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Initial-value problem of the one-dimensional wave propagation in a homogeneous random medium

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The initial-value problem of the one-dimensional wave propagation in a homogeneous random medium is treated by means of the "Laplace transform," based again on a group-theoretic consideration introduced in the preceding paper. We first define the "Fourier transform" of a random process regarded as a function on the translation group associated with the homogeneity. The inverse "Fourier transform" then gives a general representation of a nonstationary random process generated by a stationary process. Similarly, we define the "Laplace transform" of a random process vanishing on the negative coordinate axis as well as the "Laplace transform" of its derivatives. The one-dimensional wave equation together with the random initial values can be directly treated by means of the "Laplace-transform" technique and is solved approximately in two Gaussian cases where the random media are represented by the well-known O-U (Ornstein-Uhlenbeck) process and by the Z_0 process having zero spectrum at the origin. Various statistical parameters associated with the solution can be calculated from the stochastic solution by the averaging procedure. It is shown that the behavior of the average wave is quite different between the two cases and that the result is in agreement with that of the preceding paper. The average of the absolute square of the wave is also calculated using the stochastic solution, and its range of validity is discussed by comparing with the previous results.

I. INTRODUCTION

In the preceding paper,[1] which we refer to as I, we have studied the mode of wave propagation in a random medium, introducing a group-theoretic technique related to the medium homogeneity in dealing with the random-wave solution, and we have discovered many features of wave propagation in the one-dimensional (1D) random medium—especially the fact that the wave solution is in the cutoff mode in most cases.

We will give here another application of the group-theoretic consideration. We first introduce a "Fourier transform" of a random function regarded as a function on the translation group. The "inverse transform" is shown to give a general representation of a random function generated by a strictly stationary random process. By the same token, we define a "Laplace transform" of a random function vanishing on the negative coordinate axis and establish some formulas concerning the derivatives of the random function. Although an initial-value or boundary-value problem in the 1D case can be treated using the two independent solutions obtained in I, we can directly deal with the initial-value problem of the stochastic differential equation by means of "Laplace transform." The wave equation together with the initial values can then be transformed into a functional equation, which, in the case of a Gaussian random medium, can be solved by means of the Wiener-Hermite expansion as done in I.

Since the solution is represented as a random function, statistical quantities of interest can be evaluated by the averaging procedure. For a Gaussian medium described by an O-U (Ornstein-Uhlenbeck) process, the average of the wave solution with nonrandom initial values was obtained in Ref. 2 by means of the Fokker-Planck equation, which shows the exponential decrease of the average wave with distance. In this paper we calculate the average wave for both O-U process and Z_0 process and show that the result in the former case agrees with that of Ref. 2 and that the behavior of the average wave in the latter case is quite different: it increases exponentially with distance. The average of the square of the absolute value can be also evaluated using the stochastic solution. The approximate solution by this direct method is reasonably accurate within the region not very far from the initial point. By comparing the average of the absolute square with that of I, it is shown that, because of approximation in solving the equation, the calculation does not always give a correct asymptotic expression in the far region. On the other hand, using the two independent solutions of I in matching the initial condition may not always be accurate because of the approximation, but can well describe a global behavior of the solution on the entire axis. Therefore, the two methods are complementary to each other.

II. TRANSLATION OPERATOR

A. Invariant space

In paper I we introduced the translation operator D^a, $-\infty < a < \infty$, acting on a nonstationary random process $\psi(x, \omega)$ generated by the stationary process $\epsilon(T^x\omega)$, namely,

$$D^a\psi(x, \omega) = \psi(x+a, T^{-a}\omega), \quad -\infty < a < \infty . \tag{2.1}$$

