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Violation of the two-time Leggett-Garg inequalities for a harmonic oscillator

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We investigate the violation of the Leggett-Garg inequalities for a harmonic oscillator in various quantum states and with various choices of a projection operator for a dichotomic variable. We focus on the two-time quasiprobability distribution function with a dichotomic variable constructed with the position or momentum operator of a harmonic oscillator. Our results are the generalization of the previous work by Mawby and Halliwell [Phys. Rev. A **107**, 032216 (2023)], but the new points are the following. We first obtain the explicit expression for the two-time quasiprobability distribution function for the (thermal) squeezed coherent state. Second, we find the two-time quasiprobability distribution function with the dichotomic variable and the projection operator constructed in terms of the momentum operator. Third, we demonstrate that the violation of the Leggett-Garg inequalities can be boosted by adopting the dichotomic variable and the projection operator defined by a finite range of the position/momentum, in which a larger violation appears for the ground state and the squeezed state. We give an intuitive interpretation when the violation of the Leggett-Garg inequalities appears. We also present a mathematical formula to compute the quasiprobability distribution function by using the integral representation of the Heaviside function, which is useful to generalize it to a quantum field theory.

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I. INTRODUCTION

Testing quantum coherence in macroscopic systems is one of the fundamental problems in modern physics to explore the boundary between the quantum world and the classical world. The Leggett-Garg inequalities are the relations that must be satisfied from two intuitive principles in the macroscopic world [1–3]: macrorealism and noninvasive measurability. Macroscopic realism means that the physical quantity is a predetermined value regardless of the measurements. Noninvasive measurements imply that we can measure this predetermined value without disturbing the system's state. In contrast to classical mechanics, quantum mechanics breaks these two principles because of quantum superpositions and state-collapse disturbance. Experiments to verify the violation of the Leggett-Garg inequalities have been performed on spin operators in qubit systems and superconducting circuits [4–8]. Recently, they were also applied to neutron interferometers to test how far the prediction of quantum mechanics holds against macrorealism [9]. The coherence of the neutrino oscillation was tested using the violation of the Leggett-Garg inequalities [10]. These are the frontiers of testing quantum mechanics in the macroscopic world. There are further proposals, e.g., testing the quantum nature of macroscopic systems with the Leggett-Garg inequalities including gravity [11]. Theoretical research on the Leggett-Garg inequalities themselves is under debate (e.g., [12–16]).

The authors of Ref. [17] demonstrated that the Leggett-Garg inequalities can be violated in a harmonic oscillator in coherent states (see also [18,19]). Furthermore, Halliwell's group has investigated the violation of the Leggett-Garg inequalities in harmonic oscillators in thermal coherent state [20–24]. The optomechanical oscillator system is a promising method for preparing the quantum states of a massive object as tabletop experiments [25–28]. Continuous measurement cooling technology has been developed to realize such a quantum state [29–31]. Feasibility tests have demonstrated that the quantum states of a macroscopic pendulum will be realized in the near future [32–34]. As a first step toward a test of the macrorealism with a macroscopic oscillator, we investigate the violation of the two-time Leggett-Garg inequalities with a harmonic oscillator in various quantum states. The results in this paper contain generalizations of previous studies in the Ref. [24], but the new results of this paper are as follows: We investigate the violation of the two-time Leggett-Garg inequalities for one-dimensional harmonic oscillators in various quantum states using two different types of dichotomic measurement. One is the simplest choice of a continuous variable, where the boundary is chosen as a constant of the eigenvalue of the position/momentum. The other is the case that the dichotomic variable is defined by a finite range of the position/momentum. We show that the latter choice boosts the violation of the Leggett-Garg inequalities, which appear for the ground state and the squeezed states. We explain when the violation of the Leggett-Garg inequalities occurs, which is useful to understand the violation of the Leggett-Garg inequalities in an intuitive manner. We show the result with a dichotomic variable of the momentum operator, which demonstrates the dual property between the position and momentum. We also develop a formulation to calculate the quasiprobability distribution function using the integral formula of the Heaviside function.

The present paper is organized as follows: In Sec. II we briefly review the basic formulas of the two-time quasiprobability distribution function for the two-time Leggett-Garg inequalities. In Sec. III there are three subsections. Each subsection describes a generalization of the previous work in Ref. [24]. The first subsection III A shows the explicit expression for the two-time quasiprobability distribution function for the squeezed coherent state, which is obtained by extending the formulas developed in Ref. [24], where we also investigate the behaviors of the maximum violation of Leggett-Garg inequalities. Here we explain the intuitive understanding when the Leggett-Garg inequalities occur. In the second subsection III B, we explain the expression of the quasiprobability distribution function with momentum as the observed quantity. In the third subsection III C, we develop a formula to compute the quasiprobability distribution function, which is useful for quantum continuous variables by using the integral representation of the Heaviside function. We compare the results with these two different formulas. In Sec. IV we demonstrate the result with another dichotomic variable and the projection operator, in which we consider a finite region of position/momentum to construct the dichotomic variable. This leads to the violation of the Leggett-Garg inequalities in the ground state as well as the squeezed state, which even boosts the violation. Section V is devoted to summary and conclusions. In Appendix A a brief review of deriving the quasiprobability distribution function for the thermal squeezed coherent state based on the method made by Mawby and Halliwell [24] is presented. In Appendix B we present a detailed derivation of the expression (21). Throughout this paper, we use the unit $\hbar = 1$.

II. LEGGETT-GARG INEQUALITIES AND QUASIPROBABILITY DISTRIBUTION FUNCTION

In the first part of this section, we review the two-time Leggett-Garg inequalities with the two-time quasiprobability distribution function. We introduce a dichotomic variable Q, which gives ± 1 as a result of measurement. We define Q_1 and Q_2 to be the results of measurements at the time t_1 and t_2, respectively. Further, we introduce s_1 and s_2, which are to be chosen ± 1 for the measurement at the time t_1 and t_2. Under these assumptions, the inequalities $(1+s_1Q_1)(1+s_2Q_2)\geqslant 0$ hold for the four combination of s_1 and s_2.

Within a framework of macrorealism, there exists a probability function $p(Q_1, Q_2)$, which gives

$$\langle Q_1\rangle = \sum_{Q_1,Q_2} p(Q_1,Q_2)Q_1, \quad \langle Q_2\rangle = \sum_{Q_1,Q_2} p(Q_1,Q_2)Q_2,$$
$$\langle Q_1Q_2\rangle = \sum_{Q_1,Q_2} p(Q_1,Q_2)Q_1Q_2. \tag{1}$$

The probability function takes values of the range between 0 and 1, and the expectation value of $(1+s_1Q_1)(1+s_2Q_2)$ must be non-negative,

$$\langle (1+s_1Q_1)(1+s_2Q_2)\rangle \geqslant 0, \tag{2}$$

which is regarded as the Leggett-Garg inequalities of two-time.

On the basis of the framework of quantum mechanics, introducing the dichotomic quantum variable $\hat{Q}$, the corresponding variables $\hat{Q}_1$ and $\hat{Q}_2$ are defined by $\hat{Q}_1 = \hat{Q}(t_1) = e^{i\hat{H}t_1}\hat{Q}e^{-i\hat{H}t_1}$ and $\hat{Q}_2 = \hat{Q}(t_2) = e^{i\hat{H}t_2}\hat{Q}e^{-i\hat{H}t_2}$, respectively, where we assume the unitary evolution of the system described by the Hamiltonian operator $\hat{H}$. Then the quasiprobability is defined by

$$q_{s_1,s_2}(t_1,t_2) = \tfrac{1}{8}\mathrm{Tr}[(1+s_1\hat{Q}_1)(1+s_2\hat{Q}_2)\rho_0] + (1\leftrightarrow 2), \tag{3}$$

where ρ_0 is the density matrix of the initial state. Introducing the Heisenberg operator by

$$P_s(t) = e^{i\hat{H}t}P_s e^{-i\hat{H}t} = \tfrac{1}{2}e^{i\hat{H}t}(1+s\hat{Q})e^{-i\hat{H}t}, \tag{4}$$

where $P_s = (1+s\hat{Q})/2$ is regarded as a projection operator, the quasiprobability is written as

$$\begin{aligned} q_{s_1,s_2}(t_1,t_2) &= \tfrac{1}{2}\mathrm{Tr}[P_{s_1}(t_1)P_{s_2}(t_2)\rho_0] + (1\leftrightarrow 2)\\ &= \mathrm{ReTr}[P_{s_2}(t_2)P_{s_1}(t_1)\rho_0]. \end{aligned} \tag{5}$$

We note that $q_{s_1,s_2}(t_1,t_2)$ satisfies the relations of the probability [21]

$$\langle \hat{Q}(t_1)\rangle = \mathrm{Tr}[\hat{Q}(t_1)\rho_0] = \sum_{s_1,s_2=\pm 1} s_1 q_{s_1,s_2}(t_1,t_2), \tag{6}$$

$$\langle \hat{Q}(t_2)\rangle = \mathrm{Tr}[\hat{Q}(t_2)\rho_0] = \sum_{s_1,s_2=\pm 1} s_2 q_{s_1,s_2}(t_1,t_2), \tag{7}$$

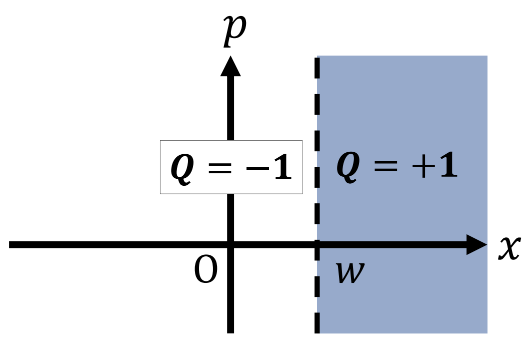

FIG. 1. Schematic explanation of how to define the dichotomic measurement in Sec. III A, Eq. (10)

$$\frac{1}{2}\langle\{\hat{Q}(t_1),\hat{Q}(t_2)\}\rangle = \frac{1}{2}\mathrm{Tr}[\{\hat{Q}(t_1),\hat{Q}(t_2)\}\rho_0] = \sum_{s_1,s_2=\pm1} s_1 s_2 q_{s_1,s_2}(t_1,t_2), \tag{8}$$

$$1 = \sum_{s_1,s_2=\pm1} q_{s_1,s_2}(t_1,t_2), \tag{9}$$

where $\{\hat{Q}(t_1),\hat{Q}(t_2)\} = \hat{Q}(t_1)\hat{Q}(t_2) + \hat{Q}(t_2)\hat{Q}(t_1)$. However, it may have negative values in quantum theory, and then we call $q_{s_1,s_2}(t_1,t_2)$ quasiprobability.

III. TWO-TIME QUASIPROBABILITY DISTRIBUTION FUNCTION OF THE DICHOTOMIC MEASUREMENT

In this section we evaluate the quasiprobability distribution function with dichotomic measurements, which is defined by a measurement value Q as shown by Fig. 1. The dichotomic measurement means that when the observable is larger than the threshold w, $Q = +1$, and when it is smaller than w, $Q = -1$, where w is a parameter. For this assumption, the measurement operator $\hat{Q}$ can be written by a sign function and the projection operator can be written by a Heaviside function:

$$\hat{Q} = \mathrm{sgn}(\hat{x} - w), \quad P_{s_i} = \theta(s_i(\hat{x} - w)). \tag{10}$$

Using the Heisenberg picture, the time-dependent operator is written as

$$\hat{Q}(t_i) = e^{i\hat{H}t_i}\mathrm{sgn}(\hat{x} - w)e^{-i\hat{H}t_i},$$
$$P_{s_i}(t_i) = e^{i\hat{H}t_i}\theta(s_i(\hat{x} - w))e^{-i\hat{H}t_i}. \tag{11}$$

In subsection III A, we first consider the two-time quasiprobability distribution function for the thermal squeezed coherent state, which is obtained by extending the formulas developed in Ref. [24]. In the second subsection III B, the expression of the quasiprobability distribution function with the dichotomic variable with the momentum operator. In the third subsection III C, we develop a formula to compute the quasiprobability distribution function, which is used to prove the validity of both the formulas.

