rangle\langle\psi|^{ABCD_1}\otimes\rho^{D_2}$ with $E_3(\rho^{A|BC|D})=E_3(\rho^{ABD})$, and this leads to

$$|\psi\rangle^{ABCD_1}=|\psi\rangle^{AB_1D_1}|\psi\rangle^{B_2C}$$

up to some local unitary operation on B_1B_2. So we get $E_3(\rho^{A|C|E})=0$ and $E_2(C|ADE)=E_2(ABC|E)=0$, which means E_3 is monogamous as desired.

We assume with no loss of generality that $E_2(|\psi\rangle^{AB|CD|EF})=E_2(|\psi\rangle^{ABCD|EF})$, then for any ensemble of ρ^{ABCDE}, $\{p_i,|\psi_i\rangle^{ABCDE}\}$, we have

$$\begin{aligned}2E_2(|\psi\rangle^{ABCD|EF})&=h(\rho^{ABCD})+h(\rho^{EF})\\&\geqslant2\sum_i p_iE_2(|\psi_i\rangle^{ABCD|E})\\&=\sum_i p_i[h(\rho^{A_iB_iC_iD_i})+h(\rho^{E_i})]\\&\geqslant2E_2(\rho^{AB|CD|E}).\end{aligned}$$

If $E_2(|\psi\rangle^{AB|CD|EF})=E_2(\rho^{AB|CD|E})$, we can get $h(\rho^{ABCD})=\sum_i p_ih(\rho^{A_iB_iC_iD_i})$ and $h(\rho^{EF})=\sum_i p_ih(\rho^{E_i})$. This yields $|\psi\rangle^{ABCDEF}=|\psi\rangle^{ABCDE_1}|\psi\rangle^{E_2F}$ up to some local unitary operation on E_1E_2 according to Theorem 1 in Ref. [49]. Therefore $E_2(ABCD|F)=0$ as desired. If $E_2(|\psi\rangle^{AB|CD|EF})=E_2(\rho^{AB|C|E})$, then $E_2(|\psi\rangle^{AB|CD|EF})=E_2(\rho^{AB|CD|E})=E_2(\rho^{AB|C|E})$, which reveals $|\psi\rangle^{ABCDEF}=|\psi\rangle^{ABCDE_1}|\psi\rangle^{E_2F}$ up to some local unitary operation as above. Note that $E_2(\rho^{AB|CD|E})=E_2(|\psi\rangle^{AB|CD|E_1})$ in such a case, and we thus can derive $|\psi\rangle^{AB|CD|E_1}=|\psi\rangle^{ABE_1C_1}|\psi\rangle^{C_2D}$ up to some local unitary operation on C_1C_2 using Theorem 1 in Ref. [49] again. Therefore, $E_2(ABEF|D)=E_2(AB|D|F)=0$, i.e., E_2 is monogamous.

In general, according to Theorem 1 in Ref. [49], if $E_k(X_1|X_2|\cdots|X_p)=E_k(X_1|X_2|\cdots|X_{p-1}|X_p')$ with $X_1|X_2|\cdots|X_p\succ^c X_1|X_2|\cdots|X_{p-1}|X_p'$ for $|\psi\rangle^{X_1X_2\cdots X_p}$, we can get

$$|\psi\rangle^{X_1X_2\cdots X_p}=|\psi\rangle^{X_1X_2\cdots X_{p_1}'}|\psi\rangle^{X_{p_2}'Z_p}\qquad(22)$$

up to some local unitary operation on $X'_{p_1}X'_{p_2}$, where $\mathcal{H}^{X'_{p_1}}\otimes\mathcal{H}^{X'_{p_2}}$ is isomorphic to a subspace of $\mathcal{H}^{X'_p}$, $Z_p=X_p-X'_p$. Therefore $E_2(|\psi\rangle^{X_1X_2\cdots X_{p-1}|X_p})=0$. The other cases can be checked similarly. This completes the proof. ■

E_3 is not coarsening monotonic, so it cannot be completely monogamous. Moreover, even though Eq. (16) holds, we cannot guarantee Eq. (17) is valid. For instance, $E_3(|\psi\rangle^{AB}|\psi\rangle^{CD})=E_3(\rho^{ABC})$ whenever $h(\rho^C)=h(\rho^A)$ but $E_2(|\psi\rangle^{ABC|D})>0$. They also are not tightly completely monogamous since they are not tightly coarsening monotonic. Moreover, we take $|\psi\rangle^{AB_1C}|\psi\rangle^{B_2D}$ with $|\psi\rangle^{AB_1C}=\frac{1}{\sqrt{3}}(|000\rangle+|101\rangle+|210\rangle)$ and $|\psi\rangle^{B_2D}=\sqrt{0.95}|20\rangle+\sqrt{0.05}|31\rangle$. If h is the von Neumann entropy, then $E_3(|\psi\rangle^{ABCD})=E_2(|\psi\rangle^{AB|CD})$ and ρ^{CD} is separable but ρ^{AB} is entangled.

Particularly, for the case of $k=2$, E_2 is a genuine entanglement measure. By Theorem 2, if the reduced function is strictly concave, E_2 is monogamous. In such a case, E_2 is monogamous means it is monogamous as a 2-entanglement, but not regarded as genuine entanglement since genuine entanglement under some given partition is meaningless. The same is true in other cases we proposed below where we do not repeat any more. In addition, according to the examples we discussed, E_2 as a genuine entanglement measure is neither completely monogamous nor tightly complete monogamous.The same is true for other cases below (if it is shown to be not coarsening monotonic/tightly coarsening monotonic on genuine entangled states) where we do not restate again.

B. The maximal sum

Let $|\psi\rangle=|\psi\rangle^{A_1A_2\cdots A_n}$ be a pure state in $\mathcal{H}^{A_1A_2\cdots A_n}$ and h be a non-negative concave function on $\mathcal{S}^X$. We define

$$E'_k(|\psi\rangle)=\begin{cases}\max_{\Gamma_k}\mathcal{P}_k^{\gamma_i}(|\psi\rangle), & \min_{\Gamma_k}\mathcal{P}_k^{\gamma_i}(|\psi\rangle)>0,\\ 0, & \min_{\Gamma_k}\mathcal{P}_k^{\gamma_i}(|\psi\rangle)=0,\end{cases}\tag{23}$$

where the maximum/minimum is taken over all feasible k-partitions in Γ_k. E'_k may be not a well-defined entanglement monotone since we cannot guarantee that it is nonincreasing on average under LOCC, and it is not even a well-defined entanglement measure.

With some abuse of terminologies, we can conclude the following properties of E'_k.

Theorem 3. If the reduced function h is subadditive, then (i) E'_k is k-monotonic, (ii) E'_k is tightly coarsening monotonic, and (iii)

$$E'_k(X_1X_2\cdots X_p)\geqslant E'_l(Y_1Y_2\cdots Y_s)\tag{24}$$

for any state $\rho^{A_1A_2\cdots A_n}\in\mathcal{S}^{A_1A_2\cdots A_n}$, where $X_1X_2\cdots X_p\succ^a Y_1Y_2\cdots Y_s$, $l\leqslant s\leqslant k\leqslant p$.