Consider a class of random functions such that

$$D^a\psi(x, \omega) = \Lambda^a\psi(x, \omega), \quad -\infty < a < \infty , \tag{2.2}$$

where Λ^a is an eigenvalue for the operator D^a. Since D^a is a one-parameter group, Λ^a must satisfy

$$\Lambda^{a+b} = \Lambda^a\Lambda^b, \quad \Lambda^0 = 1 , \tag{2.3}$$

that is, Λ^a gives a 1D representation of the group R,

$$\Lambda^a = e^{isa} , \tag{2.4}$$

where s is a complex number. We write an eigenfunction with the eigenvalue e^{isa} as $\phi(x, \omega \mid s)$, that is,

$$D^a\phi(x, \omega \mid s) = e^{isa}\phi(x, \omega \mid s), \tag{2.5}$$

and denote by $\mathfrak{D}_s$ the linear space of such eigenfunctions, which is an invariant space under D^a. Hence, $u(T^x\omega)$, a stationary process generated by $\epsilon(T^x\omega)$, is invariant under D^a,

$$D^a u(T^x\omega) = u(T^x\omega), \tag{2.6}$$

so that it belongs to $\mathfrak{D}_0$. For convenience, we say that a process satisfying (2.5) is s-stationary. An s-stationary process can be generally expressed

$$\phi(x, \omega \mid s) = e^{isx}u(T^x\omega \mid s), \tag{2.7}$$

where $u(T^x\omega \mid s)$ is a stationary process in $\mathfrak{D}_0$, i.e., 0-stationary process. As shown below, a nonstationary process generated by ϵ can be represented as a sum of s-stationary processes; that is, it is decomposed into the elements of the spaces $\mathfrak{D}_s$.

B. "Fourier transform" and "Laplace transform"

To minimize the description we develop the theory in a formal fashion; a little more rigorous theory will be given elsewhere. The random function $D^a\psi(x, \omega)$ appearing in (2.1), with x fixed, can be regarded as another random function with respect to the pair (a, ω). Now we take the Fourier transform of $D^a\psi(x, \omega)$ with respect to a,

$$\phi(x, \omega \mid s) = \int_{-\infty}^{\infty} e^{-isa}D^a\psi(x, \omega)\,da, \quad -\infty < s < \infty , \tag{2.8}$$

which we call "Fourier transform" of $\psi(x, \omega)$. As readily checked, the "Fourier transform" (2.8) satisfies (2.5), that is, it belongs to $\mathfrak{D}_s$. The "inverse transform" of (2.8) is given by[3]

$$\psi(x, \omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi(x, \omega \mid s)\,ds , \tag{2.9}$$

or, using (2.7),

$$\psi(x, \omega) = \int_{-\infty}^{\infty} e^{isx}u(T^x\omega \mid s)\,ds . \tag{2.10}$$

That is to say, $\psi(x, \omega)$ is decomposed into the sum of s-stationary processes. Eq. (2.9) or (2.10) can be considered as a general representation for a nonstationary process generated by $\epsilon(T^x\omega)$. Here we will not dwell on the applications of this representation.

For our purpose we introduce the "Laplace transform" of a random process such that

$$\psi(x, \omega) = 0, \quad x < 0 , \tag{2.11}$$

$$|\psi(x, \omega)| < Ce^{\tau x}, \quad x \to \infty , \tag{2.12}$$

for almost all ω, where C and τ are certain real constants. We write the "Laplace transform" as a "Fourier transform" in the complex domain $\mathrm{Im}\, s < -\tau$ (Ref. 4);

$$\phi(x, \omega \mid s) = \int_{-x}^{\infty} e^{-isa}D^a\psi(x, \omega)\,da, \quad \mathrm{Im}\, s < -\tau . \tag{2.13}$$

The inverse transform is then given by

$$\psi(x, \omega) = \frac{1}{2\pi}\int_{-\infty-i\rho}^{\infty-i\rho} \phi(x, \omega \mid s)\,ds, \quad \rho > \tau , \tag{2.14}$$

$$= \frac{1}{2\pi}\int_{-\infty-i\rho}^{\infty-i\rho} e^{isx}u(T^x\omega \mid s)\,ds . \tag{2.15}$$

