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Theory of waves in a homogeneous random medium

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A novel theory is developed to cope with the difficulty of the multiple-scattering problem in a random medium (RM). The theory is given for a one-dimensional homogeneous RM which is represented by a strictly stationary random process. Some possible forms of the stochastic solution are determined by a group-theoretic consideration based on the shift-invariance property of the homogeneous RM. The form of the solution has some analogy with Floquet's solution for a periodic medium. It is shown that there are two kinds of solutions in the one-dimensional RM: a travelling-wave mode and a cutoff mode. The former exists only when the power spectrum of the medium becomes zero at nearly double the wave number. Otherwise the wave is in the cutoff mode which is almost a standing wave whose envelope increases or decreases exponentially with distance. For a Gaussian RM with small fluctuations, an approximate stochastic solution given in the possible form is obtained in terms of multiple Wiener integrals with respect to the Brownian-motion process. The average and variance are calculated for the phase and amplitude of the wave in terms of the power spectrum of the RM. The law of large numbers is shown to hold concerning the fluctuations of the phase and amplitude. The average value of the wave and the transmission coefficient of a medium with finite thickness are also studied using the stochastic solution.

I. INTRODUCTION

There are a number of papers concerning this subject and diverse methods have been introduced concerning many related problems; some well-known techniques are the Fokker-Planck equation and the quantum-field-theoretic method in the theory of multiple scattering in long-distance propagation.[1] Some specific properties in the one-dimensional case were rigorously obtained: for instance, the case of Gaussian white noise,[2] Poisson-distributed random scatterers,[3,4] the O-U (Ornstein-Uhlenbeck) process,[5] a random stack of dielectric slabs,[6] etc. However, these methods, particularly suited to one dimension, are not applicable to the three-dimensional case.

In this paper we present a novel theory based on a different point of view. Although the theory is intended to be applicable to the three-dimensional (3D) case, we start with the 1D case here for comparison with the known results in above works. Therefore, the refractive index of the medium is assumed to be a strictly stationary random process on the 1D coordinate. In our analysis we make use of the medium homogeneity, that is, the shift-invariance property of the stationary process, and determine the possible form of the stochastic solution by a group-theoretic consideration based on the invariance. At this point, we meet the problem of representing the translation group by means of a random matrix, which, however, is not completely answered. The 1D representation gives a possible form of solution which bears an analogy with the well-known Floquet solution in the case of a periodic medium.[7]

To obtain a concrete solution, we assume that the random medium is a Gaussian stationary process generated by the Brownian-motion process, and accordingly the solution given in a possible form is expressed in terms of multiple Wiener integrals.[8,9] The wave equation is then transformed into a hierarchical set of nonlinear function equations to be solved approximately. In order to demonstrate some specific characteristics of the solution depending on the medium, we take up three Gaussian processes for the random medium: the O-U process, a well-known Markov process with Lorentzian spectrum; a double Markov process having zero spectrum at zero spatial frequency, which we call the Z_0 process; and a triple Markov process having zero spectrum at a nonzero frequency α, which we call the Z_α process. We point out that in solving a differential equation it is convenient to introduce the alternative representation of the multiple Wiener integrals based on the Fourier-transformed Brownian-motion process. The definitions and related formulas concerning the alternative representation as well as the three Gaussian processes are summarized in Appendix A for reference.

It is shown that, depending on the spectrum of the medium, there are two modes of solution, which we call the traveling-wave mode and the cutoff mode. The former is a traveling wave whose phase and amplitude are randomly modulated; it exists only when the power spectrum of the medium becomes zero at nearly double the wave number. Otherwise the solution is in the cutoff mode: it is almost a randomly modulated standing wave whose envelope increases or decreases exponentially with distance without energy transfer. Since the wave is in the cutoff mode in most cases, we investigate this mode in some detail. The approximate solution is obtained for small random fluctuations of the medium. The average and the variance are calculated for the phase and the amplitude of the wave in terms of the power spectrum of the medium. Using the stochastic solution, further studies of the wave are made concerning the average value of the wave and the transmission coefficient of the medium with finite thickness; they are compared with some results of the previous works. Other boundary-value problems, such as the Green's function in the random medium or the excitation problem, can be treated using the two independent stochastic solutions. The initial-value problem such as studied in Ref. 5, although particular to the 1D problem, can be treated systematically by means of the integral-transform method introduced again by a group-theoretic consideration: It is the subject of the following paper. The results obtained in this paper are experimentally verified by the computer simulation of random media, which will appear in a subsequent paper.

