Wigner non-negative states that verify the Wigner entropy conjecture

Qipeng Qian[1] and Christos N. Gagatsos[1,2,3]

[1]*Program in Applied Mathematics, The University of Arizona, Tucson, Arizona 85721, USA*

[2]*Department of Electrical and Computer Engineering, The University of Arizona, Tucson, Arizona 85721, USA*

[3]*Wyant College of Optical Sciences, The University of Arizona, Tucson, Arizona 85721, USA*

(Received 17 June 2024; accepted 10 July 2024; published 29 July 2024)

We present further progress, in the form of analytical results, on the Wigner entropy conjecture set forth by Van Herstraeten and Cerf [Phys. Rev. A **104**, 042211 (2021)] and Hertz *et al.* [J. Phys. A: Math. Theor. **50**, 385301 (2017)]. Said conjecture asserts that the differential entropy defined for non-negative, yet physical, Wigner functions is minimized by pure Gaussian states while the minimum entropy is equal to $1 + \ln \pi$. We prove this conjecture for the qubits formed by Fock states $|0\rangle$ and $|1\rangle$ that correspond to non-negative Wigner functions. In particular, we derive an explicit form of the Wigner entropy for those states lying on the boundary of the set of Wigner non-negative qubits. We then consider general mixed states and derive a sufficient condition for the conjecture's validity. Lastly, we elaborate on the states which are in accordance with our condition.

DOI: 10.1103/PhysRevA.110.012228

I. INTRODUCTION

Uncertainty relations are of fundamental interest in quantum information theory. They are closely related to the wave-particle duality in quantum mechanics and also illustrate one of the essential differences between quantum and classical mechanics. Furthermore, uncertainty relations directly put constraints on the precision of measurements and indicate inherent limitations in our understanding of quantum systems.

The exploration of uncertainty relations traces back to the foundational Heisenberg uncertainty principle [1], in which the variance of the quadrature operators $\hat{q}$ and $\hat{p}$ is used as the quantifier of uncertainty. Later studies on uncertainty relations resulted in a natural generalization of classical information-related entropies to quantum systems (for an overview of entropic uncertainty relations see, for example, Refs. [2,3]). In Ref. [4] an entropic uncertainty relation has been presented, setting a lower bound on the summation of the Shannon entropy of the probability distribution function (PDF) of the position and the Shannon entropy of the PDF of the momentum of a quantum system. Said lower bound is stronger than the Heisenberg uncertainty relation. Furthermore, considering the subadditivity of Shannon's differential entropy, one can expect that the entropy of the joint distribution of q and p, i.e., of the Wigner function, will induce an even stronger bound, which would nevertheless imply the bound in Ref. [4].

The Wigner entropy $S[W]$ is defined in Refs. [5–7] as the differential Shannon entropy of the Wigner function $W(q,p)$ of the state with non-negative Wigner function (not necessarily corresponding to a classical state),

$$S[W] = -\int dq\,dp\,W(q,p)\ln W(q,p). \tag{1}$$

It possesses several properties [5] such as additivity (for product states) and, unlike the Wehrl entropy [8], invariance under symplectic transformations. The Wigner entropy has been used in the analysis of noisy polarizers [9], high-energy physics [10], and nonequilibrium field theory [11]. In a broader view, phase space methods exploring the properties of quantum states are always of current interest; see, for example, Refs. [12–14]. In this work, we focus on the following conjecture, presented in Refs. [5–7],

Conjecture 1. For any Wigner non-negative state,

$$S[W] \geqslant 1 + \ln \pi, \tag{2}$$

while the lower bound is attained by any pure Gaussian state.

It is known that the marginals of the Wigner function coincide with probability densities of q and p, denoted as $P_q = \int dp W(q,p)$ and $P_p = \int dq W(q,p)$, respectively. As discussed before, the Bialynicki-Birula-Mycielski inequality [4] and the subadditivity of Shannon's differential entropy give

$$S[P_q] + S[P_p] \geqslant 1 + \ln \pi, \tag{3}$$

$$S[P_q] + S[P_p] \geqslant S[W]. \tag{4}$$

If Conjecture 1 is true, inequalities (3) and (4) can be written as

$$S[P_q] + S[P_p] \geqslant S[W] \geqslant 1 + \ln \pi, \tag{5}$$

i.e., we get a stronger entropic uncertainty relation for Wigner non-negative states.

In Ref. [5], Conjecture 1 was proven analytically for passive states, i.e., states whose Fock basis representation has the form $\hat{\rho}_p = \sum_{n=0}^{\infty} q_n |n\rangle\langle n|$ under the condition $q_{n+1} \leqslant q_n$, where q_n are probabilities (non-negative real numbers satisfying $\sum_{n=0}^{\infty} q_n = 1$), and provided seminumerical evidence for states that can be produced by mixing Wigner non-negative states in a balanced beam splitter. In this paper, we prove analytically Conjecture 1 for (1) qubits in the basis $\{|0\rangle, |1\rangle\}$, where $|0\rangle$ and $|1\rangle$ are Fock states, and (2) general mixed states which are Wigner non-negative and satisfy a sufficient condition. Throughout the paper we consider single-mode states.

It is worthwhile to mention that recent progress has been made in similar conjectures relating to the family of Rényi entropies for non-negative Wigner functions [15,16].

This paper is organized as follows: In Sec. II, we analyze the conditions for any qubit in the basis $\{|0\rangle, |1\rangle\}$ to have a non-negative Wigner function. We then demonstrate that the Wigner entropy attains its minimum on the boundary of the Wigner non-negative set as defined by the derived condition. We explicitly derive the form of the Wigner entropy for these states and indeed verify that the lower bound of the Wigner entropy of such Wigner non-negative qubits matches the one of Conjecture 1. In Sec. III, we consider the more general case of any mixed state and derive a sufficient condition such that Wigner non-negative states satisfy the lower bound stated in Conjecture 1. In Sec. IV, by showing a few concrete examples, we demonstrate that the set of states in accordance with our condition is nonempty and distinct from the set described in Ref. [5]. Finally, in Sec. V we summarize our results very briefly and we discuss future directions.