A. Two-time quasiprobability distribution function with the dichotomic variable of position operator

First, we evaluate the quasiprobability distribution function following the method developed in the paper by Mawby and Halliwell [24]. We first consider the squeezed coherent state, which is written as

$$\rho_0 = D(\xi)S(\zeta)\,|0\rangle\,\langle 0|\,S^\dagger(\zeta)D^\dagger(\xi) \tag{12}$$

with the squeezing operator $S(\zeta)$ defined by $S(\zeta) = e^{\frac{1}{2}(\zeta\hat{a}^{\dagger 2} - \zeta^*\hat{a}^2)}$ and the coherent operator $D(\xi)$ defined by $D(\xi) = e^{\xi\hat{a}^\dagger - \xi^*\hat{a}}$, where we also use $\xi = (x_0 + ip_0)/\sqrt{2}$ and $\zeta = re^{i\theta_0}$, and x_0 and p_0 are the dimensionless initial position/momentum expected values. Then the quasiprobability distribution function is given by

$$\begin{aligned} q_{s_1,s_2}(t_1,t_2) &= \mathrm{Re}\,\mathrm{Tr}[P_{s_2}(t_2)P_{s_1}(t_1)\rho_0] \\ &= \mathrm{Re}\langle 0|S^\dagger(\zeta)D^\dagger(\xi)\theta(s_2(\hat{x}(t_2) - w))\theta(s_1(\hat{x}(t_1) - w))D(\xi)S(\zeta)|0\rangle \\ &= \frac{1}{4}\left[1 + s_1\mathrm{erf}\left(\frac{x_{\xi(t_1)} - w}{\lambda(t_1)}\right) + s_2\mathrm{erf}\left(\frac{x_{\xi(t_2)} - w}{\lambda(t_2)}\right) + s_1 s_2\mathrm{erf}\left(\frac{x_{\xi(t_1)} - w}{\lambda(t_1)}\right)\mathrm{erf}\left(\frac{x_{\xi(t_2)} - w}{\lambda(t_2)}\right)\right] \\ &\quad + s_1 s_2 \mathrm{Re}\sum_{n=1}^{\infty} e^{-in\omega(t_2 - t_1 + \beta(t_2) - \beta(t_1))} J_{0n}\left(-\frac{x_{\xi(t_1)} - w}{\lambda(t_1)}, \infty\right) J_{n0}\left(-\frac{x_{\xi(t_2)} - w}{\lambda(t_2)}, \infty\right), \end{aligned} \tag{13}$$

where we defined $x_{\xi(t)} = \sqrt{2}\mathrm{Re}[\xi(t)] = \sqrt{2}\mathrm{Re}[\xi e^{-i\omega t}]$, $\lambda(t) = \sqrt{\sinh(2r)\cos(\theta_0 - 2\omega t) + \cosh(2r)}$, $\beta(t)$ is defined by Eq. (A12), and $J_{mn}(x_1, x_2)$ is defined as the matrix element in Ref. [23] [see also Eq. (A21) for $J_{mn}(x_1,\infty)$], and $\mathrm{erf}(z)$ is the error function.

We have also found the explicit expression for the quasiprobability for the thermal squeezed coherent state, Eq. (A16), as described in Appendix A. We define the parameter for the initial state of the coherent state by $x_0 = \sqrt{2}\mathrm{Re}[\xi] = \sqrt{2}|\xi|\cos\Theta$, $p_0 = \sqrt{2}\mathrm{Im}[\xi] = \sqrt{2}|\xi|\sin\Theta$. Here we note that x_0 and p_0 are understood as the initial values of the position and momentum of coherent oscillating motion, normalized by the factors $1/\sqrt{2m\omega}$ and $\sqrt{m\omega/2}$, respectively. Figure 2 plots the minimum value $q_{1,-1}(0,t_2)$ on the plane of x_0 and p_0, where we found the minimum value of $q_{1,-1}(0,t_2)$ by varying t_2 for each x_0 and p_0. Here we fixed $w = 0$ and the squeezed parameter $r = 1/2$ and $\theta_0 = 0$ with the temperature $T = 0$. As is predicted in Ref. [24], $q_{s_1,s_2}(0,t_2)$ achieves the minimum value $q_{s_1,s_2}(0,t_2) = -0.0284$, which is the same as that for the coherent state with $r = 0$. As is demonstrated in [24] for the coherent state, the minimum value of the quasiprobability distribution function becomes larger as the

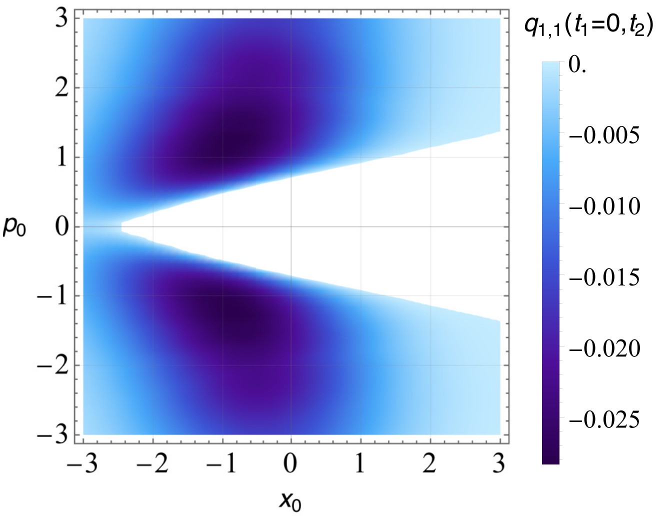

FIG. 2. Contour plot of the minimum values of $q_{s_1=+1,s_2=-1}(t_1 = 0, t_2)$, on the plane of x_0 and p_0. We fixed $w = 0$ and the squeezing parameters $r = 1/2$ and $\theta_0 = 0$. x_0 and p_0 are the dimensionless initial position and momentum values, respectively.

TABLE I. Parameters adopted for the curves in Fig. 3.

s_1	s_2	r	x_0	p_0
1	−1	0	−0.554	1.95
1	−1	0.5	−0.896	1.18
1	−1	0.6	−0.991	1.07
1	−1	0.7	−1.09	0.968
1	−1	0.8	−1.21	0.875
1	−1	0.9	−1.34	0.792
1	−1	1.0	−1.48	0.717

temperature T is increased. This property is the same for the squeezed coherent state, and this can be understood from the fact that the quasiprobability distribution function of the squeezed coherent state is obtained by that of the coherent state with an appropriate variable transformation, which is explicitly shown in Appendix A. Hereafter in this section, we fix $w = 0$, for simplicity.

The left panel of Fig. 3 exemplifies the quasiprobability distribution function, $q_{s_1,s_2}(0, t_2)$, as a function of ωt_2, where the left panel assumes $s_1 = 1$, $s_2 = -1$. Figure 3 adopted the parameters x_0 and p_0 noted in Table I, which are chosen so that the quasiprobability distribution function achieves the maximum violation, $q_{s_1,s_2}(0, t_2) = -0.0284$. With the use of Fig. 3, we explain how to understand the violation of the Leggett-Garg inequalities in an intuitive way. The violation of the Leggett-Garg inequalities appears when the position measurements give the opposite value against the expectation value. The initial values of x_0 and p_0 for the curves in this figure are roughly $-1.5 < x_0 < -0.5$ and $0.7 < p_0 < 2$. This means that the initial expectation value of the coherent oscillation motion starts from the left $x_0 < 0$ with the right moving momentum $p_0 > 0$. $q_{1,-1}(0, t_2)$ computes the quasiprobability that the result of the position measurement at $t_1 = 0$ gives $x > 0$ and the result of the position measurement at t_2 gives $x < 0$. However, the initial value of $x_0 < 0$ at t_1 is opposite to the condition $x > 0$ at $t_1 = 0$. The right panel of Fig. 3 plots the expectation value of $\langle\psi_0|\hat{x}(t)|\psi_0\rangle/2|\xi| = \cos(\omega t - \Theta)$, where the curve of the left panel assumes the same parameters adopted for the same type of curve in Fig. 3. From the right panel of Fig. 3, the expectation value of the position becomes positive after a short time. The quasiprobability distribution function $q_{1,-1}(0, t_2)$ takes the minimum negative values at the time t_2 when it becomes $\langle\psi_0|\hat{x}(t_2)|\psi_0\rangle > 0$. This is opposite to the condition that $q_{1,-1}(0, t_2)$ means at t_2. Namely, $q_{1,-1}(0, t_2)$ computes the quasiprobability that the result of the position measurement at t_2 gives $x < 0$. Thus, the violation of the Leggett-Garg inequalities appears when the measurements give opposite values against the expectation values. This property is maintained even for other s_1, s_2 pairs and can be said to be general.

B. Two-time quasiprobability distribution function with the dichotomic variable of momentum operator

In this subsection we consider the quasiprobability distribution function with the dichotomic variable constructed with the momentum operator for the squeezed coherent state. Here the dichotomic variable and the measurement operator can be written by a sign function, and the projection operator can be

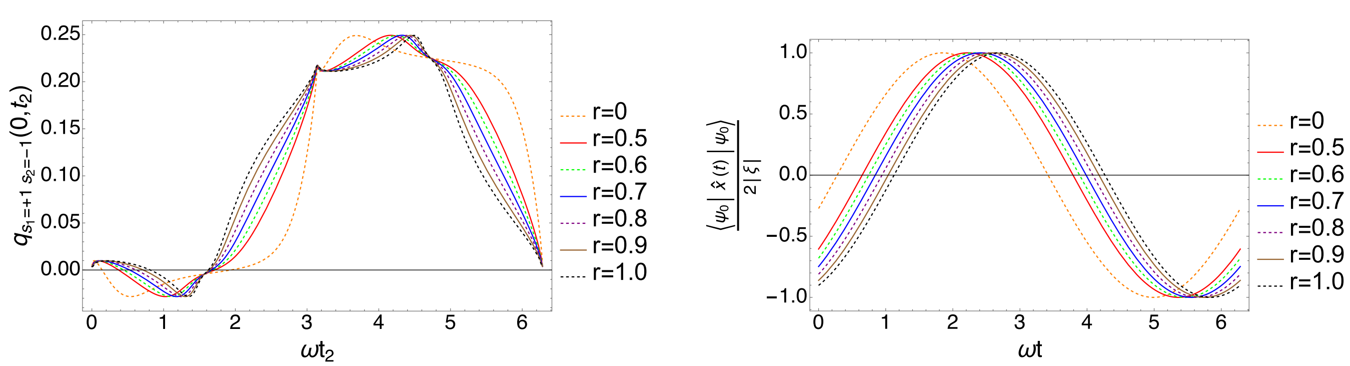

FIG. 3. The left panel shows the quasiprobability distribution function as a function of ωt adopted $s_1 = 1$, $s_2 = -1$, where we fixed $t_1 = 0$ and $\theta_0 = 0$, and the other parameters are noted in Table I. The right panel shows as the expectation values of position $\langle\psi_0|\hat{x}(t)|\psi_0\rangle/2|\xi| = \cos(\omega t - \arctan[p_0/x_0])$ as a function of ωt. The same-type curves in the left panel and in the right panel of Fig. 3 assume the same parameters for each, as described in Table I. $q_{s_1,s_2}(0, t_2)$ achieves the minimum value $q_{s_1,s_2}(0, t_2) = -0.0284$.