Since $\phi(x, \omega \mid s)$ belongs to $\mathfrak{D}_s$, s being a complex number, $\phi(x, \omega \mid s)$ is again written in the form (2.7). Now we give the formulas for the "Laplace transform" of the derivative similar to the ordinary Laplace transform. Let us denote the "Fourier transform" of $\nabla\psi(x, \omega)$ and $\nabla^2\psi(x, \omega)$ by $\phi_1(x, \omega \mid s)$ and $\phi_2(x, \omega \mid s)$, respectively. Then

$$\phi_1(x, \omega \mid s) = e^{isx}[-\psi_0(T^x\omega) + (is + \nabla_x)u(T^x\omega \mid s)], \tag{2.16}$$

$$\phi_2(x, \omega \mid s) = e^{isx}[-\psi_0'(T^x\omega) - (is + \nabla_x)\psi_0(T^x\omega) + (is + \nabla_x)^2u(T^x\omega \mid s)], \tag{2.17}$$

where $\psi_0(T^x\omega)$ and $\psi_0'(T^x\omega)$ are stationary processes derived from the random initial values,

$$\psi_0(\omega)=\psi(0,\omega)\,, \tag{2.18}$$

$$\psi_0'(\omega)=\nabla\psi(0,\omega)\,. \tag{2.19}$$

The proofs are given in the Appendix. When the initial values are independent of ω, (2.17) is a little simplified.

III. INITIAL-VALUE PROBLEM OF WAVE EQUATION

As in I, let the wave equation for a 1D random medium be

$$\nabla^2\psi(x,\omega)+k^2[1+\epsilon(T^x\omega)]\,\psi(x,\omega)=0\,, \tag{3.1}$$

where $\epsilon(T^x\omega)$ is a real stationary process in the strict sense. We assume that $\psi(x,\omega)=0$ for $x<0$ and that the initial values at $x=0$ are given by (2.18) and (2.19). Since the operator D^a commutes with ∇^2 and $\epsilon(T^x\omega)$, we can take the "Laplace transform" of the wave equation (3.1):

$$-\psi_0'(T^x\omega)-(is+\nabla_x)\psi_0(T^x\omega)+[(is+\nabla_x)^2+k^2(1+\epsilon(T^x\omega))]\,u(T^x\omega\mid s)=0\,, \tag{3.2}$$

where $u(T^x\omega\mid s)$ is given by (2.13) and (2.7). As the equation (3.2) involves only the stationary processes generated by $\epsilon(T^x\omega)$, we can solve it at $x=0$, regarding it as a functional equation for $u(\omega\mid s)$. For simplicity, we assume in the following that the initial values are real and nonrandom, and denote

$$\psi_0=\psi(0,\omega)\,, \tag{3.3}$$

$$\psi_0'=\nabla\psi(0,\omega)\,. \tag{3.4}$$

Then (3.2) is written

$$[(is+\nabla_x)^2+k^2(1+\epsilon(T^x\omega))]\,u(T^x\omega\mid s)=\psi_0'+is\psi_0\,. \tag{3.5}$$

However, the random initial values would not affect the nature of the following treatment.

As in I, we now solve (3.5) in the case where $\epsilon(T^x\omega)$ is a Gaussian process of the form (IA22):

$$\epsilon(T^x\omega)=\int_{-\infty}^{\infty}G(s_1)e^{is_1x}\,dB(s_1)\,. \tag{3.6}$$

We expand the stationary process $u(T^x\omega\mid s)$ in the following form,

$$u(T^x\omega\mid s)=F_0(s)\Big(1+\int_{-\infty}^{\infty}F_1(s_1\mid s)e^{is_1x}dB(s_1)+\int\int_{-\infty}^{\infty}F_2(s_1,s_2\mid s)e^{i(s_1+s_2)x}\,\hat{h}^{(2)}[dB(s_1),dB(s_2)]+\cdots\Big), \tag{3.7}$$

where $F_n(s_1,\ldots,s_n\mid s)$ is symmetric in $(s_1,\ldots,s_n)$ and is supposed to be analytic in s in the domain $\mathrm{Im}\,s<-\tau$ for a certain τ. Substituting (3.6) and (3.7) into (3.5) and using (IA8), (IA10), and (IA14), we obtain a set of nonlinear function equations for F_n's. The equations corresponding to the degree $n=0,1,2$ of the multiple integrals are[5]