II. QUBIT STATES

We denote the matrix representation of the density operator of a qubit formed by Fock states $|0\rangle$ and $|1\rangle$ in Bloch ball representation as

$$\rho = \tfrac{1}{2}(I + r_1\sigma_x + r_2\sigma_y + r_3\sigma_z), \tag{6}$$

where $\{\sigma_x, \sigma_y, \sigma_z\}$ are the Pauli matrices,

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{7}$$

I is the identity matrix, and $r_i \in [-1, 1]$ for $i = 1, 2, 3$ satisfy

$$r_1^2 + r_2^2 + r_3^2 \leqslant 1. \tag{8}$$

Following standard procedures (e.g., the compact formulation presented in Ref. [17, Sec. 1.5]), we can find the Wigner function corresponding to Eq. (6) to be

$$W(q,p) = \frac{1}{\pi}e^{-q^2-p^2}[(q^2+p^2)(1-r_3) + \sqrt{2}r_1 q + \sqrt{2}r_2 p + r_3]. \tag{9}$$

Since Eq. (6) under condition (8) represents a physical state, the Wigner function of Eq. (9) is naturally physical under the same condition. However, we need to identify a condition on $r_i,\ i = 1, 2, 3$, such that Eq. (9) also represents a non-negative Wigner function. To this end, we require $W(q,p) \geqslant 0$ and we derive the following condition by completing the squares with respect to q and p in the polynomial part of Eq. (9):

$$2\left(r_1^2 + r_2^2\right) + (1 - 2r_3)^2 \leqslant 1. \tag{10}$$

Under the conditions of Eqs. (8) and (10), we are able to analyze whether Conjecture 1 is true for the qubit case. However, some observations leading to simplifications are due.

First, the Wigner entropy is invariant under symplectic transformations. Therefore, we can set $r_2 = 0$ in Eq. (9) since arbitrary values of r_2 correspond to optical phase shifts, i.e., a Gaussian unitary transformation [18] (optical phase shifting corresponds to a symplectic transformation on phase space).

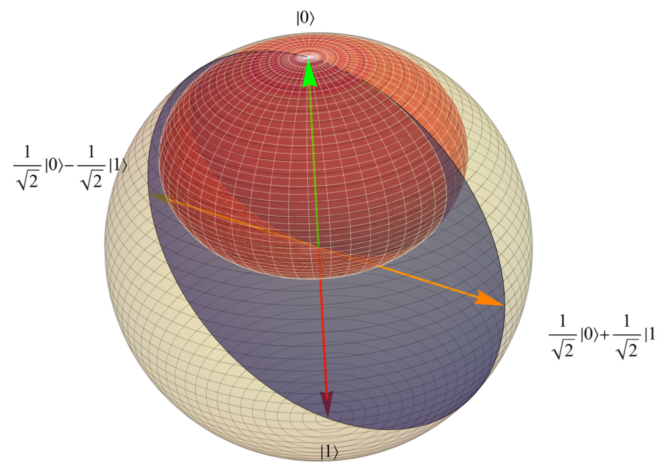

FIG. 1. The Bloch ball (yellow with black mesh lines), as defined by the $|0\rangle$ and $|1\rangle$ Fock vectors, contains all physical states, even those corresponding to partly negative Wigner functions. The ellipsoid (red with white mesh lines) contains physical states whose Wigner function is non-negative. States that belong to the same latitude are equivalent up to an optical phase. Therefore, it suffices to look only at states contained in the intersection of the disk (shown in blue) and the ellipsoid [the region defined by Eq. (13)], i.e., physical states with non-negative Wigner functions. In fact, it suffices to consider only states contained in said elliptical region only on the left of (or on the right of) the line depicting the $|0\rangle$ vector. This is because symmetric states with respect to the upward vector are equivalent up to an optical phase shifting.

Therefore, the Wigner function and the conditions we consider, respectively, become

$$W_{13}(q,p) = \frac{1}{\pi}e^{-q^2-p^2}[(q^2+p^2)(1-r_3) + \sqrt{2}r_1 q + r_3], \tag{11}$$

$$r_1^2 + r_3^2 \leqslant 1, \tag{12}$$

$$2r_1^2 + (1-2r_3)^2 \leqslant 1. \tag{13}$$

The Bloch ball with the Wigner non-negative regions of our system is depicted in Fig. 1.

Second, for any fixed q, p, r_3, the second derivative on $-W_{13}(q,p)\ln W_{13}(q,p)$ with respect to r_1 gives

$$\frac{\partial^2}{\partial r_1^2}\{-W_{13}(q,p)\ln[W_{13}(q,p)]\} = -\frac{2q^2e^{-2q^2-2p^2}}{\pi^2 W_{13}(q,p)} \leqslant 0, \tag{14}$$

implying that the Wigner entropy is concave with respect to r_1 due to the linearity of integration in Eq. (1). Thus, the minimum of the Wigner entropy can only exist at some point along the boundary defined by condition (13), i.e.,

$$2r_1^2 + (1-2r_3)^2 = 1. \tag{15}$$

Finally, using Eq. (15), we simplify further Eq. (11),

$$W_3^{\pm}(q,p) = \frac{1}{\pi}e^{-q^2-p^2}[(q^2+p^2)(1-r_3) \pm 2q\sqrt{r_3(1-r_3)} + r_3], \tag{16}$$

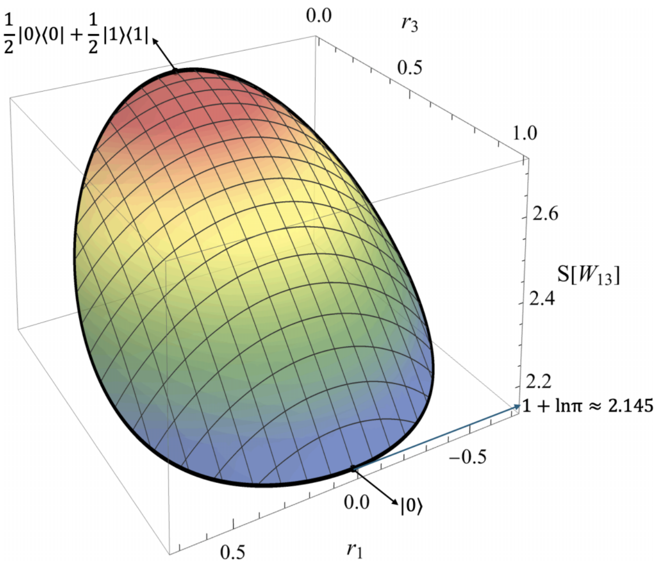

FIG. 2. The Wigner entropy $S[W_{13}]$ numerically evaluated for 20 000 unique choices of (r_1, r_3), compatible with Eqs. (12) and (13). The border of the surface is the Wigner entropy $S[W_3^{\pm}]$ [its analytical expression is given in Eq. (18)], while the upper point corresponds to the maximally mixed state $\frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$ and lower point corresponds to $|0\rangle$. The concavity with respect to r_1 and the minimization of the Wigner entropy are proven analytically in the main text.

