PHYSICAL REVIEW A VOLUME 11, NUMBER 3 MARCH 1975

# Averaged Lagrangian method applied to resonant nonlinear optics. The self-steepening of light pulses

J. A. Armstrong*
*IBM Thomas J. Watson Research Center, Yorktown Heights, New York 10598*
(Received 20 June 1974)

The averaged Lagrangian method of Whitham for treating nonlinear, dispersive wave propagation is used to formulate the problem of optical pulse propagation in a resonant medium. For the particular case of self-steepening, it is shown that this formulation takes the same mathematical form as Lighthill's treatment of the propagation of wave trains on deep water. Certain aspects of the solution are discussed and compared with alternative treatments of self-steepening. Some comments are given concerning the applicability of the averaged Lagrangian method to other problems of coherent, nonlinear optics.

## I. INTRODUCTION

One of the difficult problems of nonlinear optics is that of the propagation of a light pulse in a medium which is both strongly nonlinear and strongly dispersive. This situation arises, for example, when a suitably short and intense laser pulse propagates in a medium having a strong resonance line very close to, or at, the frequency of the light. In particular, this combination of nonlinearity and strong dispersion is responsible for the recently reported self-steepening of light pulses due to adiabatic following in Rb vapor.[1] Near, but not on, resonance the phase and group velocity depend strongly on both intensity and frequency; as a consequence the more intense parts of the pulse move faster than the weaker parts; moreover, the rapid variation in intensity produces a phase modulation which moves the instantaneous frequency nearer to resonance on the rising edge of the pulse, thus enhancing both dispersion and nonlinearity. Other phenomena occur when the resonance overlaps the carrier frequency of the light; of these, self-induced transparency[2] (SIT) is the most striking.

The traditional theory of such phenomena rests on coupled first-order nonlinear partial differential equations derived from the full wave equation by the slowly varying envelope approximation, as described, for example, in Ref. 3. There is, however, an alternative approach to such problems. This is the so-called averaged Lagrangian method of Whitham.[4] It was pointed out by Knight and Peterson[5] that the averaged Lagrangian method should be useful in optical-pulse propagation problems, but detailed applications seem not to have been made. The main purpose of this paper is to show how the averaged Lagrangian method can be used to formulate the problem of self-steepening of light pulses due to the adiabatic-following nonlinearity.[6] However, we make some comments on the applicability of the method to other problems of coherent, nonlinear optics.

The physical reason for introducing an averaged Lagrangian is the same as for making the slowly-varying-envelope approximation; i.e., there are two distinct time (and distance) scales in all these problems. This leads, in Whitham's terminology, to the desirability of "two-timing" the equations of motion to suppress the rapid time and space variations of the carrier wave. Although the motivation is similar for both the slowly-varying-envelope approximation and the averaged Lagrangian, it is not clear that these two approximation methods are fully equivalent. The precise relationship between them is a matter for further investigation.

Whitham showed how the slow variations of a wave group are governed by averaged equations of motion which can be obtained from an averaged variational principle. The method appears to be very powerful and can be described roughly as follows. First, the normal Lagrangian of the problem is written down in terms of independent, generalized coordinates. Second, this Lagrangian is averaged in time and space over one period of a certain well-defined, periodic traveling wave. These strictly periodic, traveling-wave solutions to the nonlinear equation are the basis for an approximate representation of nonlinear *pulse* propagation. That is, since these periodic waves are exact solutions of the nonlinear equations, it is in terms of slow changes in their amplitude, frequency, and wave vector that one may "best" describe nonlinear pulse propagation. Third, the resulting averaged Lagrangian density, which depends only on slowly varying quantities, is used to define an averaged action integral which is made stationary by the calculus of variations in the usual way. One, or one set, of the resulting

Euler-Lagrange equations provides the nonlinear dispersion law of the problem, and the remaining equations describe the gradual changes in the instantaneous frequency and $k$ vector of the wave packet.

The method may be better understood in terms of a simple linear example (which does not, of course, display the full power of the technique). Consider a linearly polarized electromagnetic wave interacting with a medium whose electric polarization is linear in the applied field. The appropriate generalized coordinate for the field is the vector potential $\vec{A} = A\hat{x}$. Since the medium is linear, its polarization $\vec{P} = Nqr\hat{x}$ is a suitable generalized coordinate. Here $N$ is the number density of oscillators of charge $q$, mass $m$, and displacement $r$ opposed by a force $-Kr$. The equations of motion are

$$\ddot{P} = -(K/m)P - (Nq^2/mc)\dot{A}\,, \tag{1a}$$

$$(1/c^2)\ddot{A} - A_{zz} = (4\pi/c)\dot{P}. \tag{1b}$$

These equations follow from the Lagrangian density[7]

$$L = \frac{1}{8\pi}\left[\left(\frac{\dot{A}}{c}\right)^2 - A_z^2\right] + \frac{4\pi}{\omega_p^2}\left(\frac{\dot{P}^2}{2} - \frac{\omega_0^2 P^2}{2} - \frac{\omega_p^2 \dot{A} P}{4\pi c}\right). \tag{2}$$

Here $\omega_0^2 = K/m$ and $\omega_p^2 = 4\pi N q^2/m$.

According to the "program" outlined above, the next step is to find solutions to (1) of the form $P = P(\omega t - kz)$, $A = A(\omega t - kz)$. The solutions are obviously

$$A = A_0 \cos(\omega t - kz + \gamma), \quad P = P_0 \sin(\omega t - kz + \gamma), \tag{3}$$

where the phase shift $\gamma$ will turn out to be arbitrary. The ratio $P_0/A_0$ has the well-defined value

$$P_0/A_0 = (\omega/4\pi c)(\epsilon - 1), \tag{4}$$

where it is also found that

$$c^2k^2/\omega^2 \equiv \epsilon = 1 + \omega_p^2/(\omega_0^2 - \omega^2). \tag{5}$$

The next step is to put (3) in the Lagrangian density (2) and average it over one period of the traveling wave. That is, compute

$$\mathcal{L} = (1/2\pi)\int_0^{2\pi} L\, d\psi,$$

where $\psi \equiv \omega t - kz + \gamma$. The result is

$$2\pi\mathcal{L} = \int_0^{2\pi} [(\omega^2/8\pi c^2)(1-\epsilon)A_0^2 \sin^2\psi + (2\pi/\omega_p^2)(\omega^2\cos^2\psi - \omega_0^2\sin^2\psi)P_0^2 + (\omega A_0 P_0/c)\sin^2\psi]\, d\psi, \tag{6}$$

whence

$$\mathcal{L} = (\omega^2 A_0^2/8\pi^2c^2)[\epsilon - 1 + (\epsilon - 1)^2(\omega^2 - \omega_0^2)/\omega_p^2]. \tag{7}$$

One must constantly bear in mind the definition $\epsilon \equiv (ck/\omega)^2$.

