Traveling time in a spacetime-symmetric extension of nonrelativistic quantum mechanics

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Time continues to be an intriguing physical property in the modern era. On the one hand, we have the classical and relativistic notion of time, where space and time have the same hierarchy, essential in describing events in spacetime. On the other hand, in quantum mechanics, time appears as a classical parameter, meaning that it does not have an uncertain relation with its canonical conjugate. In this work we use a recent spacetime-symmetric proposal [Phys. Rev. A **95**, 032133 (2017)] that tries to solve the unbalance in nonrelativistic quantum mechanics by extending the usual Hilbert space: the time parameter t and the position operator $\hat{X}$ in one subspace, and the position parameter x and time operator $\mathbb{T}$ in the other subspace. Time as an operator is better suited for describing tunneling processes. We then solve the 1/2-fractional integrodifferential equation for a particle subjected to strong and weak potential limits and obtain an analytical expression for the tunneling time through a rectangular barrier. Using a Gaussian energy distribution, we demonstrate that for wave packets well resolved in time, the expectation value of the operator $\mathbb{T}$ is the energy average of the classical time $T_{\text{class}} = \partial S/\partial E$, where S is the classical action, which can be real or imaginary. The imaginary classical time does not contribute to the traveling time. Furthermore, we apply our results to a Gaussian energy distribution and compare them to previous works. This work is a correction of a previous paper [Phys. Rev. A **107**, 052220 (2023)].

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I. INTRODUCTION

Time in quantum mechanics (QM) has always been a point of discussion [1–13], mostly in connection of the time of arrival in QM [13–27], including for an unification of QM and general relativity [28–37]. Contrary to what happens in relativity, where spacetime is a single entity [38], position and time are different kinds of numbers: while the position is a q-number, time is a c-number [39–41]. This hierarchy incompatibility between said quantities has led physicists to search for ways to include a time operator in QM. Though Pauli argued [42] that a bounded-from-below Hamiltonian is incompatible with a time operator canonically conjugate to it (both the Hamiltonian and the time operator must possess completely continuous spectra spanning the entire real line), there are ways to overcome it (see, for example, [1,16,20,21]). One of the most famous related works is the Page and Wootters (P&W) mechanism, together with its recent interpretations [43–46], in which the universe is in a stationary state, consistent with a Wheeler-DeWitt equation [19,47]. The apparent dynamical evolution that systems undergo is relative to the degrees of freedom of the rest of the universe, which acts like a clock.

Not considering time as an operator, interpretations of the relation between time and energy were also made by Mandelstam and Tamm [48] and Margolus and Levitin [49]. Any Δt appearing in those works must be interpreted as a time interval, not an operator uncertainty. In both cases, Δt is considered the smallest time interval for a system to evolve into an orthogonal state. However, in the former, the system has an energy *spread* ΔE, which bonds the interval by $\Delta t \Delta E \sim \hbar$. In contrast, in the latter, the system has an *average* energy $\langle E \rangle$, bounding the interval from below by $\pi\hbar/2\langle E\rangle$.

Using the idea of quantum events [50,51], it is possible to give meaning to the usual time-energy "uncertainty" relations and relate the uncertainty of a quantum measurement of time to its energy uncertainty. By requiring consistency with how time enters the fundamental laws of physics, one can also draw a picture showing that there is only one time: both classical and quantum times are manifestations of entanglement [52].

Höhn *et al.* show that there is an equivalence between the relational quantum dynamics, (a) the relational observables in the clock-neutral picture of Dirac quantization, (b) the PaW mechanism, and (c) the relational Heisenberg picture obtained via symmetry reduction using quantum reduction maps [53,54].

The spacetime-symmetric proposal [55,56] that we present and use in this paper uses similar ideas to the P&W mechanism. The system has a new Hilbert space with an *operator* time, implying an extended state for the system that depends on variables in the usual position Hilbert space and variables in the new temporal Hilbert space. One key difference to the P&W formalism is that this new Hilbert space is as intrinsic to the system as the position Hilbert space, extending regular QM, and no auxiliary systems are required. This provides a clear interpretation of the time-energy uncertainty relation and different types of experiments, where predictions of the positions of particles, the time of arrival, or both can be obtained. It is a natural subject then to examine the tunneling times in the spacetime-symmetric proposal. This proposal has shown promising results compared to the Büttiker-Landauer and

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phase-time [13,57–59] approaches to time-of-arrival problems and is the main reason we use this formulation. Worth calling attention to is that 1/2-fractional time derivatives and integrations appear in the equations to be solved. It has been demonstrated recently that the origin of the time-extended Hilbert space of the STS can be explained as a reinterpretation of the role of space and time in classical mechanics [60].

Our goal in this paper is to study weak and strong potential barriers, providing connection formulas for the wave functions in the *extended Hilbert space*, and to examine results for the tunneling times by comparing them with tunneling times obtained using the usual QM formalism. The tunneling time through a barrier is an old problem which goes back to MacColl [61]. The exact definition, applicability, and measure of tunneling times change according to the circumstances and interest: dwell times, arrival times, asymptotic phase times, delay times, and jump times, among others. It is impossible to furnish a fair review of all these works here, but we refer readers to Refs. [13,62,63]. Some particular cases will be mentioned later for comparison. It is worth noting that some works (e.g., Ref. [64]) argue that, when one considers a wave packet that includes more than one momentum component, it is not possible to talk about tunneling time, except when considering square barriers, and that the concept of tunneling time comes from a classical interpretation of a quantum phenomenon. The spacetime-symmetric proposal does not have this distinction since it asks the question, "What is the average time a particle takes to travel from one point to the other?" and this includes both tunneling and traveling above the barrier. Using a Gaussian distribution in energy, we demonstrate analytically that the time expectation (tunneling or not) of a wave packet well localized in time equals the energy average of the classical time $T_{\text{class}} = \partial S/\partial E$, where S is the action, which can be real or imaginary. In the time-delocalized wave packets, the main contribution to the time expectation comes from the energy distribution properties. Application is shown for the rectangular barrier with a constant energy distribution, which describes a wave packet relatively well localized in time.

This paper is organized as follows. In Sec. II we summarize the spacetime-symmetric proposal, and in Sec. III we obtain and solve the approximated 1/2−fractional equations for the weak and strong potential limits. In Sec. IV we apply results obtained in Sec. III for different regimes: (i) minimum energy and maximum energy of the energy distribution below the barrier height, (ii) minimum energy below the barrier height and maximum energy above the barrier height, and (iii) minimum and maximum energy above the barrier height, as well as a discussion about the role of uncertainty in time in our application, where we used a real Gaussian energy distribution. We show that, when the wave packet is well localized in time, the expected value of the time operator is an average (in the energies) of the classical time, while for a packet well localized in the energy, it is the time a particle with energy E_0 (the center of the Gaussian) takes to traverse a barrier of height V_0. We also show that, in case (i), the tunneling time is 0, while being nonzero for the other cases. Final comments are presented in Sec. V.

A previous version of this article [65], now retracted, contained errors described in the retraction notice [66]. These have been corrected in the current article. Specifically, the integration from Eq. (33) [Eq. (33) in the present manuscript] to Eq. (35) [Eq. (38) in the present manuscript] involving the Dirac delta should have surface terms appearing from the finite integration limits in the energy. This made the expected value of the time operator have an imaginary component, and this was fixed in the current paper, as shown in Eqs. (38) and (44), which presents real values.

II. THE SPACETIME-SYMMETRIC PROPOSAL

We begin our discussion by revisiting the proposal used in this paper. The spacetime-symmetric extension of QM proposed by Dias and Parisio (hereafter the STS proposal) [55,56] uses a similar idea to the P&W mechanism [43,44], in which the entire Hilbert space is divided into one subspace that refers to the *system of interest*, and another one, that refers to the *clock*. The main difference between the two is that the complete Hilbert space in the STS proposal

$$\mathcal{H} = \mathcal{H}_{\text{pos}} \otimes \mathcal{H}_{\text{time}} \tag{1}$$

(here $\mathcal{H}_{\text{pos}}$ is the usual Hilbert space of QM and $\mathcal{H}_{\text{time}}$ is a temporal *extension* of the regular theory) refers entirely to the system: $\mathcal{H}_{\text{time}}$ is as intrinsic to the system as $\mathcal{H}_{\text{pos}}$ in this approach.