where the $\pm$ corresponds to the two possible solutions of Eq. (15) with respect to r_1. Both Wigner functions of Eq. (16) are equivalent up to an optical phase. Therefore, they correspond to the same Wigner entropy. We choose to work with $W_3^+(q, p)$,

$$W_3(q, p) \equiv W_3^+(q, p). \tag{17}$$

The explicit form of the Wigner entropy on the boundary defined by Eq. (15) (see Appendix A) is

$$S_b \equiv S[W_3^{\pm}] = e^{-\frac{r_3}{1-r_3}}(1 - r_3) + r_3 + \ln\frac{\pi}{r_3} + \mathrm{Ei}\left(-\frac{r_3}{1 - r_3}\right), \tag{18}$$

where $\mathrm{Ei}(x) = \int_{-\infty}^{x} dt \frac{e^t}{t}$ is the exponential integral function and the subscript b denotes that we work on the boundary of the Wigner non-negative qubits. In Fig. 2, we plot the Wigner entropy $S[W_{13}]$. We find that the entropy S_b is minimized at $r_3 = 1$ and its minimum value is $1 + \ln\pi$, which is consistent with Conjecture 1. For $r_3 = 1$ and $r_2 = 0$, per Eq. (8) we get $r_1 = 0$, which means that the state minimizing the Wigner entropy is $|0\rangle$, which is the only pure Gaussian state in the Bloch ball. The details on the minimization are provided in Appendix A.

III. A SUBSET OF WIGNER NON-NEGATIVE STATES

In this section, for a restricted set of physical and Wigner non-negative (in general mixed) states $\hat{\rho}$, we find that the Wigner entropy is in accordance with Conjecture 1.

Let $k \in \mathbb{R}$ and $k \geqslant 1$. For any non-negative Wigner function $W(q, p)$, we define the functional,

$$\mu_k[W] = k\pi^{k-1} \int dq dp W^k(q, p). \tag{19}$$

We note that $\mu_1[W] = 1$ (normalization) and $\mu_2[W]$ is the purity of the state corresponding to the Wigner function $W(q, p)$. We pose the following conjecture:

Conjecture 2.

$$\left.\frac{\partial \mu_k[W]}{\partial k}\right|_{k\to 1} \leqslant 0. \tag{20}$$

The left-hand side of Eq. (20) is equal to $1 + \ln\pi - S[W]$. Therefore, Conjecture 2 is equivalent to Conjecture 1. As discussed before, for $k = 1$ we get the normalization property of Wigner functions,

$$\mu_1[W] = \int dp dq W(q, p) = 1. \tag{21}$$

Therefore, Conjecture 2 is true if the following sufficient condition holds:

$$\mu_k[W] \leqslant 1 \tag{22}$$

for $k \in \mathbb{R}$ and $k \geqslant 1$. Furthermore, utilizing the fact that

$$\int dq dp W_0^k(q, p) = \frac{1}{k\pi^{k-1}}, \tag{23}$$

where $W_0(q, p)$ denotes the Wigner function of $|0\rangle$, we can rewrite Eq. (22) as

$$\nu_k[W] \leqslant \nu_k[W_0], \tag{24}$$

where

$$\nu_k[W] = \int dq dp W^k(q, p) \tag{25}$$

for $k \in \mathbb{R}$ and $k \geqslant 1$.

We now derive a sufficient condition such that Eq. (24) is true. For any (generally mixed) state $\hat{\rho}$, its Wigner function has the form (see Appendix B)

$$W(q, p) = W_0(q, p)P(q, p), \tag{26}$$

where $P(q, p) = \sum_{a,b=0}^{\infty} c_{ab} q^a p^b$ is a polynomial in q, p. To prove Eq. (26), one can start by writing $\hat{\rho}$ on the Fock basis and then calculate the generic Wigner function using well-known methods (see, for example, Ref. [17]). Since Wigner functions are real-valued, $c_{ab} \in \mathbb{R}$ for any $(a, b) \in \mathbb{N}^2$. We introduce coefficients $\tilde{c}_{ab}$ such that Eq. (26) is rewritten as

$$W(q, p) = W_0(q, p) \sum_{a,b=0}^{\infty} \frac{\pi \tilde{c}_{ab}}{\Gamma\left(\frac{1+a}{2}\right)\Gamma\left(\frac{1+b}{2}\right)} q^a p^b, \tag{27}$$

where $\Gamma(x) = \int_0^{\infty} s^{x-1} e^{-s} ds$ and since $W(q, p)$ is normalized, the coefficients $\tilde{c}_{ab}$ must satisfy

$$\sum_{a,b=0}^{\infty} \tilde{c}_{ab} = 1. \tag{28}$$

We define $F_{ab}(q, p)$ as

$$F_{ab}(q, p) = W_0(q, p) \frac{\pi}{\Gamma\left(\frac{1+a}{2}\right)\Gamma\left(\frac{1+b}{2}\right)} q^a p^b. \tag{29}$$

In Appendix C we prove that

$$\|F\|_k^k \leqslant \nu_k[W_0], \tag{30}$$

where $\|F\|_k = (\int dqdp|F_{ab}(q,p)|^k)^{\frac{1}{k}}$ for any $(a,b) \in \mathbb{N}^2$ and $k \in \mathbb{R}$ and $k \geqslant 1$.

Let us impose the following condition:

Condition 1.

$$\sum_{a,b=0}^{\infty} |\tilde{c}_{ab}| = 1.$$

For a non-negative Wigner function $W(q,p)$ satisfying Condition 1, we utilize Eq. (30) and the triangle inequality to get

$$\nu_k[W] \equiv \|W\|_k^k \leqslant \left[\sum_{a,b=0}^{\infty} |\tilde{c}_{ab}| \|F_{ab}(q,p)\|_k\right]^k \tag{31}$$

$$\leqslant \left[\sum_{a,b=0}^{\infty} |\tilde{c}_{ab}| (\nu_k[W_0])^{\frac{1}{k}}\right]^k \tag{32}$$

$$= [(\nu_k[W_0])^{\frac{1}{k}}]^k = \nu_k[W_0]. \tag{33}$$

Therefore, Condition 1 is a sufficient condition for Conjecture 2 and thus Conjecture 1 to hold.

IV. EXAMPLES

We provide a few examples demonstrating that the set of states satisfying our conditions is nonempty and distinct from the set of passive states explored in Ref. [5]. In particular, we give three examples to show how the set of states explored in Sec. III intersects with both the sets of passive states and of Fock-diagonal states.

Example 1. Consider states of the form $p_0|0\rangle\langle 0| + p_1|1\rangle\langle 1|$ with $0 \leqslant p_i \leqslant 1,\ i = 0, 1$, and $\sum_{i=0}^{1} p_i = 1$. All states of this form which additionally satisfy $p_1 \leqslant \frac{1}{2}$ (e.g., passive states) satisfy Condition 1.