Proof. (i) The k-monotonicity of E'_k is clear since for any k-partition $X_1|X_2|\cdots|X_k$ of $A_1A_2\cdots A_n$ and $(k-1)$-partition $Y_1|Y_2|\cdots|Y_{k-1}$ of $A_1A_2\cdots A_n$ that satisfies $X_1|X_2|\cdots|X_k\succ^b Y_1|Y_2|\cdots|Y_{k-1}$ we have

$$E'_k(|\psi\rangle^{X_1|X_2|\cdots|X_k})\geqslant E'_{k-1}(|\psi\rangle^{Y_1|Y_2|\cdots|Y_{k-1}})$$

due to the subadditivity of h. (ii) If the reduced function h is subadditive, the tight coarsening monotone is clear by definition. (iii) We only need to show

$$E'_k(X_1X_2\cdots X_p)\geqslant E'_k(Y_1Y_2\cdots Y_k)$$

holds for pure state with $X_1X_2\cdots X_p\succ^a Y_1Y_2\cdots Y_k$. If h is subadditive, then for any $|\psi\rangle^{A_1A_2\cdots A_n}\in\mathcal{H}^{A_1A_2\cdots A_n}$ and any p-partition $X_1X_2\cdots X_p$ of $A_1A_2\cdots A_n$, we assume with no loss of generality that $Y_1Y_2\cdots Y_k=X_1X_2\cdots X_k$. It turns out that

$$\begin{aligned}E'_k(|\psi\rangle^{X_1X_2\cdots X_p})&\geqslant\frac{1}{2}\left[\sum_{i=1}^{k-1}h(\rho^{X_i})+h(\rho^{X_k\cdots X_p})\right]\\&=\frac{1}{2}\left[\sum_{i=1}^{k-1}h(\rho^{X_i})+h(\rho^{X_1\cdots X_{k-1}})\right]\\&\geqslant\frac{1}{2}\sum_j p_j\left[\sum_{i=1}^{k-1}h\big(\rho_j^{X_i}\big)+h\big(\rho_j^{X_1\cdots X_{k-1}}\big)\right]\\&=\frac{1}{2}\sum_j p_j\left[\sum_{i=1}^{k-1}h\big(\rho_j^{X_i}\big)+h\big(\rho_j^{X_k}\big)\right]\\&\geqslant E'_k(\rho^{X_1X_2\cdots X_k})\end{aligned}$$

for any decomposition $\rho^{X_1X_2\cdots X_k}=\sum_j p_j|\psi_j\rangle\langle\psi_j|^{X_1X_2\cdots X_k}$, where $h(\rho_j^X)=\mathrm{Tr}_{\overline{X}}|\psi_j\rangle\langle\psi_j|^{X_1X_2\cdots X_k}$. This completes the proof. ■

Although Eq. (24) is valid for E'_k, we cannot derive that it is coarsening monotonic. In general, tightly coarsening monotonicity is stronger than the coarsening monotonicity, so we thus conjecture that E'_k is coarsening monotonic. In such a sense, if E'_k is nonincreasing under LOCC (resp. nonincreasing on average under LOCC), it is a a complete k-EM (resp. k-EMo) provided that h is subadditive.

By Theorem 2, E'_k is weakly monogamous if h is strictly concave since $E'_k=E_k$ for the state under any k-partition. But Eq. (15) may be not valid for E'_k since it may not be a k-EMo. Even if it is a k-EMo/k-EM, we cannot derive that it is monogamous in general. Obviously, $E'_3(|\psi\rangle^{AB}|\psi\rangle^{CD})=E'_3(\rho^{ABC})$ whenever $h(\rho^C)=h(\rho^A)$ but $E'_2(|\psi\rangle^{ABC|D})>0$. Let $|\psi\rangle^{ABCD}=|\psi\rangle^{ABC}|\psi\rangle^{D}$ with $|\psi\rangle^{ABC}=\frac{1}{\sqrt{3}}(|000\rangle+|101\rangle+|210\rangle)$, then $E'_3(|\psi\rangle^{ABC}|\psi\rangle^{D})=E'_2(|\psi\rangle^{BC|AD})$ and ρ^{AD} is separable but ρ^{BC} is entangled.

C. The arithmetic mean

Let $|\psi\rangle=|\psi\rangle^{A_1A_2\cdots A_n}$ be a pure state in $\mathcal{H}^{A_1A_2\cdots A_n}$ and h be a non-negative concave function on $\mathcal{S}^X$. We define

$$\bar{E}_k(|\psi\rangle)=\begin{cases}\frac{\sum_{\gamma_i}\mathcal{P}_k^{\gamma_i}(|\psi\rangle)}{|\Gamma_k|}, & \min_{\Gamma_i}\mathcal{P}_k^{\gamma_i}(|\psi\rangle)>0,\\ 0, & \min_{\Gamma_i}\mathcal{P}_k^{\gamma_i}(|\psi\rangle)=0,\end{cases}\tag{25}$$

where the minimum is taken over all feasible k-partitions in Γ_k.

Theorem 4. $\bar{E}_k$ is a k-EMo, and it is k-monotonic if the reduced function h is subadditive.

Proof. It is easy to see that $\bar{E}_k$ is an entanglement monotone. Let γ_i^k be a k-partition in Γ_k. Then for any$(k-1)$-partition γ_i^{k-1} that is more coarsened than γ_i^k, i.e., $\gamma_i^k\succ^b$

γ_i^{k-1}, we have

$$\mathcal{P}_k^{\gamma_i^k}(|\psi\rangle) \geqslant \mathcal{P}_{k-1}^{\gamma_i^{k-1}}(|\psi\rangle)$$

for any $|\psi\rangle \in \mathcal{H}^{A_1A_2\cdots A_n}$ if h is subadditive. One can check that $\bar{E}_k(|\psi\rangle) \geqslant \bar{E}_{k-1}(|\psi\rangle)$ is equivalent to $\mathcal{P}_k^{\gamma_i^k}(|\psi\rangle) \geqslant \mathcal{P}_{k-1}^{\gamma_i^{k-1}}(|\psi\rangle)$, i.e., $\bar{E}_k$ is k-monotonic provided that h is subadditive. ■