Finally we briefly refer to possible generalizations of the present theory. The method introduced here is also applicable to a lossy random medium immediately and to the other non-Gaussian random media, such as a medium with random point scatterers; in this case the multiple Wiener integral with respect to the Poisson process can be useful.[9] In the 3D case, the refractive index is then assumed to be a homogeneous or a homogeneous and isotropic random field. The underlying group is the group of motions in the 3D space (Euclidean-motion group). These problems will be studied in later works.

II. SOLUTION OF THE WAVE EQUATION IN AN HOMOGENEOUS RANDOM MEDIUM

Let the 1D wave equation be

$$\nabla^2\psi(x,\omega)+k^2n^2(x,\omega)\psi(x,\omega)=0,\quad \nabla\equiv\frac{d}{dx}, \tag{2.1}$$

where ω is the probability parameter denoting a sample point in the sample space Ω. Let the square of the random refractive index be

$$n^2(x,\omega)=1+\epsilon(x,\omega), \tag{2.2}$$

where $\epsilon(x,\omega)$ is the small fluctuating part with zero mean: It is assumed to be a real stationary process in the strict sense. We denote by $\mathfrak{B}$ the smallest σ algebra of ω sets, by P the probability measure on $\mathfrak{B}$, $\langle\ \rangle$ the average over Ω, i.e., the integration with respect to P, and by R the line $-\infty<x<\infty$ as well as the additive group of the real numbers x on the line.

A translation of a sample function $\epsilon(x,\omega)$ by the distance a induces a measure-preserving transformation T^a in Ω carrying ω into $T^a\omega$ [10]:

$$\epsilon(x+a,\omega)=\epsilon(x,T^a\omega). \tag{2.3}$$

We call the transformation T^a the shift. Clearly, the T^a, $-\infty<a<\infty$, form a one-parameter group:

$$T^{a+b}=T^aT^b,\ T^0=1\quad\text{(identity)}. \tag{2.4}$$

By a nonlinear functional of $\epsilon(x,\omega)$, we mean a random variable generated by ϵ (a $\mathfrak{B}$-measurable function on Ω). Let $u(\omega)$ be such a random variable. We say that a stationary process $u(x,\omega)$ given in the form

$$u(x,\omega)=u(T^x\omega) \tag{2.5}$$

is a stationary process generated by ϵ. (However, all stationary processes derivable from ϵ cannot be given in this form.) In this sense we also write

$$\epsilon(x,\omega)=\epsilon(T^x\omega), \tag{2.6}$$

where $\epsilon(\omega)$ on the right-hand side is a random variable $\epsilon(\omega)\equiv\epsilon(0,\omega)$.

Let a nonstationary process $\psi(x,\omega)$ be a random variable generated by ϵ, such that it is measurable with respect to the coordinate parameter x. We introduce the translation operator by the definition

$$D^a\psi(x,\omega)=\psi(x+a,T^{-a}\omega),\quad -\infty<a<\infty. \tag{2.7}$$

Clearly, the D^a, $-\infty<a<\infty$ form a group;

$$D^{a+b}=D^aD^b,\quad D^0=1. \tag{2.8}$$

Thus the operator D^a gives a representation of the group R in the space of random processes generated by ϵ.