Example 2. Consider states of the form $p_0|0\rangle\langle 0| + p_1|1\rangle\langle 1| + p_2|2\rangle\langle 2|$ with $0 \leqslant p_i \leqslant 1$, $i = 0, 1, 2$, and $\sum_{i=0}^{2} p_i = 1$. All such states with

$$p_1 \leqslant \tfrac{1}{2}, \tag{34}$$

$$p_1 - 2p_2 \geqslant 0 \tag{35}$$

satisfy Condition 1. This set of states does not necessarily include passive states. For example, for

$$p_0 = p_2 = \tfrac{1}{4}, \tag{36}$$

$$p_1 = \tfrac{1}{2} \tag{37}$$

the state is not passive but still satisfies Condition 1 and thus satisfies Conjecture 1.

Example 3. Consider the state of $p_0|0\rangle\langle 0| + p_1|1\rangle\langle 1| + p_2|2\rangle\langle 2| + p_3|3\rangle\langle 3|$ with $0 \leqslant p_i \leqslant 1$, $i = 0, 1, 2, 3$, and $\sum_{i=0}^{3} p_i = 1$. All such states with

$$p_1 - 2p_2 + 3p_3 \geqslant 0, \tag{38}$$

$$p_2 - 3p_3 \geqslant 0, \tag{39}$$

$$p_0 + p_2 \geqslant \tfrac{1}{2} \tag{40}$$

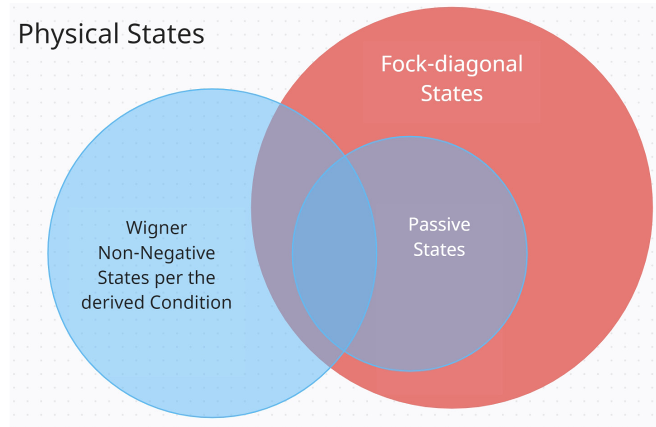

FIG. 3. Graphical explanation of the relation between the states satisfying Condition 1, passive and (more generally) Fock-diagonal states.

satisfy Condition 1. We note that when $p_0 = p_1 = p_2 = p_3 = \frac{1}{4}$, the state is passive but does not satisfy Condition 1.

From the examples above, we observe that our set only intersects with the set of passive states but does not contain it. Moreover, it directly follows that if a state is Fock-diagonal and Wigner non-negative, the state is not necessarily in compliance with Condition 1. In Appendix D, we delve deeper into the relationship between our set and the set of Fock-diagonal states. We find that our Condition 1 does not imply that a state $\hat{\rho}$ is Fock-diagonal but it does imply it when $\rho_{n,m} = 0$ if $|n-m|$ is odd, where $\rho_{n,m}$ is the element of the matrix representation of $\hat{\rho}$ on the Fock basis.

V. CONCLUSIONS

In this paper, we proved that the newly introduced Conjecture 1 [5–7] holds true for two cases: for (generally mixed) qubits formed by Fock states $|0\rangle$ and $|1\rangle$ and for states that satisfy Condition 1. Therefore, for Wigner non-negative states, we presented progress toward a stronger position-momentum uncertainty relation compared to the one derived in the seminal work [4] and expanded the results of Ref. [5]. Moreover, the entropic uncertainty relation considered in this work, subsumes [5] the Wehrl entropy inequality for the always non-negative Q functions. We note that the Wehrl entropy is minimized for coherent states while it is not in general invariant under symplectic transformations, e.g., the Wehrl entropy of a coherent state and a squeezed state are not in general equal.

The relationship between the set defined by Condition 1 and the sets of passive states and Fock-diagonal states is depicted in Fig. 3. The difficulty of proving Conjecture 1 for all $W(q,p) \geqslant 0$ lies in lacking a computationally useful criterion for Wigner non-negativity which also excludes non-physical states: the condition $W(q,p) \geqslant 0$ merely imposes non-negativity on the the function, while one would need to take into account the condition

$$\int dqdpW(q,p)W_{|\psi\rangle}(q,p) \geqslant 0 \tag{41}$$

for all pure states $|\psi\rangle$, as well to ensure that $W(q, p)$ corresponds to a physical state. One way forward could be to consider a set of functions $\tilde{W}(q, p)$ that includes all physical Wigner functions, plus a subset of functions that are non-negative but do not correspond to physical states. For example, this can be done by considering only a (convenient) subset of pure states satisfying Eq. (41). We note that an approach leading to the conclusion that the Wigner non-negative state minimizing the Wigner entropy is a pure state would prove Conjecture 1 in general. This would be an immediate consequence of Hudson's theorem stating that any pure state with a non-negative Wigner function is necessarily a Gaussian state [19].

Lastly, we envision future works elaborating on Conjecture 1 for states defined across multiple modes, entropy power inequalities [3], or even on the properties for the complex-valued Wigner entropy, i.e., for partly negative Wigner functions, in the direction of Refs. [7,20,21].

ACKNOWLEDGMENTS

The authors are thankful to Z. V. Herstraeten, M. G. Jabbour, A. Hertz, and N. Cerf for numerous fruitful discussions. C.N.G. and Q.Q. acknowledge financial support from the National Science Foundation, FET, Award No. 2122337.

APPENDIX A: MINIMIZATION OF EQ. (18)

In this Appendix, we prove that the Wigner entropy S_b of Eq. (18) attains its minimum at $r_3 = 1$ and the corresponding value is $1 + \ln \pi$.