In general, $\bar{E}_k$ is not coarsening monotonic. For example, we take $|\psi\rangle^{A_1A_2}|\psi\rangle^{A_3A_4}|\psi\rangle^{A_5}$, then

$$\begin{aligned}\bar{E}_4(|\psi\rangle^{A_1A_2}|\psi\rangle^{A_3A_4}|\psi\rangle^{A_5}) &= \frac{1}{20}\left[6\sum_i h(\rho^{A_i}) + \sum_{i<j} h(\rho^{A_iA_j}))\right]\\ &\leqslant \frac{9}{20}\sum_i h(\rho^{A_i}) < \frac{1}{2}\sum_i h(\rho^{A_i})\\ &= \bar{E}_4(|\psi\rangle^{A_1A_2}|\psi\rangle^{A_3A_4})\end{aligned}$$

if h is subadditive. Let

$$|\Phi\rangle^{ABC} = |\phi\rangle^{AB_1}|\phi\rangle^{B_2C} \tag{26}$$

where both $|\phi\rangle^{AB_1}$ and $|\phi\rangle^{B_2C}$ are entangled. We take h are the von Neumann entropy $S(\cdot)$. If $S(\rho^C) < S(\rho^A)/2$, then

$$\begin{aligned}\bar{E}_2(|\Phi\rangle^{ABC}) &= 2[S(\rho^A) + S(\rho^C)]/3 < S(\rho^A)\\ &= \bar{E}_2(\rho^{AB}) = \bar{E}_2(|\Phi\rangle^{A|BC}).\end{aligned}$$

For any given $|\psi\rangle^{ABCD}$, we cannot guarantee $\bar{E}_3(|\psi\rangle^{ABCD}) \geqslant \bar{E}_3(|\psi\rangle^{AB|C|D})$ since

$$\begin{aligned}&\tfrac{1}{6}[h(\rho^A) + h(\rho^{BC}) + h(\rho^D) + h(\rho^{AB}) + h(\rho^C)\\ &\quad + h(\rho^D) + h(\rho^{AC}) + h(\rho^B) + h(\rho^D) + h(\rho^{AD})\\ &\quad + h(\rho^B) + h(\rho^C) + h(\rho^A) + h(\rho^B)\\ &\quad + h(\rho^{CD}) + h(\rho^A) + h(\rho^{BD}) + h(\rho^C)]\\ &= \tfrac{1}{6}[3h(\rho^A) + 3h(\rho^B) + 3h(\rho^C) + 3h(\rho^D)\\ &\quad + 2h(\rho^{AB}) + 2h(\rho^{AC}) + 2h(\rho^{AD})]\\ &< h(\rho^{AB}) + h(\rho^C) + h(\rho^D)\end{aligned}$$

occurs possibly. Namely, it is not tightly coarsening monotonic either.

By definition, using analogous arguments as the proof of Theorem 2, we have the following proposition.

Proposition 1. If h is strictly concave, $\bar{E}_k$ is monogamous.

D. The geometric mean

In Ref. [36], two k-EMs called k-geometric multipartite concurrence and q-k-geometric multipartite concurrence, respectively, are proposed. In fact, the way of defining k-EM therein is valid for any non-negative concave function h. Let $|\psi\rangle = |\psi\rangle^{A_1A_2\cdots A_n}$ be a pure state in $\mathcal{H}^{A_1A_2\cdots A_n}$ and h be a non-negative concave function on $\mathcal{S}^X$. We define

$$E_k^G(|\psi\rangle) = \left[\prod_{\gamma_i\in\Gamma_k} \mathcal{P}_k^{\gamma_i}(|\psi\rangle)\right]^{1/|\Gamma_k|}. \tag{27}$$

Note here that $\mathcal{P}_k^{\gamma_i}$ is different from that of $\mathcal{P}_k$ associated with concurrence and the q-concurrence in Ref. [36]: $2\mathcal{P}_k^{\gamma_i}(|\psi\rangle) = k[\mathcal{P}_k(|\psi\rangle)]^2$ when we take $h = \sqrt{2(1-\mathrm{Tr}\rho^2)}$ or $h = \sqrt{2(1-\mathrm{Tr}\rho^q)}$, $q > 1$.

Theorem 5. E_k^G is a k-EMo.

Proof. Let $|\psi\rangle = |\psi\rangle^{A_1A_2\cdots A_n}$ be a pure state in $\mathcal{H}^{A_1A_2\cdots A_n}$ and ε be an n-partite stochastic LOCC with the output states $\{p_j, |\phi_j\rangle^{A_1'A_2'\cdots A_n'}\}$. Then

$$\begin{aligned}E_k^G(|\psi\rangle) &= \left[\prod_{\gamma_i\in\Gamma_k} \mathcal{P}_k^{\gamma_i}(|\psi\rangle)\right]^{1/|\Gamma_k|}\\ &\geqslant \left[\prod_{\gamma_i\in\Gamma_k}\left(\sum_j p_j\mathcal{P}_k^{\gamma_i}(|\phi_j\rangle)\right)\right]^{1/|\Gamma_k|}\\ &\geqslant \sum_j p_j\left[\prod_{\gamma_i\in\Gamma_k} \mathcal{P}_k^{\gamma_i}(|\phi_j\rangle)\right]^{1/|\Gamma_k|}\\ &= \sum_j p_j E_k^G(|\phi_j\rangle^{A_1'A_2'\cdots A_n'}),\end{aligned}$$

where the second inequality is true because the geometric mean function $f = (\prod_i^n x_i)^{1/n}$ is a concave function [57], i.e., E_k^G is nonincreasing on average under LOCC. This completes the proof. ■

Proposition 2. E_k^G is k-monotonic if h is subadditive.

Proof. Let $|\psi\rangle = |\psi\rangle^{A_1A_2\cdots A_n}$ be a pure state in $\mathcal{H}^{A_1A_2\cdots A_n}$. For any $\gamma_i \in \Gamma_k$, we denote by γ_i^b the $(k-1)$-partition in Γ_{k-1} with $\gamma_i \succ^b \gamma_i^b$. It follows that

$$\mathcal{P}_{k-1}^{\gamma_i^b}(|\psi\rangle) \leqslant \mathcal{P}_k^{\gamma_i}(|\psi\rangle)$$

for any γ_i^b due to the subadditive of h. We thus can conclude

$$\begin{aligned}E_k^G(|\psi\rangle) &= \left[\prod_{\gamma_i\in\Gamma_k} \mathcal{P}_k^{\gamma_i}(|\psi\rangle)\right]^{1/|\Gamma_k|}\\ &\geqslant \left[\prod_{\gamma_j\in\Gamma_{k-1}} \mathcal{P}_{k-1}^{\gamma_j}(|\psi\rangle)\right]^{1/|\Gamma_{k-1}|}\\ &= E_{k-1}^G(|\psi\rangle)\end{aligned}$$

since $\mathcal{P}_{k-1}^{\gamma_j}(|\psi\rangle) = \mathcal{P}_{k-1}^{\gamma_i^b}(|\psi\rangle)$ for some γ_i^b. ■

Note that if E_k^G satisfies Eq. (9), then for any $\gamma_i \in \Gamma_k$, we have $\mathcal{P}_k^{\gamma_i}(|\psi\rangle) = \sum_j p_j\mathcal{P}_k^{\gamma_i}(|\phi_j\rangle)$ with notations defined as in the proof of Theorem 5 just by replacing the LOCC with the associated partial trace. We thus can conclude that E_k^G is also monogamous if the associated reduced function is strictly concave.