In this new space we define the time operator $\mathbb{T}$ with eigenkets $|t\rangle$ as

$$\mathbb{T}|t\rangle = t|t\rangle, \tag{2}$$

where t is the eigenvalue associated with $|t\rangle$. The set of eigenkets $\{|t\rangle\}$ resolve an identity $\mathbb{I} = \int_{-\infty}^{\infty} dt\, |t\rangle\langle t|$. We then define the energy operator $\mathbb{H}$ through the commutation relation [67]

$$[\mathbb{T}, \mathbb{H}] = -i\hbar, \tag{3}$$

which gives us naturally the time-energy uncertainty relation $\Delta\mathbb{T}\Delta\mathbb{H} \geqslant \hbar/2$. We want to emphasize that, since the STS proposal considers an extension of the Hilbert space of the system of interest, this uncertainty relation relates the energy *of the system* and a time operator that acts *on the system*. This does not happen when you consider, for example, the P&W mechanism, where an auxiliary system takes the role of a clock [43]. The price paid for this in the STS proposal is that we do not have the commutation relation $[x, \mathbb{P}] \propto i\hbar$ since in the new Hilbert space x is a classical parameter.

The complete state of the system is given by

$$||\Psi\rangle = \iint dx\, dt\, \Psi(x\,\&\,t)|x\rangle \otimes |t\rangle. \tag{4}$$

The double ket notation indicates that this state belongs to both Hilbert spaces. The way we write the argument of $\Psi(x\,\&\,t) \equiv (\langle x| \otimes \langle t|)||\Psi\rangle$ is such as to remind us that, in this proposal, position and time will be on equal footing, but with some caveats that will be made clear later.

The square modulus of $\Psi(x\,\&\,t)$ is related to the wave functions in their respective spaces as

$$\mathcal{P}(x,t)\, dx\, dt = |\Psi(x\,\&\,t)|^2\, dx\, dt = |\psi(x|t)|^2 f(t)\, dx\, dt = |\phi(t|x)|^2 g(x)\, dx\, dt, \tag{5}$$

where $\mathcal{P}(x,t)\,dx\,dt$ is the probability of finding a particle in the length interval $[x, x+dx]$ and the time interval $[t, t+dt]$. The notation in $\psi(x|t)$, the usual wave function, means that $|\psi(x|t)|^2dx$ is the probability of finding the particle in the length interval $[x, x+dx]$ *given that* the clock reads t. Through Bayes' rule [68], it implies that $f(t)\,dt$ is the probability of finding the particle in the time interval $[t, t+dt]$, and analogously for $\phi(t|x)$ and $g(x)$. The functions $f(t)$ and $g(x)$ cannot be given through the equations of the systems alone; these depend on the type of experiment, the settings of the laboratory, etc. [55].

The STS proposal, then, tells us that if the experiment does not require predictions on time (for instance, the fringes at the end of the run of a double-slit experiment), all we need is the usual wave function $\psi(x|t)$. If we need only predictions of time (e.g., tunneling through a potential barrier), all that is required is $\phi(t|x)$. In cases where predictions of both position *and* time are needed, the complete wave function $\Psi(x\,\&\,t)$ should be used.

The "dynamics" in $\mathcal{H}_{\text{time}}$ is given by

$$\mathbb{P}|\phi(x)\rangle = -i\hbar\frac{\partial}{\partial x}|\phi(x)\rangle, \tag{6}$$

with $\mathbb{P}$, the Momentum operator in $\mathcal{H}_{\text{time}}$, defined as

$$\mathbb{P} \equiv \sigma_z\sqrt{2m(\mathbb{H} - V(x,\mathbb{T}))}, \tag{7}$$

$\sigma_z = \text{diag}(1,-1)$. When projected on $\langle t|$, this leads us to

$$\sigma_z\sqrt{2m\left(i\hbar\frac{\partial}{\partial t} - V(x,t)\right)}\phi(t|x) = -i\hbar\frac{\partial}{\partial x}\phi(t|x), \tag{8}$$

where $\phi(t|x) = \langle t|\phi(x)\rangle$. The quotation marks in "dynamics" mean that $\mathbb{P}$ generates variations not in time, as the usual QM does, but in the (classical) position parameter. Compare this to the usual QM, where the Hamiltonian is the generator of variations in the (classical) parameter time. Because of the presence of σ_z, $\phi(t|x)$ is a pseudo-spinor with components

$$\phi(x|t) = \begin{pmatrix}\phi^+(x|t)\\ \phi^-(x|t)\end{pmatrix}. \tag{9}$$

As such, the square modulus then is given by $|\phi(t|x)|^2 = \phi^\dagger(t|x)\phi(t|x)$.

A comment about this proposal is necessary. In the usual QM, $\hat{X}$, $\hat{P}$, and $\hat{H}$ are operators acting on $\mathcal{H}_{\text{pos}}$, and t is a classical parameter. The point of the STS proposal is that the extended Hilbert space *symmetrizes* operators and parameters: on the one hand, we have position and momentum as operators, and the generator of the dynamics, the Hamiltonian, as a function of these two, with the label t acting as a parameter; on the other hand, we have *time* and *energy* as operators, and the *momentum* is the generator of the "dynamics," while still having a classical parameter: in this case, the position x of the particle. This is why, for time-of-flight experiments, all we need is $\phi(t|x)$: the measuring devices are classical objects, meaning that we have, in principle, arbitrary precision of *where* the device is located. Then x has to act as a classical parameter.

Of course, if we consider the detectors lightweight and behave quantum mechanically [69], the uncertainty in the detector's position would be significant, and we would not be able to apply only the STS formalism. Since this will not be the case in the present work, we do not have to worry.

A. Expectation values in the spacetime-symmetric proposal

In the usual QM formalism, experimental results from measuring a quantity that has an operator $\hat{A}$ related to it are compared to the expectation value via

$$\langle\hat{A}\rangle(t) = \frac{\langle\psi(t)|\hat{A}|\psi(t)\rangle}{\langle\psi(t)|\psi(t)\rangle}, \tag{10}$$

which corresponds to averaging measurements of $\hat{A}$ in an ensemble of identically prepared systems, given that we measure at time t. We usually do not write the denominator because the wave function is normalized to the unit, and its normalization is a constant:

$$i\hbar\frac{\partial}{\partial t}\langle\psi(t)|\psi(t)\rangle = \langle\psi(t)|(\hat{H} - \hat{H}^\dagger)|\psi(t)\rangle = 0, \tag{11}$$

because of the hermiticity of the Hamiltonian [39–41].

Now, consider the expectation value in $\mathcal{H}_{\text{time}}$. We have, as in $\mathcal{H}_{\text{pos}}$,

$$\langle\mathbb{B}\rangle(t) = \frac{\langle\phi(x)|\mathbb{B}|\phi(x)\rangle}{\langle\phi(x)|\phi(x)\rangle}, \tag{12}$$

having a similar interpretation of the average of measurements of $\mathbb{B}$, given that the measurement happened at the position x. However, in contrast to what happens in $\mathcal{H}_{\text{pos}}$, the denominator is generally not constant.

The physical interpretation, given by Ref. [56], is that in the usual QM, the particle is expected to exist in some position, regardless of the instant of the measurement. This is different in the extended space, in general. Consider the double-slit experiment: there are points in space where the particle never arrives, independent of how long we wait. If we mirror the interpretation, the difference is clear: the particle *should* exists in *some* instant of time, *independent* of the position of the measurement. This does not happen in general; dark regions on the fringes illustrate this. Some regions are forbidden, no matter how long we wait for the particle to arrive. This means that whenever we use the STS expectation values, we have to carry the factor $\langle\phi(x)|\phi(x)\rangle$ throughout the calculations.

III. WEAK AND STRONG POTENTIAL APPROXIMATIONS

We need to solve Eq. (8) to obtain the wave function in the extended space. This is difficult because of the appearance of a derivative operator inside the square root. We can, however, consider the two extreme cases of weak and strong potentials, which enables us to obtain approximate equations in these limits that can be applied, for instance, to scattering and tunneling problems.