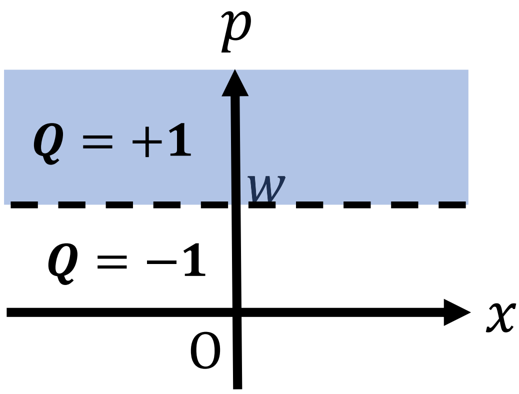

FIG. 4. Schematic explanation of how to define the dichotomic measurement in Sec. III B, Eq. (14).

written by a Heaviside function (see Fig. 4):

$$\hat{Q} = \mathrm{sgn}(\hat{p} - w), \quad P_{s_i} = \theta(s_i(\hat{p} - w)). \tag{14}$$

For the Heisenberg operators $\hat{x}(t)$ and $\hat{p}(t)$, which unitarily evolve through $\hat{H}$,

$$\hat{p}(t) = \hat{x}(t + \pi/2\omega), \tag{15}$$

then the quasiprobability distribution function is written as

$$\begin{aligned} q_{s_1,s_2}(t_1,t_2) &= \mathrm{Re\,Tr}[P_{s_2}(t_2)P_{s_1}(t_1)\rho_0] \\ &= \mathrm{Re}\langle 0|S^\dagger(\zeta)D^\dagger(\xi)\theta(s_2(\hat{p}(t_2) - w)) \\ &\quad \times \theta(s_1(\hat{p}(t_1) - w))D(\xi)S(\zeta)|0\rangle \\ &= \mathrm{Re}\langle 0|S^\dagger(\zeta)D^\dagger(\xi)\theta\left(s_2\left(\hat{x}\left(t_2 + \frac{\pi}{2\omega}\right) - w\right)\right) \\ &\quad \times \theta\left(s_1\left(\hat{x}\left(t_1 + \frac{\pi}{2\omega}\right) - w\right)\right)D(\xi)S(\zeta)|0\rangle, \end{aligned} \tag{16}$$

where $\rho_0 = D(\xi)S(\zeta)|0\rangle\langle 0|S^\dagger(\zeta)D^\dagger(\xi)$. From the second line to the third line, we used Eq. (15). The derivation is shown in Appendix A, and we have

$$\begin{aligned} q_{s_1,s_2}(t_1,t_2) = \frac{1}{4}\Bigg[&1 + s_1\mathrm{erf}\left(\frac{x_{\xi(t_1+\pi/2\omega)} - w}{\lambda\left(t_1 + \frac{\pi}{2\omega}\right)}\right) + s_2\mathrm{erf}\left(\frac{x_{\xi(t_2+\pi/2\omega)} - w}{\lambda\left(t_2 + \frac{\pi}{2\omega}\right)}\right) + s_1 s_2\mathrm{erf}\left(\frac{x_{\xi(t_1+\pi/2\omega)} - w}{\lambda\left(t_1 + \frac{\pi}{2\omega}\right)}\right)\mathrm{erf}\left(\frac{x_{\xi(t_2+\pi/2\omega)} - w}{\lambda\left(t_2 + \frac{\pi}{2\omega}\right)}\right)\Bigg] \\ &+ s_1 s_2 \mathrm{Re}\sum_{n=1}^{\infty} e^{-in\omega[t_2 - t_1 + \beta(t_2+\pi/2\omega) - \beta(t_1+\pi/2\omega)]} J_{0n}\left(-\frac{x_{\xi(t_1+\pi/2\omega)} - w}{\lambda\left(t_1 + \frac{\pi}{2\omega}\right)}, \infty\right) J_{n0}\left(-\frac{x_{\xi(t_2+\pi/2\omega)} - w}{\lambda\left(t_2 + \frac{\pi}{2\omega}\right)}, \infty\right). \end{aligned} \tag{17}$$

Also note that $x_{\xi(t+\pi/\omega 2)}$ is equivalent to $p_{\xi(t)}$ which appears when the coherent operator acts on the momentum operator as $D^\dagger(\xi(t))\hat{p}D(\xi(t)) = \hat{p} + p_{\xi(t)}$, where we defined $p_{\xi(t)} = \sqrt{2}\mathrm{Im}[\xi(t)] = \sqrt{2}\mathrm{Im}[\xi e^{-i\omega t}]$. Thus, Eq. (15) is useful to find the quasiprobability distribution function with the dichotomic variable with the measurement of momentum.

The panels of Fig. 5 demonstrate the contour of the minimum value of $q_{1,-1}(0,t_2)$ on the plane x_0 and p_0. The left panel assumes $w = 0$, $r = 1/2$, and $\theta_0 = 0$, while the right panel assumes $w = 0$, $r = 1/2$, and $\theta_0 = \pi$. The minimum value of $q_{1,-1}(0,t_2)$ in Fig. 5 is -0.0284, which is the same as that in Fig. 2. One can see that the right panel of Fig. 5 is given by rotating Fig. 2 by the factor $\pi/2$ around the origin. This can be understood by the relation (15), and the projection operators and the squeezed coherent states of these two models are related by the rotation in phase space. The period of the rotation of the squeezed state in phase space is π/ω, and the period of the rotation of the coherent motion in phase space is $2\pi/\omega$. The result in this subsection might be useful from the perspective of testing the violation of the Leggett-Garg

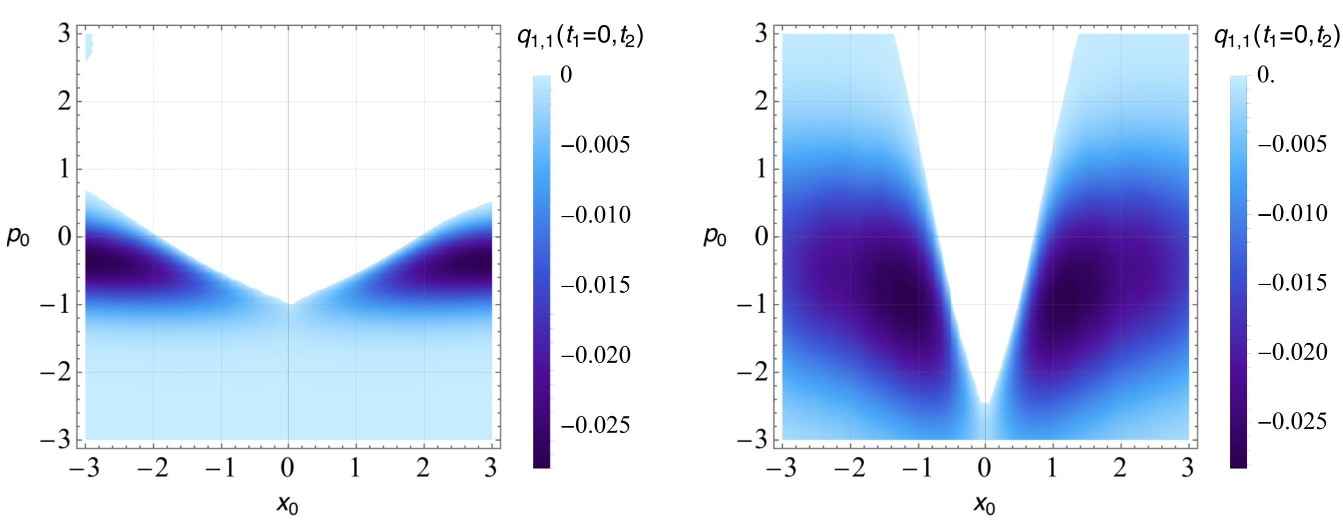

FIG. 5. The contour of the minimum value of the Leggett-Garg violation $q_{1,-1}(t_1 = 0, t_2)$ with Eq. (17) on the plane of x_0 and p_0. t_2 takes different values at each point. t_2 is a variable parameter to minimize $q_{1,-1}(t_1 = 0, t_2)$. In these panels we fixed $w = 0$, $r = 1/2$, and $\theta_0 = 0$ ($\theta_0 = \pi$) in the left (right) panel.

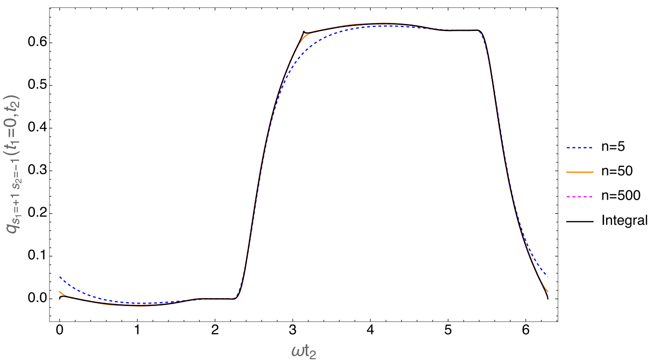

FIG. 6. Comparison of the numerical results of Eq. (21) and Eq. (13). In the numerical evaluation of Eq. (13), we increased the maximum number of the sum with respect to n up to $n = 5$ (blue dashed curve), $n = 50$ (orange solid curve), $n = 500$ (red dashed curve), where the red dashed curve is not seen. The black solid curve plots the numerical integration of Eq. (21), which overlaps the red dashed curve, i.e., Eq. (13) with $n = 500$. Thus, the two formulas (21) and (13) coincide as long as the sum with respect to n is taken sufficiently large. Here we fixed $x_0 = 0.550$, $p_0 = 1.925$, $r = 1$, $\theta_0 = \pi/3$, $s_1 = 1$, and $s_2 = -1$.

inequality for a harmonic oscillator with measurements of momentum.

C. Another calculation method of quasiprobability distribution function and its applicability

In this subsection we develop a different prescription for computing the quasiprobability distribution function, which can be useful to generalize it to a system of quantum field theory as demonstrated in the accompanying paper [35]. We restart from the expression of the quasiprobability distribution function, $q_{s_1,s_2}(t_1,t_2) = \mathrm{ReTr}[P_{s_2}(t_2)P_{s_1}(t_1)\rho_0]$, where we consider the initial state as a coherent squeezed state, $\rho_0 = D(\xi)S(\zeta)\,|0\rangle\,\langle 0|\,S^\dagger(\zeta)D^\dagger(\xi)$. Using the mathematical formula $\theta'(x-\alpha) = -d\theta(x-\alpha)/d\alpha = \delta(x-\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dp\, e^{-ip(x-\alpha)}$, we have

$$\theta(x-\alpha) = \int_\alpha^\infty dc \frac{1}{2\pi}\int_{-\infty}^{\infty} dp\, e^{-ip(x-c)}. \tag{18}$$

Further, using the creation and annihilation operators, $\hat{a}^\dagger$ and $\hat{a}$, the position operator of x is written as $\hat{x} = (\hat{a}+\hat{a}^\dagger)/\sqrt{2m\omega}$, and we have

$$P_s(t) = e^{i\hat{H}t}\theta(s\hat{x})e^{-i\hat{H}t}$$
$$= \int_0^\infty dc \int_{-\infty}^{\infty} \frac{dp}{2\pi} \exp\left[ip\left(-s\frac{\hat{a}e^{-i\omega t}+\hat{a}^\dagger e^{i\omega t}}{\sqrt{2m\omega}} + c\right)\right]. \tag{19}$$