In general, E_k^G is neither coarsening monotonic nor tightly coarsening monotonic. In order to see this, we let $|\psi\rangle^{ABCD} = |\psi\rangle^{ABC}|\psi\rangle^D$ with $|\psi\rangle^{ABC}$ being the three-qubit GHZ state. Obviously,

$$\begin{aligned}E_3^G(|\psi\rangle^{ABC}|\psi\rangle^D) &= \frac{\sqrt{3}}{\sqrt{2}}h(\rho^A) < \frac{3}{2}h(\rho^A)\\ &= E_3^G(|\psi\rangle^{ABC}) = E_3^G(|\psi\rangle^{A|B|CD}).\end{aligned}$$

TABLE II. Comparing of E_k, E'_k, $\bar{E}_k$, and E_k^G with the assumption that the reduced functions are strictly concave and subadditive. k-M, CoM, and TCoM signify the measure is k-monotonic, coarsening monotonic, and tightly coarsening monotonic, respectively.

k-EM	k-M	CoM	TCoM	Complete	M	CM	TCM
E_k	✓	×	×	×	✓	×	×
E'_k [a]	✓	✓ [b]	✓	✓ [b]	× [c]	×	×
$\bar{E}_k$	✓	×	×	×	✓	×	×
E_k^G	✓	×	×	×	✓	×	×

[a]We assume here that E'_k is a k-EMo.
[b]We prove only Eq. (24) in this paper, and we conjecture that it is CoM.
[b]It is weakly monogamous.

In addition, for the state as in Eq. (26) with $S(\rho^C) < S(\rho^A)/2$, we have

$$E_2^G(|\Phi\rangle^{ABC}) < E_2^G(\rho^{AB}) = E_2^G(|\Phi\rangle^{A|BC}).$$

We clearly have

$$E_k \leqslant E_k^G \leqslant \bar{E}_k \leqslant E'_k, \quad E_n = E_n^G = \bar{E}_n = E'_n \tag{28}$$

hold for any $\rho \in \mathcal{S}^{A_1A_2\cdots A_n}$, and all these measures are faithful. In conclusion, theses measures are monogamous (weakly monogamous for E'_k) if and only if the associated reduced functions are strictly concave, while they are k-monotonic if and only if the associated reduced functions are subadditive. We summarize these properties in Table II for more clarity. So far we have known that the von Neumann entropy, the Tsallis q-entropy for $q > 1$, and the reduced function of concurrence are both strictly concave and subadditive [7,35,48,58,59], so the k-EMs defined in this way with these reduced functions are better than the other ones.

We now consider the cases of taking h as the reduced function of concurrence and the von Neumann entropy as illustrated examples, respectively.

Example 1. We take the k-entanglement measure with the reduced function h as that of the concurrence, i.e., $h(\rho) = \sqrt{2(1-\mathrm{Tr}\rho^2)}$, and denote the corresponding four kinds of measures by C_k, C'_k, $\bar{C}_k$, and C_k^G, respectively. By definition, C_k, $\bar{C}_k$, and C_k^G are monogamous while C'_k is weakly monogamous as h is strictly concave. C_k, C'_k, $\bar{C}_k$, and C_k^G are k-monotonic since h is subadditive. For the four-qubit GHZ state $|GHZ_4\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle)$, it is immediate that

$$\begin{aligned} C_3(ABCD) &= C_3(A|B|CD) = \tfrac{3}{2}, \\ C_2(ABCD) &= C_2(AB|CD) = C_2(A|B|CD) \\ &= C_2(A|BCD) = 1. \end{aligned}$$

It is also true for $C'_{2,3}$, $\bar{C}_{2,3}$, and $C_{2,3}^G$ as $C_{2,3}(ABCD) = C'_{2,3}(ABCD) = \bar{C}_{2,3}(ABCD) = C_{2,3}^G(ABCD)$. For $|W_4\rangle = \frac{1}{2}(|1000\rangle + |0100\rangle + |0010\rangle + |0001\rangle)$,

$$\rho^A = \rho^B = \rho^C = \rho^D = \begin{pmatrix} \frac{3}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}$$

and

$$\rho^{XY} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

for any $X, Y \in \{A, B, C, D\}$, $X \neq Y$. It follows that

$$\begin{aligned} C_3(ABCD) &= C_3(A|B|CD) = \frac{\sqrt{3}+1}{2}, \\ C'_3(ABCD) &= C'_3(A|B|CD) = \frac{\sqrt{3}+1}{2}, \\ \bar{C}_3(ABCD) &= \bar{C}_3(A|B|CD) = \frac{\sqrt{3}+1}{2}, \\ C_3^G(ABCD) &= C_3^G(A|B|CD) = \frac{\sqrt{3}+1}{2}, \\ C_2(ABCD) &= C_2(A|BCD) = \frac{\sqrt{3}}{2} \\ &< C_2(AB|CD) = 1, \\ C'_2(ABCD) &= C'_2(AB|CD) = 1, \\ \bar{C}_2(ABCD) &= \frac{3}{7} + \frac{2\sqrt{3}}{7} < \bar{C}_2(AB|CD) = 1, \\ C_2^G(ABCD) &= \frac{\sqrt[7]{9}}{\sqrt[7]{16}} < C_2^G(AB|CD) = 1. \end{aligned}$$

Example 2. If h is the von Neumann entropy, $h(\rho) = S(\rho) = -\mathrm{Tr}(\rho\log_2\rho)$, then the associated E_k, $\bar{E}_k$, and E_k^G are monogamous while E'_k is weakly monogamous since S is strictly concave, and all these measures are k-monotonic since S is subadditive. For $|\mathrm{GHZ}_4\rangle$,

$$\begin{aligned} E_3(ABCD) &= E_3(A|B|CD) = \tfrac{3}{2}, \\ E_2(ABCD) &= E_2(AB|CD) = E_2(A|B|CD) \\ &= E_2(A|BCD) = 1. \end{aligned}$$

It is also true for $E'_{2,3}$, $\bar{E}_{2,3}$, and $E_{2,3}^G$ as $E_{2,3}(ABCD) = E'_{2,3}(ABCD) = \bar{E}_{2,3}(ABCD) = E_{2,3}^G(ABCD)$. For $|W_4\rangle$,

$$\begin{aligned} E_3(ABCD) &= E_3(A|B|CD) = \frac{3}{2} - \frac{3}{4}\log_2 3, \\ E'_3(ABCD) &= E'_3(A|B|CD) = \frac{3}{2} - \frac{3}{4}\log_2 3, \\ \bar{E}_3(ABCD) &= \bar{E}_3(A|B|CD) = \frac{3}{2} - \frac{3}{4}\log_2 3, \\ E_3^G(ABCD) &= E_3^G(A|B|CD) = \frac{3}{2} - \frac{3}{4}\log_2 3, \\ E_2(ABCD) &= E_2(A|BCD) = 1 - \frac{3}{4}\log_2 3 \\ &< E_2(AB|CD) = 1, \\ E'_2(ABCD) &= E'_2(AB|CD) = 1, \end{aligned}$$

$$\bar{E}_2(ABCD) = 1 - \frac{3}{7}\log_2 3 < \bar{E}_2(AB|CD) = 1,$$

$$E_2^G(ABCD) = \frac{\sqrt[7]{4 - 3\log_2 3}}{\sqrt[7]{4}} < E_2^G(AB|CD) = 1.$$