If $\psi(x,\omega)$ is a solution to (2.1), $D^a\psi(x,\omega)$ also becomes a solution since the operator D^a commutes with ∇^2 and $\epsilon(x,\omega)$, and can be expressed as a linear combination of the two independent solutions $\psi_1(x,\epsilon)$ and $\psi_2(x,\omega)$. Hence

$$D^a\psi_i(x,\omega)=\sum_{j=1}^{2}C_a^{ij}(\omega)\psi_j(x,\omega),\quad i=1,2, \tag{2.9}$$

where $C_a^{ij}(\omega)$ denotes a random variable generated by ϵ. In terms of matrix notation, (2.9) is expressed as

$$D^a\underline{\Psi}(x,\omega)=\underline{C}_a(\omega)\underline{\Psi}(x,\omega), \tag{2.10}$$

$$\underline{\Psi}(x,\omega)=\begin{pmatrix}\psi_1(x,\omega)\\ \psi_2(x,\omega)\end{pmatrix},\quad \underline{C}_a(\omega)=\begin{pmatrix}C_a^{11}(\omega) & C_a^{12}(\omega)\\ C_a^{21}(\omega) & C_a^{22}(\omega)\end{pmatrix}. \tag{2.11}$$

Corresponding to (2.8) the matrix $\underline{C}_a(\omega)$ satisfies the following equations:

$$\underline{C}_{a+b}(\omega)=\underline{C}_a(T^{-b}\omega)\underline{C}_b(\omega), \tag{2.12}$$

$$\underline{C}_0(\omega)=\underline{I}, \tag{2.13}$$

where $\underline{I}$ is the unit matrix. Therefore we see that the random matrix $\underline{C}_a(\omega)$ gives a representation of the group D^a or R by means of (2.12) and that the representation in this form is closely related to the representation by the operator factors so called in Ref. 11, whose structure, however, is little known. Further relations following from (2.12) are

$$\underline{C}_a^{-1}(\omega)=\underline{C}_{-a}(T^{-a}\omega), \tag{2.14}$$

$$\underline{C}_a(T^{-b}\omega)\underline{C}_b(\omega)=\underline{C}_b(T^{-a}\omega)\underline{C}_a(\omega). \tag{2.15}$$

Equation (2.12) is a function equation for $\underline{C}_a(\omega)$. Assuming that $\underline{C}_a(\omega)$ is differentiable, we easily transform (2.12) into a differential equation,

$$\frac{d\underline{C}_a(\omega)}{da}=\underline{\lambda}(T^{-a}\omega)\underline{C}_a(\omega), \tag{2.16}$$

where $\underline{\lambda}(T^{-a}\omega)$ is the matrix of stationary processes which is induced by the shift T^{-a} applied to the random matrix,

$$\underline{\lambda}(\omega)=\lim_{a\to 0}\frac{\underline{C}_a(\omega)-\underline{I}}{a}. \tag{2.17}$$

We call $\underline{\lambda}(\omega)$ the infinitesimal generator of $\underline{C}_a(\omega)$, which is the differential coefficient at the origin. Equation (2.16) is the differential equation for $\underline{C}_a(\omega)$ with given $\underline{\lambda}(T^{-a}\omega)$ and the initial condition (2.13). The representation $\underline{C}_a(\omega)$ is defined on the basis of $\underline{\Psi}(x,\omega)$. However, the representation basis can be expressed in terms of $\underline{C}_a(\omega)$ by itself: in fact,

$$\underline{\Psi}(x,\omega)=\underline{C}_x(T^x\omega)\underline{u}(T^x\omega), \tag{2.18}$$

where $\underline{u}(T^x\omega)$ is a D^a invariant column vector of the stationary processes derived from the random initial value

$$\underline{u}(\omega)\equiv\underline{\Psi}(0,\omega). \tag{2.19}$$

Let $\underline{U}(\omega)$ be a random matrix which has the inverse for almost all ω. Define a transformation of $\underline{C}_a(\omega)$ by

$$\underline{\tilde{C}}_a(\omega)=\underline{U}(T^{-a}\omega)\underline{C}_a(\omega)\underline{U}^{-1}(\omega). \tag{2.20}$$