A. Weak potential

Since the generator of the "dynamics" in $\mathcal{H}_{\text{time}}$ is a function of the operators $\mathbb{T}$ and $\mathbb{H}$, we expand the momentum operator $\mathbb{P}$ in a Taylor series up to first order. For this, we consider the actuation of $\mathbb{H}$ to be greater than that of $V(x,\mathbb{T})$ in the sense that we can expand the momentum operator in a Taylor series, meaning that the particle rarely will have significant potential energy. Mathematically,

$$\mathbb{P} = \sigma_z\sqrt{2m(\mathbb{H} - V(x, \mathbb{T}))},$$
$$\simeq \sigma_z\sqrt{2m\mathbb{H}}\left[1 - \frac{1}{4}\left(\frac{1}{\mathbb{H}}V(x, \mathbb{T}) + V(x, \mathbb{T})\frac{1}{\mathbb{H}}\right)\right], \quad (13)$$

where we used $\sqrt{1+\lambda} \simeq 1 + \lambda/2$ for sufficiently small λ, and since $\mathbb{H}$ does not commute with $\mathbb{T}$, we symmetrize the expansion. For simplicity, from now on, we will consider the potential to be independent of time, which gives us

$$\mathbb{P} \simeq \sigma_z\sqrt{2m}\left[\mathbb{H}^{1/2} - \frac{1}{2}\frac{V(x)}{\mathbb{H}^{1/2}}\right]. \quad (14)$$

When projected on $\langle t|$, the operators $\mathbb{H}^{1/2}$ and $1/\mathbb{H}^{1/2} \equiv \mathbb{H}^{-1/2}$ produce 1/2-*fractional* derivatives and integrals, which can be defined as the Caputo fractional derivative [70] and the Riemmann-Liouville fractional integral, respectively [71–75]. Then we will have

$$-i\hbar\partial_x\phi(t|x) = \sigma_z\sqrt{2mi\hbar}\partial_t^{1/2}\phi(t|x)$$
$$- \sigma_z\sqrt{\frac{m}{2i\hbar}}V(x)\partial_t^{-1/2}\phi(t|x). \quad (15)$$

This fractional partial differential equation can, in principle, be solved using different methods, such as the Laplace transform of fractional derivatives and integrals [71–75]. For now, we will focus on the case of a constant potential $V(x) = V_0$. It is then possible to separate the equation into temporal and spatial parts if we consider $\phi(t|x) = F(t)G(x)$:

$$pG(x) = -\sigma_z i\hbar\partial_x G(x), \quad (16a)$$
$$pF(t) = \sqrt{2m}\left[\sqrt{i\hbar}\partial_t^{1/2} - \frac{V_0}{2\sqrt{i\hbar}}\partial_t^{-1/2}\right]F(t), \quad (16b)$$

p being the constant of separation, and we made use of the linearity of the fractional derivatives and integrals [71–75]. We use the ansatz

$$G^{\pm}(x) = \exp\left[\pm\frac{i}{\hbar}px\right],$$
$$F(t) = \exp[-i\omega t], \quad (17)$$

where $G^{\pm}(x)$ are the $\pm$ spatial components of the spinor, together with the fractional derivative property [55,71,72]

$$\partial_t^{\alpha}\exp[\beta t] = \beta^{\alpha}\exp[\beta t], \quad (18)$$

to obtain

$$p = \sqrt{2m\hbar\omega}\left(1 - \frac{V_0}{2\hbar\omega}\right), \quad (19)$$

that is, the first-order approximation of a particle with momentum $p = \sqrt{2m(E - V_0)}$ and energy $E = \hbar\omega$. Thus, the momentum in the STS proposal is consistent with known results from classical mechanics (CM) and QM, at least in the weak and constant potential approximation.

Using this approximation, we can solve for E and arrive at

$$E = \frac{p^2}{2m} + V_0. \quad (20)$$

If we apply this to the case $V_0 = 0$, we obtain the solution for the free particle obtained in Ref. [55],

$$\phi^{\pm}(t|x) = \exp\left[-\frac{i}{\hbar}\frac{p^2}{2m}t \pm \frac{i}{\hbar}px\right], \quad (21)$$

as expected.

B. Strong potential

Considering a Taylor series expansion of the momentum operator with a strong, time-independent potential, we can write

$$\mathbb{P} \simeq \sigma_z\sqrt{-2mV(x)}\left[1 - \frac{\mathbb{H}}{2V(x)}\right], \quad (22)$$

which leads us, in a similar way to Eq. (15), to

$$\sqrt{-2mV(x)}\left[1 - \frac{i\hbar\partial_t}{2V(x)}\right]\phi(t|x) = -\sigma_z i\hbar\partial_x\phi(t|x). \quad (23)$$

Curiously, in the strong potential approximation, the order of the derivatives is the same, losing the fractional properties. Separating this equation enables us to write

$$i\hbar\partial_t F(t) = EF(t), \quad (24a)$$
$$\sigma_z i\hbar\partial_x G(x) = \sqrt{\frac{-m}{2V(x)}}[E - 2V(x)]G(x), \quad (24b)$$

where, as before, $\phi(t|x) = F(t)G(x)$, and E is the separation constant. Equation (24a) is trivial, giving us

$$F(t) = \exp\left(-\frac{i}{\hbar}Et\right), \quad (25)$$

compatible with known results from the usual QM [39–41]. Since we are considering a strong potential, we notice that the term on the right-hand side of Eq. (24b), multiplying $G(x)$, is a Taylor series expansion for small $E/V(x)$, and we can rewrite it as

$$\sqrt{\frac{-m}{2V(x)}}[E - 2V(x)] \simeq -\sqrt{2m(E - V(x))}, \quad (26)$$

as can be checked, giving us

$$G^{\pm}(x) = \exp\left[\pm\frac{i}{\hbar}\int_{x_0}^{x}dx'\sqrt{2m(E - V(x'))}\right]$$
$$= \exp\left(\pm\frac{i}{\hbar}S(E, x)\right), \quad (27)$$

where $S(E, x)$ is the classical action and x_0 depends on the boundary conditions. $S(E, x)$ is also called abbreviated action functional and is related to the usual action by a Legendre transformation $\tilde{S}(x, t) = S(E, x) - Et$ (see [76] for details). Note that $\partial S(E, x)/\partial E = t = T_{\text{class}}$ provides the classical time, which we will use later. The constant potential is trivial and gives us, up to a multiplication constant,

$$G(x) = \exp\left[\pm\frac{i}{\hbar}px\right], \quad (28)$$

with $p = \sqrt{2m(E - V_0)} \in \mathbb{C}$ being the momentum of the system, which again coincides with the CM and QM momenta relations, subject to a constant potential with intensity V_0.

Notice that we obtained the relation $p = \sqrt{2m(E - V_0)}$ without any *ad hoc* hypothesis; the momentum was obtained through the dynamics of the STS proposal, as opposed to Ref. [56]. Our results confirm their findings.

IV. RESULTS

A. Tunnelling time

As in Ref. [56], we define the time of travel (or, in the specific case we want to tackle in this section, tunneling time) as the difference between the expectation values:

$$T_{\rm STS}(x_{\rm i} \to x_{\rm f}) = \langle \mathbb{T} \rangle(x_{\rm f}) - \langle \mathbb{T} \rangle(x_{\rm i}), \tag{29}$$

with $\langle \mathbb{T} \rangle(x)$ given by Eq. (12). A comment is in order. As argued in Ref. [64], the time a quantum object takes to tunnel through a barrier is ill-posed since it is not generally possible to demarcate the tunneling and nontunneling regions, except for the rectangular barrier. The authors argue that the correct question is "how long does it take a quantum particle to cross a barrier?" Equation (29) is even more generic since it asks "how long, on average, does it take for a particle to move from $x_{\rm i}$ to $x_{\rm f}$," and this includes smooth potentials, like a Gaussian barrier [77]. Besides, expressions like the phase time and the Larmor times apply for monochromatic waves [13,55,58,62,64], while Eq. (29) and the tunnelling flight time from Ref. [64] can be used for wave packets.