VII. k-EM FROM THE PRODUCT OF THE REDUCED FUNCTIONS

A. The minimal product

Let $|\psi\rangle = |\psi\rangle^{A_1A_2\cdots A_n}$ be a pure state in $\mathcal{H}^{A_1A_2\cdots A_n}$ and h be a non-negative concave function on $\mathcal{S}^X$. For any $\gamma_i \in \Gamma_k$, let

$$\mathcal{Q}_k^{\gamma_i}(|\psi\rangle) \equiv \left[\prod_{t=1}^{k} h(\rho^{X_{t(i)}})\right]^{1/k}, \quad 2 \leqslant k < n. \tag{29}$$

We define

$$E_{G-k}(|\psi\rangle) = \min_{\Gamma_k} \mathcal{Q}_k^{\gamma_i}(|\psi\rangle), \tag{30}$$

where the minimum is taken over all feasible k-partition Γ_k. By definition, E_{G-k} is not faithful, and whenever

$$2 \leqslant k \leqslant \begin{cases} \frac{n}{2} + 1, & \text{if } n \text{ is even,} \\ \frac{n+1}{2} + 1, & \text{if } n \text{ is odd,} \end{cases} \tag{31}$$

$E_{G-k}(\rho) = 0$ if and only if ρ is genuinely entangled, $\rho \in \mathcal{S}^{A_1A_2\cdots A_n}$.

Theorem 6. E_{G-k} is a k-EMo.

Proof. For any given $|\psi\rangle^{A_1A_2\cdots A_n} \in \mathcal{H}^{A_1A_2\cdots A_n}$, we assume $E_{G-k}(|\psi\rangle^{A_1A_2\cdots A_n}) = [\prod_{i=1}^k h(\rho^{X_i})]^{1/k}$ for some k-partition $X_1|X_2|\cdots|X_k$ of $A_1A_2\cdots A_n$. For arbitrarily given n-partite stochastic LOCC ε, we let the output state under ε is $\{p_j, |\phi_j\rangle^{A'_1A'_2\cdots A'_n}\}$ and denote the X part marginal state of $|\phi_j\rangle^{A'_1A'_2\cdots A'_n}$ by σ_j^X. Thus,

$$\begin{aligned}
&E_{G-k}(|\psi\rangle^{A_1A_2\cdots A_n}) \\
&= \left[\prod_{i=1}^{k} h(\rho^{X_i})\right]^{1/k} \\
&\geqslant \left[\sum_j p_j h(\sigma_j^{X'_1}) \sum_j p_j h(\sigma_j^{X'_2}) \cdots \cdot \sum_j p_j h(\sigma_j^{X'_k})\right]^{1/k} \\
&\geqslant \sum_i p_i \left[h(\sigma_i^{A'B'}) h(\sigma_i^{C'}) h(\sigma_i^{D'})\right]^{1/k} \\
&\geqslant \sum_i p_i E_{G-k}(|\phi_i\rangle^{A'_1A'_2\cdots A'_n}) \\
&\geqslant E_{G-k}(\sigma^{A'_1A'_2\cdots A'_n}),
\end{aligned}$$

where the second inequality is true because the geometric mean function $f = (\prod_i^n x_i)^{1/n}$ is concave. ■

$E_{G-k} \geqslant E_{G-2}$ for any concave function h. For any given $|\psi\rangle^{ABCD} \in \mathcal{H}^{ABCD}$, we assume $E_{G-3}(|\psi\rangle^{ABCD}) = [h(\rho^A)h(\rho^{BC})h(\rho^D)]^{1/3}$ and $E_{G-2}(|\psi\rangle^{ABCD}) = h(\rho^A)$ with no loss of generality. It follows that $h(\rho^A) = \min\{h(\rho^{AB}), h(\rho^{AC}), h(\rho^{AD}), h(\rho^A), h(\rho^B), h(\rho^C), h(\rho^D)\}$, which reveals $E_{G-3}(|\psi\rangle^{ABCD}) \geqslant E_{G-2}(|\psi\rangle^{ABCD})$. But E_{G-k} is not k-monotonic in general. For example, we take $|W\rangle^{ABCED} = \frac{1}{\sqrt{5}}(|10000\rangle + |01000\rangle + |00100\rangle + |00010\rangle + |00001\rangle)$, then

$$\rho^A = \rho^B = \rho^C = \rho^D = \rho^E = \begin{pmatrix} 4/5 & 0 \\ 0 & 1/5 \end{pmatrix}$$

and any two-partite marginal state is

$$\begin{pmatrix} 3/5 & 0 & 0 & 0 \\ 0 & 1/5 & 1/5 & 0 \\ 0 & 1/5 & 1/5 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

This leads to

$$\begin{aligned}
E_{G-4}(|W\rangle^{ABCDE}) &= [h^3(\rho^A)h(\rho^{AB})]^{1/4} \\
&< [h^2(\rho^A)h(\rho^{AB})]^{1/3} \\
&= E_{G-3}(|W\rangle^{ABCDE}),
\end{aligned}$$

which is equivalent to $h^4(\rho^A) < h(\rho^{AB})$ in general; e.g., we take $h(\rho) = 1 - \mathrm{Tr}\rho^2$, then $h^4(\rho^A) = (8/25)^4 < 12/25 = h(\rho^{AB})$ obviously.

For $|\psi\rangle^{ABC}|\psi\rangle^D$, $E_{G-3}(|\psi\rangle^{ABC}|\psi\rangle^D) = 0$ but $E_{G-3}(|\psi\rangle^{ABC}) > 0$ whenever $|\psi\rangle^{ABC}$ is genuinely entangled, namely, there exists $|\psi\rangle^{ABCD}$ which violates Eq. (13). Let $|\Psi\rangle = |\psi\rangle^{ABC_1}|\psi\rangle^{C_2D}$ with $|\psi\rangle^{ABC_1}$ as in Eq. (20) and $|\psi\rangle^{C_2D} = \frac{1}{\sqrt{2}}(|20\rangle + |31\rangle)$, then $|\Psi\rangle = |\psi\rangle^{ABC_1}|\psi\rangle^{C_2D}$ is a genuinely entangled pure state in $\mathcal{H}^{ABCD}$. Taking $h(\rho) = \sqrt{2(1 - \mathrm{Tr}\rho^2)}$, we get

$$E_{G-2}(\rho^{AB}) = C(\rho^{AB}) \approx 0.9899 > 0.5838 \approx E_{G-3}(|\Psi\rangle);$$

i.e., E_{G-k} is not coarsening monotonic.

It is straightforward that E_{G-k} is not tightly coarsening monotonic. In fact, we have

$$E_k(X_1|X_2|\cdots|X_p) \leqslant E_k(Y_1|Y_2|\cdots|Y_q) \tag{32}$$

holds for all k-entangled state $\rho \in \mathcal{S}^{X_1X_2\cdots X_p}$ whenever $X_1|X_2|\cdots|X_p \succ^b Y_1|Y_2|\cdots|Y_q$ with $k \leqslant q \leqslant p$.