This averaged Lagrangian is a function of $A_0^2$, $\omega$, and $K$, each of which was strictly constant in the above derivation. The essence of the method is that now, once $\mathcal{L}(\omega, k, A_0^2)$ has been obtained, $\omega$, $k$, and $A_0^2$ are to be reinterpreted as quantities which vary slowly in time and space. One can therefore write an action integral which involves only slowly varying quantities,

$$J = \int \mathcal{L}(\omega, k, A_0^2)\, dz\, dt. \tag{8}$$

Whitham showed in Ref. 4 that the evolution of the wave group will be such that $J$ is stationary; and from the requirement that $\delta J = 0$ there follow the Euler-Lagrange equations

$$\partial\mathcal{L}/\partial A_0^2 = 0, \tag{9}$$

and

$$\frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial\omega} - \frac{\partial}{\partial z}\frac{\partial\mathcal{L}}{\partial k} = 0. \tag{10}$$

In obtaining (10) it must be borne in mind that, when $\omega$ and $k$ are allowed to be slowly varying, they are related by

$$\omega = \partial\psi/\partial t\,, \quad k = -\partial\psi/\partial z\,. \tag{11}$$

To continue with the example, we apply (9) and (10) to the Lagrangian (7). Variation with respect to $A_0^2$ gives

$$\epsilon = 1 + \omega_p^2/(\omega_0^2 - \omega^2), \tag{12}$$

which is the well-known dispersion relation. From (11) it follows that

$$\frac{\partial\omega}{\partial z} + \frac{\partial k}{\partial t} = 0, \tag{13}$$

which together with (10) defines the pulse reshaping. Whitham shows in Ref. 4 that (10) and (13) can be recast in the form

$$\frac{\partial}{\partial t}A_0^2 + \frac{\partial}{\partial z}(v_g A_0^2) = 0, \tag{14a}$$

$$\frac{\partial\omega}{\partial t} + v_g\frac{\partial\omega}{\partial z} = 0, \tag{14b}$$

where the group velocity

$$v_g = \frac{\partial\mathcal{L}/\partial k}{\partial\mathcal{L}/\partial\omega}\,,$$

and use has been made of the fact that in any linear problem $\mathcal{L}$ can be written $\mathcal{L} = F(\omega, k)A_0^2$.

Equations (14) govern pulse reshaping due to dispersion in any medium whose response function is linear in the driving field. They show, for example, that the instantaneous frequency (or $k$ vector) of a wave group propagates as a simple wave at the normally defined, frequency-dependent group velocity. Note that (14b) may be nonlinear in $\omega$ even though the problem is linear in $A_0$. Equation (14a) embodies energy conservation and, once (14b) has been solved, can be used to calculate the reshaping of the pulse.

This simple example has provided an introduction to the averaged Lagrangian method, whose full power will be more apparent when it is used to formulate the nonlinear self-steepening problem. Section II contains the derivation of the averaged Lagrangian for pulse propagation in a medium of two-level atoms. Section III contains the equations which are derived from the averaged Lagrangian, as well as their transformation to a form amenable to analytic solution in the case where the optical carrier is well outside the inhomogeneous linewidth of the resonance. Section IV contains a discussion of the method, and some aspects of the solution.

## II. AVERAGED LAGRANGIAN FOR A TWO-LEVEL ATOM IN AN OPTICAL FIELD

The material system of interest in the experiment on self-steepening[1] is essentially a gas of two-level atoms. As is well known, the Schrödinger equation for such a system interacting with an electromagnetic field may be described by the so-called "vector model"[8] whose equations are

$$d\vec{r}/dt = \vec{\omega} \times \vec{r}, \tag{15}$$

where $r_1 = \rho_{12} + \rho_{21}$, $r_2 = i(\rho_{12} - \rho_{21})$, $r_3 = \rho_{11} - \rho_{22}$, $\omega_1 = -\mu E_x/\hbar$, $\omega_2 = -\mu E_y/\hbar$, and $\omega_3 = \omega_0$. The $\rho_{ij}$ are the elements of the density matrix for the two-level system. The upper and lower states, labeled 1 and 2 respectively, are separated in energy by $\hbar\omega_0$. The real number $\mu$ is the one nonzero matrix element of the electric-dipole raising operator

$$(\mu_+)_{\text{op}} = \begin{pmatrix} 0 & \mu \\ 0 & 0 \end{pmatrix}.$$

Equation (15) is in the lab frame; so $E_x$ and $E_y$ include the rapid as well as the slow time and space dependences.

Since we are interested in the behavior of light pulses much shorter than the relaxation times of the two-level system, Eq. (15) does not contain any dissipative terms. Then since $\vec{r}\cdot d\vec{r}/dt = \vec{r}\cdot\vec{\omega}\times\vec{r} = 0$, we have the well-known fact that $\vec{r}\cdot\vec{r} = 1$. It follows that $r_1$, $r_2$, and $r_3$ are not independent quantities and hence are *not suitable* for generalized coordinates in a Lagrangian formulation.[7] Since the endpoint of the vector $\vec{r}$ is confined to a sphere, it is convenient to take as generalized coordinates the angles $\theta$ and $\phi$, where $\theta$ is the angle between $\vec{r}$ and the $\hat{r}_3$ axis, and $\phi$ is the azimuth of the projection of $\vec{r}$ on the $(r_1, r_2)$ plane measured from the positive $\hat{r}_1$ axis. We then have

$$r_1 = \sin\theta\cos\phi, \quad r_2 = \sin\theta\sin\phi, \quad r_3 = \cos\theta. \tag{16}$$

In terms of $\theta$ and $\phi$, Eq. (15) takes the new form

$$\dot{\theta} = (\mu/\hbar)(E_x \sin\phi - E_y\cos\phi), \tag{17a}$$

$$\dot{\phi} = \omega_0 + (\mu/\hbar)\cot\theta\,(E_x\cos\phi + E_y\sin\phi). \tag{17b}$$

It may be *verified by inspection* that Eqs. (17) come from the following Lagrangian:

$$L_{\text{system}} = (\dot{\phi} - \omega_0)\cos\theta + (\mu/\hbar)\sin\theta\,(E_x\cos\phi + E_y\sin\phi). \tag{18}$$

For example, the equation for $\dot{\theta}$ comes from variation of $L_{\text{system}}$ with respect to $\phi$ as follows:

$$\frac{\partial L}{\partial\dot{\phi}} = \cos\theta, \qquad \frac{d}{dt}\frac{\partial L}{\partial\dot{\phi}} = -\sin\theta\,\dot{\theta},$$

$$\frac{\partial L}{\partial\phi} = -\frac{\mu}{\hbar}\sin\theta\,(E_x\sin\phi - E_y\cos\phi),$$

and since the prescription is

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\phi}} = \frac{\partial L}{\partial\phi},$$

we have (17a) immediately.