Hence, the quasiprobability is expressed as

$$q_{s_1,s_2}(t_1,t_2) = \mathrm{Re\,Tr}\Big[\iint_0^\infty dc_1 dc_2 \iint_{-\infty}^{\infty} \frac{dp_1 dp_2}{(2\pi)^2} \times e^{-ip_2 s_2(\hat{a}e^{-i\omega t_2}+\hat{a}^\dagger e^{i\omega t_2})+ip_2c_2} \times e^{-ip_1 s_1(\hat{a}e^{-i\omega t_1}+\hat{a}^\dagger e^{i\omega t_1})+ip_1c_1}\rho_0\Big], \tag{20}$$

which leads to

$$q_{s_1,s_2}(t_1,t_2) = \mathrm{Re}\left\{\frac{1}{2\pi}\frac{e^{-\delta/2}}{\sqrt{B}}\int_0^{\pi/2} du\left[\frac{1}{\sigma} - \frac{\sqrt{2\pi}\,b e^{\beta^2/2\sigma}}{2\sigma^{3/2}} \times \mathrm{erfc}\left(\frac{\beta}{\sqrt{2\sigma}}\right)\right]\right\}, \tag{21}$$

where we defined as follows:

$$\gamma = \xi\cosh|\zeta| - \xi^* e^{i\theta_0}\sinh|\zeta|, \tag{22}$$

$$E(t) = e^{-i\omega t}\cosh r + e^{i\omega t}e^{-i\theta_0}\sinh r, \tag{23}$$

$$\mathcal{E}(t) = E(t)\gamma + E^*(t)\gamma^*, \tag{24}$$

$$B = |E(t_1)|^2|E(t_2)|^2 - [E^*(t_1)E(t_2)]^2, \tag{25}$$

$$\sigma = \frac{1}{B}[|E(t_2)|^2\cos^2 u + |E(t_1)|^2\sin^2 u - 2s_1s_2E(t_2)E^*(t_1)\sin u\cos u], \tag{26}$$

$$\beta = \frac{1}{B}(-|E(t_2)|^2 s_1\mathcal{E}(t_1)\cos u - |E(t_1)|^2 s_2\mathcal{E}(t_2)\sin u + E(t_2)E^*(t_1)[s_2\mathcal{E}(t_1)\sin u + s_1\mathcal{E}(t_2)\cos u]), \tag{27}$$

$$\delta = \frac{1}{B}(|E(t_2)|^2\mathcal{E}(t_1)^2 + |E(t_1)|^2\mathcal{E}(t_2)^2 - 2E(t_2)E^*(t_1)\mathcal{E}(t_1)\mathcal{E}(t_2)). \tag{28}$$

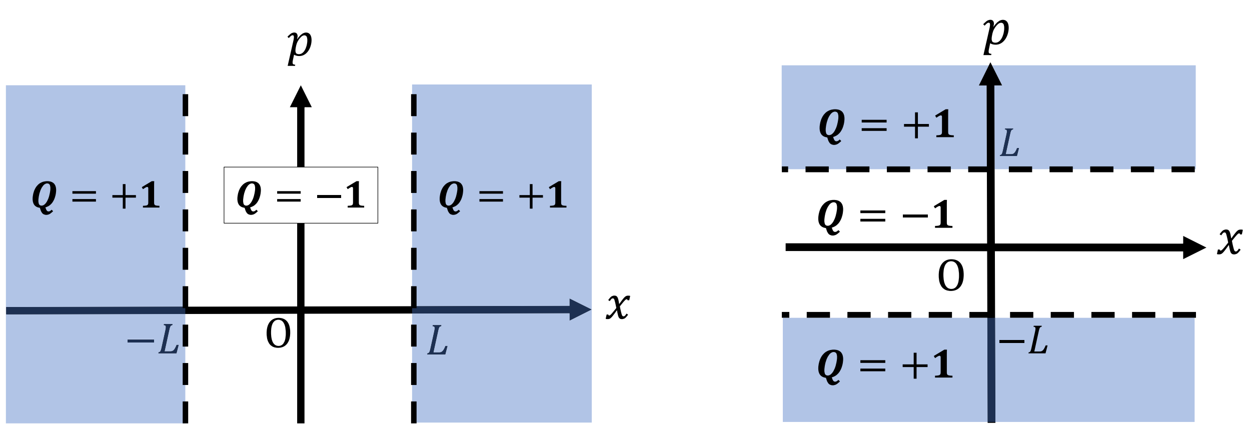

FIG. 7. The left panel shows how the dichotomic variable Q is defined in the phase space in Sec. IV A, Eq. (29). The right panel does the same in Sec. IV B, Eq. (33).

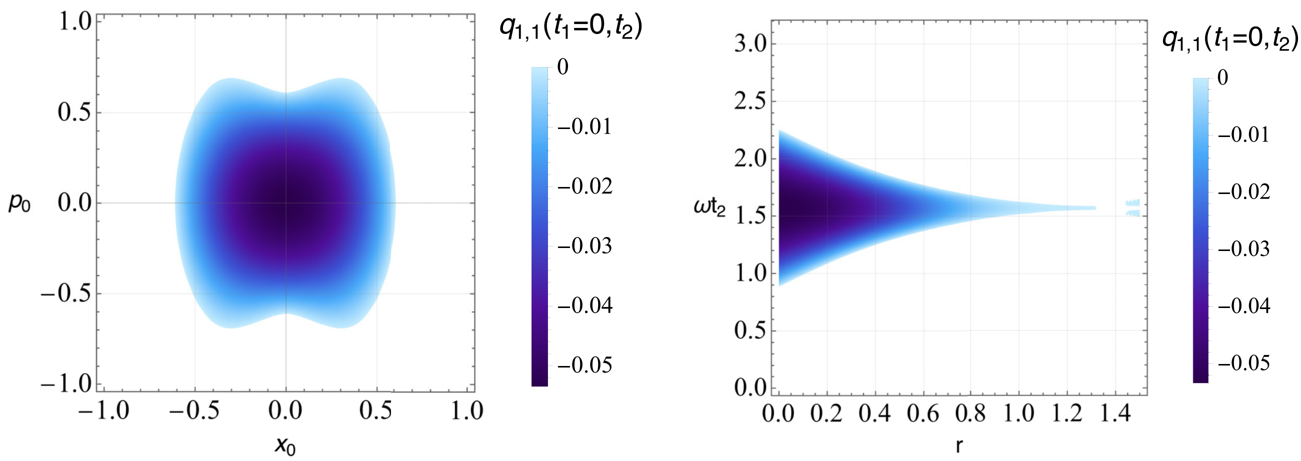

FIG. 8. The left panel is the contour of the minimum value of $q_{1,1}(t_1=0,t_2)$ of Eq. (32) on the plane of the coherent parameters x_0 and p_0, where we fixed $r=0$, $\theta_0=0$, and $L=1$, and t_2 is is a free parameter to the minimum value of quasiprobability distribution function for each set of x_0 and p_0. The right panel is the contour of $q_{1,1}(t_1=0,t_2)$ on the plane of r and ωt_2, where we fixed $L=1$, $x_0=0$, and $p_0=0$. Each panel assumes $\theta_0=0$, $s_1=1$, and $s_2=1$.

The derivation of Eq. (21) is described in Appendix B, where we performed the Gaussian integration for the quasiprobability distribution function (20). The numerical calculation of Eq. (21) gives the same result as that in Sec. III A, as long as the calculation guarantees convergence. Figure 6 demonstrates that the numerical sum of Eq. (13) reproduces the numerical integration of Eq. (21) as the maximum value of the sum with respect to n in Eq. (13) increases. Thus, Fig. 6 demonstrates that the same result is obtained from the two different formulas (21) and (13). This method can be extended to a quantum scalar field theory by replacing x in Eq. (18) with a coarse-grained field operator, as demonstrated in the accompanying paper [35].

IV. LARGER VIOLATION OF LEGGETT-GARG INEQUALITIES WITH EXTENDED DICHOTOMIC VARIABLE

A. Extended dichotomic variable with the position operator

In this subsection we investigate a larger violation of the Leggett-Garg inequalities by introducing the dichotomic variables and measurement operators defined as follows:

$$\hat{Q}=\mathrm{sgn}(\hat{x}-L)+\mathrm{sgn}(-\hat{x}-L)+1, \tag{29}$$

where $L(>0)$ is the parameter, which determines the region at which the measurement value switches. Following the definition $P_{s_i}=(1+s_i\hat{Q})/2$, the projection operator is written as

$$P_{s_i}=\theta(s_i(\hat{x}-L))+\theta(-s_i(\hat{x}+L))+\tfrac{1}{2}(s_i-1). \tag{30}$$

The dichotomic variable defined by the above projection operator is understood as follows. When the result of a measurement of the position of a harmonic oscillator is $|x|>L$, we assign $Q=1$. On the other hand, when the result of a measurement of the position is $|x|\leqslant L$, we assign $Q=-1$. Therefore, the projection operator P_s with $s=1$ gives the projection of the region $|x|>L$, while P_s with $s=-1$ does the projection of the region $|x|\leqslant L$. In the phase space, we can understand the definition of the dichotomic variable and the projection operator as shown by the left panel of Fig. 7.

We consider the squeezed coherent state as the initial state, $\rho_0=|\xi,\zeta\rangle\langle\xi,\zeta|$, where $|\xi,\zeta\rangle$ is defined by $|\zeta\rangle=D(\xi)S(\zeta)\,|0\rangle$. In this case, we evaluate the following formula as the quasiprobability distribution function:

$$\begin{aligned}q_{s_1,s_2}(t_1,t_2)=\mathrm{Re}\big[&\langle 0|\,S^\dagger(\zeta)D^\dagger(\xi)e^{iHt_2}\big\{\theta(s_2(\hat{x}-L))+\theta(-s_2(\hat{x}+L))+\tfrac{1}{2}(s_2-1)\big\}e^{-iHt_2}\\&\times e^{iHt_1}\big\{\theta(s_1(\hat{x}-L))+\theta(-s_1(\hat{x}+L))+\tfrac{1}{2}(s_1-1)\big\}e^{-iHt_1}D(\xi)S(\zeta)\,|0\rangle\big].\end{aligned} \tag{31}$$

Using the method developed in the previous section, we have

$$\begin{aligned}q_{s_1,s_2}(t_1,t_2)=\frac{1}{4}&\left\{1+s_1\left[1+\mathrm{erf}\left(\frac{x_{\xi(t_1)}-L}{\lambda(t_1)}\right)-\mathrm{erf}\left(\frac{x_{\xi(t_1)}+L}{\lambda(t_1)}\right)\right]\right\}\left\{1+s_2\left[1+\mathrm{erf}\left(\frac{x_{\xi(t_2)}-L}{\lambda(t_2)}\right)-\mathrm{erf}\left(\frac{x_{\xi(t_2)}+L}{\lambda(t_2)}\right)\right]\right\}\\&+s_1s_2\mathrm{Re}\left\{\sum_{n=1}^{\infty}e^{-in\omega(t_2-t_1+\beta(t_2)-\beta(t_1))}\left[J_{0n}\left(\frac{x_{\xi(t_1)}-L}{\lambda(t_1)},\infty\right)J_{n0}\left(\frac{x_{\xi(t_2)}-L}{\lambda(t_2)},\infty\right)-J_{0n}\left(\frac{x_{\xi(t_1)}-L}{\lambda(t_1)},\infty\right)J_{n0}\right.\right.\\&\times\left.\left.\left(\frac{x_{\xi(t_2)}+L}{\lambda(t_2)},\infty\right)-J_{0n}\left(\frac{x_{\xi(t_1)}+L}{\lambda(t_1)},\infty\right)J_{n0}\left(\frac{x_{\xi(t_2)}-L}{\lambda(t_2)},\infty\right)+J_{0n}\left(\frac{x_{\xi(t_1)}+L}{\lambda(t_1)},\infty\right)J_{n0}\left(\frac{x_{\xi(t_2)}+L}{\lambda(t_2)},\infty\right)\right]\right\}.\end{aligned} \tag{32}$$