Under the change of variables,

$$q' = \sqrt{1 - r_3}q \quad \text{and} \quad p' = \sqrt{1 - r_3}p, \tag{A1}$$

and using the condition of Eq. (15), Eq. (17) and its corresponding Wigner entropy become

$$W_3'(q', p') = \frac{1}{\pi} e^{-\frac{q'^2+p'^2}{1-r_3}} [p'^2 + (q' + \sqrt{r_3})^2], \tag{A2}$$

$$S_b \equiv S[W_3'] = -\frac{1}{1 - r_3} \int dq'dp' W_3' \ln(W_3'). \tag{A3}$$

With further change of variables,

$$q' = R \sin\theta \quad \text{and} \quad p' = R \cos\theta, \tag{A4}$$

satisfying $dq'dp' = RdRd\theta$, Eq. (A2) becomes,

$$W_3''(R, \theta) = \frac{1}{\pi} e^{-\frac{R^2}{1-r_3}} [R^2 + 2\sqrt{r_3}R \sin\theta + r_3], \tag{A5}$$

and the Wigner entropy S_b of Eq. (A3) in the new variables can be computed as

$$S_b \equiv S[W_3''] = -\frac{1}{1 - r_3} \int_0^\infty \int_0^{2\pi} dRd\theta R \frac{1}{\pi} e^{-\frac{R^2}{1-r_3}} (R^2 + 2\sqrt{r_3}R \sin\theta + r_3) \ln(W_3) \tag{A6}$$

$$= -\frac{1}{1 - r_3} \int_0^\infty \int_0^{2\pi} dRd\theta R \frac{1}{\pi} e^{-\frac{R^2}{1-r_3}} (R^2 + 2\sqrt{r_3}R \sin\theta + r_3) \left[\left(-\ln\pi - \frac{R^2}{1 - r_3} \right) + \ln(R^2 + r_3 + 2\sqrt{r_3}R \sin\theta) \right] \tag{A7}$$

$$= -\frac{1}{1 - r_3} \int_0^\infty \int_0^{2\pi} dRd\theta R \frac{1}{\pi} e^{-\frac{R^2}{1-r_3}} \left\{ (R^2 + r_3) \left(-\ln\pi - \frac{R^2}{1 - r_3} \right) \right. \tag{A8}$$

$$+ 2\sqrt{r_3}R \sin\theta \left(-\ln\pi - \frac{R^2}{1 - r_3} \right) + (R^2 + r_3) \ln(R^2 + r_3 + 2\sqrt{r_3}R \sin\theta)$$

$$\left. + 2\sqrt{r_3}R \sin\theta \ln(R^2 + r_3 + 2\sqrt{r_3}R \sin\theta) \right\}. \tag{A9}$$

We have the following useful relations pertaining to the previous integrals:

$$\int_0^{2\pi} d\theta \sin\theta = 0, \tag{A10}$$

$$\int_0^{2\pi} d\theta \ln(R^2 + r_3 + 2\sqrt{r_3}R \sin\theta) = 2\pi \left(\ln \frac{R^2 + r_3}{2} + \ln \frac{R^2 + r_3 + \sqrt{(R^2 - r_3)^2}}{R^2 + r_3} \right), \tag{A11}$$

$$\int_0^{2\pi} d\theta \sin\theta \ln(R^2 + r_3 + 2\sqrt{r_3}R \sin\theta) = 2\pi \frac{R^2 + r_3 - \sqrt{(R^2 - r_3)^2}}{2\sqrt{r_3}R}, \tag{A12}$$

which when used in Eq. (A9) obtains

$$S_b = 2e^{-\frac{r_3}{1-r_3}} (1 - r_3) + r_3 - \frac{2U}{1 - r_3}, \tag{A13}$$

where

$$\frac{2U}{1 - r_3} = -e^{-\frac{r_3}{1-r_3}} (r_3 - 1) - \text{Ei}\left(\frac{r_3}{-1 + r_3} \right) - \ln \frac{\pi}{r_3}. \tag{A14}$$

Therefore,

$$S_b = e^{-\frac{r_3}{1-r_3}} (1 - r_3) + r_3 + \ln \frac{\pi}{r_3} + \text{Ei}\left(-\frac{r_3}{1 - r_3} \right). \tag{A15}$$

Then, we calculate the derivative

$$\frac{d}{dr_3}S_b = e^{-\frac{r_3}{1-r_3}}\frac{r_3-2}{1-r_3} + 1 - \frac{1}{r_3} + \frac{e^{-\frac{r_3}{1-r_3}}}{r_3(1-r_3)} \quad \text{(A16)}$$

$$= e^{-\frac{r_3}{1-r_3}}\frac{r_3^2-2r_3+1}{r_3(1-r_3)} - \frac{1-r_3}{r_3} \quad \text{(A17)}$$

$$= e^{-\frac{r_3}{1-r_3}}\frac{1-r_3}{r_3} - \frac{1-r_3}{r_3} \quad \text{(A18)}$$

$$= \frac{1-r_3}{r_3}(e^{-\frac{r_3}{1-r_3}} - 1) \quad \text{(A19)}$$

$$\leqslant 0. \quad \text{(A20)}$$

It is clear that $\frac{d}{dr_3}S_b = 0$ is only possible when $r_3 = 0, 1$. By L'Hôpital's rule, we have

$$\lim_{r_3\to 0} \frac{(1-r_3)(e^{-\frac{r_3}{1-r_3}}-1)}{r_3} = \frac{1+\frac{e^{-\frac{r_3}{1-r_3}}(r_3-2)}{1-r_3}}{1} = -1, \quad \text{(A21)}$$

$$\lim_{r_3\to 1} \frac{(1-r_3)(e^{-\frac{r_3}{1-r_3}}-1)}{r_3} = \frac{0(0-1)}{1} = 0. \quad \text{(A22)}$$

Therefore, we conclude that S_b obtains its minimum value $1+\ln\pi$ at $r_3 = 1$.

APPENDIX B: PROOF OF EQ. (26)

In this Appendix, we prove that the Wigner function $W(q, p)$ for any N-mode, generally mixed state $\hat{\rho}$ can be written as $W(\mathbf{q},\mathbf{p}) = W_0(\mathbf{q},\mathbf{p})P(\mathbf{q},\mathbf{p})$, and we give the expressions for $W_0(\mathbf{q},\mathbf{p})$ and $P(\mathbf{q},\mathbf{p})$.

We write the state $\hat{\rho}$ on the Fock basis

$$\hat{\rho} = \sum_{\mathbf{n},\mathbf{m}} c_{\mathbf{nm}}|\mathbf{n}\rangle\langle\mathbf{m}|, \quad \text{(B1)}$$

where $|\mathbf{n}\rangle = |n_1,\ldots,n_N\rangle$.

The Wigner characteristic function for the state ρ is

$$X_{\hat{\rho}}(\boldsymbol{\eta}) = \mathrm{tr}[\hat{D}(\boldsymbol{\eta})\hat{\rho}] = \sum_{\mathbf{n},\mathbf{m}} c_{\mathbf{nm}}\langle\mathbf{m}|\hat{D}(\boldsymbol{\eta})|\mathbf{n}\rangle, \quad \text{(B2)}$$

where $\boldsymbol{\eta} = (\eta_1,\ldots,\eta_N)$, $\eta_i = (q_{\eta_i} + ip_{\eta_i})/\sqrt{2}$, and $\hat{D}(\boldsymbol{\eta})$ is the N-mode displacement operator.