We turn now to the total Lagrangian for the coupled system of electromagnetic field plus two-level atoms. As in the linear example, the field must be described by its vector potential. In the present discussion, where we are concerned with circularly polarized light, the Lagrangian density for the field is

$$L_{\text{field}} = \frac{1}{8\pi}\left[\frac{1}{c^2}(\dot{A}_x^2 + \dot{A}_y^2) - \left(\frac{\partial A_x}{\partial z}\right)^2 - \left(\frac{\partial A_y}{\partial z}\right)^2\right]. \tag{19}$$

The total Lagrangian density for the problem is

$$L_{\text{total}} = L_{\text{field}} + \tfrac{1}{2}N\hbar\,[(\dot{\phi} - \omega_0)\cos\theta - (\mu/\hbar c)\sin\theta\,(\dot{A}_x\cos\phi + \dot{A}_y\sin\phi)]. \tag{20}$$

We have replaced $E_i$ by $-\dot{A}_i/c$ in (18), and have chosen the multiplicative factor $\frac{1}{2}N\hbar$ so as to give the correct source term in Maxwell's equations. In this connection we note that the $x$ and $y$ components of the electric polarization in the lab frame are

$$\left.\begin{matrix} P_x \\ \\ P_y \end{matrix}\right\} = \tfrac{1}{2}N\mu \sin\theta \times \left\{\begin{matrix} \cos\phi \\ \\ \sin\phi \end{matrix}\right. . \tag{21}$$

The full field equations, which follow from the Lagrangian (20) are

$$\begin{aligned} \frac{1}{c^2}\frac{\partial^2 A_x}{\partial t^2} - \frac{\partial^2 A_x}{\partial z^2} &= \frac{2\pi N\mu}{c}\frac{\partial}{\partial t}\sin\theta\cos\phi, \\ \frac{1}{c^2}\frac{\partial^2 A_y}{\partial t^2} - \frac{\partial^2 A_y}{\partial z^2} &= \frac{2\pi N\mu}{c}\frac{\partial}{\partial t}\sin\theta\sin\phi. \end{aligned} \tag{22}$$

We next average $L_{\text{total}}$ over one cycle of the exact, periodic, traveling-wave solution of Eqs. (17) and (22). Despite the strong nonlinearity of this system, we can indeed find all of the exact traveling-wave solutions. The system (17) and (22) is *solved exactly and uniquely* by the set

$$\begin{aligned} &A_x = A_0 \sin(\omega t - kz), \quad A_y = -A_0\cos(\omega t - kz), \\ &\phi = \omega t - kz + \begin{pmatrix} 0 \\ \pi \end{pmatrix}, \quad \theta = \mp\tan^{-1}\frac{\mu\omega A_0}{\hbar c(\omega_0 - \omega)}, \\ &\left(\frac{ck}{\omega}\right)^2 = 1 \mp \frac{2\pi N\mu^2}{\hbar(\omega_0-\omega)}\left[1 + \left(\frac{\mu\omega A_0}{c\hbar\,(\omega_0-\omega)}\right)^2\right]^{-1/2}. \end{aligned} \tag{23}$$

For each frequency $\omega$, the solution is twofold degenerate; the zero phase in $\phi$ goes with the negative sign in $\theta$ and the plus sign in the dispersion relation, whereas the $\pi$ phase in $\phi$ goes with the opposite signs in the last two equations of (23). Some discussion of these exact circularly polarized solutions was given by Bullough in Ref. 9, who found them independently, and discussed them in the context of self-induced transparency. Equations (23) are for what is called the "sharp-line case," but they are easily generalized to include inhomogeneous broadening, for which there is a distribution of resonance frequencies $g(\omega_0)$, normalized such that $\int_{-\infty}^{\infty} g(\omega_0)\,d\omega_0 = 1$. For this case Eqs. (22) are modified in that the right-hand side in each equation is integrated over the distribution $g(\omega_0)$:

$$\begin{aligned} \frac{1}{c^2}\frac{\partial^2 A_x}{\partial t^2} - \frac{\partial^2 A_x}{\partial x^2} &= \frac{2\pi N\mu}{c}\frac{\partial}{\partial t}\int_{-\infty}^{\infty} g(\omega_0)\sin\theta\cos\phi\, d\omega_0, \\ \frac{1}{c^2}\frac{\partial^2 A_x}{\partial t^2} - \frac{\partial^2 A_g}{\partial z^2} &= \frac{2\pi N\mu}{c}\frac{\partial}{\partial t}\int_{-\infty}^{\infty} g(\omega_0)\sin\theta\sin\phi\, d\omega_0. \end{aligned} \tag{24}$$

Equations (17) are not altered by inhomogeneous broadening. The exact, traveling-wave solutions of (17) and (24) are the set

$$\begin{aligned} &A_x = A_0\sin(\omega t - kz), \quad A_y = -A_0\cos(\omega t - kz), \\ &\phi = \omega t - kz + \begin{pmatrix} 0 \\ \pi \end{pmatrix}, \quad \theta = \mp\tan^{-1}[\mu\omega A_0/\hbar c(\omega_0 - \omega)], \\ &(ck/\omega)^2 = 1 \mp (2\pi N\mu^2/\hbar)\int g(\omega_0) \\ &\qquad \times [(\omega_0-\omega)^2 + (\mu\omega A_0/c\hbar)^2]^{-1/2}\, d\omega_0. \end{aligned} \tag{25}$$

If, as is the case in the self-steepening experiments of Ref. 1, the input light frequency is many inhomogeneous linewidths off resonance, Eqs. (25) reduce to (23). On the other hand, for self-induced transparency experiments, inhomogeneous broadening plays an important role, and (25) must usually be used.

It may be appropriate to emphasize that the solutions obtained in Eqs. (23) and (25) are in no way dependent on an approximation to the Bloch equations (15). In particular, the adiabatic-following approximation[6] has *not* been used. However, it is true that the motion of the pseudomoment (for each $\Delta\omega$) described in (23) is the "prototype" of the motion described as adiabatic following by Grischkowsky. But the adiabatic following approximation holds only when the whole two-level spectrum is far off-resonance compared to its linewidth, i.e., when only single values of $\Delta\omega$ and $\theta$ are required to describe the whole set of two-level atoms. The solution given in (21) holds for more general conditions, and in particular, describes a motion of the pseudomoments for which $r_3$ may be a function of $\Delta\omega$, a situation reminiscent of that found by Bullough[9] for a particular example of SIT.