The solutions that led to Eqs. (17), (20), (25), and (27) are eigenfunctions of $\mathbb{P}$, with eigenvalues $p = \sqrt{2mE}$ outside the barrier or $p = \sqrt{2m(E - V_0)}$ inside the barrier. When preparing systems for experiments in the usual QM, we generally consider a wave packet, a linear combination of eigenfunctions of the Hamiltonian $\hat{H}$. In the same manner, since $\mathbb{P}$ is a linear operator, linear combinations of solutions of Eq. (8) are also solutions of the same equation. In this manner, the wave packet is written as

$$\phi^{\pm}(t|x) = \int_{E_{\rm min}}^{E_{\rm max}} dE\, C_E^{\pm} \exp\left(-\frac{i}{\hbar}Et\right) G^{\pm}(E, x), \tag{30}$$

where the limits $E_{\rm min}$ and $E_{\rm max}$ must be chosen such that we meet the conditions of strong and/or weak potential, depending on the region, and $C_E^{\pm}$ is the energy distribution for the wave packet. The discrete case is straightforward. The correct way of writing the wave packet should be in terms of eigenfunctions and eigenvalues of $\mathbb{P}$. Since we know the relation between p and E [e.g., $p = p(E)$], this is, at heart, just a change of variables in the integration, with the distributions $C_E^{\pm}$ having to change accordingly [55]. We also changed the notation from $G^{\pm}(x)$ to $G^{\pm}(E, x)$ to emphasize the energy dependence of the spatial part. We are making an abuse of notation using the same $\phi^{\pm}(t|x)$ as before, but since from now on, we will only work with the wave packet, there should be no confusion.

Using the completeness relation $\int_{-\infty}^{\infty} dt\, |t><t| = \mathbb{I}$, we can write the expectation value of $\mathbb{T}$ as

$$\langle \mathbb{T} \rangle(x) = \frac{\langle \phi(x)|\mathbb{T}|\phi(x)\rangle}{\langle \phi(x)|\phi(x)\rangle} = \frac{\int_{-\infty}^{\infty} dt\, t\, \rho(t|x)}{\int_{-\infty}^{\infty} dt\, \rho(t|x)}, \tag{31}$$

where

$$\rho(t|x) = \phi^{\dagger}(t|x)\phi(t|x) = \left| \int_{E_{\rm min}}^{E_{\rm max}} dE\, C_E^{+} \exp\left(-\frac{i}{\hbar}Et\right) G^{+}(E, x) \right|^2 + \left| \int_{E_{\rm min}}^{E_{\rm max}} dE\, C_E^{-} \exp\left(-\frac{i}{\hbar}Et\right) G^{-}(E, x) \right|^2, \tag{32}$$

with $\langle t|\phi(x)\rangle = \phi(t|x)$, as used in Eq. (8). Equation (31) is very similar, for instance, to Eq. (4) combined with Eq. (3) from Ref. [64]. However, there are some differences: first, the position Y of the "screen" is far from the interaction region. Equation (29), together with Eq. (31), can be used right at the interfaces since the STS considers the position x to be a classical parameter. Second, in Eq. (31), the limits in the integration are $(-\infty, +\infty)$, instead of $(0, +\infty)$ in Eqs (3) and (4) of Ref. [64].

To calculate the expectation value in Eq. (31), we can write the numerator as

$$N \equiv \int_{-\infty}^{\infty} dt\, t\, \rho(t|x) = \sum_{r=\pm} \int_{-\infty}^{\infty} dt\, t \left[\int_{E_{\rm min}}^{E_{\rm max}} dE\, C_E^r F(t) G^r(E, x) \right] \times \left[\int_{E_{\rm min}}^{E_{\rm max}} dE'\, C_{E'}^r F'(t) G^r(E', x) \right]^*, \tag{33}$$

where the prime denotes that we need to substitute $E \to E'$ in the argument of the second integral, and we introduced the notation $r = \pm$ to write compactly both the $+$ and $-$ solutions in the above expression. The temporal integral can be rewritten as

$$\int_{-\infty}^{\infty} dt\, t\, \exp\left[-\frac{i}{\hbar}(E - E')t\right] = -2\pi i\hbar^2 \partial_{E'} \delta(E' - E), \tag{34}$$

where we made use of

$$t\, \exp\left[-\frac{i}{\hbar}(E - E')t\right] = -i\hbar \partial_{E'} \exp\left[-\frac{i}{\hbar}(E - E')t\right],$$

and the integral representation of the Dirac delta [78]

$$2\pi\delta(x - a) = \int_{-\infty}^{\infty} dp\, \exp\left[ip(x - a)\right].$$

Since the integration limits on the E and E' integrals are not infinite, we must integrate by parts and use the Dirac delta. The integral in E', $I_{E'}$, is written as

$$I_{E'} = \int_{E_{\rm min}}^{E_{\rm max}} dE' \left[C_{E'}^r G^r(E', x)\right]^* \partial_{E'} \delta(E' - E) = \left[C_{E'}^r G^r(E', x)\right]^* \delta(E' - E) \Big|_{E'=E_{\rm min}}^{E_{\rm max}} - \int_{E_{\rm min}}^{E_{\rm max}} dE' \partial_{E'} \left[C_{E'}^r G^r(E', x)\right]^* \delta(E' - E) = \left[C_{E_{\rm max}}^r G^r(E_{\rm max}, x)\right]^* \delta(E_{\rm max} - E)$$

$$- \left[C^r_{E_{\min}} G^r(E_{\min}, x)\right]^* \delta(E_{\min} - E) - \partial_E \left[C^r_E G^r(E, x)\right]^*. \quad (35)$$

Using the value of the Heaviside step function at zero to be equal to $1/2$, such that

$$\int_0^a dz\, f(z)\delta(z) = \int_{-a}^0 dz\, f(z)\delta(z) = \frac{1}{2} f(0), \quad (36)$$

we can integrate the contributions of the first two terms in $I_{E'}$:

$$I_{E_{\max}} = \int_{E_{\min}}^{E_{\max}} dE C^r_E G^r(E, x)\delta(E_{\max} - E) = \frac{1}{2} C^r_{E_{\max}} G^r(E_{\max}, x), \quad (37)$$

and equivalently to the term containing $\delta(E_{\min} - E)$. Putting everything together, we will have

$$N = -2\pi i\hbar^2 \sum_{r=\pm} \left[\frac{1}{2} \left|C^r_{E_{\max}} G^r(E_{\max}, x)\right|^2 - \frac{1}{2} \left|C^r_{E_{\min}} G^r(E_{\min}, x)\right|^2 - \int_{E_{\min}}^{E_{\max}} dE C^r_E G^r(E, x) \partial_E \left[C^r_E G^r(E, x)\right]^* \right]. \quad (38)$$

We can write the quantity $|C^r_{E_{\max}} G^r(E_{\max}, x)|^2 - |C^r_{E_{\min}} G^r(E_{\min}, x)|^2$ using the fact that, on the one hand,

$$\int_{E_{\min}}^{E_{\max}} d[|h|^2] = \left|C^r_{E_{\max}} G^r(E_{\max}, x)\right|^2 - \left|C^r_{E_{\min}} G^r(E_{\min}, x)\right|^2, \quad (39)$$

where $h = C^r_E G^r(E, x)$, while on the other hand

$$d[|h|^2] = d[hh^*] = h^* dh + h\, dh^*, \quad (40)$$

to write

$$N = \pi i\hbar^2 \sum_{r=\pm} \int_{E_{\min}}^{E_{\max}} dE \left[C^r_E G^r(E, x) \partial_E \left[C^r_E G^r(E, x)\right]^* - \left[C^r_E G^r(E, x)\right]^* \partial_E \left[C^r_E G^r(E, x)\right]\right]. \quad (41)$$

We can then use the linearity of the derivative to write $h^* \partial_E h = [h \partial_E h^*]^*$, and arrive at

$$N = -2\pi\hbar^2 \sum_{r=\pm} \text{Im} \left[\int_{E_{\min}}^{E_{\max}} dE\, C^r_E G^r(E, x) \times \frac{\partial}{\partial_E} \left[C^r_E G^r(E, x)\right]^* \right]. \quad (42)$$

Similarly, we can write the denominator as

$$D \equiv \int_{-\infty}^{\infty} dt\, \rho(t|x) = 2\pi\hbar \sum_{r=\pm} \int_{E_{\min}}^{E_{\max}} dE \left|C^r_E G^r(E, x)\right|^2, \quad (43)$$

which finally brings us to

$$\langle \mathbb{T} \rangle(x) = -\hbar \sum_{r=\pm} \text{Im} \left[\int_{E_{\min}}^{E_{\max}} dE\, \partial_E \left[C^r_E G^r(E, x)\right]^* C^r_E \times G^r(E, x) \right] \Big/ \sum_{r=\pm} \int_{E_{\min}}^{E_{\max}} dE \left|C^r_E G^r(E, x)\right|^2. \quad (44)$$

Using Eq. (27), the numerator of the above equation can be rewritten as

$$N_1 = -\hbar \sum_{r=\pm} \text{Im} \left[\int_{E_{\min}}^{E_{\max}} dE\, e^{-2r\, \text{Im}[S(E,x)]/\hbar} \times \left[C^r_E \partial_E \left(C^r_E\right)^* - \frac{ir}{\hbar} \left|C^r_E\right|^2 T^*_{\text{class}}(E, x) \right] \right], \quad (45)$$

where $N_1 = N/(2\pi\hbar)$, $T_{\text{class}}(E, x) = \partial S(E, x)/\partial E$ is the classical time, which is real for energies above the barrier and imaginary for energies below the barrier. Notice that, since this expression takes the imaginary part, for a *real* energy distribution (e.g., a Gaussian distribution centered in some energy E_0), the integral with the derivative of the distribution is real and does not contribute to the final expected value $\langle \mathbb{T} \rangle(x)$. In the same manner, for imaginary times $T_{\text{class}}(E, x) = i\mathcal{T}$, $\mathcal{T}$ being a real number, the remaining integral also does not contribute. Therefore, the expectation value $\langle \mathbb{T} \rangle(x)$ is proportional to an energy average of the classical time $T_{\text{class}}(E, x)$, weighted by the energy distribution. This is our first main result.