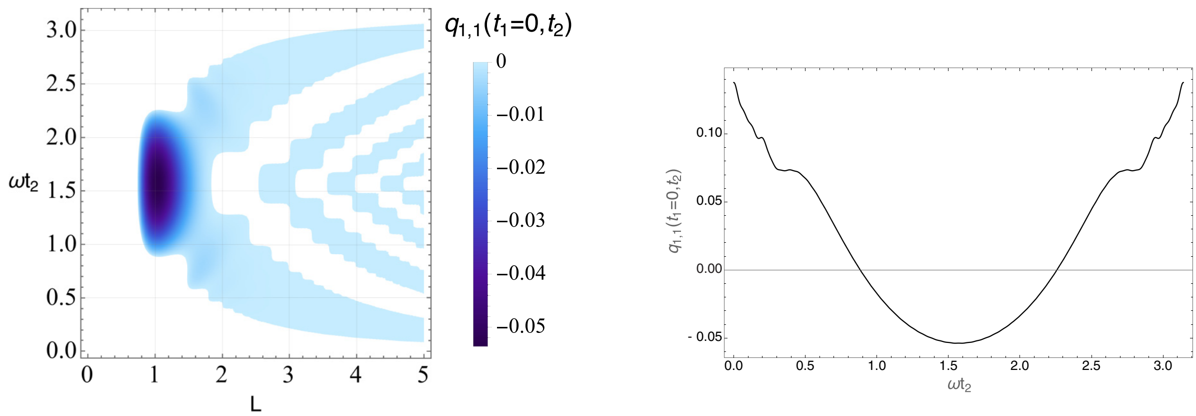

FIG. 9. The left panel is the contour of the minimum value of $q_{1,1}(t_1 = 0, t_2)$ of Eq. (32) on the plane of L and ωt_2, where we fixed $\theta_0 = 0$. L is a dimensionless threshold value. The right panel is the plot of the quasiprobability $q_{1,1}(t_1 = 0, t_2)$ of Eq. (32) as function of ωt_2, where we fixed $x_0 = 0$, $p_0 = 0$, $r = 0$, $\theta_0 = 0$, and $L = 1.02$, which achieves the minimum value of $q_{1,1}(t_1 = 0, t_2)$ in the left panel.

The left panel of Fig. 8 plots the contour of the minimum value of the quasiprobability $q_{1,1}(0, t_2)$ of Eq. (32) on the plane of x_0 and p_0 with fixed $r = 0$ and $\theta_0 = 0$. The right plane of Fig. 8 plots the contour on the plane of r and ωt_2 with fixed $L = 1$, $x_0 = 0$, and $p_0 = 0$. In these panels, we fixed $\theta_0 = 0$ and $s_1 = s_2 = 1$. Thus, for the coherent state, the Leggett-Garg inequality is clearly violated for $s_1 = 1$ and $s_2 = 1$, and we also found a very small violation for $s_1 = -1$ and $s_2 = 1$ with a nonzero value of r, which is not explicitly shown here.

We note that the quasiprobability distribution function takes the negative values smaller than -0.05 for $x_0 \lesssim 0.5$ and $p_0 \lesssim 0.5$ in the left panel of Fig. 8. We also note that the quasiprobability distribution function takes the negative values smaller than -0.05 for and $r \lesssim 0.5$ at $\omega t_2 \sim \pi/2$ in the right panel of Fig. 8. The minimum value of the quasiprobability distribution function appears when $x_0 = p_0 = r = 0$, i.e., the ground state within the squeezed coherent states $\rho_0 = D(\xi)S(\zeta)|0\rangle\langle 0|S^\dagger(\zeta)D^\dagger(\xi)$. This means that the Leggett-Garg inequality is violated when the initial state is in the ground state $\rho_0 = |0\rangle\langle 0|$. These values of the quasiprobability are smaller than those of the previous section. Thus, the violation is boosted by the choice of the projection operator (29). The clear violation for the dichotomic variable with Eq. (29) appears only for $s_1 = s_2 = 1$ when the measurements at t_1 and t_2 give $|x| > L$. This will be explained in an intuitive way that the violation of the Leggett-Garg inequality appears when the position measurement gives the opposite value against the expectation value, which comes from the broadening feature of the wave function on the basis of the superposition principle.

The left panel of Fig. 9 shows the minimum value of the quasiprobability $q_{1,1}(t_1 = 0, t_2)$ on the plane of L and t_2. Here we fixed $s_1 = s_2 = 1$ and $\theta_0 = 0$. The right panel of Fig. 9 plots the quasiprobability $q_{1,1}(t_1 = 0, t_2)$ as a function of ωt_2, which achieves the minimum value of the quasiprobability in the left panel of Fig. 9. We note that the period of the quasiprobability of Eq. (32) is π/ω. Figure 10 shows the minimum value of the quasiprobability distribution function $q_{1,1}(t_1 = 0, t_2)$ on the plane of L and r, where t_2 is the free parameter. Here we fixed $s_1 = s_2 = 1$ and $\theta_0 = 0$. Figure 10 shows that the relatively strong violation of the Leggett-Garg inequality appears for $0.8 \lesssim L \lesssim 1.2$ and $r \lesssim 0.5$. Under the condition $\theta_0 = 0$, the minimum value of the quasiprobability distribution function is -0.0538, which is 43% of the Lüders bound, $-1/8$ (see Refs. [21,36]). The minimum value appears for $r = 0$ and $L = 1.03$ at $\omega t_2 = 1.55$. Thus, the choice of the dichotomic variable using the rectangular measurement enables us to detect the violation of the Leggett-Garg inequality in the ground state and the squeezed state, which even boosts the violation of the amplitude. From Fig. 10, it can be seen that under the condition of $L \lesssim 0.6$, no violation appears for any r. This can be explained by the fact that the distribution of the expectation value of the position is in the region $-L < x < L$ at the time of measurements, and there are no surprises in the results of measurements.

B. Extended dichomotic variable with the momentum operator

In this subsection, we investigate the violation of the Leggett-Garg inequalities by introducing a similar dichotomic variable and measurement operators with the momentum

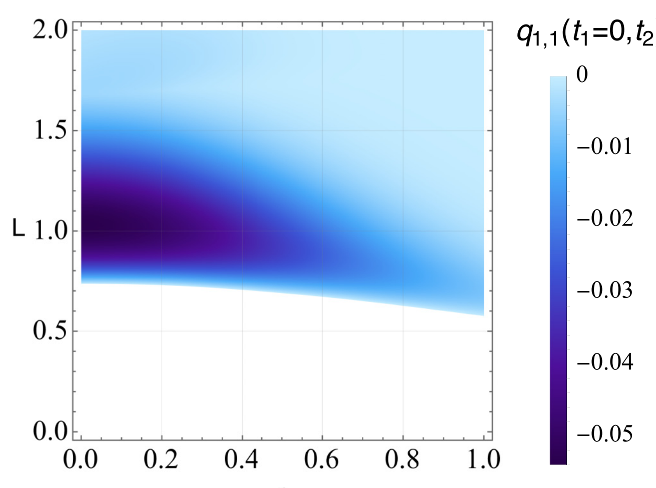

FIG. 10. Contour plot of the quasiprobability distribution function $q_{1,1}(t_1 = 0, t_2)$ on plane of r and L, which t_2 is free parameter.

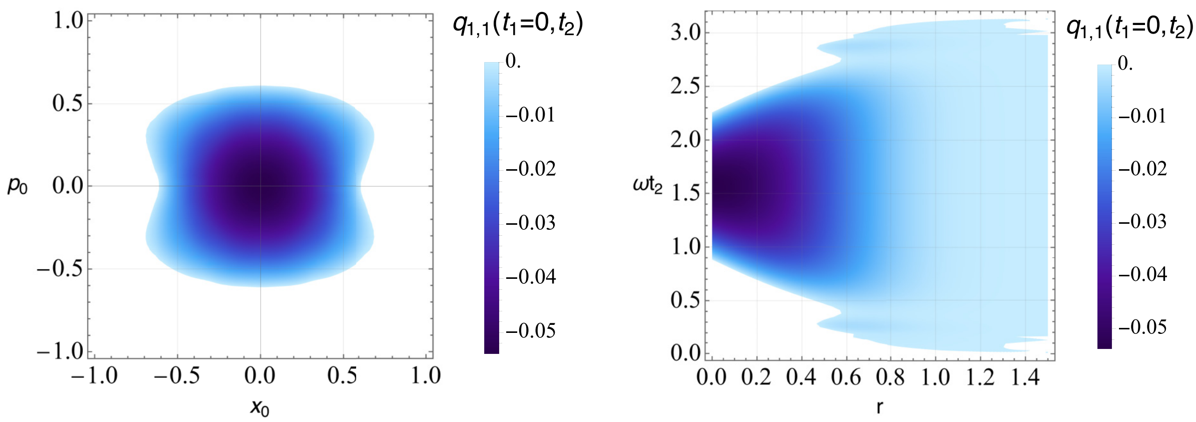

FIG. 11. Same as Fig. 8 but for the dichotomic variable and the projection operator for Eq. (33), where we fixed $L = 1.02$. In the left panel, we fixed $r = \theta_0 = 0$. In the right panel, we fixed $x_0 = p_0 = \theta_0 = 0$.

operator,

$$\hat{Q} = \text{sgn}(\hat{p} - L) + \text{sgn}(-\hat{p} - L) + 1,$$
$$P_{s_i} = \theta(s_i(\hat{p} - L)) + \theta(-s_i(\hat{p} + L)) + \tfrac{1}{2}(s_i - 1), \qquad (33)$$

with a dimensionless threshold value $L(> 0)$. The right panel of Fig. 7 shows the definition of the dichotomic variable. The expression of the quasiprobability distribution function can be evaluated in a similar way to the previous subsection, and we have

$$\begin{aligned} q_{s_1,s_2}(t_1,t_2) = \frac{1}{4}&\left\{1 + s_1\left[1 + \text{erf}\left(\frac{x_{\xi(t_1+\pi/2\omega)} - L}{\lambda\left(t_1 + \frac{\pi}{2\omega}\right)}\right) - \text{erf}\left(\frac{x_{\xi(t_1+\pi/2\omega)} + L}{\lambda\left(t_1 + \frac{\pi}{2\omega}\right)}\right)\right]\right\} \\ &\times\left\{1 + s_2\left[1 + \text{erf}\left(\frac{x_{\xi(t_2+\pi/2\omega)} - L}{\lambda\left(t_2 + \frac{\pi}{2\omega}\right)}\right) - \text{erf}\left(\frac{x_{\xi(t_2+\pi/2\omega)} + L}{\lambda\left(t_2 + \frac{\pi}{2\omega}\right)}\right)\right]\right\} + s_1 s_2 \text{Re}\left\{\sum_{n=1}^{\infty} e^{-in\omega[t_2 - t_1 + \beta(t_2+\pi/2\omega) - \beta(t_1+\pi/2\omega)]}\right. \\ &\times\left[J_{0n}\left(\frac{x_{\xi(t_1+\pi/2\omega)} - L}{\lambda\left(t_1 + \frac{\pi}{2\omega}\right)}, \infty\right)J_{n0}\left(\frac{x_{\xi(t_2+\pi/2\omega)} - L}{\lambda\left(t_2 + \frac{\pi}{2\omega}\right)}, \infty\right) - J_{0n}\left(\frac{x_{\xi(t_1+\pi/2\omega)} - L}{\lambda\left(t_1 + \frac{\pi}{2\omega}\right)}, \infty\right)J_{n0}\left(\frac{x_{\xi(t_2+\pi/2\omega)} + L}{\lambda\left(t_2 + \frac{\pi}{2\omega}\right)}, \infty\right)\right. \\ &\left.\left. - J_{0n}\left(\frac{x_{\xi(t_1+\pi/2\omega)} + L}{\lambda\left(t_1 + \frac{\pi}{2\omega}\right)}, \infty\right)J_{n0}\left(\frac{x_{\xi(t_2+\pi/2\omega)} - L}{\lambda\left(t_2 + \frac{\pi}{2\omega}\right)}, \infty\right) + J_{0n}\left(\frac{x_{\xi(t_1+\pi/2)} + L}{\lambda\left(t_1 + \frac{\pi}{2\omega}\right)}, \infty\right)J_{n0}\left(\frac{x_{\xi(t_2+\pi/2\omega)} + L}{\lambda\left(t_2 + \frac{\pi}{2\omega}\right)}, \infty\right)\right]\right\}, \end{aligned} \qquad (34)$$

where we adopted the initial state $\rho_0 = D(\xi)S(\zeta)|0\rangle\langle 0| S^\dagger(\zeta)D^\dagger(\xi)$. Equation (34) is obtained from Eq. (32) by replacing $\hat{p}(t)$ with $\hat{x}(t + \frac{\pi}{2\omega})$ following the relation (15). The left panel of Fig. 11 is given by rotating the left panel of Fig. 8 by the factor $\pi/2$, but the right panel of Fig. 11 is very different from that of Fig. 8. In the right panel of Fig. 11, the maximum violation occurs at $t_2 = \pi/2\omega$ and $r = 0$, which is the same property as that of Fig. 8. If we choose the squeezing parameter $\theta_0 = \pi$ in the right panel of Fig. 11, the contour plot on the plane of r and ωt_2 perfectly matches the right panel of Fig. 8. These behaviors can be understood by the relation (15), and the projection operators and the squeezed coherent states of these two models are related by the rotation in phase space, as discussed in Sec. III B.