Theorem 7. If h is strictly concave, E_{G-k} is monogamous.

Proof. We consider E_{G-3} at first. If $E_{G-3}(|\psi\rangle^{A|B|CD}) = E_{G-3}(\rho^{ABC})$, then for any decomposition of $\rho^{ABC} = \sum_i p_i|\psi_i\rangle\langle\psi_i|^{ABC}$, from

$$\begin{aligned}
E_{G-3}(|\psi\rangle^{A|B|CD}) &= [h(\rho^A)h(\rho^B)h(\rho^{CD})]^{1/3} \\
&= [h(\rho^A)h(\rho^B)h(\rho^{AB})]^{1/3} \\
&\geqslant \left[\sum_i p_i h(\rho_i^A) \sum_i p_i h(\rho_i^B) \sum_i p_i h(\rho_i^{AB})\right]^{1/3} \\
&\geqslant \sum_i p_i \left[h(\rho_i^A) h(\rho_i^{\mathrm{B}}) h(\rho_i^{AB})\right]^{1/3} \\
&= \sum_i p_i \left[h(\rho_i^A) h(\rho_i^{\mathrm{B}}) h(\rho_i^{C})\right]^{1/3} \\
&\geqslant \sum_i p_i E_{G-3}(|\psi_i\rangle^{ABC}) \\
&\geqslant E_{G-3}(\rho^{ABC})
\end{aligned}$$

we get $h(\rho^A) = \sum_i p_i h(\rho_i^A)$, $h(\rho^B) = \sum_i p_i h(\rho_i^{\mathrm{B}})$, $h(\rho^{CD}) = h(\rho^{AB}) = \sum_i p_i h(\rho_i^{AB}) = \sum_i p_i h(\rho_i^C)$. This indicates $|\psi\rangle^{ABCD} = |\psi\rangle^{ABC_1}|\psi\rangle^{C_2D}$ up to some local unitary operation on C_1C_2 since h is strictly concave. So $E_{G-2}(\rho^{AB|D})=0$.

In general, if $|\psi\rangle^{X_1X_2\cdots X_p}$ admits Eq. (9), then for any decomposition $\{p_i,\,|\psi_i\rangle^{X_1|X_2|\cdots|X_{s-1}|X_s'|X_{s+1}'|\cdots|X_p'}\}$ of $\rho^{X_1|X_2|\cdots|X_{s-1}|X_s'|X_{s+1}'|\cdots|X_p'}$,

$$h(\rho^{X_t})=\sum_i p_i h(\rho_i^{X_t}),\quad 1\leqslant t\leqslant s-1$$

and

$$h(\rho^{X_r})=\sum_i p_i h(\rho_i^{X_r'}),\quad s\leqslant r\leqslant p.$$

This reveals $|\psi\rangle^{X_1X_2\cdots X_p}$ admits the form

$$|\psi\rangle^{X_1X_2\cdots X_p}=|\psi\rangle^{X_1X_2\cdots X_{s-1}X_{s_1}'}|\psi\rangle^{X_{s_2}'Z_sX_{(s+1)_1}'}\times|\psi\rangle^{X_{(s+1)_2}'Z_{s+1}X_{(s+2)_1}'}\otimes\cdots\otimes|\psi\rangle^{X_{p_2}'Z_p},\tag{33}$$

which satisfies Eq. (10) and Eq. (11). The proof is completed. ■

E_{G-k} is not coarsening monotonic, so they cannot be completely monogamous. E_{G-k} always violates the tightly coarsening monotone, so it is not tightly complete monogamous. Moreover, even if the equality holds in Eq. (32), for example, for the W state $|W\rangle=\frac{1}{\sqrt{3}}(|100\rangle+|010\rangle+|001\rangle)$, we have $E_{G-2}(|W\rangle)=h(\rho^A)=E_{G-2}(|W\rangle^{A|BC})$, but ρ^{BC} is entangled.

E_{G-k} is not completely monogamous as a genuine entanglement measure since it is not coarsening monotonic on genuine entangled states, and it is not tightly complete monogamous as a genuine entanglement measure since it is not tightly coarsening monotonic on genuine entangled states. For example, for $|W\rangle^{ABCD}=\frac{1}{2}(|1000\rangle+|0100\rangle+|0010\rangle+|0001\rangle)$, we have $E_{G-3}(ABCD)=E_{G-3}(AB|C|D)$, but ρ^{AB} is not separable.

B. The maximal product

Let $|\psi\rangle=|\psi\rangle^{A_1A_2\cdots A_n}$ be a pure state in $\mathcal{H}^{A_1A_2\cdots A_n}$ and h be a non-negative concave function on $\mathcal{S}^X$. We define

$$E_{G-k}'(|\psi\rangle)=\begin{cases}\max_{\Gamma_k}\mathcal{Q}_k^{\gamma_i}(|\psi\rangle), & \min_{\Gamma_k}\mathcal{P}_k^{\gamma_i}(|\psi\rangle)>0,\\ 0, & \min_{\Gamma_k}\mathcal{P}_k^{\gamma_i}(|\psi\rangle)=0,\end{cases}\tag{34}$$

where the maximum/minimum is taken over all feasible k-partitions in Γ_k. E_{G-k}' may be not a well-defined entanglement monotone since we cannot guarantee that it nonincreasing on average under stochastic LOCC, and it is not even a well-defined entanglement measure.

By definition,

$$E_{G-k}'(X_1|X_2|\cdots|X_p)\geqslant E_{G-k}'(Y_1|Y_2|\cdots|Y_q)\tag{35}$$

holds for all k-entangled state $\rho\in\mathcal{S}^{X_1X_2\cdots X_p}$ whenever $X_1|X_2|\cdots|X_p\succ^b Y_1|Y_2|\cdots|Y_q$ with $k\leqslant q\leqslant p$, but it is not tightly coarsening monotonic. For example, we take $|W\rangle^{ABCD}$ as in Sec. VII A, then

$$E_{G-3}'(|W\rangle^{ABCD})<E_{G-2}'(|W\rangle^{AB|CD}).$$

$E_{G-k}'=E_{G-k}$ for the state under any k-partition, so E_{G-k}' is also weakly monogamous if the associated reduced function is strictly concave. If we take $|\psi\rangle^{ABCD}=|\psi\rangle^{ABC}|\psi\rangle^{D}$, where $|\psi\rangle^{ABC}=|\psi\rangle^{ABC_1}$ with $|\psi\rangle^{ABC_1}$ as in Eq. (20), we can also obtain $E_{G-3}'<E_{G-2}'$. So E_{G-k}' is not k-monotonic. For the state in Eq. (20), we can easily check that

$$E_{G-3}'(ABCD)\approx 0.587542<E_{G-2}'(AB)$$

when we take $h(\rho)=\sqrt{2(1-\mathrm{Tr}\rho^2)}$, i.e., E_{G-k}' is not coarsening monotonic.