Then $\underline{\tilde{C}}_a(\omega)$ is shown to satisfy the same multiplicative law as $\underline{C}_a(\omega)$:

$$\underline{\tilde{C}}_{a+b}(\omega)=\underline{\tilde{C}}_a(T^{-b}\omega)\underline{\tilde{C}}_b(\omega). \tag{2.21}$$

We say that $\underline{\tilde{C}}_a(\omega)$ is equivalent to $\underline{C}_a(\omega)$. The corresponding transformation of the representation basis is given by

$$\underline{\tilde{\Psi}}(x,\omega)=\underline{U}(\omega)\underline{\Psi}(x,\omega), \tag{2.22}$$

which satisfies

$$D^a\underline{\tilde{\Psi}}(x,\omega)=\underline{\tilde{C}}_a(\omega)\underline{\tilde{\Psi}}(x,\omega). \tag{2.23}$$

Then the infinitesimal generator $\underline{\tilde{\lambda}}$ of $\underline{\tilde{C}}_a(\omega)$ is transformed according to

$$\underline{\tilde{\lambda}}(\omega)=\left[\frac{d\underline{U}(T^{-a}\omega)}{da}\right]_{a=0}\underline{U}^{-1}(\omega)+\underline{U}(\omega)\underline{\lambda}(\omega)\underline{U}^{-1}(\omega). \tag{2.24}$$

It is well known that if $\underline{C}_a(\omega)$ satisfying (2.12) is nonrandom—that is, independent of ω—it is always reducible to a 1D form, $C_a=e^{isa}$ (s; a complex number), by an equivalent transformation. However, it is still an open question whether or not the random matrix satisfying (2.12) or $\underline{\lambda}(\omega)$ defined by (2.17) is reducible to a 1D representation by a suitable transformation (2.20) or (2.24).

Substituting (2.18) into (2.1) gives the equation for $\underline{\lambda}(T^x\omega)$ and $\underline{u}(T^x\omega)$:

$$\{[\underline{\lambda}(T^x\omega)+\nabla]^2+k^2[1+\epsilon(T^x\omega)]\underline{I}\}\underline{u}(T^x\omega)=0. \tag{2.25}$$

Since (2.25) involves only the stationary processes generated by ϵ, it may be solved only at $x=0$. Equation (2.25) can be solved once a relation is set between $\underline{\lambda}(\omega)$ and $\underline{u}(\omega)$. In particular, when the elements $u_1(\omega)$ and $u_2(\omega)$ of $\underline{u}(\omega)$ never vanish for almost all ω, $\underline{u}(\omega)$ can be incorporated into $\underline{\lambda}(\omega)$ by a suitable equivalent transformation [e.g., $\underline{U}(\omega)$ with the diagonal elements u_1^{-1} and u_2^{-1}]. Then, putting $u_1=u_2=1$, (2.25) becomes the equation only for $\underline{\lambda}(T^x\omega)$. We note, however, that even if $\lambda(\omega)$ is obtained from (2.25), integrating (2.16) is almost as difficult as integrating the original equation (2.1). In what follows, therefore, we assume that the equation allows a 1D solution for $\underline{\lambda}(\omega)$ and we look for $\psi(x,\omega)$ in the 1D form.

For the 1D representation, (2.16) can be easily integrated,

$$C_a(\omega)=\exp\left(\int_0^a\lambda(T^{-x}\omega)\,dx\right), \tag{2.26}$$

where $C_a(\omega)$ and $\lambda(\omega)$ stand for $\underline{C}_a(\omega)$ and $\underline{\lambda}(\omega)$ in the 1D form. Correspondingly, the 1D base is written

$$\psi(x,\omega)=\exp\left(\int_0^x\lambda(T^a\omega)\,da\right)u(T^x\omega), \tag{2.27}$$

where $u(T^x\omega)$ is a stationary process derived from the random initial value

$$u(\omega)\equiv\psi(0,\omega). \tag{2.28}$$

Equation (2.27) gives a possible form of the solution, which is analogous to the well-known Floquet solution.[12] The equation for λ and u becomes