Utilizing the representation of the displacement operator on the Fock basis [17] and the fact that $\langle\mathbf{m}|\hat{D}(\boldsymbol{\eta})|\mathbf{n}\rangle = \prod_{i=1}^N\langle m_i|\hat{D}(\eta_i)|n_i\rangle$, Eq. (B2) can be written as

$$X_{\hat{\rho}}(\boldsymbol{\eta}) = e^{-\frac{|\boldsymbol{\eta}|^2}{2}}\sum_{\mathbf{n},\mathbf{m}} c_{\mathbf{nm}}\prod_{i=1}^N \tilde{f}_{n_i,m_i}(\eta_i),$$

where

$$\tilde{f}_{n_i,m_i}(\eta_i) = \theta(m_i-n_i)\sqrt{\frac{n_i!}{m_i!}}\eta^{m_i-n_i}L_{n_i}^{(m_i-n_i)}(|\eta_i|^2) + \theta(n_i-m_i)\sqrt{\frac{m_i!}{n_i!}}(-\eta^*)^{n_i-m_i}L_{m_i}^{(n_i-m_i)}(|\eta_i|^2) + \delta_{n_im_i}L_n(|\eta_i|^2) \quad \text{(B3)}$$

and $L_n^{(m-n)}(.)$ is the associated Laguerre polynomial of the nth order. For $|x| \neq 0$ we define $\theta(.)$ as $\theta(|x|) = 1$, $\theta(-|x|) = 0$, and $\theta(0) = 0$.

We can write Eq. (B3) as

$$X_{\hat{\rho}}(\boldsymbol{\eta}) = X_{|\mathbf{0}\rangle\langle\mathbf{0}|}(\boldsymbol{\eta})\sum_{\mathbf{n},\mathbf{m}} c_{\mathbf{nm}}\prod_{i=1}^N \tilde{f}_{n_i,m_i}(\eta_i). \quad \text{(B4)}$$

Performing a $2N$-dimensional Fourier transform $(q_{\eta_i}, p_{\eta_i}) \to (q_i, p_i)$, $i = 1,\ldots,N$ on Eq. (B4), we find the Wigner function,

$$W(\mathbf{q},\mathbf{p}) = W_0(\mathbf{q},\mathbf{p})P(\mathbf{q},\mathbf{p}), \quad \text{(B5)}$$

where $W_0(\mathbf{q},\mathbf{p}) \equiv W_{|\mathbf{0}\rangle\langle\mathbf{0}|}(\mathbf{q},\mathbf{p})$ is the Wigner function of an N-mode vacuum state and

$$P(\mathbf{q},\mathbf{p}) = \sum_{\mathbf{n},\mathbf{m}} c_{\mathbf{nm}}\prod_{i=1}^N f_{n_i,m_i}(q_i,p_i), \quad \text{(B6)}$$

where

$$f_{n_i,m_i}(q_i,p_i) = \theta(m_i-n_i)\sqrt{\frac{n_i!}{m_i!}}(-1)^{n_i}(\sqrt{2}q_i + i\sqrt{2}p_i)^{m_i-n_i}L_{n_i}^{(m_i-n_i)}\big(2q_i^2+2p_i^2\big) \quad \text{(B7)}$$

$$+\theta(n_i-m_i)\sqrt{\frac{m_i!}{n_i!}}(-1)^{m_i}(\sqrt{2}q_i - i\sqrt{2}p_i)^{n_i-m_i}L_{m_i}^{(n_i-m_i)}\big(2q_i^2+2p_i^2\big) + (-1)^{n_i}L_{n_i}\big(2q_i^2+2p_i^2\big)\delta_{n_im_i}. \quad \text{(B8)}$$

Note that $P(\mathbf{q},\mathbf{p})$ can be strictly a polynomial, or a polynomial (series) expansion of a function, e.g., for the case of the thermal state.

APPENDIX C: PROOF OF EQ. (30)

First, by plugging in $F_{ab}(q, p)$ to Eq. (30), we get

$$\ln\left(\pi^{k-1}k^{-\frac{a+b}{2}k}\frac{\Gamma(\frac{1+ak}{2})\Gamma(\frac{1+bk}{2})}{[\Gamma(\frac{1+a}{2})\Gamma(\frac{1+b}{2})]^k}\right) \leqslant 0, \quad \text{(C1)}$$

which is equivalent to

$$(k-1)\ln\pi - \frac{a+b}{2}k\ln k + \ln\left[\Gamma\left(\frac{1+ak}{2}\right)\right] + \ln\left[\Gamma\left(\frac{1+bk}{2}\right)\right] - k\ln\left[\Gamma\left(\frac{1+a}{2}\right)\right] - k\ln\left[\Gamma\left(\frac{1+b}{2}\right)\right] \leqslant 0. \quad \text{(C2)}$$

Denoting by $f(a,b,k)$ the left-hand side of the above inequality and using the result from Ref. [22],

$$\psi(x) \leqslant \ln x - \frac{1}{2x}, \quad \text{(C3)}$$

where $\psi(x) = \frac{d}{dx}\{\ln[\Gamma(x)]\}$, we get

$$\frac{\partial f(a,b,k)}{\partial k} = \ln\pi - \frac{a+b}{2}(1+\ln k) + \frac{a}{2}\psi\left(\frac{1+ak}{2}\right) + \frac{b}{2}\psi\left(\frac{1+bk}{2}\right) - \ln\left[\Gamma\left(\frac{1+a}{2}\right)\right] - \ln\left[\Gamma\left(\frac{1+b}{2}\right)\right] \quad \text{(C4)}$$

$$\leqslant \ln\pi - \frac{a+b}{2}(1+\ln k) + \frac{a}{2}\ln\frac{1+ak}{2} - \frac{a}{2(1+ak)}$$

$$+ \frac{b}{2}\ln\frac{1+bk}{2} - \frac{b}{2(1+bk)} - \ln\left[\Gamma\left(\frac{1+a}{2}\right)\right] - \ln\left[\Gamma\left(\frac{1+b}{2}\right)\right] \quad \text{(C5)}$$

$$=: g(a,b,k). \quad \text{(C6)}$$

We can then calculate

$$\frac{\partial g(a,b,k)}{\partial k} = -\frac{a+b}{2k} + \frac{a^2}{2(1+ak)} + \frac{a^2}{2(1+ak)^2} + \frac{b^2}{2(1+bk)} + \frac{b^2}{2(1+bk)^2} \quad \text{(C7)}$$

$$= -\frac{a}{2k(1+ak)^2} - \frac{b}{2k(1+bk)^2} < 0, \quad \text{(C8)}$$

where we always work with $k \in \mathbb{R}$ and $k \geqslant 1$.