We now calculate the averaged Lagrangian by integrating (20) over one cycle in the quantity $\psi = \omega t - kz$, using the dependence of $A$, $\theta$, and $\phi$ on $t$ and $z$ given in (25). The result is the following averaged Lagrangian:

$$\begin{aligned} \mathcal{L} &= \frac{1}{2\pi}\int_0^{2\pi} L_{\text{total}}\, d\psi \\ &= \frac{1}{8\pi}\left(\frac{\omega A_0}{c}\right)^2\left[1 - \left(\frac{ck}{\omega}\right)^2\right] \\ &\quad \pm \frac{N\hbar}{2}\int_{-\infty}^{\infty} g(\omega_0)\left[(\omega_0-\omega)^2 + \left(\frac{\mu\omega A_0}{\hbar c}\right)^2\right]^{1/2} d\omega_0. \end{aligned} \tag{26}$$

Equivalently, since $\omega A_0/c = -E_0$, we can write the averaged Lagrangian as

$$\begin{aligned} \mathcal{L}(\omega, k, E_0^2) &= \frac{1}{8\pi}E_0^2\left[1 - \left(\frac{ck}{\omega}\right)^2\right] \\ &\quad \pm \frac{N\hbar}{2}\int g(\Delta\omega)\left[(\Delta\omega)^2 + \left(\frac{\mu E_0}{\hbar}\right)^2\right]^{1/2} d(\Delta\omega) \end{aligned} \tag{27}$$

where $\Delta\omega \equiv \omega_0 - \omega$. In the derivation of $\mathcal{L}$ each of the quantities $\omega, k, E_0^2$ was strictly constant, corresponding to the perfectly periodic wave over which we averaged. Now however, Whitham's method requires that we relax this condition and allow $\omega$, $k$, and $E_0$ to be slowly varying functions of space and time. However, we still require $\omega$ and $k$ to be given by $\omega = \partial\psi/\partial t$, $k = -\partial\psi/\partial z$, where $\psi$ is the instantaneous phase of the wave group. This interconnection of $\omega$ and $k$ must be remembered in writing down the Euler-Lagrange equations which ensure the stationarity of the "averaged action," $\int \mathcal{L}\, dt\, dz$. Variation of $\mathcal{L}$ with respect to $E_0^2$ gives

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{E}_0^2} = 0 = \frac{\partial \mathcal{L}}{\partial E_0^2},$$

*from which we regain the nonlinear dispersion relation*

$$(ck/\omega)^2 = \epsilon = 1 \pm (2\pi N\mu^2/\hbar) \times \int_{-\infty}^{\infty} g(\Delta\omega)[(\Delta\omega)^2 + (\mu E_0/\hbar)^2]^{-1/2}\, d(\Delta\omega). \tag{28}$$

Variation of $\mathcal{L}$ with respect to the phase $\psi$ gives

$$-\frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \omega} + \frac{\partial}{\partial z}\frac{\partial \mathcal{L}}{\partial k} = 0. \tag{29}$$

The derivatives $\mathcal{L}_\omega$ and $\mathcal{L}_k$ are fairly complicated and will be written down later; here we note only that (29) describes the evolution of the instantaneous frequency and wave vector as the pulse is reshaped by the nonlinear medium. Carrying out the indicated total derivatives in (29), we arrive at a nonlinear second-order partial differential equation for the phase of the wave group:

$$\mathcal{L}_{\omega\omega}\psi_{tt} - 2\mathcal{L}_{\omega k}\psi_{zt} + \mathcal{L}_{kk}\psi_{zz} = 0. \tag{30}$$

Note that for this nonlinear problem knowledge of the phase $\psi$ determines *all* properties of the wave packet. The quantities $\omega$ and $k$ are, of course, known from the derivatives of $\psi$, and the amplitude $E_0(t)$ of the group can be calculated from the dispersion relation (28) when $\omega$ and $k$ are known. This is simple only for the sharp-line case, however.

This approach is different in spirit from the slowly-varying-envelope-and-phase approach, which treats amplitude and phase as independent quantities and does not make use of a nonlinear dispersion relation. Of course, not all nonlinear pulse propagation problems may have an associated nonlinear dispersion relation; when they do not, the present method probably will not be useful.

Before applying Eqs. (27) and (30) to the specific problem of pulse self-steepening, it is appropriate to give more attention to the significance of the nonlinear dispersion relation (28) and its applicability to various problems of coherent nonlinear optics. The general behavior of $\epsilon(\omega, E_0)$ is sketched in Figs. 1 and 2 for (a) the sharp-line case, $g(\omega_0) = \delta(\omega_0 - \bar{\omega})$; and (b) the case of inhomogeneous broadening, for which $g(\omega_0)$ is a smooth, more or less bell-shaped function with a single maximum and a narrow fractional width.

Figure 1 gives the shape of the dispersion curves, and Fig. 2 gives the frequency derivative of the nonlinear dielectric constant. In linear dispersion theory $d\epsilon/d\omega$ determines the group delay of a wave packet of steady shape; in nonlinear cases, $d\epsilon/d\omega$ will in general depend on intensity, leading to a distortion of the envelope of a propagating pulse. Moreover, in the nonlinear case the relative sign of $d^2\epsilon/d\omega^2$ and $\epsilon(\omega, E_0) - \epsilon(\omega, 0)$ is important in determining the qualitative nature of the pulse distortion.

Consideration of the sharp-line dielectric constant, Eq. (23), shows that Fig. 1(a) should be thought of as consisting of four branches rather than two, as might first be supposed. This may be seen from the following table, which lists the difference $\phi - \omega t + kz$ along with $r_3 = \cos\theta$ for each

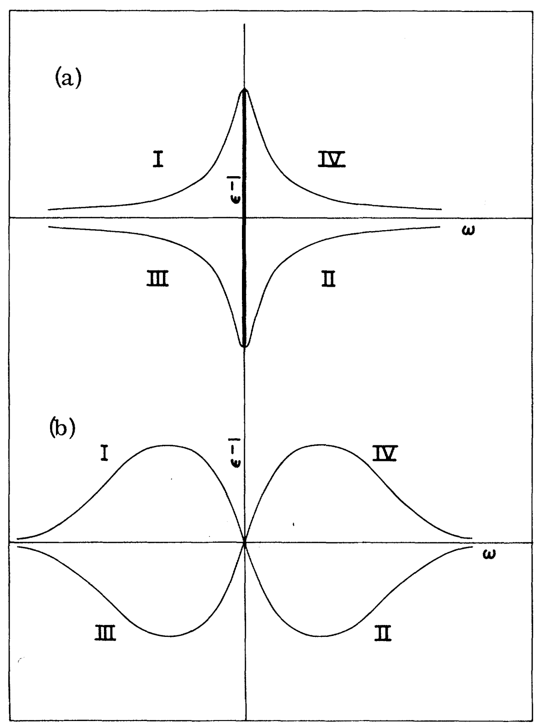


FIG. 1. Dispersion of $\epsilon$; (a) the sharp-line case and (b) for inhomogeneous broadening.