$$\{[\lambda(T^x\omega)+\nabla]^2+k^2[1+\epsilon(T^x\omega)]\}u(T^x\omega)=0, \tag{2.29}$$

which can be solved putting $x=0$, once, for instance, a relation is set between λ and u by an initial condition. As before, if $u(\omega)>0$ for almost all ω, u can be incorporated into λ by an equivalent transformation, so that putting $u\equiv 1$ we have

$$\psi(x,\omega)=\exp\left(\int_0^x \lambda(T^a\omega)\,da\right), \quad \psi(0,\omega)=1, \tag{2.30}$$

$$\nabla\lambda(T^x\omega)+\lambda^2(T^x\omega)+k^2[1+\epsilon(T^x\omega)]=0. \tag{2.31}$$

Equation (2.31) can be solved at $x=0$ as a functional equation for $\lambda(\omega)$.

III. TRAVELING-WAVE MODE

Putting $\lambda_0\equiv\langle\lambda\rangle$, $\lambda(\omega)=\lambda_0+\tilde{\lambda}(\omega)$, we see that (2.30) describes a traveling-wave $e^{\lambda_0 x}$ (λ_0; a complex constant) which is modulated randomly by the fluctuating part $\tilde{\lambda}(\omega)$. Hence we call (2.30) the traveling-wave mode and (2.31) the random dispersion equation. We look for such a solution, assuming that it exists.[13] It is possible to deal with complex λ as it is in (2.31), but, in order to avoid some errors in the approximation, we put

$$\lambda=\alpha+i\beta \tag{3.1}$$

and deal with two real stationary processes α and β (we suppress the argument $T^x\omega$ or ω in the following equations). Then, (2.31) is transformed into the set of equations

$$\alpha^2-\beta^2+\nabla\alpha+k^2(1+\epsilon)=0, \tag{3.2}$$

$$2\alpha\beta+\nabla\beta=0. \tag{3.3}$$

As shown easily, (3.3) corresponds to the energy conservation for the traveling wave (2.30). Bearing in mind the stationarity, we represent α and β satisfying (3.3) in terms of a single stationary process γ:

$$\alpha=\nabla\gamma, \tag{3.4}$$

$$\beta=\beta_0 e^{-2\gamma}, \tag{3.5}$$

where β_0 is a real constant such that $\langle\gamma\rangle=0$. We first note that, because of the stationarity, (3.4) gives the relation

$$\alpha_0\equiv\langle\alpha\rangle=0, \tag{3.6}$$

which means that the amplitude of the traveling wave neither increases nor decreases exponentially with x; this is because of the energy conservation (3.3). The stationary process γ satisfies the nonlinear differential equation

$$\nabla^2\gamma+(\nabla\gamma)^2-\beta_0^2 e^{-4\gamma}+k^2(1+\epsilon)=0, \tag{3.7}$$

which can be solved at $x=0$. As a measure of the magnitude of ϵ, we introduce a parameter σ by

$$\langle\epsilon^2\rangle=\sigma^2. \tag{3.8}$$

If $\sigma=0$, i.e., $\epsilon\equiv 0$ for almost all ω, we have $\gamma\equiv 0$ and then (3.7) reduces to $k^2-\beta_0^2=0$, the ordinary dispersion equation for free space. We may, therefore, assume that, if σ is small enough, γ is also small. Then (3.7) is approximated by

$$\nabla^2\gamma+(\nabla\gamma)^2-\beta_0^2\,(1-4\gamma+8\gamma^2)+k^2(1+\epsilon)=0. \tag{3.9}$$