C. The arithmetic mean

Let $|\psi\rangle=|\psi\rangle^{A_1A_2\cdots A_n}$ be a pure state in $\mathcal{H}^{A_1A_2\cdots A_n}$ and h be a non-negative concave function on $\mathcal{S}^X$. We define

$$\bar{E}_{G-k}(|\psi\rangle)=\begin{cases}\frac{\sum_{\gamma_i}\mathcal{Q}_k^{\gamma_i}(|\psi\rangle)}{|\Gamma_k|}, & \min_{\Gamma_i}\mathcal{P}_k^{\gamma_i}(|\psi\rangle)>0,\\ 0, & \min_{\Gamma_i}\mathcal{P}_k^{\gamma_i}(|\psi\rangle)=0,\end{cases}\tag{36}$$

where the minimum is taken over all feasible k-partitions in Γ_k.

Theorem 8. $\bar{E}_{G-k}$ is a k-EMo.

Proof. For any $|\psi\rangle\in\mathcal{H}^{A_1A_2\cdots A_n}$, $\gamma_i\in\Gamma_k$, we let the output state under some n-partite stochastic LOCC be $\{p_j,|\phi_j\rangle\}$ and denote the X part marginal state of $|\phi_j\rangle$ by σ_j^X. It ie easy to see

$$\begin{aligned}\bar{E}_{G-k}(|\psi\rangle)&=\frac{1}{|\Gamma_k|}\sum_{\gamma_i}\mathcal{Q}_k^{\gamma_i}(|\psi\rangle)\\&=\frac{1}{|\Gamma_k|}\sum_{\gamma_i}\left[\prod_{t=1}^{k}h\big(\rho^{X_{t(i)}}\big)\right]^{1/k}\\&\geqslant\frac{1}{|\Gamma_k|}\sum_{\gamma_i}\left\{\prod_{t=1}^{k}\left[\sum_j p_jh\big(\sigma_j^{X_{t(i)}}\big)\right]\right\}^{1/k}\\&\geqslant\frac{1}{|\Gamma_k|}\sum_{\gamma_i}\left\{\sum_j p_j\left[\prod_{t=1}^{k}h\big(\sigma_j^{X_{t(i)}}\big)\right]^{1/k}\right\}\\&=\frac{1}{|\Gamma_k|}\sum_j p_j\left\{\sum_{\gamma_i}\left[\prod_{t=1}^{k}h\big(\sigma_j^{X_{t(i)}}\big)\right]^{1/k}\right\}\\&=\sum_j p_j\bar{E}_{G-k}(|\phi_j\rangle),\end{aligned}$$

where the second inequality is true because the geometric mean function $f=(\prod_i^n x_i)^{1/n}$ is concave, i.e., $\bar{E}_{G-k}$ is a k-EMo. ■

If a k-entanglement measure is k-monotonic, it is faithful, but not vice versa. We show below that although $\bar{E}_{G-k}$ is not k-monotonic, it is faithful.

Proposition 3. $\bar{E}_{G-k}$ is faithful, i.e., $\bar{E}_{G-k}(\rho)=0$ if and only if ρ is k-separable, $\rho\in\mathcal{S}^{A_1A_2\cdots A_n}$.

Proof. We discuss the case of $n=4$ and $k=3$ for a pure state at first. If $\bar{E}_{G-3}(|\psi\rangle)=0$, $|\psi\rangle\in\mathcal{H}^{ABCD}$, we have

$$\begin{aligned}h_Ah_{BC}h_D&=h_{AB}h_Ch_D=h_{AC}h_Bh_D\\&=h_Ah_Bh_{CD}=h_Ah_{BD}h_C=h_{AD}h_Bh_C=0,\end{aligned}$$

which yields that $|\psi\rangle$ must be 3-separable. In general, if $\bar{E}_{G-k}(|\psi\rangle^{A_1A_2\cdots A_n})=0$, i.e., $\sum_{\gamma_i}\mathcal{Q}_k^{\gamma_i}(|\psi\rangle^{A_1A_2\cdots A_n})=0$, then $\mathcal{Q}_k^{\gamma_i}(|\psi\rangle^{A_1A_2\cdots A_n})=0$ for any $\gamma_i\in\Gamma_k$. So there exists some

TABLE III. Comparing of E_{G-k}, E'_{G-k}, and $\bar{E}_{G-k}$ with the assumption that the reduced functions is strictly concave.

k-EM	k-M	CoM	TCoM	Complete	M	CM	TCM
E_{G-k}	×	×	×	×	✓	×	×
E'_{G-k} [a]	×	×	×	×	× [b]	×	×
$\bar{E}_{G-k}$	×	×	×	×	✓	×	×

[a]We assume here that E'_{G-k} is a k-EMo.
[b]It is weakly monogamous.

$t_1(i_1)$ such that $h(\rho^{X_{t_1(i_1)}})=0$, which reveals that $|\psi\rangle^{A_1A_2\cdots A_n}$ is biseparable. In the partitions that exclude $X_{t_1(i_1)}$ as a subsystem, we can find some $t_2(i_2)$ such that $h(\rho^{X_{t_2(i_2)}})=0$, which reveals that $|\psi\rangle^{A_1A_2\cdots A_n}$ is 3-separable. The process can continue until we find at last some $t_{k-1}(i_{k-1})$ such that $h(\rho^{X_{t_{k-1}(i_{k-1})}})=0$, which reveals that $|\psi\rangle^{A_1A_2\cdots A_n}$ is k-separable. ■

We can easily check that $\bar{E}_{G-k}$ is also monogamous if the associated reduced function is strictly concave. Let $|\psi\rangle^{ABC}$ be the three-qubit GHZ state. Then

$$\bar{E}_{G-3}(|\psi\rangle^{ABC}|\psi\rangle^{D}) = h(\rho^A)/2 < 6h(\rho^A)/7 = \bar{E}_{G-2}(|\psi\rangle^{ABC}|\psi\rangle^{D}).$$

Namely, E'_{G-k} is not k-monotonic. In addition, for any genuinely entangled $|\psi\rangle^{ABC}$, we have

$$\begin{aligned}\bar{E}_{G-3}(|\psi\rangle^{ABC}|\psi\rangle^{D}) &= \tfrac{1}{6}h_Ah_Bh_C < h_Ah_Bh_C\\ &= \bar{E}_{G-3}(A|B|CD) = \bar{E}_{G-3}(\rho^{ABC})\\ &= \bar{E}_{G-3}(|\psi\rangle^{ABC})\end{aligned}$$

in general, where $\bar{E}_{G-3}(A|B|CD)$ denotes $\bar{E}_{G-3}(|\psi\rangle^{ABC}|\psi\rangle^{D})$ under the partition $A|B|CD$. For the state in Eq. (20),

$$\begin{aligned}\bar{E}_{G-3}(ABCD) < E'_{G-3}(ABCD) &< E'_{G-2}(AB)\\ &< E'_{G-2}(A|BCD)\end{aligned}$$

[e.g., we take $h(\rho)=\sqrt{2(1-\mathrm{Tr}\rho^2)}$]. So we conclude that $\bar{E}_{G-k}$ is neither coarsening monotonic nor tightly coarsening monotonic, and thus it is not completely monogamous nor tightly complete monogamous.