V. SUMMARY AND CONCLUSION

In the present paper, we have investigated the violation of the two-time Leggett-Garg inequalities for testing the quantum nature of a harmonic oscillator in the various quantum states. The new points are the following. (i) We first obtained the explicit expression for the two-time quasiprobability distribution function for the thermal squeezed coherent state as a generalization of the work by Mawby and Halliwell [24], with which we have explicitly shown that the squeezed coherent thermal states do not increase the violation of the Leggett-Garg inequalities as predicted in the previous paper in Ref. [24]. (ii) We also extended the result to the two-time quasiprobability distribution function with the dichotomic variable and the projection operator defined by the momentum operator. (iii) We found that the violation of the Leggett-Garg inequalities can be boosted by adopting the dichotomic variable and the projection operator with a finite range of position/momentum measurements, in which the larger violation appears for the ground state or the squeezed state.[1]

[1]We note that larger violations of the Leggett-Garg inequalities close to the Lüders bound are discussed in [37].

(iv) We developed a formula to compute the quasiprobability distribution function, which is useful for quantum continuous variables by using the integral representation of the Heaviside function, which can be generalized to a formulation with a field theory, as reported in the accompanying paper [35]. The numerical results of these two formulas are used to demonstrate the consistency.

It is known that the two-time Leggett-Garg inequalities and the three-time Leggett-Garg inequalities are necessary and sufficient to prove macrorealism [20]. Therefore, it might be useful to investigate the three-time Leggett-Garg inequalities. Application of the Leggett-Garg inequalities to realistic optomechanical experiments [29,30] should be investigated in the future. To this end, we further need to extend the formulation for the system taking the impacts of noises of environments, feedback control, and quantum filtering process into account. The method to realize the projection operators assumed in the present paper is also left as a future investigation.

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APPENDIX A: TWO-TIME QUASIPROBABILITY FOR THE THERMAL SQUEEZED COHERENT STATE

Here we derive the expression for the quasiprobability distribution when the initial state is the thermal squeezed coherent state, whose density matrix is written as

$$\rho_0 = \frac{1}{1+N_{th}}\sum_{m=0}^{\infty}\left(\frac{N_{th}}{1+N_{th}}\right)^m D(\xi)S(\zeta)|m\rangle\langle m|S(\zeta)^\dagger D(\xi)^\dagger, \tag{A1}$$

where $N_{th} = [\exp(\hbar\omega/k_B T) - 1]^{-1}$, and T is the temperature. In this case, the two-time quasiprobability is given by

$$q_{s_1,s_2}(t_1,t_2) = \frac{1}{1+N_{th}}\sum_{m=0}^{\infty}\left(\frac{N_{th}}{1+N_{th}}\right)^m \mathrm{Re}\big[\langle m|S^\dagger(\zeta)D^\dagger(\xi)e^{i\hat{H}t_2}\theta(s_2\hat{x})e^{-i\hat{H}(t_2-t_1)}\theta(s_1\hat{x})e^{-i\hat{H}t_1}D(\xi)S(\zeta)|m\rangle\big]. \tag{A2}$$

Using the formula

$$e^{-i\hat{H}t}D(\xi)S(\zeta) = D(\xi(t))S(\zeta(t))e^{-i\hat{H}t} \tag{A3}$$

with defined $\xi(t) = \xi e^{-i\omega t}$ and $\zeta(t) = \zeta e^{-2i\omega t}$, the right-hand side of Eq. (A2) is written as

$$q_{s_1,s_2}(t_1,t_2) = \frac{1}{1+N_{th}}\sum_{m=0}^{\infty}\left(\frac{N_{th}}{1+N_{th}}\right)^m \mathrm{Re}\big[e^{i(t_2-t_1)/2}\langle m|S^\dagger(\zeta(t_2))D^\dagger(\xi(t_2))\theta(s_2\hat{x})e^{-i\hat{H}(t_2-t_1)}\theta(s_1\hat{x})D(\xi(t_1))S(\zeta(t_1))|m\rangle\big]. \tag{A4}$$

Further, since we can write

$$\theta(s\hat{x})D(\xi(t)) = D(\xi(t))\theta(s(\hat{x}+x_{\xi(t)})), \tag{A5}$$

$$S(\zeta(t_2))^\dagger e^{-i\hat{H}(t_2-t_1)}S(\zeta(t_1)) = S(\zeta(t_2))^\dagger S(\zeta(t_2))e^{-i\hat{H}(t_2-t_1)} = e^{-i\hat{H}(t_2-t_1)}, \tag{A6}$$

$$D(\xi(t_2))^\dagger e^{-i\hat{H}(t_2-t_1)}D(\xi(t_1)) = D(\xi(t_2))^\dagger D(\xi(t_2))e^{-i\hat{H}(t_2-t_1)} = e^{-i\hat{H}(t_2-t_1)}, \tag{A7}$$

where $x_{\xi(t)} = \sqrt{2}\mathrm{Re}[\xi(t)]$, we have

$$q_{s_1,s_2}(t_1,t_2) = \frac{1}{1+N_{th}}\sum_{m=0}^{\infty}\left(\frac{N_{th}}{1+N_{th}}\right)^m \mathrm{Re}\big[e^{i\omega(t_2-t_1)/2}\langle m|S^\dagger(\zeta(t_2))\theta(s_2(\hat{x}+x_{\xi(t_2)}))e^{-i\hat{H}(t_2-t_1)}\theta(s_1(\hat{x}+x_{\xi(t_1)}))S(\zeta(t_1))|m\rangle\big]. \tag{A8}$$

Using the properties of the unitary operator $S(\zeta)$ and the Bogoliubov transformation, we have

$$\begin{aligned}\theta(s(\hat{x}+x_{\xi(t)}))S(\zeta(t)) &= S(\zeta(t))S^\dagger(\zeta(t))\theta(s(\hat{x}+x_{\xi(t)}))S(\zeta(t))\\ &= S(\zeta(t))S^\dagger(\zeta(t))\theta\left(s\left(\frac{\hat{a}+\hat{a}^\dagger}{\sqrt{2m\omega}}+x_{\xi(t)}\right)\right)S(\zeta(t))\\ &= S(\zeta(t))\theta\left(s\left(\frac{\hat{a}\cosh r+\hat{a}^\dagger e^{i(\theta_0-2\omega t)}\sinh r+\hat{a}^\dagger\cosh r+\hat{a}e^{-i(\theta_0-2\omega t)}\sinh r}{\sqrt{2m\omega}}+x_{\xi(t)}\right)\right)\\ &= S(\zeta(t)\theta\left(s\left(A(t)\hat{x}+B(t)\frac{\hat{p}}{m\omega}+x_{\xi(t)}\right)\right),\end{aligned} \tag{A9}$$

where we defined

$$A(t) = \cosh r + \cos(\theta_0 - 2\omega t)\sinh r, \quad B(t) = \sin(\theta_0 - 2\omega t)\sinh r, \tag{A10}$$

and $\zeta = re^{i\theta_0}$. Next, consider a polar coordinate transformation of the linear combination of the position and momentum operators in phase space. We defined $\lambda(t) = \sqrt{A(t)^2 + B(t)^2} = \sqrt{\sinh(2r)\cos(\theta_0 - 2\omega t) + \cosh(2r)}$, $A(t)$ and $B(t)$ can be rewritten as

$$A(t) = \lambda(t)\cos\beta(t), \quad B(t) = \lambda(t)\sin\beta(t), \tag{A11}$$

$$\beta(t) = \arctan\left(\frac{B(t)}{A(t)}\right) = \arctan\left[\frac{\sin(\theta_0 - 2\omega t)\sinh r}{\cosh r + \cos(\theta_0 - 2\omega t)\sinh r}\right]. \tag{A12}$$

Then, using

$$A(t)\hat{x} + B(t)\frac{\hat{p}}{m\omega} = \lambda(t)\left[\hat{x}\cos\beta(t) + \frac{\hat{p}}{m\omega}\sin\beta(t)\right] = \lambda(t)\frac{\hat{a}e^{-i\beta(t)} + \hat{a}^\dagger e^{i\beta(t)}}{\sqrt{2m\omega}} = \lambda(t)e^{i\hat{H}\beta(t)/\omega}\hat{x}e^{-i\hat{H}\beta(t)/\omega} = \lambda(t)\hat{x}(\beta(t)), \tag{A13}$$

and $\theta(s(\lambda(t)e^{i\hat{H}\beta(t)/\omega}\hat{x}e^{-i\hat{H}\beta(t)/\omega} + x_{\xi(t)})) = e^{i\hat{H}\beta(t)/\omega}\theta(s(\lambda(t)\hat{x} + x_{\xi(t)}))e^{-i\hat{H}\beta(t)/\omega}$, we have the quasiprobability

$$\begin{aligned}
q_{s_1,s_2}(t_1,t_2) &= \frac{1}{1+N_{th}}\sum_{m=0}^{\infty}\left(\frac{N_{th}}{1+N_{th}}\right)^m \mathrm{Re}\Big[e^{i\omega(t_2-t_1)/2}\langle m|\theta(s_2(\lambda(t_2)e^{i\hat{H}\frac{\beta(t_2)}{\omega}}\hat{x} \\
&\quad\times e^{-i\hat{H}\frac{\beta(t_2)}{\omega}} + x_{\xi(t_2)}))e^{-i\hat{H}(t_2-t_1)}\theta(s_1(\lambda(t_1)e^{i\hat{H}\frac{\beta(t_1)}{\omega}}\hat{x}e^{-i\hat{H}\frac{\beta(t_1)}{\omega}} + x_{\xi(t_1)}))|m\rangle\Big] \\
&= \frac{1}{1+N_{th}}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(\frac{N_{th}}{1+N_{th}}\right)^m \mathrm{Re}\left[e^{-in\omega(t_2-t_1)}e^{-in(\beta(t_2)-\beta(t_1))}\langle m|\theta\left(s_2\left(\hat{x} + \frac{x_{\xi(t_2)}}{\lambda(t_2)}\right)\right)|n\rangle\langle n|\theta\left(s_1\left(\hat{x} + \frac{x_{\xi(t_1)}}{\lambda(t_1)}\right)\right)|m\rangle\right].
\end{aligned} \tag{A14}$$