Denote $h(a,b)$ as

$$h(a,b) = g(a,b,1) \quad \text{(C9)}$$

$$= \ln\pi - \frac{a+b}{2} + \frac{a}{2}\ln\frac{1+a}{2} - \frac{a}{2(1+a)} + \frac{b}{2}\ln\frac{1+b}{2} - \frac{b}{2(1+b)} - \ln\left[\Gamma\left(\frac{1+a}{2}\right)\right] - \ln\left[\Gamma\left(\frac{1+b}{2}\right)\right]. \quad \text{(C10)}$$

When $a > 0$, we apply results from Ref. [22],

$$\psi(x) > \ln x - \frac{1}{2x} - \frac{1}{12x^2}, \quad \text{(C11)}$$

to get

$$\frac{\partial h(a,b)}{\partial a} = -\frac{1}{2} + \frac{1}{2}\ln\frac{1+a}{2} + \frac{a}{2(1+a)} - \frac{1}{2(1+a)^2} - \frac{1}{2}\psi\left(\frac{1+a}{2}\right) \quad \text{(C12)}$$

$$= \frac{1}{2}\left[-1 + \ln\frac{1+a}{2} + \frac{a}{1+a} - \frac{1}{(1+a)^2} - \psi\left(\frac{1+a}{2}\right)\right] \quad \text{(C13)}$$

$$< \frac{1}{2}\left(-1 + \ln\frac{1+a}{2} + \frac{a}{1+a} - \frac{1}{(1+a)^2} - \ln\frac{1+a}{2} + \frac{1}{1+a} + \frac{1}{3(1+a)^2}\right) \quad \text{(C14)}$$

$$= -\frac{1}{3(1+a)^2} \quad \text{(C15)}$$

$$< 0. \quad \text{(C16)}$$

Similarly for b, we have $\frac{\partial h(a,b)}{\partial b} < 0$.

Now, setting a or b equal to 0, we also have

$$\left.\frac{\partial h(a,b)}{\partial a}\right|_{a=0} < 0 \quad \text{(C17)}$$

$$\left.\frac{\partial h(a,b)}{\partial b}\right|_{b=0} < 0. \quad \text{(C18)}$$

Thus, for any $a, b \in \mathbb{N}$, we have

$$h(a,b) \leqslant \max\{h(0,0), h(0,1), h(1,0), h(1,1)\} \tag{C19}$$

$$= \max\left\{0, \frac{1}{2}\left(\ln\pi - \frac{3}{2}\right), \ln\pi - \frac{3}{2}\right\} \tag{C20}$$

$$\leqslant 0. \tag{C21}$$

Then, combining with $\frac{\partial g(a,b,k)}{\partial k} < 0$, for any $a, b \in \mathbb{N}$ and $k \in \mathbb{R}$ and $k \geqslant 1$, we have

$$g(a,b,k) \leqslant g(a,b,1) = h(a,b) \leqslant 0, \tag{C22}$$

which implies $\frac{\partial f(a,b,k)}{\partial k} \leqslant 0$.

Thus, we conclude that for any $(a,b) \in \mathbb{N}^2$ and $k \in \mathbb{R}$ and $k \geqslant 1$, we have

$$f(a,b,k) \leqslant f(a,b,1) = 0, \tag{C23}$$

which prove Eq. (30) for any $(a,b) \in \mathbb{N}^2$ and $k \in \mathbb{R}$ and $k \geqslant 1$.

APPENDIX D: FOCK BASIS PROPERTIES FOR STATES SATISFYING CONDITION 1

In this Appendix, we show that Condition 1 does not imply that the state $\hat{\rho}$ is Fock-diagonal but it implies $\hat{\rho}_{n,m} = 0$ if $|n-m|$ is odd.

Let $W(q,p)$ be a Wigner function that satisfies Condition 1 and $\chi_W(\eta)$, $\eta \in \mathbb{C}$, be its corresponding Wigner characteristic function. Without loss of generality, we can assume $n - m = l \geqslant 0$ and get

$$\langle n|\hat{\rho}|m\rangle = \frac{1}{\pi}\int \chi_W(\eta)\langle n|\hat{D}^\dagger(\eta)|m\rangle d^2\eta \tag{D1}$$

$$= \frac{1}{\pi}\int \chi_W(\eta)e^{-\frac{|\eta|^2}{2}}\langle n|e^{-\eta\hat{a}^\dagger}e^{\eta^*\hat{a}}|m\rangle d^2\eta \tag{D2}$$

$$= \frac{1}{\pi}\int \chi_W(\eta)e^{\frac{|\eta|^2}{2}}\left(\sum_{s=0}^{\infty}\frac{(-\eta^*)^s}{s!}\hat{a}^s|n\rangle\right)^\dagger\left(\sum_{t=0}^{\infty}\frac{(\eta^*)^t}{t!}\hat{a}^t|m\rangle\right)d^2\eta \tag{D3}$$

$$= \frac{1}{\pi}\int \chi_W(\eta)e^{-\frac{|\eta|^2}{2}}\left(\sum_{s=0}^{n}\frac{(-\eta^*)^s}{s!}\sqrt{\frac{n!}{(n-s)!}}|n-s\rangle\right)^\dagger\left(\sum_{t=0}^{m}\frac{(\eta^*)^t}{t!}\sqrt{\frac{m!}{(m-t)!}}|m-t\rangle\right)d^2\eta \tag{D4}$$

$$= \frac{1}{\pi}\int \chi_W(\eta)e^{-\frac{|\eta|^2}{2}}\left(\sum_{t=0}^{m}\frac{(-\eta)^{t+l}(\eta^*)^t}{(t+l)!t!}\frac{\sqrt{(m+l)!m!}}{(m-t)!}\right)d^2\eta \tag{D5}$$

$$= \frac{1}{\pi}\int \chi_W(\eta)e^{-\frac{|\eta|^2}{2}}(-\eta)^l\left(\sum_{t=0}^{m}\frac{(-1)^t|\eta|^{2t}}{(t+l)!t!}\frac{\sqrt{(m+l)!m!}}{(m-t)!}\right)d^2\eta. \tag{D6}$$

The case for $n - m < 0$ can be easily handled by switching m to n and t to s. Since the (inverse) Fourier transform conserves the parity of the function, we have that $\chi_W(\eta)$ is an even function of η. Together with the fact that $(\sum_{t=0}^{m}\frac{(-1)^t|\eta|^{2t}}{(t+l)!t!}\frac{\sqrt{(m+l)!m!}}{(m-t)!})$ is also an even function of η, the parity of the integrand only depends on l. Therefore, when l is odd, which in turn means the integrand is odd in η, we have $\langle n|\hat{\rho}|m\rangle = 0$.