Before discussing an application, let us provide additional general statements about Eqs. (43) and (45) using a *real* Gaussian wave packet for the energy distribution. For simplicity, we assume $C^-_E = 0$ and

$$C^+_E = C = \mathcal{A} \exp\left(-\frac{\sigma^2(E - E_0)^2}{\hbar^2}\right), \quad (46)$$

which is a Gaussian energy distribution centered in E_0, $\mathcal{A}$ is a normalization factor and $\hbar/\sigma$ is the width in the energy. The component C^-_E being zero means that, since the expressions in Eq. (54) are plane waves, the waves traveling from right to left on the real x axis are discarded. Therefore, while the numerator becomes

$$N_1 = -\hbar\, \text{Im} \left\{ \int_{E_{\min}}^{E_{\max}} dE\, e^{-\frac{2}{\hbar} \text{Im}[S(E,x)] - \frac{-2\sigma^2(E-E_0)^2}{\hbar^2}(E-E_0)^2} \times \left[-\frac{i}{\hbar} \frac{\partial}{\partial E} [S(E, x)]^* \right] \right\} |\mathcal{A}|^2, \quad (47)$$

which is zero for energies below V_0, the denominator, for energies above the barrier becomes

$$D = \frac{|\mathcal{A}|^2 \hbar \sqrt{\pi}}{2\sigma\sqrt{2}} \left[\text{erf}\left(\frac{\sqrt{2}(E_{\max} - E_0)\sigma}{\hbar}\right) - \text{erf}\left(\frac{\sqrt{2}(E_{\min} - E_0)\sigma}{\hbar}\right) \right], \quad (48)$$

since $|G(E, x)| = 1$ for this case. We discuss two limiting physical situations:

(i) For $\sigma \to 0$ (wave packet well localized in time and delocalized in energy), the contribution of the energy distribution in the integration is negligible. In this limit, for energies above the barrier, we have

$$\langle \mathbb{T} \rangle^{(\sigma\to 0)}(x) = \frac{N^{(\sigma\to 0)}}{D^{(\sigma\to 0)}} \simeq \frac{\int dE\, T_{\text{class}}(E,x)}{\Delta E} = \frac{\Delta S}{\Delta E}, \quad (49)$$

where $\Delta S = S_f - S_i$, with $S_f = S(E_{\max}, x)$ and $S_i = S(E_{\min}, x)$ and we used the approximation $\text{erf}(\text{ax}) \simeq \frac{2\text{ax}}{\sqrt{\pi}}$. The quantum time is the energy average of the classical time, namely, $\overline{T}_{\text{class}} = \Delta S/\Delta E$.

(ii) For increasing σ (wave packet well localized in energy and delocalized in time), the integral in Eq. (47) goes to 0 faster and faster (unless $E = E_0$, case when only this energy contributes to the integral). Then we will have $N_1 = 0$ for $E_0 < V_0$. For $E_0 > V_0$ we have

$$N_1 = \frac{\partial}{\partial E} S(E,x)\bigg|_{E=E_0}, \quad (50)$$

which is the classical time for a particle with energy E_0 in a constant potential V_0. The denominator $\mathcal{D}$ goes to unity in this regime.

If, on the other hand, our distribution is complex, we can write $C_E^{\pm} = |C_E^{\pm}| e^{i\varphi(E)}$, where $\varphi(E)$ is the energy-dependent phase of the complex distribution. Then we can see that the denominator $D_1 = D/(2\pi\hbar)$ of Eq. (44) is kept the same [it does not depend on the phase $\varphi(E)$], while the numerator is given by

$$\begin{aligned} N_1 = &-\hbar \sum_{r=\pm} \text{Im}\left\{ \int_{E_{\min}}^{E_{\max}} dE\, e^{-2r\,\text{Im}\,[S(E,x)]/\hbar} \right. \\ &\left. \times \left[|C_E^{\pm}|\partial_E |C_E^{\pm}| + i|C_E^{\pm}|^2 \left(\partial_E \varphi(E) - \frac{r}{\hbar} T^*_{\text{class}} \right) \right] \right\} \\ = &-\hbar \sum_{r=\pm} \text{Im}\left[i \int_{E_{\min}}^{E_{\max}} dE\, e^{-2r\,\text{Im}\,[S(E,x)]/\hbar} \right. \\ &\left. \times |C_E^{\pm}|^2 \left(\partial_E \varphi(E) - \frac{r}{\hbar} T^*_{\text{class}} \right) \right], \end{aligned} \quad (51)$$

where we used the fact that

$$\int_{E_{\min}}^{E_{\max}} dE\, e^{-2r\,\text{Im}\,[S(E,x)]/\hbar} |C_E^{\pm}| \partial_E |C_E^{\pm}| \quad (52)$$

is purely real. Thus, we find that the phase of the distribution affects both the expected value of the operator $\mathbb{T}$ and the time of travel defined in Eq. (29). The presence of this phase introduces a great difficulty in the treatment, since it will depend on the preparation and the experimental setup. Without this information, it becomes a free parameter of the theory, and we will not consider it in the remaining of this work.

B. Toy model: Rectangular potential barrier

The toy model we use for our main result is the textbook potential barrier:

$$V(x) = \begin{cases} V_0 = \text{const}, & 0 < x < L, \\ 0, & \text{everywhere else}. \end{cases} \quad (53)$$

V_0 is such that we can use the strong potential limit of Sec. III B for this region, and L is the length of the barrier.

We want the wave function to be continuous in the interfaces $x = 0$ and $x = L$ for all instants of time, following the same principles as in the usual QM [39–41]. Since $F(t)$ has the same form for all regions, the temporal connection is trivial and implies that the energies $E = \hbar\omega$ must be equal in all regions. Then, for the spatial part, we consider

$$G^{\pm}(x) = \begin{cases} A_1^{\pm} \exp\left[\pm \frac{i}{\hbar} p_1 x\right], & x < 0, \\ A_2^{\pm} \exp\left[\pm \frac{i}{\hbar} p_2 x\right], & 0 < x < L, \\ A_3^{\pm} \exp\left[\pm \frac{i}{\hbar} p_1 x\right], & L < x, \end{cases} \quad (54)$$

with

$$\begin{aligned} p_1 &= \sqrt{2mE}, \\ p_2 &= \sqrt{2m(E - V_0)}. \end{aligned} \quad (55)$$

Connecting the wave function at the interfaces, we have

$$\begin{aligned} A_2^{\pm} &= A_1^{\pm}, \\ A_3^{\pm} &= A_1^{\pm} \exp\left[\pm \frac{i}{\hbar}(p_2 - p_1)L\right]. \end{aligned} \quad (56)$$

Combining Eqs. (54) and (56), together with $F(t) = \exp(-iEt/\hbar)$, we have the total wave function for the rectangular barrier.

Equation (44) allows us to predict tunneling times and dwell times whenever the potential is sufficiently strong and constant. Because, in principle, we can position the probes with arbitrary precision in this treatment, the time it takes for the particle to tunnel, on average, is given by

$$T_{\text{STS}}(0 \to L) = \langle \mathbb{T} \rangle(L) - \langle \mathbb{T} \rangle(0) \quad (57)$$

for a potential barrier located between $x = 0$ and $x = L$ [56].