$$\Big(k^2-s^2+k^2\int_{-\infty}^{\infty}G^*(s_1)F_1(s_1\mid s)\,ds_1\Big)F_0(s)=\psi_0+is\psi_0'\,, \tag{3.8}$$

$$[(s+s_1)^2-k^2]F_1(s_1\mid s)-2k^2\int_{-\infty}^{\infty}G^*(s_2)F_2(s_1,s_2\mid s)\,ds_2=k^2G(s_1)\,, \tag{3.9}$$

$$[(s+s_1+s_2)^2-k^2]F_2(s_1,s_2\mid s)-\tfrac{1}{2}k^2[G_1(s_1)F_1(s_2\mid s)+G(s_2)F_1(s_1\mid s)]-3k^2\int_{-\infty}^{\infty}G^*(s_3)F_3(s_1,s_2,s_3\mid s)\,ds=0\,. \tag{3.10}$$

These equations can be solved approximately provided that the excitation term $G(s_1)$ or, what is the same thing, the parameter

$$\sigma^2=\langle\epsilon^2\rangle=\int_{-\infty}^{\infty}|G(s_1)|^2\,ds_1 \tag{3.11}$$

is small enough. We solve for F_n's so as to ensure the analyticity in the domain $\mathrm{Im}\,s<-\tau$ for a certain τ. Once F_n's are obtained, we get $u(T^x\omega\mid s)$ by (3.7) and in turn $\psi(x,\omega)$ by (2.15). By means of the stochastic solution for $\psi(x,\omega)$ so obtained, we can evaluate some statistical parameters concerning $\psi(x,\omega)$, such as the average, covariance, average of square amplitude, etc. We give here the expressions for $\langle\psi\rangle$ and $\langle|\psi|^2\rangle$ by means of (3.7) and (IA14):

$$\langle\psi(x,\omega)\rangle=\frac{1}{2\pi}\int_{-\infty-i\tau}^{\infty-i\tau}e^{isx}F_0(s)\,ds\,, \tag{3.12}$$

$$\langle|\psi(x,\omega)|^2\rangle=\left|\frac{1}{2\pi}\int_{-\infty-i\tau}^{\infty-i\tau}F_0(s)e^{isx}\,ds\right|^2+\int_{-\infty}^{\infty}ds_1\left|\frac{1}{2\pi}\int_{-\infty-i\tau}^{\infty-i\tau}F_1(s_1\mid s)F_0(s)e^{isx}\,ds\right|^2+\int\int_{-\infty}^{\infty}ds_1\,ds_2\left|\frac{1}{2\pi}\int_{-\infty-i\tau}^{\infty-i\tau}F_2(s_1,s_2\mid s)F_0(s)e^{isx}\,ds\right|^2+\cdots\,. \tag{3.13}$$

Owing to the analyticity of F_n's, we have

$$\langle \psi(x,\omega)\rangle = 0, \quad x<0, \tag{3.14}$$

$$\langle |\psi(x,\omega)|^2\rangle = 0, \quad x<0. \tag{3.15}$$

IV. AVERAGE WAVE

The average of wave solution has been studied in many papers. To compare with some results in the one-dimensional case of Paper I and Ref. 2, we take up two random media described again by the O-U process and Z_0 process. For these processes we have [see (IA26) and (IA29)]

$$G(s) = \sigma\left(\frac{\kappa}{\pi}\right)^{1/2}\frac{1}{\kappa+is} \quad \text{(O-U)}, \tag{4.1}$$

$$G(s) = \sigma\left(\frac{2\kappa}{\pi}\right)^{1/2}\frac{is}{(\kappa+is)^2} \quad (Z_0), \tag{4.2}$$

the poles of which lie on the upper-half plane. We first look for the approximate solution for $F_0(s)$ to evaluate $\langle \psi(x,\omega)\rangle$ by means of (3.12). In the first approximation we obtain

$$F_1(s_1\,|\,s) \cong \frac{k^2 G(s_1)}{(s+s_1)^2 - k^2} \tag{4.3}$$

from (3.9) neglecting F_2. We insert (4.3) into the integral term of (3.8):

$$M(s) \equiv k^4 \int_{-\infty}^{\infty} \frac{|G(s_1)|^2\, ds_1}{(s_1+s)^2 - k^2} = \frac{\sigma^2 k^4}{(s-i\kappa)^2 - k^2} \quad \text{(O-U)}, \tag{4.4}$$

$$= \sigma^2 k^4 \frac{s^2 + \kappa^2 - k^2}{[(s-i\kappa)^2 - k^2]^2} \quad (Z_0). \tag{4.5}$$