When k is even, $\langle n|\hat{\rho}|m\rangle$ can be nonzero. The example below shows this. Consider a more general form of the Wigner function, $p_0|0\rangle\langle 0| + p_1|1\rangle\langle 1| + p_2|2\rangle\langle 2| + c|0\rangle\langle 2| + c^*|2\rangle\langle 0|$, and let $c = c_1 + ic_2$, $c_1 = \mathrm{Re}c$ and $c_2 = \mathrm{Im}c$, then we have

$$W(q,p) = W_0[(1-2p_1) + (2p_1 - 4p_2 + 2\sqrt{2}c_1)q^2 + 4p_2q^2p^2 + (2p_1 - 4p_2 - 2\sqrt{2}c_1)p^2 + 2p_2q^4 + 2p_2p^4 - 4\sqrt{2}c_2qp]. \tag{D7}$$

Therefore, all states with

$$p_1 \leqslant \tfrac{1}{2}, \quad p_1 - 2p_2 - \sqrt{2}c_1 \geqslant 0, \quad c_2 \leqslant 0 \tag{D8}$$

satisfy Condition 1. So, when $p_0 = \frac{1}{3}$, $p_1 = \frac{1}{2}$, $p_2 = \frac{1}{6}$, $c = \frac{\sqrt{2}}{16} - i$, the state is not Fock-diagonal but still satisfies Condition 1.

[1] W. Heisenberg, Über den anschaulichen Inhalt der quantentheoretischen Kinematik und Mechanik, Z. Phys. **43**, 172 (1927).

[2] P. J. Coles, M. Berta, M. Tomamichel, and S. Wehner, Entropic uncertainty relations and their applications, Rev. Mod. Phys. **89**, 015002 (2017).

[3] A. Hertz and N. J. Cerf, Continuous-variable entropic uncertainty relations, J. Phys. A: Math. Theor. **52**, 173001 (2019).

[4] I. Białynicki-Birula and J. Mycielski, Uncertainty relations for information entropy in wave mechanics, Commun. Math. Phys. **44**, 129 (1975).

[5] Z. Van Herstraeten and N. J. Cerf, Quantum Wigner entropy, Phys. Rev. A **104**, 042211 (2021).

[6] A. Hertz, M. G. Jabbour, and N. J. Cerf, Entropy-power uncertainty relations: towards a tight inequality for all Gaussian pure states, J. Phys. A: Math. Theor. **50**, 385301 (2017).

[7] A. Hertz, Exploring continuous-variable entropic uncertainty relations and separability criteria in quantum phase space, Ph.D. thesis, Université libre de Bruxelles, 2018.

[8] A. Wehrl, On the relation between classical and quantum-mechanical entropy, Rep. Math. Phys. **16**, 353 (1979).

[9] M. Aifer, N. M. Myers, and S. Deffner, Thermodynamics of quantum information in noisy polarizers, PRX Quantum **4**, 020343 (2023).

[10] J. Barata, J.-P. Blaizot, and Y. Mehtar-Tani, Quantum to classical parton dynamics in QCD media, Phys. Rev. D **108**, 014039 (2023).

[11] D. Reiche, J.-T. Hsiang, and B.-L. Hu, Quantum thermodynamic uncertainty relations, generalized current fluctuations and nonequilibrium fluctuation-dissipation inequalities, Entropy **24**, 1016 (2022).

[12] M. Gärttner, T. Haas, and J. Noll, Detecting continuous-variable entanglement in phase space with the q distribution, Phys. Rev. A **108**, 042410 (2023).

[13] Z. Van Herstraeten, M. G. Jabbour, and N. J. Cerf, Continuous majorization in quantum phase space, Quantum **7**, 1021 (2023).

[14] S. J. C. Salazar, H. G. Laguna, and R. P. Sagar, Phase-space quantum distributions and information theory, Phys. Rev. A **107**, 042417 (2023).

[15] M. Kalka, B. J. Spisak, D. Woźniak, M. Wołoszyn, and D. Kołaczek, Dynamical entropic measure of nonclassicality of phase-dependent family of Schrödinger cat states, Sci. Rep. **13**, 16266 (2023).

[16] N. C. Dias and J. N. Prata, On a recent conjecture by Z. Van Herstraeten and N. J. Cerf for the quantum Wigner entropy, Ann. Henri Poincaré **24**, 2341 (2023).

[17] A. Ferraro, S. Olivares, and M. G. A. Paris, *Gaussian States in Quantum Information*, Napoli Series on Physics and Astrophysics (Bibliopolis, Napoli, 2005).

[18] This can be proven as follows: Use spherical coordinates in Eq. (6), i.e., for $|r| \in [0, 1]$, $\theta \in [0, \pi]$, $\phi \in [0, 2\pi)$, we write $r_1 = |r| \sin\theta \cos\phi$, $r_2 = |r| \sin\theta \sin\phi$, and $r_3 = |r| \cos\theta$. Then, applying the (Gaussian unitary) phase shift operator $e^{-i\phi\hat{n}}$ on the state of Eq. (6), i.e., calculating $e^{-i\phi\hat{n}}\rho e^{i\phi\hat{n}}$ and using the fact that the Pauli operators are represented as matrices on the $\{|0\rangle, |1\rangle\}$ (Fock states) basis, we find the resulting state to be $|r|(I + \sin\theta\sigma_1 + \cos\theta\sigma_3)/2$ or equivalently in Cartesian coordinates $(I + r_1\sigma_1 + r_3\sigma_3)/2$.

[19] R. L. Hudson, When is the Wigner quasi-probability density non-negative? Rep. Math. Phys. **6**, 249 (1974).

[20] A. J. Pizzimenti, P. Dhara, Z. V. Herstraeten, S. Cheng, and C. N. Gagatsos, Exploring the possibility of a complex-valued non-Gaussianity measure for quantum states of light, arXiv:2303.00880.

[21] N. J. Cerf, A. Hertz, and Z. Van Herstraeten, Complex-valued Wigner entropy of a quantum state, Quantum Stud.: Math. Found. **11**, 331 (2024).

[22] L. Gordon, A stochastic approach to the gamma function, Am. Math. Mon. **101**, 858 (1994).