C. Application of tunneling time: A Gaussian distribution for a wave packet moving to the right

For an application of Eq. (44), we will consider the same distribution as in the previous discussion in Sec. IV A, that is, $C_E^- = 0$ and

$$C_E^+ = C = \mathcal{A} \exp\left(-\frac{\sigma^2 (E - E_0)^2}{\hbar^2}\right), \quad (58)$$

where $\hbar/\sigma$ is the spreading of the energy (that is, the inverse of the precision in the time), E_0 is the center of the Gaussian, and $\mathcal{A}$ is the normalization constant. This distribution means that, as discussed in the previous section, waves traveling from right to left on the real x axis are discarded.

Figure 1 displays $\rho(t|x)$ for the considered distribution and the rectangular potential barrier between $x = 0 \to 1$. The wave packet moves from left to right (from most negative x to most positive x). Notice that the absolute values can be very high since we have a non-normalized wave function.

For the specific distribution we are working on, the expectation value of $\mathbb{T}$ can be written as $\langle \mathbb{T} \rangle(x) = \mathcal{N}/\mathcal{D}$, where

$$\mathcal{N} = \text{Im} \int_{E_{\min}}^{E_{\max}} dE\, i\, x\, \partial_E(p^*) e^{-2x\,\text{Im}(p)/\hbar - 2\sigma^2 (E-E_0)^2/\hbar^2} \quad (59)$$

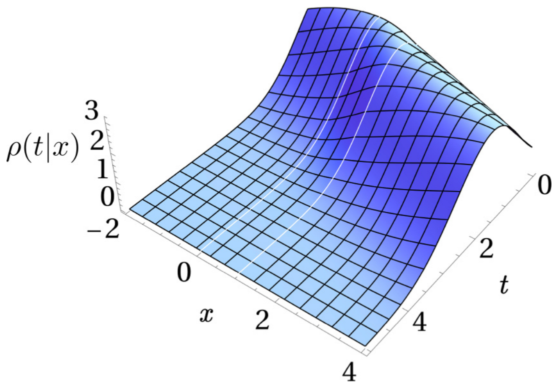

FIG. 1. $\rho(t|x)$ for $m = \hbar = L = 1$ and $V_0 = 2$, in arbitrary units. The energy distribution is given by $C_E^+ = \exp[-\sigma^2(E - E_0)^2/\hbar]$ and $C_E^- = 0$, where $\sigma = 1$, $E_0 = 4$, and the integration limits ranging from $E = 0$ to $E = 10$, meaning that plane wave moves initially from left to right. Quantities are in arbitrary units.

and

$$\mathcal{D} = \int_{E_{\min}}^{E_{\max}} dE\, [e^{-2x\,\mathrm{Im}(p)/\hbar - 2\sigma^2(E-E_0)^2/\hbar^2}]. \tag{60}$$

Clearly, $\langle \mathbb{T} \rangle(0) = 0$, so that the object of interest, Eq. (57), is equal to

$$T_{\mathrm{STS}}(0 \to L) = \langle \mathbb{T} \rangle(L). \tag{61}$$

Because of the presence of the imaginary part in Eq. (59), we can consider three separate cases:

(i) $E_{\min} < E_{\max} < V_0$: In this case, $T_{\mathrm{STS}}(0 \to L) = 0$, since the momenta are imaginary for this range of energies, such that the classical time T_{class} is also imaginary, giving a real integral in Eq. (59).

(ii) $E_{\min} < V_0 < E_{\max}$: In this case, the numerator is written as

$$\mathcal{N} = \int_{V_0}^{E_{\max}} dE \left[e^{-2\sigma^2(E-E_0)^2/\hbar^2} \frac{Lm}{\sqrt{2m(E - V_0)}} \right]. \tag{62}$$

For energies lower than the barrier, p is complex, and similarly to case 1, the integral is real. The quantity $Lm/\sqrt{2m(E - V_0)}$ is the classical time. The denominator is given by

$$\mathcal{D} = \int_{E_{\min}}^{V_0} dE\, e^{-2L\sqrt{2m(V_0-E)}/\hbar - 2\sigma^2(E-E_0)^2/\hbar^2} + \int_{V_0}^{E_{\max}} dE\, e^{-2\sigma^2(E-E_0)^2/\hbar^2}. \tag{63}$$

(iii) $V_0 < E_{\min} < E_{\max}$: This is the simplest case, providing us with an average (in the energies) of the classical times:

$$T_{\mathrm{STS}}(0 \to L) = \frac{\int_{E_{\min}}^{E_{\max}} dE \left[e^{-2\sigma^2(E-E_0)^2/\hbar^2} \frac{Lm}{\sqrt{2m(E-V_0)}} \right]}{\int_{E_{\min}}^{E_{\max}} dE\, e^{-2\sigma^2(E-E_0)^2/\hbar^2}} = \frac{\int_{E_{\min}}^{E_{\max}} dE\, |C_E|^2 T_{\mathrm{class}}}{\int_{E_{\min}}^{E_{\max}} dE\, |C_E|^2}. \tag{64}$$

To compare our results with others works, we can also rewrite the above expressions if one defines $k_0 = \sqrt{2mV_0}/\hbar$ (the wave number related to the barrier), $\tau_0 = mL/\hbar k_0$ (the characteristic time of the barrier), $\Gamma = Lk_0$ (the barrier intensity), $\sigma_0 = \sigma V_0/\hbar$ (which provides how precise the energy of the system is compared to the intensity of the barrier), $\lambda = E/V_0$, and $\lambda_0 = E_0/V_0$ (giving the dimensionless energy in the integral and the dimensionless center of the Gaussian distribution, respectively). Then, for each case, we will have

(i) $\lambda_{\min} < \lambda_{\max} < 1$:

$$\frac{T_{\mathrm{STS}}(0 \to L)}{\tau_0} = 0. \tag{65}$$

(ii) $\lambda_{\min} < 1 < \lambda_{\max}$:

$$\frac{T_{\mathrm{STS}}(0 \to L)}{\tau_0} = \frac{\mathcal{N}_2}{\mathcal{D}_2}, \tag{66}$$

where

$$\mathcal{N}_2 = \int_1^{\lambda_{\max}} d\lambda\, e^{-2\sigma_0^2(\lambda-\lambda_0)^2} (\sqrt{\lambda - 1})^{-1} \tag{67}$$

and

$$\mathcal{D}_2 = \int_{\lambda_{\min}}^{1} d\lambda\, e^{-2\Gamma\sqrt{1-\lambda} - 2\sigma_0^2(\lambda-\lambda_0)^2} + \int_1^{\lambda_{\max}} d\lambda\, e^{-2\sigma_0^2(\lambda-\lambda_0)^2}. \tag{68}$$

(iii) $1 < \lambda_{\min} < \lambda_{\max}$:

$$\frac{T_{\mathrm{STS}}(0 \to L)}{\tau_0} = \frac{\int_{\lambda_{\min}}^{\lambda_{\max}} d\lambda\, e^{-2\sigma_0^2(\lambda-\lambda_0)^2} (\sqrt{\lambda - 1})^{-1}}{\int_{\lambda_{\min}}^{\lambda_{\max}} d\lambda\, e^{-2\sigma_0^2(\lambda-\lambda_0)^2}}. \tag{69}$$

Equation (66) can be compared, for different values of σ_0, to the characteristic times obtained from the precession of spin in an infinitesimal field in the $\hat{z}$ direction (for more details, check further Ref. [58]), also known as Larmor time τ_y (which coincides with the dwell time τ_{D}) and the phase time τ_ϕ [62]. The tunneling time, in units of the characteristic barrier time $\tau_0 = mL/\hbar k_0$, is shown in Fig. 2 as a function of $k/k_0 = \lambda^2$. For $k/k_0 < 1$, we have energies below the barrier, and for $k/k_0 > 1$, energies above the barrier. The times τ_z and τ_ϕ are for monochromatic waves, while $T_{\mathrm{STS}}(0 \to L)$ has a bandwidth, such that $k/k_0 = \lambda_0^2 = (E_0/V_0)^2$ is related to the center of the Gaussian energy distribution from Eq. (58). For the top row of Fig. 2, we have $\sigma_0 = 0.5$, middle row $\sigma_0 = 2$, and bottom row $\sigma_0 = 5$. Distinct strengths of the barrier $\Gamma = k_0 L \equiv p_0 L/\hbar$ are used: $\Gamma = \pi/3$ in the first column [Figs. 2(a), 2(d), and 2(g)], $\Gamma = \pi/3$ for the second column [Figs. 2(b), 2(e), and 2(h)], and $\Gamma = \pi/3$ for the third column [Figs. 2(c), 2(f), and 2(i)]. We can see that, for $k/k_0 > 1$, our results act like an average of the times τ_y and τ_ϕ, while not agreeing at all for energies below the barrier ($k/k_0 < 1$). We notice that for $k/k_0 < 1$, the energy dependence of T_{STS} for $\sigma_0 = 0.5$ is almost monotonic [see Figs. 2(a), 2(b), and 2(c)]. However, for $\sigma_0 = 5.0$ [see Figs. 2(g), 2(h), and 2(i)], T_{STS} strongly varies inside the energy interval. This results from the better energy resolution for larger σ_0 values.