We denote (4.4) and (4.5) by $M_1(s)$ and $M_2(s)$, respectively. On determining the contour of integration relative to the poles, we have assumed that s lies well below the real axis to ensure the analyticity. Substituting (4.4) and (4.5) into (3.8), we have

$$F_0(s) \cong \frac{-(\psi_0' + is\psi_0)}{s^2 - k^2 - M_1(s)} \quad \text{(O-U)}, \tag{4.6}$$

$$F_0(s) \cong \frac{-(\psi_0' + is\psi_0)}{s^2 - k^2 - M_2(s)} \quad (Z_0). \tag{4.7}$$

The poles of (4.6) and (4.7) are determined by the algebraic equations

$$(s^2 - k^2)[(s-i\kappa)^2 - k^2] - \sigma^2 k^4 = 0 \quad \text{(O-U)}, \tag{4.8}$$

$$(s^2 - k^2)[(s-i\kappa)^2 - k^2]^2 - \sigma^2 k^4 [s^2 + \kappa^2 - k^2] = 0 \quad (Z_0). \tag{4.9}$$

Denoting the four roots of (4.8) by $k_1 \sim k_4$, we obtain

$$\left.\begin{aligned}
k_1 &\cong k - \frac{\sigma^2 k^3(\kappa - 2ik)}{2\kappa(\kappa^2 + 4k^2)},\\
k_2 &\cong -k + \frac{\sigma^2 k^3(\kappa + 2ik)}{2\kappa(\kappa^2 + 4k^2)},\\
k_3 &\cong k + i\kappa - \frac{\sigma^2 k^3(\kappa + 2ik)}{2\kappa(\kappa^2 + 4k^2)},\\
k_4 &\cong -k + i\kappa + \frac{\sigma^2 k^3(\kappa - 2ik)}{2\kappa(\kappa^2 + 4k^2)},
\end{aligned}\right\} \quad \text{(O-U)}. \tag{4.10}$$

assuming σ is small. The imaginary parts of (4.10) are all positive; hence, we have at least $-\tau < \mathrm{Im}\, k_1$. Similarly we obtain the six roots $k_1 \sim k_6$ of (4.9):

$$\left.\begin{aligned}
k_1 &\cong k + \frac{\sigma^2 k^3(\kappa^2 - 4k^2 - 4i\kappa k)}{2(\kappa^2 + 4k^2)^2},\\
k_2 &\cong -k - \frac{\sigma^2 k^3(\kappa^2 - 4k^2 + 4i\kappa k)}{2(\kappa^2 + 4k^2)^2},\\
k_3, k_4 &\cong k + i\kappa \pm \frac{\sigma k}{2}\left(\frac{2k}{2k + i\kappa}\right)^{1/2},\\
k_5, k_6 &\cong -k + i\kappa \pm \frac{\sigma k}{2}\left(\frac{2k}{2k - i\kappa}\right)^{1/2},
\end{aligned}\right\} \quad (Z_0). \tag{4.11}$$

where the pairs k_3, k_4 and k_5, k_6 are degenerate when $\sigma = 0$. We see that the roots k_1 and k_2 lie on the lower half-plane, whereas $k_3 \sim k_6$ are on the upper half-plane: hence, $-\tau < \mathrm{Im}\, k_1 < 0$.