By separating the term $m = 0$ from the other terms of $m \neq 0$ in the sum of m, we have

$$\begin{aligned}
q_{s_1,s_2}(t_1,t_2) = \frac{1}{1+N_{th}}\mathrm{Re}\Bigg[&\langle 0|\theta\left(s_2\left(\hat{x} + \frac{x_{\xi(t_2)}}{\lambda(t_2)}\right)\right)|0\rangle\langle 0|\theta\left(s_1\left(\hat{x} + \frac{x_{\xi(t_1)}}{\lambda(t_1)}\right)\right)|0\rangle \\
&+ \sum_{m=1}^{\infty}\left(\frac{N_{th}}{1+N_{th}}\right)^m e^{im(\omega(t_2-t_1)+\beta(t_2)-\beta(t_1))}\langle m|\theta\left(s_2\left(\hat{x} + \frac{x_{\xi(t_2)}}{\lambda(t_2)}\right)\right)|0\rangle\langle 0|\theta\left(s_1\left(\hat{x} + \frac{x_{\xi(t_1)}}{\lambda(t_1)}\right)\right)|m\rangle \\
&+ \sum_{n=1}^{\infty}e^{-in(\omega(t_2-t_1)+\beta(t_2)-\beta(t_1))}\langle 0|\theta\left(s_2\left(\hat{x} + \frac{x_{\xi(t_2)}}{\lambda(t_2)}\right)\right)|n\rangle\langle n|\theta\left(s_1\left(\hat{x} + \frac{x_{\xi(t_1)}}{\lambda(t_1)}\right)\right)|0\rangle \\
&+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{N_{th}}{1+N_{th}}\right)^m e^{i(m-n)(\omega(t_2-t_1)+\beta(t_2)-\beta(t_1))}\langle m|\theta\left(s_2\left(\hat{x} + \frac{x_{\xi(t_2)}}{\lambda(t_2)}\right)\right)|n\rangle\langle n|\theta\left(s_1\left(\hat{x} + \frac{x_{\xi(t_1)}}{\lambda(t_1)}\right)\right)|m\rangle\Bigg].
\end{aligned} \tag{A15}$$

Calculating the final term of Eq. (A15), we have to separate the term $m = n$ from the other terms of $m \neq n$, we have

$$\begin{aligned}
q_{s_1,s_2}(t_1,t_2) = \frac{1}{1+N_{th}}\mathrm{Re}\Bigg\{&\frac{1}{4}\left(1 + s_1\mathrm{erf}\left(\frac{x_{\xi(t_1)} - w}{\lambda(t_1)}\right) + s_2\mathrm{erf}\left(\frac{x_{\xi(t_2)} - w}{\lambda(t_2)}\right) + s_1 s_2\mathrm{erf}\left(\frac{x_{\xi(t_1)} - w}{\lambda(t_1)}\right)\mathrm{erf}\left(\frac{x_{\xi(t_2)} - w}{\lambda(t_2)}\right)\right) \\
&+ s_1 s_2\Bigg\{\sum_{m=1}^{\infty}\left(\frac{N_{th}}{1+N_{th}}\right)^m e^{im\omega(t_2-t_1)+im(\beta(t_2)-\beta(t_1))}J_{m0}\left(-\frac{x_{\xi(t_2)} - w}{\lambda(t_2)}, \infty\right)J_{0m}\left(-\frac{x_{\xi(t_1)} - w}{\lambda(t_1)}, \infty\right) \\
&+ \sum_{n=1}^{\infty}e^{-in\omega(t_2-t_1)-in(\beta(t_2)-\beta(t_1))}J_{0n}\left(-\frac{x_{\xi(t_2)} - w}{\lambda(t_2)}, \infty\right)J_{n0}\left(-\frac{x_{\xi(t_1)} - w}{\lambda(t_1)}, \infty\right) \\
&+ \sum_{m=1}^{\infty}\sum_{n=1, m\neq n}^{\infty}\left(\frac{N_{th}}{1+N_{th}}\right)^m J_{mn}\left(-\frac{x_{\xi(t_2)} - w}{\lambda(t_2)}, \infty\right)J_{nm}\left(-\frac{x_{\xi(t_1)} - w}{\lambda(t_1)}, \infty\right)\Bigg\}\Bigg\} + ee(s_1, s_2, t_1, t_2),
\end{aligned} \tag{A16}$$

where the last term $ee(s_1, s_2, t_1, t_2)$ is defined depending on the values of s_1 and s_2, as follows:

$$ee(s_1=+1, s_2=+1, t_1, t_2) = \int_{-(x_{\xi(t_2)}-\omega)/\lambda(t_2)}^{\infty} dx \sum_{n=1}^{\infty} \left(\frac{N_{th}}{1+N_{th}}\right)^n \psi_n^\dagger(x)\psi_n(x) \int_{-(x_{\xi(t_1)}-\omega)/\lambda(t_1)}^{\infty} dy \psi_n^\dagger(y)\psi_n(y), \quad \text{(A17)}$$

$$ee(s_1=+1, s_2=-1, t_1, t_2) = \int_{-\infty}^{-(x_{\xi(t_2)}-\omega)/\lambda(t_2)} dx \sum_{n=1}^{\infty} \left(\frac{N_{th}}{1+N_{th}}\right)^n \psi_n^\dagger(x)\psi_n(x) \int_{-(x_{\xi(t_1)}-\omega)/\lambda(t_1)}^{\infty} dy \psi_n^\dagger(y)\psi_n(y), \quad \text{(A18)}$$

$$ee(s_1=-1, s_2=+1, t_1, t_2) = \int_{-(x_{\xi(t_2)}-\omega)/\lambda(t_2)}^{\infty} dx \sum_{n=1}^{\infty} \left(\frac{N_{th}}{1+N_{th}}\right)^n \psi_n^\dagger(x)\psi_n(x) \int_{-\infty}^{-x_{\xi(t_1)}/\lambda(t_1)} dy \psi_n^\dagger(y)\psi_n(y), \quad \text{(A19)}$$

$$ee(s_1=-1, s_2=-1, t_1, t_2) = \int_{-\infty}^{-(x_{\xi(t_2)}-\omega)/\lambda(t_2)} dx \sum_{n=1}^{\infty} \left(\frac{N_{th}}{1+N_{th}}\right)^n \psi_n^\dagger(x)\psi_n(x) \int_{-\infty}^{-(x_{\xi(t_1)}-\omega)/\lambda(t_1)} dy \psi_n^\dagger(y)\psi_n(y). \quad \text{(A20)}$$

Here we defined $x_{\xi(t)} = \sqrt{2}\mathrm{Re}[\xi(t)] = \sqrt{2}\mathrm{Re}[\xi e^{-i\omega t}]$, $\lambda(t) = \sqrt{\sinh(2r)\cos(\theta_0 - 2\omega t) + \cosh(2r)}$, $\beta(t)$ is defined by Eq. (A12), and $J_{mn}(x_1, x_2)$ is defined as the matrix element in Ref. [23] as

$$J_{mn}(x_1, \infty) = \int_{x_1}^{\infty} dx \langle m|x\rangle \langle x|n\rangle = \begin{cases} \frac{1}{2(\epsilon_n-\epsilon_m)}[-\psi_m'(x)\psi_n(x) + \psi_n'(x)\psi_m(x)] & (m \neq n) \\ \frac{1}{2}[1-\mathrm{erf}(x)] & (m = n = 0) \end{cases}, \quad \text{(A21)}$$

where $|m\rangle$ and $|n\rangle$ are the energy eigenstates of the harmonic oscillator with the non-negative integers m and n, and (z) is the error function. Now we consider the case of $x_1 = x$ and $x_2 = \infty$, and $\psi_j(x)$ and ϵ_j with $j = 0, 1, 2, \dots$ are the energy eigenfunction $\psi_j(x) = \langle x|j\rangle$ and the corresponding energy eigenvalue, respectively, and the prime denotes the differentiation w.r.t the argument, i.e., $\psi'(x) = d\psi(x)/dx$.

The temperature dependence of the quasiprobability distribution function was shown in Ref. [24], which demonstrated that the violation becomes weak as the temperature increases. At $k_BT/\hbar\omega \sim 1$, the minimum value of the quasiprobability distribution function reaches zero. We find the same behavior for the thermal squeezed coherent state. This is also explained by the fact that the quasiprobability distribution function of the squeezed coherent state can be obtained by replacing a parameter of the coherent state Ref. [24] and that it is true for the case including the thermality.

The quasiprobability distribution function of the coherent squeezing state can be expressed with that for the coherent state with replacing the parameters with those at the different measurement time, which was first pointed out in [24]. It is useful to show the relationship by finding the explicit expression of $\beta(t)$. Finally, we clarify the relationship between the parameters of the squeezed coherent state and the coherent state mentioned in Ref. [24]. Here we show an explicit expression of that relationship. The quasiprobability distribution function for a coherent state is written as [24]

$$q_{s_1,s_2}(t_1,t_2) = \frac{1}{4}\left[1 + s_1\mathrm{erf}\left(x_{\xi(t_1)}\right) + s_2\mathrm{erf}\left(x_{\xi(t_2)}\right) + s_1s_2\mathrm{erf}\left(x_{\xi(t_1)}\right)\mathrm{erf}\left(x_{\xi(t_2)}\right)\right] + s_1s_2\mathrm{Re}\left[\sum_{n=1}^{\infty} e^{-in\omega(t_2-t_1)}J_{0n}\left(x_{\xi(t_1)},\infty\right)J_{n0}\left(x_{\xi(t_2)},\infty\right)\right]. \quad \text{(A22)}$$

Comparing Eq. (A16) for the squeezed coherent state and Eq. (A22) for the coherent state, the following relation between the two expressions holds. Namely, Eq. (A16) can be written using the quasiprobability for the coherent state (A22) with replacing ωt_i by $\omega t_i + \beta(t_i)$, where $i = 1,\ 2$ and $\beta(t)$ is defined by Eq. (A12). This can be read from

$$x_{\xi(t_i+\beta(t_i)/\omega)} = \mathrm{Re}\left[\sqrt{2}\xi e^{-i\omega t_i - i\beta(t_i)}\right] = \mathrm{Re}[(x_0 + ip_0)(\cos(\omega t_i + \beta(t_i)) - i\sin(\omega t_i + \beta(t_i)))] = \frac{x_0'\cos\omega t_i + p_0'\sin\omega t_i}{\lambda(t_i)}, \quad \text{(A23)}$$

where we defined $\lambda(t_i) = \sqrt{\sinh(2r)\cos(2\omega t_i - \theta_0) + \cosh(2r)}$, $x_0' = x_0(\cosh r + \sinh r\cos\theta_0) + p_0\sinh r\sin\theta_0$, $p_0' = x_0\sinh r\sin\theta_0 + p_0(\cosh r - \sinh r\cos\theta_0)$. Further, by defining $\xi' = (x_0' + ip_0')/\sqrt{2}$ we have

$$x_{\xi(t_i+\beta(t_i)/\omega)} = \mathrm{Re}\left[\sqrt{2}\frac{\xi' e^{-i\omega t_i}}{\lambda(t_i)}\right] = x_{\xi'(t_i)/\lambda(t_i)}. \quad \text{(A24)}$$

Thus, the quasiprobability distribution function for the squeezed coherent state is given by the quasiprobability distribution function for the coherent state with the replacement $t_i \to t_i + \beta(t_i)/\omega$ and $\xi \to \xi'$. This explains that the maximum violation for the coherent state and the squeezed coherent state becomes equivalent when t_2, x_0, and p_0 are taken as free movable parameters. Such a relation was shown by using the identity (B5) in Ref. [24]. This same relation also holds for the thermal squeezed coherent state and the thermal coherent state.