Figure 3 compares $T_{\mathrm{STS}}(0 \to L)$ with the experimental results from Ref. [77] for Fig. 3(a) $\sigma_0 = 0.5$, Fig. 3(b) $\sigma_0 = 1$, and Fig. 3(c) $\sigma_0 = 2$. Even though the authors use a Gaussian barrier in the experiment, we see that $T_{\mathrm{STS}}(0 \to L)$ agrees with the measure of τ_y for energies above the barrier and $\sigma_0 \approx 1$. For energies below the barrier, the agreement between

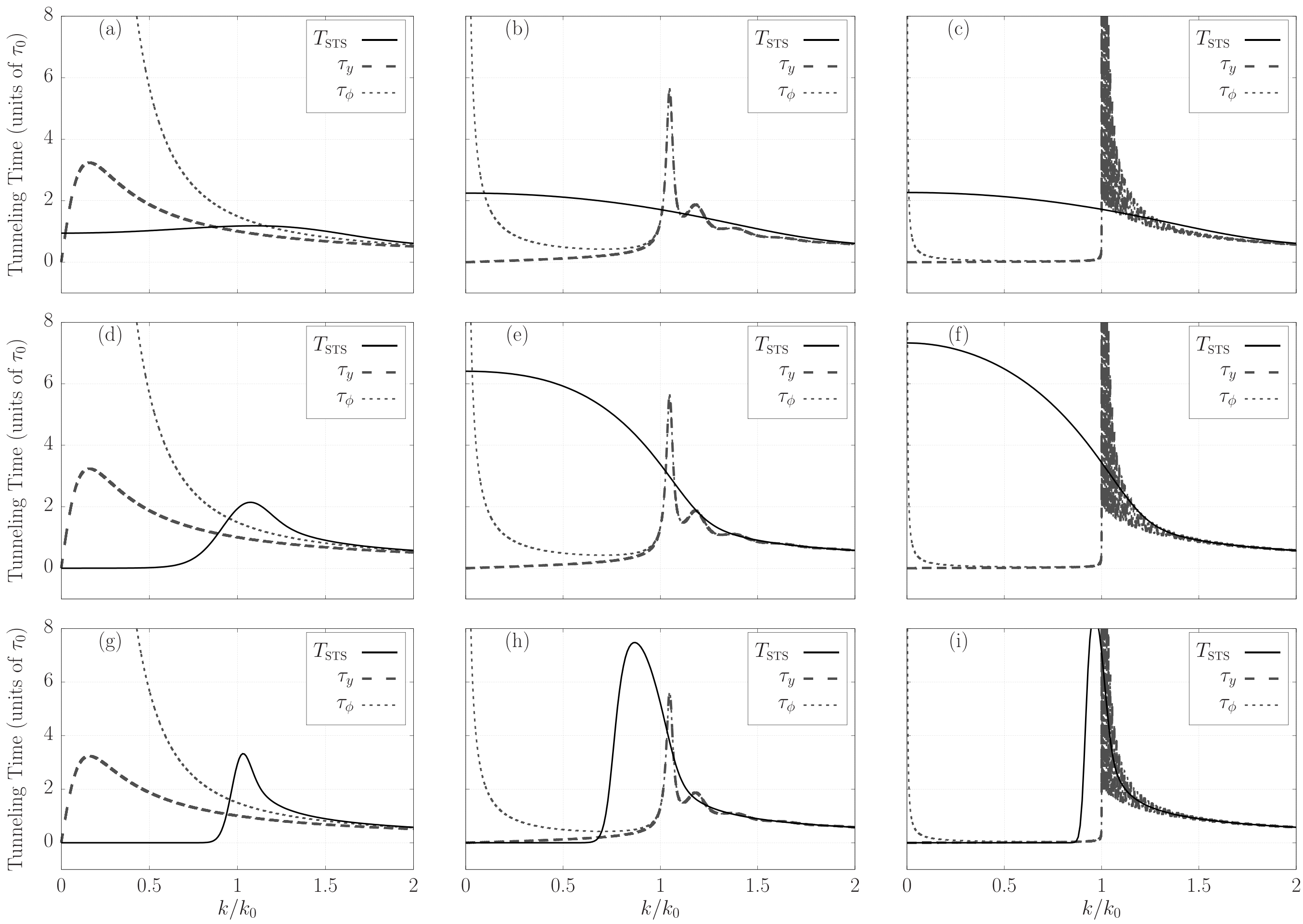

FIG. 2. Comparison between Eq. (66) for different values of σ_0 and the travel times $\tau_y = \tau_{\rm D}$ and τ_ϕ, as obtained in Ref. [58], in units of the characteristic time $\tau_0 = mL/\hbar k_0$ of the barrier. For the first row (a)–(c), we used $\sigma_0 = 0.5$. For the second row (d)–(f), $\sigma_0 = 2$. For the third row (g)–(i), $\sigma_0 = 5$. For $T_{\rm STS}$, $k = \sqrt{2mE_0}/\hbar$ [the wave number related to the center of the Gaussian energy distribution given by Eq. (58)] and $k = \sqrt{2mE}/\hbar$ for the other times. In all cases $k_0 = \sqrt{2mV_0}/\hbar$, and the limits of integration are $E_{\rm min} = 0$ and $E_{\rm max} = 10V_0$, that is, $\lambda_{\rm min} = 0$ and $\lambda_{\rm max} = 10$. The results do not change in the plotted region if we increase $\lambda_{\rm max}$. The quantity $\Gamma = k_0 L$ gives us the barrier's strength. We have $\Gamma = \pi/10$ in (a), (d), and (g), $\Gamma = 3\pi$ in (b), (e), and (h) and $\Gamma = 30\pi$ in (c), (f), and (i). We notice that for energies below the barrier height ($k/k_0 < 1$), our results differ greatly, while, except for weak barriers, for energies above the barrier ($k/k_0 > 1$), $T_{\rm STS}$ acts like an average of τ_y and τ_ϕ.

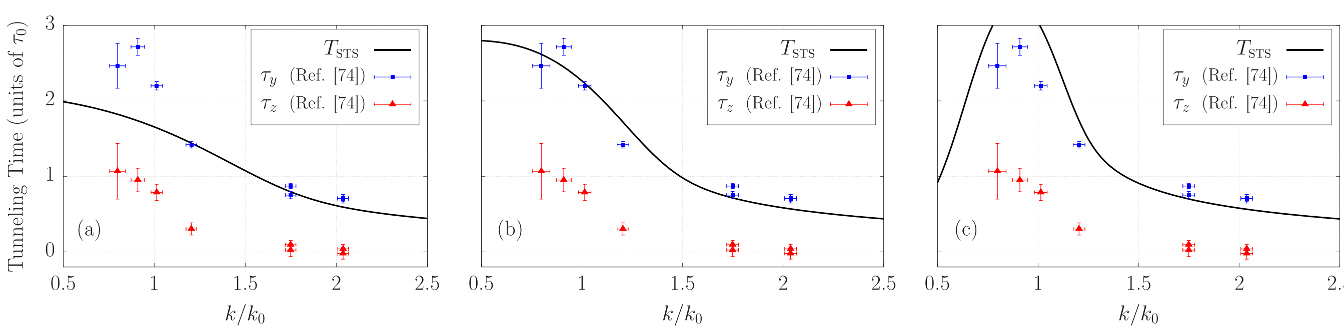

FIG. 3. Comparisons between (66) and the experimental data of Fig. 4(c) from Ref. [77], in units of τ_0, for different values of σ_0. We have (a) $\sigma_0 = 0.5$, (b) $\sigma_0 = 1$, and (c) $\sigma_0 = 2$. The barrier intensity is π (for a barrier with corresponding velocity 5.1 mm/s) and barrier length of 1.3μm, such that $\tau_0 = mL/\hbar k_0 = L/v \simeq 2.5 \times 10^{-4}$ s. The limits of integration are $\lambda_{\rm min} = 0$ and $\lambda_{\rm max} = 10$. The results do not change if we increase $\lambda_{\rm max}$ in the plotted region. Blue squares are measurements of τ_y, while red triangles are measurements of τ_z. We obtain a good agreement with τ_y when the width of the energy distribution $\hbar/\sigma$ is close to the barrier height V_0, such that $\sigma_0 \approx 1$.