Substituting (4.6) together with (4.10) into (3.12), we obtain

$$\langle \psi(x,\omega)\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{isx}\frac{-(\psi_0' + is\psi_0)}{s^2 - k^2 - M_1(s)}\, ds \quad \text{(O-U)}, \tag{4.12}$$

$$= -\frac{i}{2k}\left[(\psi_0' + ik_1\psi_0)e^{ik_1x} - (\psi_0' + ik_2\psi_0)e^{ik_2x}\right], \quad x>0, \tag{4.13}$$

$$= 0, \quad x<0,$$

where we have shown only the first two dominant terms arising from k_1 and k_2; the other two terms due to k_3 and k_4 are of the order of σ^2. Equations (4.12) and (4.10) agree with the results of Ref. 2 by the Fokker-Planck method. Similarly, using (4.7) and (4.11) in (3.12), we get

$$\langle \psi(x,\omega)\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{isx}\frac{-(\psi_0' + is\psi_0)}{s^2 - k^2 - M_2(s)}\, ds \quad (Z_0), \tag{4.14}$$

the first two dominant terms of which are again written in the form (4.13), where k_1 and k_2 are given by (4.11). The other four terms due to $k_3 \sim k_6$ have the order of σ but at $x = 0$ they are reduced to the order of σ^2.

We note that (4.13) is reduced to the correct nonrandom solution when $\sigma = 0$ in both cases, and that the initial conditions (3.3) and (3.4) are met up to the order of σ^2:

$$\langle \psi(0,\omega)\rangle = \psi_0 + O(\sigma^2), \qquad (4.15)$$

$$\langle \nabla\psi(0,\omega)\rangle = \psi_0' + O(\sigma^2). \qquad (4.16)$$

We further note that, since $\operatorname{Im} k_1 = \operatorname{Im} k_2 > 0$ for the O-U process, the amplitude of (4.13) decreases exponentially as $x\to\infty$ in such a way that

$$|\langle \psi(x,\omega)\rangle| \sim \exp\left(-\frac{\sigma^2 k^4}{\kappa(\kappa^2+4k^2)}x\right) \quad \text{(O-U)}, \qquad (4.17)$$

and on the other hand that, since $\operatorname{Im} k_1 = \operatorname{Im} k_2 < 0$ for the Z_0 process, the amplitude of (4.13) increases exponentially as $x\to\infty$ in such a way that

$$|\langle \psi(x,\omega)\rangle| \sim \exp\left(\frac{2\sigma^2\kappa k^4}{(\kappa^2+4k^2)^2}x\right) \quad (Z_0). \qquad (4.18)$$

This striking difference between the two random media was already physically explained in I in terms of the spectra of these two processes. As a matter of fact, (4.17) agrees with (I4.65), whereas (4.18) equals $e^{\alpha_0 x}$, α_0 being given by (I4.56).

V. AVERAGE OF SQUARE AMPLITUDE

We evaluate $\langle |\psi(x,\omega)|^2\rangle$ by means of (3.13) neglecting F_2 term. The first term of (3.13), however, is just the square of the average obtained in the previous section, so that we calculate the second term. To obtain F_1, we go into a one-step higher approximation than in Sec. IV. Neglecting F_3 in (3.10), we obtain

$$F_2(s_1,s_2\,|\,s) \cong \frac{k^2}{2}\,\frac{G(s_1)F_1(s_2\,|\,s)+G(s_2)F_1(s_1\,|\,s)}{(s+s_1+s_2)^2-k^2} \qquad (5.1)$$

and substitute this into (3.9):

$$\left((s+s_1)^2-k^2-k^4\int_{-\infty}^{\infty}\frac{|G(s_2)|^2\,ds_2}{(s+s_1+s_2)^2-k^2}\right)F_1(s_1\,|\,s) - k^4G(s_1)\int_{-\infty}^{\infty}\frac{G^*(s_2)F_1(s_2\,|\,s)}{(s+s_1+s_2)^2-k^2}\,ds_2 = k^2G(s_1). \qquad (5.2)$$

Equation (4.3) may be used to evaluate the second integral in the left-hand side, which, however, can be neglected compared to $k^2G(s_1)$ on the right-hand side. Hence

$$F_1(s_1\,|\,s) \cong \frac{k^2G(s_1)}{(s+s_1)^2-k^2-M(s+s_1)}. \qquad (5.3)$$

We notice that the integral in the denominator has the same form as the integral of (4.4) and (4.5). Then (5.3) can be written

$$F_1(s_1\,|\,s) \cong \frac{k^2G(s_1)}{(s+s_1)^2-k^2-M_1(s+s_1)} \quad \text{(O-U)}, \qquad (5.4)$$

$$F_1(s_1\,|\,s) = \frac{k^2G(s_1)}{(s+s_1)^2-k^2-M_2(s+s_1)} \quad (Z_0), \qquad (5.5)$$