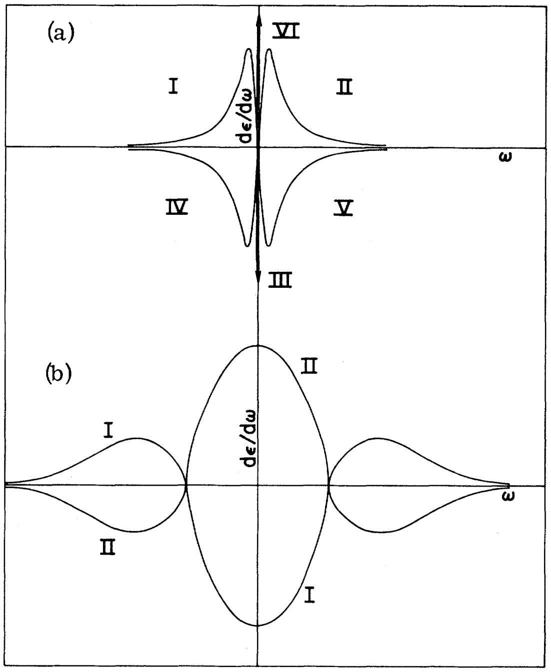


FIG. 2. Dispersion of the group delay for (a) the sharp-line case and (b) for inhomogeneous broadening.

branch:

| Branch | $r_3 = \cos\theta$ | $\phi - \omega t - kz$ |
|---|---|---|
| I | $<0$ | 0 |
| II | $<0$ | $\pi$ |
| III | $>0$ | 0 |
| IV | $>0$ | $\pi$ |

Below resonance the pseudomoment is in-phase with the circularly polarized driving field, whereas above resonance it is 180° out of phase. In each case, the population difference can be either negative or positive, corresponding to net ground- or upper-state population, respectively. But if one wishes to describe an experiment in which $r_3$ is always negative, it is appropriate to use branches I and II of Fig. 1(a). These are separated by a discontinuity in the sharp-line case, but merge smoothly in the case of inhomogeneous broadening, to give the familiar looking dispersion curve I-II of Fig. 1(b). For population inversion, one has branches III and IV of Fig. 1(b).

The group-delay-determining quantity $d\epsilon/d\omega$ is shown in Figs. 2(a) and 2(b). In the sharp-line case the discontinuity in $\epsilon$ gives rise to a $\delta$ function in $d\epsilon/d\omega$ so that, for $r_3 < 0$, $d\epsilon/d\omega$ is given by branches I, II, and III, while for $r_3 > 0$, it is given by IV, V, and VI. When smeared out by inhomogeneous broadening, the curves I and II of Fig. 2(b) apply for $r_3$ negative or positive, respectively.

Note that for $r_3$ negative, the $d\epsilon/d\omega$ is negative on resonance, but positive for $r_3$ positive. In both cases $d\epsilon/d\omega$ is intensity dependent.

*For the self-steepening experiments of Ref. 1, one is always off-resonance far enough for the sharp-line case to apply with $r_3$ negative.* For self-induced transparency the broadened case is appropriate, with a positive, but intensity-independent group delay. Just how this last feature is to be extracted from the averaged Lagrangian method is still under study.

## III. THE SELF-STEEPENING PROBLEM

Since most of the rest of this paper will be concerned with the off-resonance situation characteristic of the self-steepening experiments done in rubidium vapor,[1] we will specialize to that case by writing the averaged Lagrangian and the dispersion relation in a form which automatically gives the correct choice of signs for the case where $r_3$ is always negative. These are

$$\mathcal{L} = \left(\frac{1}{8\pi}\right) E_0^2(1-\epsilon) + \left(\frac{N\hbar}{2}\right)(\Delta\omega)\left[1 + \left(\frac{\mu E_0}{\hbar\Delta\omega}\right)^2\right]^{1/2}, \tag{31}$$

$$\epsilon = \left(\frac{ck}{\omega}\right)^2 = 1 + \frac{2\pi N\mu^2}{\hbar\Delta\omega}\left[1 + \left(\frac{\mu E_0}{\hbar\Delta\omega}\right)^2\right]^{-1/2}, \tag{32}$$

where $\Delta\omega \equiv \omega_0 - \omega$.

The above averaged Lagrangian is to be used in evaluating the partial derivatives which occur in Eq. (30). As it stands, Eq. (30) is highly nonlinear, since its coefficients are functions of $\psi_t$ and $\psi_z$. It may be transformed into a linear equation for a new dependent variable as a function of new independent variables $\omega$ and $k$. This so-called Legendre transformation is

$$\Phi(\omega, k) = kz - \omega t - \psi(z,t), \quad \partial\Phi/\partial\omega = -t, \quad \partial\Phi/\partial k = z. \tag{33}$$

(The phase $\Phi$ must not be confused with the angle $\phi$.) This changes (30) into

$$\mathcal{L}_{\omega\omega}\Phi_{kk} - 2\mathcal{L}_{\omega k}\Phi_{\omega k} + \mathcal{L}_{kk}\Phi_{\omega\omega} = 0, \tag{34}$$

which is a linear partial differential equation with variable coefficients. The boundary conditions for this problem are as follows: at $z=0$, the input to the nonlinear medium, we know (a) that the phase varies as, for example, $\omega_{\text{initial}}t$, and (b) that the amplitude is a given function of $t$, say $E_0(t)$. The inverse of this function is called $t_0(E_0)$. Since the dispersion relation (32) gives $E_0$ as a function of