For further investigations in the following, we consider the case where the process ϵ is generated by the Brownian-motion process. Then $\gamma(\omega)$, as well as $\epsilon(\omega)$, is regarded as a nonlinear functional of the Brownian-motion process[14] (see Appendix A). Their Wiener-Hermite expansions are [cf. (A19)]

$$\epsilon(\omega)=\int_{-\infty}^{\infty}G_1(s)\,dB(s)+\int\int_{-\infty}^{\infty}G_2(s,s')\hat{h}^{(2)}\times[dB(s),dB(s')]+\cdots, \tag{3.10}$$

$$G_1^*(s)=G_1(-s), \quad G_2^*(s,s')=G_2(-s,-s'); \tag{3.11}$$

$$\gamma(\omega)=\int_{-\infty}^{\infty}\Gamma_1(s)\,dB(s)+\int\int_{-\infty}^{\infty}\Gamma_2(s,s')\hat{h}^{(2)}\times[dB(s),dB(s')]+\cdots, \tag{3.12}$$

$$\Gamma_1^*(s)=\Gamma_1(-s), \quad \Gamma_2^*(s,s')=\Gamma_2(-s,-s'), \tag{3.13}$$

such that

$$\langle\epsilon^2\rangle=\int_{-\infty}^{\infty}|G_1(s)|^2\,ds+2\int\int_{-\infty}^{\infty}|G_2(s,s')|^2\,ds\,ds'+\cdots=\sigma^2, \tag{3.14}$$

$$\langle\gamma^2\rangle=\int_{-\infty}^{\infty}|\Gamma_1(s)|^2\,ds+2\int\int_{-\infty}^{\infty}|\Gamma_2(s,s')|^2\,ds\,ds'+\cdots<\infty. \tag{3.15}$$

Using (A8), (A10), and (A14), the random dispersion equation (3.9) for λ is transformed into a set of function equations for the Γ's. The first three equations corresponding to degree $n=0,1,$ and 2 are

$$k^2-\beta_0^2+\int_{-\infty}^{\infty}(s^2-8\beta_0^2)|\Gamma_1(s)|^2\,ds+2\int\int_{-\infty}^{\infty}[(s+s')^2-8\beta_0^2]|\Gamma_2(s,s')|^2\,ds\,ds'+\cdots=0, \tag{3.16}$$

$$(s^2-4\beta_0^2)\Gamma_1(s)+2\int_{-\infty}^{\infty}(s'-8\beta_0^2)\Gamma_1(s')\Gamma_2(s,-s')\,ds'-32\beta_0^2\Gamma_1(s)\int_{-\infty}^{\infty}|\Gamma_1(s')|^2\,ds'+\cdots=k^2G_1(s), \tag{3.17}$$

$$[(s+s')^2-4\beta_0^2]\,\Gamma_2(s,s')+(ss'+8\beta_0^2)\Gamma_1(s)\Gamma_1(s')+\cdots=k^2G_2(s,s'). \tag{3.18}$$

They form a set of dispersion equations; for instance, (3.16) is a dispersion equation for the propagation constant β_0 which is perturbed by the integral terms due to the random medium.

Assuming $G_1 = G$ is of first order and $G_n \equiv 0 (n \geq 2)$, we look for an approximate solution such that $\langle \gamma^2 \rangle \to 0$ as $\sigma \to 0$. We see that Γ_1 is of the order σ but Γ_2 is of σ^2, as far as the amplitude is concerned. Omitting the details of the calculation, we simply refer to the first-order approximation,

$$\Gamma_1(s) \cong k^2 G(s)/(s^2 - 4\beta_0^2), \tag{3.19}$$

which, however, does not satisfy (3.15) and the above requirement, unless

$$G(2\beta_0) = 0, \tag{3.20}$$

where $2\beta_0 \cong 2k$ by (3.16). Thus at this stage of approximation, we have the condition $|G(2k)|^2 = 0$ for the existence of the traveling-wave mode (in the higher-order argument, we need more conditions on G_n, $n \geq 2$; see also Ref. 13). We generally have $|G(2k)|^2 \neq 0$ for an arbitrary value of k. Then we have a cutoff mode and the medium becomes "dark" at most values of k (see Sec. IV E). For Z_α process (A32) whose spectrum vanishes at $s = \alpha$ the mode becomes a traveling wave when $k = \frac{1}{2}\alpha$ (in the σ^2-order approximation) so that the medium becomes "transparent" at this particular wave number. This is clearly demonstrated by a computer simulation of the random medium, which will be published in a later work.