Again we notice that the denominator of (5.4) or (5.5) has the same form as that of $F_0(s)$, namely, (4.6) or (4.7). Therefore, the zeros of the denominator as a function of s agree with $k_i - s_1$ [$i = 1\sim 4$ for (4.10) and $i = 1\sim 6$ for (4.11)].

Using (5.4) and (5.5), we evaluate approximately the second term of (3.13) as follows. As in Sec. IV, the terms due to k_i, $i\geq 3$, give higher-order terms in σ. So, in order to eliminate those higher-order terms, we approximate F_0 and F_1 by

$$F_0(s) = \frac{-(\psi_0'+is\psi_0)}{(s-k_1)(s-k_2)}, \qquad (5.6)$$

$$F_1(s_1\,|\,s) = \frac{k^2G(s_1)}{(s+s_1-k_1)(s+s_1-k_2)}, \qquad (5.7)$$

where, by (4.10) and (4.11), k_1 and k_2 can be written in the form

$$k_1 = k' + ik'', \qquad k_2 = -k' + ik'', \qquad (5.8)$$

$$k' = k - \frac{\sigma^2k^3}{2(\kappa^2+4k^2)}, \quad k'' = \frac{\sigma^2k^4}{\kappa^2+4k^2}, \quad \text{(O-U)}, \qquad (5.9)$$

$$k' = k + \frac{\sigma^2k^3(\kappa^2-4k^2)}{2(\kappa^2+4k^2)^2}, \quad k'' = -\frac{\sigma^2 2\kappa k^4}{(\kappa^2+4k^2)^2}, \quad (Z_0). \qquad (5.10)$$

Thus,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}F_1(s_1\,|\,s)F_0(s)e^{isx}\,ds = -ik^2G(s_1)e^{-k''x}\left(\frac{(\psi_0'+ik_1\psi_0)e^{ik'x}-[\psi_0'+i(k_2-s_1)\psi_0]e^{-i(k'+s_1)x}}{2k's_1(s_1+2k')} - \frac{(\psi_0'+ik_2\psi_0)e^{-ik'x}-[\psi_0'+i(k_1-s_1)\psi_0]e^{i(k'-s_1)x}}{2k's_1(s_1-2k')}\right), \qquad (5.11)$$

which is to be substituted into

$$\langle|\psi|^2\rangle - |\langle\psi\rangle|^2 = \int_{-\infty}^{\infty}ds_1\left|\frac{1}{2\pi}\int_{-\infty}^{\infty}F_1(s_1\,|\,s)F_0(s)e^{isx}\,ds\right|^2. \qquad (5.12)$$

It should be noticed that the apparent poles at $s_1 = 0, \pm 2k'$ are not the true poles of (5.11) as a whole. Hence, upon integrating (5.12), we can move the contour off (e.g., below) the real axis, and then integrate it termwise by means of residue calculus. Because of the double poles, we obtain the terms proportional to $xe^{-2k''x}$ which can become dominant at large x. However, we will not give here the full expression for (5.12) at length. Instead we examine only the dominant terms at large x:

$$\langle |\psi|^2\rangle - |\langle\psi\rangle|^2 \sim \tfrac{1}{8}\pi x e^{-2k''x}\{[\,|\psi_0' + ik_1\psi_0|^2 + |\psi_0' + ik_2\psi_0|^2][\,|G(2k)|^2 + |G(0)|^2\,] + \mathrm{Re}(\psi_0' + ik_1\psi_0)(\psi_0' + ik_2\psi_0)^* e^{2ikx}\,|G(0)|^2\}, \tag{5.13}$$

where we have put $k' = k$. By (5.9), $k'' > 0$ for the O-U process, so that (5.13) is still decreasing at large x despite the factor x.