APPENDIX B: QUASIPROBABILITY DISTRIBUTION FUNCTION USING A GAUSSIAN INTEGRAL FORMULA

Using Gaussian integration, we start the expression of quasiprobability distribution function,

$$q_{s_1,s_2}(t_1,t_2) = \mathrm{ReTr}\big[P_{s_2}(t_2)P_{s_1}(t_1)\rho\big] \tag{B1}$$

$$= \mathrm{Re\,Tr}\bigg[\iint_0^\infty dc_1dc_2 \iint_{-\infty}^\infty \frac{dp_1dp_2}{(2\pi)^2} e^{-ip_2s_2(\hat{a}e^{-i\omega t_2}+\hat{a}^\dagger e^{i\omega t_2})+ip_2c_2} e^{-ip_1s_1(\hat{a}e^{-i\omega t_1}+\hat{a}^\dagger e^{i\omega t_1})+ip_1c_1}\rho_0\bigg]. \tag{B2}$$

The initial state and projection operator are defined as follows:

$$\rho_0 = D(\xi)S(\zeta)\,|0\rangle\,\langle 0|\,S^\dagger(\zeta)D^\dagger(\xi), \tag{B3}$$

$$P_s(t) = e^{i\hat{H}t}\theta(s\hat{x})e^{-i\hat{H}t} = \int_0^\infty dc\int_{-\infty}^\infty \frac{dp}{2\pi}\exp\left[ip\left(-s\frac{\hat{a}e^{-i\omega t}+\hat{a}^\dagger e^{i\omega t}}{\sqrt{2m\omega}}+c\right)\right], \tag{B4}$$

respectively, where we redefined the integral variables as $p_i/\sqrt{2m\omega}\to p_i$ and $c_i\sqrt{2m\omega}\to c_i$ with $i=1,2$. By using the formula

$$D(\xi)S(\zeta) = S(\zeta)D(\gamma), \tag{B5}$$

where $\gamma = \xi\cosh|\zeta| - \xi^* e^{i\theta_0}\sinh|\zeta|$ with $\zeta = |\zeta|e^{i\theta_0}$, the quasiprobability distribution function is written as

$$q_{s_1,s_2}(t_1,t_2) = \mathrm{Re}[\langle 0|D^\dagger(\gamma)S^\dagger(\zeta)e^{i\hat{H}t_2}\theta(s_2\hat{x})e^{-i\hat{H}t_2}e^{i\hat{H}t_1}\theta(s_1\hat{x})e^{-i\hat{H}t_1}S(\zeta)D(\gamma)|0\rangle]$$

$$= \int_0^\infty dc_1\int_0^\infty dc_2\int_{-\infty}^\infty\frac{dp_1}{2\pi}\int_{-\infty}^\infty\frac{dp_2}{2\pi}e^{i(p_1c_1+p_2c_2)}\langle 0|D^\dagger(\gamma)e^{-ip_2s_2(E(t_2)\hat{a}+E^*(t_2)\hat{a}^\dagger)}e^{-ip_1s_1(E(t_1)\hat{a}+E^*(t_1)\hat{a}^\dagger)}D(\gamma)|0\rangle \tag{B6}$$

with defined $E(t) = e^{-i\omega t}\cosh r + e^{i\omega t}e^{-i\theta_0}\sinh r$.

Repeatedly using the BCH formula, $e^{A+B} = e^Ae^Be^{-[A,B]/2}$ and $e^Ae^B = e^{[A,B]}e^Be^A$, which hold for the operators A and B satisfying $[A,B] = \mathrm{const}$, we have

$$q_{s_1,s_2}(t_1,t_2) = \mathrm{Re}\int_0^\infty dc_1\int_0^\infty dc_2\int_{-\infty}^\infty\frac{dp_1}{2\pi}\int_{-\infty}^\infty\frac{dp_2}{2\pi}e^{ip_1c_1}e^{ip_2c_2}e^{-ip_1s_1(E(t_1)\gamma+E^*(t_1)\gamma^*)-ip_2s_2(E(t_2)\gamma+E^*(t_2)\gamma^*)}$$

$$\times\exp\left[-\frac{1}{2}\big[|E(t_2)|^2p_2^2+|E(t_1)|^2p_1^2+2p_1p_2s_1s_2E(t_2)E^*(t_1)\big]\right]. \tag{B7}$$

The integration in Eq. (B7) with respect to p_1 and p_2 can be performed as

$$\int_{-\infty}^\infty dp_1\int_{-\infty}^\infty dp_2\exp\left[-\frac{1}{2}\mathbf{p}^{\mathbf{T}}\boldsymbol{A}\boldsymbol{p}+\boldsymbol{\rho}^{\mathbf{T}}\cdot\boldsymbol{p}\right] = \frac{2\pi}{\sqrt{\det\boldsymbol{A}}}\exp\left[\frac{1}{2}\boldsymbol{\rho}^T\boldsymbol{A}^{-1}\boldsymbol{\rho}\right], \tag{B8}$$

where $\boldsymbol{A}$ and ρ are read

$$\boldsymbol{A} = \begin{pmatrix}|E(t_1)|^2 & s_1s_2E(t_2)E^*(t_1)\\ s_1s_2E(t_2)E^*(t_1) & |E(t_2)|^2\end{pmatrix}, \tag{B9}$$

$$\boldsymbol{\rho}^T = (ic_1 - is_1[E(t_1)\gamma+E^*(t_1)\gamma^*],\, ic_2 - is_2[E(t_2)\gamma+E^*(t_2)\gamma^*]). \tag{B10}$$

Further, the integration with respect to c_1 and c_2 can be written by setting $c_1 = c\cos u$ and $c_2 = c\sin u$, and we have

$$q_{s_1,s_2}(t_1,t_2) = \mathrm{Re}\frac{1}{2\pi}\int_0^{\pi/2}du\int_0^\infty dcc\frac{c}{\sqrt{B}}e^{-\frac{C}{2}}, \tag{B11}$$

where we defined

$$B = \det\boldsymbol{A} = |E(t_1)|^2|E(t_2)|^2 - [E^*(t_1)E(t_2)]^2, \tag{B12}$$

$$C = \frac{1}{B}(|E(t_2)|^2[c\cos u+s_1\mathcal{E}(t_1)]^2+|E(t_1)|^2[c\sin u+s_2\mathcal{E}(t_2)]^2-2s_1s_2E^*(t_1)E(t_2)[c\cos u-s_1\mathcal{E}(t_1)][c\sin u-s_2\mathcal{E}(t_2)]), \tag{B13}$$

$$\mathcal{E}(t) = E(t)\gamma+E^*(t)\gamma^*. \tag{B14}$$

By introducing the quantities

$$\sigma = \frac{1}{B}(|E(t_2)|^2\cos^2u+|E(t_1)|^2\sin^2u-2s_1s_2E(t_2)E^*(t_1)\sin u\cos u), \tag{B15}$$

$$\beta = \frac{1}{B}(-|E(t_2)|^2 s_1 \mathcal{E}(t_1)\cos u - |E(t_1)|^2 s_2 \mathcal{E}(t_2)\sin u + E(t_2)E^*(t_1)(s_2\mathcal{E}(t_1)\sin u + s_1\mathcal{E}(t_2)\cos u)), \tag{B16}$$

$$\delta = \frac{1}{B}(|E(t_2)|^2\mathcal{E}(t_1)^2 + |E(t_1)|^2\mathcal{E}(t_2)^2 - 2E(t_2)E^*(t_1)\mathcal{E}(t_1)\mathcal{E}(t_2),), \tag{B17}$$

C can be written as $C = \sigma c^2 + 2\beta c + \delta$, and we finally have

$$q_{s_1,s_2}(t_1,t_2) = \mathrm{Re}\big[\langle 0|D^\dagger(\gamma)S^\dagger(\zeta)e^{iHt_2}\theta(s_2\hat{x})e^{-iHt_2}e^{iHt_1}\theta(s_1\hat{x})e^{-iHt_1}S(\zeta)D(\gamma)|0\rangle\big]$$
$$= \mathrm{Re}\left\{\frac{1}{2\pi}\frac{e^{-\delta/2}}{\sqrt{B}}\int_0^{\pi/2}du\left[\frac{1}{\sigma} - \frac{\sqrt{2\pi}be^{\beta^2/2\sigma}}{2\sigma^{3/2}}\mathrm{erfc}\left(\frac{\beta}{\sqrt{2\sigma}}\right)\right]\right\}, \tag{B18}$$

where we used the mathematical formula

$$\int_0^\infty dc\, ce^{-(\sigma c^2+2\beta\sigma+\delta)/2} = e^{-\delta/2}\left[\frac{1}{\sigma} - \frac{\sqrt{\pi/2}\beta e^{\beta^2/2\sigma}}{2\sigma^{3/2}}\mathrm{erfc}\left(\frac{\beta}{\sqrt{2\sigma}}\right)\right], \tag{B19}$$

where $\mathrm{erfc}(z)$ is the complementary error function. As we will show in the next section, the integration can be evaluated numerically using the software *Mathematica*.

At the end of this section, we remark on a useful generalization of the above formula. We may adopt the following projection operator by constructing the dichotomic variable as $Q = \mathrm{sgn}(\hat{x} - \bar{x}(t))$ instead of $Q = \mathrm{sgn}(\hat{x})$:

$$P_s = \frac{1}{2}(1 + s \times \mathrm{sgn}(\hat{x} - \bar{x}(t))) = \theta(s(\hat{x} - \bar{x}(t))) = \int_0^\infty dc\int_{-\infty}^\infty \frac{dp}{2\pi}e^{ip[-s(\hat{x}-\bar{x}(t))+c]}, \tag{B20}$$

where $\bar{x}(t)$ is an arbitrary function of time. In this case, we have

$$\langle 0|D^\dagger(\gamma)S^\dagger(\zeta)e^{iHt_2}\theta(s_2(x-\bar{x}(t_2)))e^{-iHt_2}e^{iHt_1}\theta(s_1(\hat{x}-\bar{x}(t_1)))e^{-iHt_1}S(\zeta)D(\gamma)|0\rangle$$
$$= \int_0^\infty dc_1\int_0^\infty dc_2\int_{-\infty}^\infty\frac{dp_1}{2\pi}\int_{-\infty}^\infty\frac{dp_2}{2\pi}e^{ip_1c_1}e^{ip_2c_2}e^{-ip_1s_1(E(t_1)\gamma+E^*(t_1)\gamma^*-\bar{x}(t_1))-ip_2s_2(E(t_2)\gamma+E^*(t_2)\gamma^*-\bar{x}(t_2))}$$
$$\times\exp\left[-\frac{1}{2}\big[|E(t_2)|^2p_2^2 + |E(t_1)|^2p_1^2 + 2p_1p_2s_1s_2E(t_2)E^*(t_1)\big]\right]. \tag{B21}$$

Comparing this expression (B21) and Eq. (B7), we find that (B21) is reproduced from (B7) with replacing $E(t)\gamma + E^*(t)\gamma^*$ with $E(t)\gamma + E^*(t)\gamma^* - \bar{x}(t)$. This leads to an important implication. For example, when the state of the harmonic oscillator is in the coherent state, i.e., $\zeta = 0$, we have $E(t)\gamma + E^*(t)\gamma^* = 2|\xi|\cos(\omega t - \Theta)$, where we assumed $\xi = |\xi|e^{i\Theta}$. Therefore, the same quasiprobability distribution function is predicted between the case taking the coherent state with $\xi = |\xi|e^{i\Theta}$ and $\bar{x}(t) = 0$ and the case taking the ground state but with $\bar{x}(t) = -2|\xi|\cos(\omega t - \Theta)$. This means that it is possible to observe the violation of the Leggett-Garg inequalities in the system of a harmonic oscillator in the ground state by choosing $\bar{x}(t)$ properly. This also makes it possible to observe a similar violation of the inequalities in a harmonic oscillator in a squeezed state, although there appears no violation for the harmonic oscillator in the ground state or the squeezed state when we adopt $\hat{Q} = \mathrm{sgn}(\hat{x})$, i.e., $\bar{x}(t) = 0$ in the above formulas.

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