$\omega$ and $k$, we therefore have $t_0$ as a function of $\omega$ and $k$. Hence by (33) we know $\Phi_\omega$ as a function of $\omega, k$ along a line $\omega = \omega_{\text{initial}}$ in the $(\omega, k)$ plane. Along this same line $\Phi_k = z = 0$, and in addition we may take $\Phi = 0$ without loss of generality. Knowledge of the function and its first derivatives along a line is sufficient (under conditions detailed below) to determine $\Phi$ in a neighboring region of the $\omega, k$ plane, which in turn allows one to compute $E(x, t)$ in a well-defined part of the $z, t$ plane. When the coefficients of (34) are explicitly evaluated it is found that (34) is of the *elliptic* type although it has hyperbolic boundary conditions.[10] The ellipticity of (34) reflects the fact that the dependence of the dielectric constant $\epsilon(\omega, E_0^2)$ on frequency and wave intensity is such that $d^2\epsilon/d\omega^2$ has the opposite sign from $\epsilon(\omega, E_0) - \epsilon(\omega, 0)$. In other physical situations these two quantities may have the same sign, in which case the nonlinear propagation takes on an entirely different character.[11-13]

The physical significance of the ellipticity or hyperbolicity of (34) may be described roughly as follows (see Ref. 13 for more detail): When the partial differential equation (34) is elliptic, any smooth solution will be *unstable* against small amplitude perturbations. Any such perturbation will grow exponentially at least initially, causing a change in pulse shape and height.

On the other hand, when (34) is hyperbolic its solutions of interest have somewhat the character of simple waves.[13] The characteristic feature of such waves derives from the fact that their envelope velocity is intensity dependent. This causes an initial pulse to steepen (either at the front or at the back, depending on whether velocity increases or decreases with increasing intensity); this steepening occurs in simple cases without any increase in pulse height.

Much of the remainder of the paper is concerned with aspects of the solution of (34) for the case of self-steepening. The manipulations and approximations used are analogous to those of Ref. 12 in which Lighthill applies the averaged Lagrangian method to wave trains on deep water, and the reader should consult that paper for a fuller discussion of some of the mathematical details. The important result, for those who wish to skip details, is given in Eq. (42), which expresses the somewhat surprising result that our self-steepening problem can be put into the form of the axially symmetric Laplace equation.

We note first that the dispersion relation (32) can be solved for $E_0^2(\omega, k)$, so that $E_0^2$ can be eliminated from the averaged Lagrangian and the resulting $\mathcal{L}(\omega, k)$ used to evaluate the coefficients in (34). The form of the nonlinear dispersion relation and the averaged Lagrangian suggest that, rather than $\omega$ and $k$ as independent variables, we might better use

$$\nu = (\omega_0 - \omega)/\delta, \tag{35a}$$

$$\tau = \left(\frac{\mu E_0}{\hbar\Delta\omega}\right)^2 \cong -\frac{2\hbar\delta c\nu}{\pi N\mu^2\omega}\,[k - k_0(\omega)], \tag{35b}$$

where $k_0$ is the zero-intensity wave vector, calculated from (32). That is, rather than the absolute frequency $\omega$, what really counts is how far removed the laser frequency is from the resonance frequency; this difference controls both the nonlinearities and the dispersion. The variable $\nu$ is normalized to the carrier-frequency offset at $z=0$, $\delta = \omega_0 - \omega_{\text{initial}}$. Similarly, it is not the absolute strength of the field, $E$, which is significant, rather it is $E$ relative to the "saturation" value $E_s \equiv \hbar\Delta\omega/\mu$. Or, in the language of magnetic resonance, what counts is the ratio of the Rabi-precession frequency $\mu E/\hbar$ to the frequency offset from resonance, $\Delta\omega \equiv \omega_0 - \omega$. In (35) the quantity $k_0(\omega)$ equals $(\omega/c)(1 + \pi N\mu^2/\hbar\Delta\omega)$, and the new independent variable $\tau$ describes not only the ratio of field strengths, but also the extent to which the $k$ vector differs from its low-intensity value due to the strength of the laser field. The sign $\cong$ in (35) indicates that the relation is correct to first order in the difference $[k - k_0(\omega)]$, which, for the dilute vapors and modest intensities of the self-steepening experiments, is a good approximation indeed.

In terms of these new independent variables the averaged Lagrangian (31) takes the rather simple form

$$\mathcal{L} = \frac{N\hbar\delta}{4}\,\frac{(2+\tau)\nu}{(1+\tau)^{1/2}}\,. \tag{36}$$

The differential equation and the boundary conditions transform under (35) to

$$\mathcal{L}_{\tau\tau}\Phi_{\nu\nu} - 4\mathcal{L}_{\nu\tau}\Phi_{\nu\tau} + (\mathcal{L}_{\nu\nu} - J\mathcal{L}_\tau)\Phi_{\tau\tau} - [J\mathcal{L}_{\tau\tau} + (4/\nu)\mathcal{L}_{\nu\tau}]\,\Phi_\tau = 0,$$

$$t = (\Phi_\nu + K\Phi_\tau)/\delta, \quad z = \alpha'\nu\Phi_\tau, \tag{37}$$

where $\alpha' \equiv -2\hbar\delta c/\pi N\mu^2\omega$, $K \equiv \tau/\nu - \alpha'\nu F_\nu$, $J \equiv 2\tau/\nu^2 + \alpha'\nu F_{\nu\nu}$, and where $F \equiv k_0$. The occurrence of $F_\nu$ and $F_{\nu\nu}$ in the transformed equations brings both the group velocity and its dispersion explicitly into the formulation of the problem (since $v_g^{-1} = -F_\nu/\delta$).

The partial derivatives of $\mathcal{L}$ occurring in (37) are simply evaluated using (36), and we find, using the explicit form of $F(\nu)$ given above,

$$\nu^2\left(\frac{2-\tau}{1+\tau}\right)\Phi_{\nu\nu} - 8\tau\nu\Phi_{\nu\tau} + 4\tau(2-\tau)\Phi_{\tau\tau} + 2\left(\frac{4-8\tau-3\tau^2}{1+\tau}\right)\Phi_\tau = 0,$$

$$t = (\Phi_\nu + K\Phi_\tau)/\delta, \quad z = \alpha'\nu\Phi_\tau. \tag{38}$$

The only approximation which has been made in deriving (38) from (30) is that mentioned in connection with (35b).

Before making the simplifying assumptions required for solving (38), we remark that its discriminant $\Delta$ is

$$\Delta = \frac{16\nu^2\tau}{1+\tau}(5\tau^2 + 4\tau - 4). \tag{39}$$

Thus, for $\tau = (\mu E_0/\hbar\Delta\omega)^2 < 2(\sqrt{6}-1)/5 = 0.5798$, the discriminant is negative and the differential equation is elliptic. Hence, in view of the hyperbolic initial conditions, the problem is "improperly posed" in the sense of Ref. 10.