In view of the result in I, a general solution is composed of the two independent solutions, the increasing and decreasing cutoff modes, and accordingly, the asymptotic form is determined by the increasing part. Therefore, the correct asymptotic expression should be given by $\langle |\psi|^2\rangle \sim e^{2\alpha_0 x}$, α_0 (>0) being given by (I 4.55) for the O-U process and (I 4.56) for the Z_0 process. Thus we see that (5.13) cannot always be employed for very large x and that (5.12) can be used in a reasonably small range close to the initial point $x = 0$. This is due to the approximation in solving the function equations for F_n's.

APPENDIX

Formulas (2.16) and (2.17) can be demonstrated either by the partial integration or by using the delta function. We give here the proof by the latter method which is shorter than by the former.

Using the delta function, we put the initial value (3.3) into the integrand:

$$\int_{-\infty}^{\infty} e^{-isa} D^a \nabla\psi(x,\omega)\,da = \int_{-\infty}^{\infty} e^{-isa} D^a[\psi_0(\omega)\delta(x)]\,da + \int_{-x+0}^{\infty} e^{-isa} D^a \nabla\psi(x,\omega)\,da\,,$$

which can be written

$$e^{isx}(is + \nabla_x)u(T^x\omega\,|\,s) = \int_{-\infty}^{\infty} e^{-isa}\psi_0(T^{-a}\omega)\delta(x+a)\,da + \phi_1(x,\omega\,|\,s)$$
$$= e^{isx}\psi_0(T^x\omega) + \phi_1(x,\omega\,|\,s)\,,$$

yielding (2.16). Similarly,

$$\int_{-\infty}^{\infty} e^{-isa} D^a \nabla^2\psi(x,\omega)\,da = \int_{-\infty}^{\infty} e^{-isa} D^a[\psi_0(\omega)\delta'(x) + \psi_0'(\omega)\delta(x)]\,da + \int_{-x+0}^{\infty} e^{-isa} D^a \nabla^2(x,\omega)\,da\,,$$

$$e^{isx}(is + \nabla_x)^2 u(T^x\omega\,|\,s) = \nabla_x[e^{isx}\psi_0(T^x\omega)] + e^{isx}\psi_0'(T^x\omega) + \phi_2(x,\omega\,|\,s)$$
$$= e^{isx}[(is + \nabla_x)\psi_0(T^x\omega) + \psi_0'(T^x\omega)] + \phi_2(x,\omega\,|\,s)\,,$$

which gives (2.17).

[1]H. Ogura, preceding paper, Phys. Rev. A 11, 942 (1975).

[2]U. Frisch, *Probabilitistic Method in Applied Mathematics*, edited by A. T. Bharuha-Reid (Academic, New York, 1968), Vol. 1.

[3]A more plausible argument is as follows. $\psi(x,\omega)$ can be interpreted as a slowly-increasing generalized function in $\mathcal{S}_x'$ [see L. Schwartz, *Méthodes Mathématiques pour les Sciences Physique* (Herman, Paris, 1961)] for almost all ω. Then $\psi(a, T^{-a}\omega)$ belongs to $\mathcal{S}_a'$. Its Fourier transform gives $u(\omega|s) = \phi(0,\omega|s)$, which again belongs to $\mathcal{S}_s'$. In its inverse Fourier transform, we change the variables $a \to x$, $\omega \to T^x\omega$ to obtain (2.10).

[4]In much the same way as in Ref. 3, we regard $\psi(x,\omega)$ as a generalized function in $\mathfrak{D}_+'$ (again see Schwartz, Ref. 3) for almost all ω, such that $e^{-\rho x}\psi(x,\omega) \in \mathcal{S}_x'$, $\rho > \tau$. Then, $\psi(a, T^{-a}\omega)e^{-\rho a} \in \mathcal{S}_a'$, so that we can repeat the above argument.

[5]If $\epsilon(T^x\omega)$ has higher-degree terms in the Wiener-Hermite expansion as in (I 3.10), we have more excitation terms due to $G_2, G_3, \ldots$ in these equations. Similarly, if the initial values are random variables generated by $\epsilon(T^x\omega)$, and are expanded in terms of multiple Wiener integrals, we again have the other source of excitation.