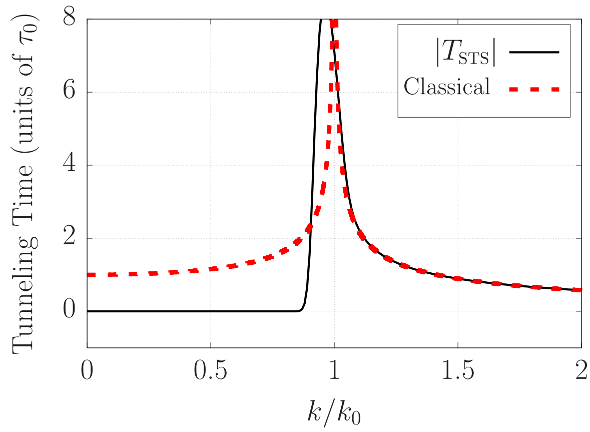

FIG. 4. Comparisons between Eq. (66) in absolute values and the "classical" time $|mL/\sqrt{2m(E_{\max}-V_0)}|$ for a barrier intensity $k_0L = 3\pi$ and $\sigma_0 = 10$, in units of $\tau_0 = mL/\hbar k_0$. The limits of integration are $\lambda_{\min} = 0$ and $\lambda_{\max} = 10$. The results do not change if we increase $\lambda_{\max}$.

$T_{\rm STS}(0 \to L)$ and τ_y tends to ameliorate for larger values of σ_0 (better resolution in the energy), namely, Figs. 3(b) and 3(c).

We observe that the time the particle travels through the barrier region for energies above the barrier is a classical-like contribution. Figure 4 shows a comparison of Eq. (66) with a classical particle with energy $E_{\max}$ and the time of travel $mL/\sqrt{2m(E_{\max}-V_0)}$. Our result agrees well with the classical time outside the barrier while disagreeing completely with the tunneling portion. Thus, our results signal that the classical time is the most probable time. We would like to reemphasize that Figs 2–4 were obtained using the limits of $V \ll E$ and $V \gg E$, and should be taken with care in the region that this approximations are not valid anymore.

We could also, through Eqs. (62) and (63), compare the effects that the presence of a potential barrier has on a particle traveling above it. For this, consider, as done at the end of Sec. IV A, the two limiting cases of a wave packet well resolved in time ($\sigma \to 0$) and a wave packet well resolved in energy ($\sigma \to \infty$). We then compare it to the free-particle case, $V_0 = 0$. In the free-particle and the traveling-above-the-barrier cases, we set $E_{\min} = 0$.

(i) For the well-resolved in-time wave packet, we have

$$T_{\rm STS}^{\sigma\to 0}(0 \to L) = L\frac{\sqrt{2m(E_{\max}-V_0)}}{E_{\max}}. \tag{70}$$

Comparing this to the free particle's time of travel, we obtain, for $\sigma \to 0$

$$\frac{T_{\rm STS}^{\rm free}}{T_{\rm STS}} = \frac{L\sqrt{2mE_{\max}}/E_{\max}}{L\sqrt{2m(E_{\max}-V_0)}/E_{\max}} = \sqrt{\frac{E_{\max}}{E_{\max}-V_0}} > 1, \tag{71}$$

showing that the particle travels faster in the barrier region in the presence of the potential.

This is compatible with known results from [79] and references therein, where the tunneling time is shorter than the time a free particle would take to cross the same region.

(ii) When $\sigma \to \infty$, we have two cases that need to be treated separately: $E_0 < V_0$ and $E_0 > V_0$. When σ becomes large, the exponential in the Gaussian energy distribution goes to 0 quickly, unless $E = E_0$, in which case the exponential equals unity. Then the only contribution comes from $E = E_0$, that is, the exponential acts like a Dirac delta, and for $E_0 < V_0$ we have $T_{\rm STS}^{\sigma\to\infty}(0 \to L) = 0$ since E_0 is not in the region of integration. For $E_0 > V_0$, we will have $\mathcal{D} \sim 1$ and

$$T_{\rm STS}^{\sigma\to\infty}(0 \to L) \sim \frac{Lm}{\sqrt{2m(E_0-V_0)}}. \tag{72}$$

Comparing this with the free particle,

$$\frac{T_{\rm STS}^{\rm free}}{T_{\rm STS}} = \frac{Lm/\sqrt{2mE_0}}{Lm/\sqrt{2m(E_0-V_0)}} = \sqrt{\frac{E_0-V_0}{E_0}} < 1. \tag{73}$$

Consequently, when we do not have precision in time, the free particle goes faster through the potential region, on average.

V. FINAL REMARKS

This work summarizes the main ideas of a recent proposal that tries to include and understand a time operator in QM. The proposal is spacetime-symmetric (STS) and allows for predicting times-of-flight and tunneling times. Using a *real* Gaussian energy distribution, we demonstrate that for wave packets well resolved in time, the expectation value for the operator $\mathbb{T}$ is the energy average of the classical time $T_{\rm class} = \partial S/\partial E$ for energies above the barrier.

We apply the proposal for a particle with energy E under weak and strong constant potentials, namely, a rectangular barrier with length L and intensity V_0. Connection formulas between distinct regions of motion are provided to obtain an explicit expression for the tunneling time through a barrier. Using a wave packet with a Gaussian distribution of energy, we show that, when considering a distribution well resolved in time (spread in energy),the tunneling time in the STS proposal is in agreement with previous times, such as $\tau_y = \tau_{\rm D}$ and τ_ϕ, for energies above the barrier and sufficient precision in time, from Ref. [58]. Furthermore, we provide, to first order, the average of classical times of flight for an ensemble of particles with momenta $\sqrt{2m(V_0-E)}$ for energies above the barrier.

The STS proposal is promising. It encompasses travel times for both classically forbidden and classically allowed regions, giving average times even for wave packets and arbitrary potentials. Apart from helping our general understanding of the time in QM, it could assist in using fractional derivatives and integrals in physics and their interpretations in QM [72,80–84], or other areas. They can be used to model power-law nonlocality, power-law long-term memory or fractal properties (Ref. [85] and references therein), anomalous diffusion processes in complex media [86], and propagation of acoustical waves in biological tissue [87], to name a few applications. Especially we can see, when comparing the dynamical equations for the weak versus strong potentials, that the order of the time derivative varies from 1/2 to 1, respectively, an artifact of the Taylor series expansions. It also could motivate further studies, giving more insights into the symmetries between spacetime and energy momentum.

It should be noted that recently an important issue was addressed concerning the STS approach in Ref. [88]. There the authors propose that the energy probability amplitudes may be different between regular QM and the STS extension. For example, in regular QM, C_E is linked to the spatial Fourier transform of the Schrödinger wave function at time t, providing energy information spread in space at that specific time t. Meanwhile, in the STS approach, the energy distribution gives energy information at a specific position x. This distinction could enhance the compatibility of the STS approach with other models, for instance, the data presented in Ref. [77].

Generally speaking, solving Eq. (8) is the main challenge. One possible way to do it is using the Fourier transform of the square-root operator. In Ref. [75], Sec. 28.2 gives the treatment for powers of the operator $-\Delta x + \partial_t$, but for different integrodifferential operators. In principle, this could be expanded to the momentum operator in the STS proposal and give solutions beyond the scope of constant potentials. We could then compare predictions with, for instance, the toy model for the Stark problem of Ref. [89]. Possible problems of the inverse Fourier transform convergence could be avoided by limiting the integration frequencies, the barrier acting as a filter, as justified by Eq. (12.5) of Ref. [13] [60].

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