This ellipticity of the partial differential equations describing self-steepening was noted in Ref. 1. In that paper it was incorrectly stated that the discriminant of the reduced equations was "usually negative." In fact, the discriminant of the Eqs. (2a) and (2b) of Ref. 1 is negative for *all* intensities. That result is not in conflict with (39) above, although it might seem that (39) implies that for high intensities the discriminant of (30) changes sign and the equation becomes hyperbolic. The point is that, as pointed out in (35), the connection between $\tau$ and $k$ used in deriving (38) holds only when $\tau$ is small; hence the value of $\Delta$ derived above ceases to be significant when $\tau$ is large enough for $\Delta$ to change sign.

The conditions under which the partial differential equation for dispersive, nonlinear pulse propagation is either elliptic or hyperbolic were set forth clearly by Lighthill.[12] As mentioned previously, what is important is the relative sign of the quantities $(\epsilon - \epsilon_0)$ and $d^2\epsilon_0/d\omega^2$, where $\epsilon(\omega, E)$ is the nonlinear dielectric constant and $\epsilon_0(\omega) = \epsilon(\omega, 0)$ is the low intensity, linear dielectric constant. In the notation of self-focusing, $(\epsilon - \epsilon_0) \approx (\omega/c)(n_2 E^2)$, and $d^2\epsilon_0/d\omega^2$ is proportional to $(d/d\omega)(1/v_g)$. If these two quantities are of the same sign, Eq. (34) is hyperbolic, whereas if they are of opposite sign, (34) is elliptic.

In the present case the dispersion of the group velocity and the nonlinearity of the index of refraction are both due to the two-level resonance and, in the adiabatic-following limit, are of opposite sign. In other cases, e.g., self-steepening due to the Kerr effect in a liquid with normal dispersion,[11] the nonlinearity and the group velocity dispersion are due to different processes and can have the same sign. This makes the governing equation hyperbolic, and the self-steepening which occurs is qualitatively different from what occurs due to adiabatic following.

Equation (38) can be simplified by the following good approximations. It is known from experiment and from numerical calculations[1] that the parameter $\nu = (\omega_0 - \omega)/(\omega_0 - \omega_{\text{initial}})$ remains between 1 and 0.9 during the entire evolution of the self-steepened pulses; we therefore set $\nu = 1$ in the coefficients of (38). On the other hand, the variable $\tau = (\mu E/\hbar\Delta\omega)^2$ was everywhere less than about 0.06. We therefore drop $\tau$ everywhere in the coefficients of (38) where it is added to numbers of order unity. The result is

$$\Phi_{\nu\nu} - 4\tau\Phi_{\nu\tau} + 4\tau\Phi_{\tau\tau} + 4\Phi_\tau = 0, \tag{40}$$

a simpler equation whose discriminant is $\Delta' = 16(\tau^2 - 1)$. In view of the smallness of $\tau$ noted above ($\tau^2 < 0.0036$), the discriminant of (40) is very little different from that of the equation obtained by dropping the $\Phi_{\nu\tau}$ altogether in (40), namely

$$\Phi_{\nu\nu} + 4\tau\Phi_{\tau\tau} + 4\Phi_\tau = 0. \tag{41}$$

Introduction of the variable $s = \sqrt{\tau}$ transforms (38) into

$$\Phi_{\nu\nu} + \Phi_{ss} + (1/s)\Phi_s = 0,$$

$$z = \frac{\alpha'\nu\Phi_s}{2s}, \quad t = \frac{\Phi_\nu + K\Phi_s/2s}{\delta}. \tag{42}$$

But (42) is the Laplace equation for axial symmetry, and thus the averaged Lagrangian method has led naturally to a formulation of the self-steepening problem in terms of well-known equations. Moreover, Lighthill obtained precisely the same Eq. (42) in his treatment of wave groups on deep water.[12]

## IV. DISCUSSION

Much of the program promised in the Introduction has now been carried out. We have given the Lagrangian formulation of the resonant pulse propagation problem, have derived Whitham's averaged Lagrangian, and have shown that the optical problem has the same mathematical structure as the self-steepening of wave trains on deep water.

In this final section we deal with the following matters: First, we make some comments on the method of solving (42) for particular pulse shapes and on the type of analytical results which can be obtained. Second, we compare the "shock distance" which results from the present formulation with the expression derived by Kadomtsev and Karpman[13] for the distance over which small perturbations in amplitude are amplified by $e^1$. Third, we comment on those aspects of recent experiments which can be understood on the basis of the present theory. Fourth, we mention the difficulties which lie in the way of application of the averaged Lagrangian method to other nonlinear optical problems of interest, such as self-induced transparency.

The combination of an elliptic differential equation with hyperbolic boundary conditions can be solved by Garabedian's method of complex-valued characteristics,[14] using the Green's function for the axially symmetric potential equation. The solution for $\Phi$ therefore appears as an integral in the complex plane; moverover, it turns out to be a singular integral. The evaluation of such integrals requires care and experience. In Ref. 12 Lighthill carries through the calculation in some detail for the case of an input pulse shape which is Lorentzian. No purpose would be served by repeating his arguments here; instead we will summarize his results and give some of our own, referring the reader to his paper for the details. A pulse which is initially symmetric around its peak remains symmetric as it distorts. The peak propagates at the low-intensity group velocity of the medium, and the pulse "draws in" so that its energy is compressed into a shorter space, or time. As this happens, the height of the peak increases and the peak distorts in such a way as to cause a cusp to appear at the peak. This occurs at a finite distance $z_{\text{crit}}$, at which point the slope of $\omega_{\text{inst}}$ vs time in the moving frame becomes infinite. Beyond this distance the variations in the pulse shape and instantaneous frequency have become so rapid that higher derivatives, not included in the present analysis, must be taken into account.

All of these qualitative results, obtained specifically for a Lorentzian pulse, apply as well to any *smooth, symmetric* input pulse with a single peak. What changes quantitatively with pulse shape is: (a) a numerical factor in the expression for the critical distance, and (b) the ratio $K$ by which the peak intensity has increased at $z_{\text{crit}}$ compared to its initial value. We have applied Lighthill's technique to an input pulse of somewhat more realistic shape for optical experiments, namely $E_0(t) = E_{\text{peak}} \operatorname{sech}(t/T)$, and find the critical distance to be

$$z_{\text{crit}} = \frac{0.558cT(\hbar\Delta\omega/\mu E_{\text{peak}})}{c/v_g - 1}. \tag{43}$$

For this sech pulse the factor $K$ is 2.25, whereas for the Lorentzian pulse Lighthill found $K = 1.93$. The changes with pulse shape are clearly not drastic.