By definition,

$$E_{G-k} \leqslant \bar{E}_{G-k} \leqslant E'_{G-k}, \quad E_{G-n} = \bar{E}_{G-n} = E'_{G-n}, \tag{37}$$

$\bar{E}_{G-k}$ is faithful, but E'_{G-k} and E_{G-k} are not faithful. For the case of $k=2$, $E_2=E_{G-2}$ coincide with the GMEM $\varepsilon^{(n)}_{g''}$ in Ref. [35], and $E'_2=E'_{G-2}$ coincide with the GMEM $\varepsilon^{(n)}_{g'}$ in Ref. [35]. We compare these product-based measures in Table III. Compared to the sum-based k-entanglement monotones investigated in the previous section, the former class is better than the product-based three ones here as they are even not k-monotonic. In both classes, the monogamy is related to whether the reduced function is strictly concave, which is the same as the bipartite entanglement measures [48]. In addition to the von Neumann entropy, the Tsallis q-entropy with $q>1$, and the reduced function of concurrence, the Rényi α-entropy with $0<\alpha<1$, the reduced function of the negativity, and the reduced functions of the fidelity-based distance entanglement measures [22] are also strictly concave [22,29,30,35], which can export monogamous product-based k-entanglement monotones.

Finally, we present the cases of taking h as the reduced function of concurrence and the von Neumann entropy as illustrated examples, respectively.

Example 3. We take k-entanglement measure with $h(\rho)=\sqrt{2(1-\mathrm{Tr}\rho^2)}$, and denote the corresponding three kinds of measures by C_{G-k}, C'_{G-k}, and $\bar{C}_{G-k}$, respectively. By definition, C_{G-k} and $\bar{C}_{G-k}$ are monogamous while C'_{G-k} is weakly monogamous as h is strictly concave. For $|\mathrm{GHZ}_4\rangle$,

$$\begin{aligned}C_{G-3}(ABCD) &= C_{G-3}(A|B|CD)\\ &= C_{G-2}(ABCD) = C_{G-2}(AB|CD)\\ &= C_{G-2}(A|B|CD) = C_{G-2}(A|BCD) = 1.\end{aligned}$$

It is also true for $C'_{G-2,G-3}$, $\bar{C}_{G-2,G-3}$, and $C^{G}_{G-2,G-3}$ as $C_{G-2,G-3}(ABCD) = C'_{G-2,G-3}(ABCD) = \bar{C}_{G-2,G-3}(ABCD)$. For $|W_4\rangle$,

$$\begin{aligned}C_{G-3}(ABCD) &= C_{G-3}(A|B|CD) = \sqrt[3]{3/4},\\ C'_{G-3}(ABCD) &= C'_{G-3}(A|B|CD) = \sqrt[3]{3/4},\\ \bar{C}_{G-3}(ABCD) &= \bar{C}_{G-3}(A|B|CD) = \sqrt[3]{3/4},\\ C_{G-2}(ABCD) &= C_{G-2}(A|BCD) = \sqrt{3}/2\\ &< C_{G-2}(AB|CD) = 1,\\ C'_{G-2}(ABCD) &= C'_{G-2}(AB|CD) = 1,\\ \bar{C}_{G-2}(ABCD) &= \frac{3}{7} + \frac{2\sqrt{3}}{7} < \bar{C}_{G-2}(AB|CD) = 1.\end{aligned}$$

Example 4. If $h(\rho)=S(\rho)=-\mathrm{Tr}(\rho\log_2\rho)$, then the associated E_{G-k}, and $\bar{E}_{G-k}$ are monogamous while E'_{G-k} is weakly monogamous since S is strictly concave. For $|\mathrm{GHZ}_4\rangle$,

$$\begin{aligned}E_{G-3}(ABCD) &= E_{G-3}(A|B|CD)\\ &= E_{G-2}(ABCD) = E_{G-2}(AB|CD)\\ &= E_{G-2}(A|B|CD) = E_{G-2}(A|BCD) = 1.\end{aligned}$$

It is also true for $E'_{G-2,G-3}$, $\bar{E}_{G-2,G-3}$, and $E^{G}_{G-2,G-3}$ as $E_{G-2,G-3}(ABCD) = E'_{G-2,G-3}(ABCD) = \bar{E}_{G-2,G-3}(ABCD)$. For $|W_4\rangle$,

$$\begin{aligned}E_{G-3}(ABCD) &= E_{G-3}(A|B|CD) = \left(1-\tfrac{3}{4}\log_2 3\right)^{2/3},\\ E'_{G-3}(ABCD) &= E'_{G-3}(A|B|CD) = \left(1-\tfrac{3}{4}\log_2 3\right)^{2/3},\\ \bar{E}_{G-3}(ABCD) &= \bar{E}_{G-3}(A|B|CD) = \left(1-\tfrac{3}{4}\log_2 3\right)^{2/3},\\ E_{G-2}(ABCD) &= E_{G-2}(A|BCD) = 1-\tfrac{3}{4}\log_2 3\\ &< E_{G-2}(AB|CD) = 1,\\ E'_{G-2}(ABCD) &= E'_{G-2}(AB|CD) = 1,\\ \bar{E}_{G-2}(ABCD) &= 1-\tfrac{3}{7}\log_2 3 < \bar{E}_{G-2}(AB|CD) = 1.\end{aligned}$$

VIII. CONCLUSION

In this paper we have established the concept of monogamy, completeness, complete monogamy, and the tightly complete monogamy for the k-entanglement measure as a multipartite entanglement measure. Consequently, the monogamy, complete monogamy, and tightly complete monogamy are revealed clearly: they are just corresponding to the three types of coarser relation of the system partition, respectively. More generally, our method represents a natural starting point for building a general theory of monogamy relation since our approach is based on the coarser relation of the partition of the system, which is the most basic one that makes it is feasible for any quantum correlation. It can make us realize the monogamy relation more profoundly.

We then proposed two general ways of quantifying the k-entanglement via the reduced functions: one is the sum-based and the other is product-based. These two classes include a great many measures since the reduced function can choose any concave one. All these measures are shown to be monogamous or weakly monogamous according to our definition whenever the reduced function is strictly concave, which is the same as the bipartite entanglement measures [48]. But they are not complete in general, and they are neither completely monogamous nor tightly complete monogamous. This is also the first example that there exists a measure of quantum correlation that is monogamous but not completely monogamous, which is contrary to the mutual information and the complete multipartite entanglement measure induced by the partial norm of entanglement as they are completely monogamous but not monogamous [30,35,51]. Thus, the monogamy and the complete monogamy seem independent of each other. Together with the framework of complete monogamy we proposed before [21,29,50,51], we have established a thorough theory on revealing the distribution of multipartite entanglement which may shed a new light on the multipartite quantum resource theory.

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