It is interesting to note a kind of quasilinearity in the solutions of this highly nonlinear problem. That is, for a given pulse shape the fractional increase in peak height at $z_{\text{crit}}$ is independent of the input intensity and pulse width, although, of course, $z_{\text{crit}}$ itself decreases with increasing intensity. This remark embodies an important and somewhat surprising result of the present theory. As we have seen, $z_{\text{crit}}$ is the distance at which the self-steepened pulse shape becomes singular when linear dispersion, nonlinear pulse velocity, and self-phase modulation are taken into account. But the authors of Refs. 13 and 15, in considering this same problem, calculated the rate at which a small perturbation of the pulse amplitude grows due to the inherent instability of pulse shapes in a medium where $\epsilon - \epsilon_0$ and $d^2\epsilon_0/d\omega^2$ have different signs. From the fact that small perturbations tend to grow exponentially, according to the linearized theory, the authors of Ref. 15 inferred that very short, very intense pulses could be produced by passing optical pulses through nonlinear media such as near-resonant alkali metal vapors. But when we compare our expression for $z_{\text{crit}}$ with the distance for growth of a perturbation by $e^1$ derived in Ref. 15, we find that they are identical within a numerical factor of order unity.

Thus, the analytical treatment of the fully nonlinear problem has shown that a singular pulse shape is reached in what amounts to a single $e$-folding distance of the linearized theories. And therefore it is *not* likely that, as speculated in Ref. 15, nonlinear self-steepening in alkali vapors would lead to dramatic decreases in pulse width and increases in pulse intensity. It is true that sharp fronts can be produced,[1] but this occurs before any dramatic change in pulse duration or height. When $(\epsilon - \epsilon_0)$ and $d^2k_0/d\omega^2$ have the same sign, the governing differential equation is hyperbolic, rather than elliptic and even symmetric input pulses develop sharp leading edges (Ref. 11).

The fact that experiments on alkali vapors[1] show sharp fronts as well as increased peak heights, whereas the theory so far describes only symmetrically reshaped pulses, shows the direction in which further theoretical work is needed, viz. on asymmetrically shaped input pulses for the elliptic problem. However, despite the fact that the theory has so far been applied only to symmetric input pulses, the distance observed in Ref. 1 for the formation of a steep front is, within 30–50%, the same as the distance which would be predicted from (43), assuming the input to be symmetric.

Finally we turn again to the application of the averaged Lagrangian method to "on-resonance" phenomena such as self-induced transparency. Here inhomogeneous broadening plays an important role, and in general we expect Eqs. (27) and (28) to give the averaged Lagrangian and dielectric "constant," and Figs. 1(b) and 2(b) to be appropriate.

As has been remarked several times, the relative sign of $(\epsilon - \epsilon_0)$ and $d^2\epsilon_0/d\omega^2$ determines whether (34) is elliptic or hyperbolic. Consideration of (32) and Figs. 1(b) and 2(b) shows that, for symmetric, inhomogeneously broadened lines $(\epsilon - \epsilon_0)$ has a constant sign on each side of line center, but $d^2\epsilon_0/d\omega^2$ *changes* sign about one inhomogeneous linewidth off resonance on either side. Thus, starting from a frequency well below resonance where, as we have seen (34) is elliptic, and increasing the optical carrier frequency one will move into a region, when $d^2\epsilon_0/d\omega^2 = 0$, of hyperbolic behavior which will persist until line center. At line center, the Legendre transformation used to go from (30) to (34) breaks down because both $(\epsilon - \epsilon_0)$ and $d^2\epsilon_0/d\omega^2$ are zero for all intensities. Thus line center is a singular point and will require special considerations, which will have to be based on the highly nonlinear Eq. (30). We recall, however, that the set of periodic traveling-wave solutions given in (25) is the unique, exact solution to (17) and (24) and thus will be the starting point for a treatment of self-induced transparency by the averaged Lagrangian method. This topic will be taken up in a subsequent paper.

## ACKNOWLEDGMENTS

The author has benefitted from conversations with D. Grischkowsky, E. Courtens, P. D. Gerber, and J. C. Slonczewski, and from a thorough reading of the manuscript by D. Grischkowsky. I am indebted to Professor R. K. Bullough for many discussions of these matters and for pointing out the relevance of Ref. 16, in which corrections are made to Whitham's equations so as to include additional effects of dispersion via higher-order derivatives. These corrections will have an effect only near the singularity in pulse shape, and will be included in a subsequent paper.

---

*Work partially supported by the U. S. Office of Naval Research under Contract No. N00014-70-C-0187.

[1]D. Grischkowsky, E. Courtens, and J. A. Armstrong, Phys. Rev. Lett. 31, 422 (1973).

[2]S. L. McCall and E. L. Hahn, Phys. Rev. 183, 457 (1969).

[3]E. Courtens, in *The Laser Handbook*, edited by F. T. Arecchi and E. O. Schulz DuBois (North-Holland, Amsterdam, 1972), p. 1261ff.

[4]G. B. Whitham, J. Fluid Mech. 44, Pt. 2, 373 (1970).

[5]B. W. Knight and G. A. Peterson, Proc. Natl. Acad. Sci. USA 65, 841 (1970).

[6]D. Grischkowsky, in Proceedings of Physics of Quantum Electronics Session, Crystal Mountain, Washington, July 1973 (Addison-Wesley, Reading, Mass., to be published).

[7]H. Goldstein, *Classical Mechanics* (Addison-Wesley, Cambridge, Mass., 1953), Chaps. 1 and 11.

[8]R. P. Feynman, F. L. Vernon, Jr., and R. W. Hellwarth, J. Appl. Phys. 28, 49 (1957).

[9]R. K. Bullough, lectures given at the "Physics of Quantum Electronics" 1970 Summer School at Prescott, Arizona (unpublished).

[10]R. Courant and D. Hilbert, *Methods of Mathematical Physics* (Interscience, New York, 1962), Vol. II, p. 226ff.

[11]F. Shimizu, IEEE J. Quantum Electron. QE-8, 851 (1972).

[12]M. J. Lighthill, J. Inst. Math. Applics. 1, 269 (1965).

[13]B. B. Kadomtsev and V. I. Karpman, Usp. Fiz. Nauk. 103, 193 (1971) [Sov. Phys.-Usp. 14, 40 (1971)].

[14]P. R. Garabedian, J. Math Phys. 36, 192 (1958).

[15]B. Ya. Zel'dovich and I. I. Sobel'man, Zh. Eksp. Teor. Fiz. Pis'ma Red. [JETP Lett. 13, 129 (1971)].

[16]V. H. Chu and C. C. Mei, J. Fluid Mech. 41, 873 (1970).