Whenever a traveling wave exists, the boundary-value problems can be treated as in an ordinary transmission line in terms of the two complex-conjugate solutions of the form (2.30). However, we omit further studies on this mode.

IV. CUTOFF-WAVE MODE

A. Approximate method of solution

When $|G(2k)|^2 \neq 0$, we look for a solution of the form (2.27) instead of the traveling-wave solution (2.30). In this case the stationary process $u(T^x\omega)$ can become zero for some x or ω. As implicated by the heading, we have the solution whose amplitude increases or decreases exponentially with x; it corresponds to an unstable solution of the Mathew equation (a forbidden state in the case of Schrödinger equation). Therefore, the following approximate method of solution is somewhat similar to that of Hill's equation for obtaining such a solution.[7,15] The method described here is valid when σ is reasonably small; otherwise the method has to be modified.

Again we assume that $\epsilon(T^x\omega)$ is a Gaussian process of the form (A22), which we rewrite as a sum of band-limited processes,

$$\epsilon(x, \omega) = \sum_{n=-\infty}^{\infty} \epsilon_n(x, \omega) e^{inkx}, \tag{4.1}$$

where

$$\epsilon_n(x, \omega) = \int_{-k/2}^{k/2} G(nk+s) e^{isx}\, dB_n(s, \omega), \quad \epsilon_n^* = \epsilon_{-n}, \tag{4.2}$$

$$dB_n(s) \equiv dB(nk+s), \quad dB_n^*(s) = dB_{-n}(-s),$$
$$-\tfrac{1}{2}k \leq s < \tfrac{1}{2}k, \tag{4.3}$$

$$\langle dB_n(s)\, dB_m^*(s') \rangle = \delta_{nm}\delta(s-s')\, ds\, ds'; \tag{4.4}$$

$\epsilon_n(x, \omega)$ is a complex Gaussian process with bandwidth k. The transformation properties of dB_n and ϵ_n under D^a are

$$D^a\, dB_n(s, \omega) = dB(s, T^{-a}\omega) = e^{-i(nk+s)a}\, dB_n(s, \omega), \tag{4.5}$$

$$D^a \epsilon_n(x, \omega) = e^{-inka} \epsilon_n(x, \omega). \tag{4.6}$$

We denote by $\mathfrak{D}_{-nk}$ the linear space of random processes subject to the transformation like (4.6) under the operator D^a. $\epsilon_n(x, \omega)$, therefore, is not a stationary process unless $n = 0$.

We expand the solution ψ also in a similar form,

$$\psi(x, \omega) = \sum_{n=-\infty}^{\infty} A_n(x, \omega) e^{inkx}, \tag{4.7}$$

where $A_n(x, \omega)$ is a slowly varying narrow-band process,[16] whose bandwidth is assumed to be sufficiently narrower than k; the assumption is valid when σ is small enough. However, it is shown in Appendix B that in the lowest-order approximation the solution can be expressed as

$$\psi(x, \omega) = A_1(x, \omega) e^{ikx} + A_{-1}(x, \omega) e^{-ikx}, \tag{4.8}$$

where the narrow-band processes A_1 and A_{-1} satisfy the set of equations

$$\begin{aligned} &(2i/k) A_1' + \epsilon_0 A_1 + \epsilon_2 A_{-1} = 0, \\ &-(2i/k) A_{-1}' + \epsilon_0 A_{-1} + \epsilon_2^* A_1 = 0, \end{aligned} \tag{4.9}$$

the prime denoting the differentiation and the arguments x and ω being suppressed. In the higher-order approximation, A_n, $n \neq \pm 1$, can be incorporated into the equations as correction terms.

Eliminating A_{-1} from (4.9) yields the second-order differential equation

$$A_1'' - \frac{\epsilon_2'}{\epsilon_2} A_1' + \frac{k^2}{4}\left[\frac{2i}{k}\left(\frac{\epsilon_2'}{\epsilon_2}\epsilon_0 - \epsilon_0'\right) + \epsilon_0^2 - |\epsilon_2|^2\right] A_1 = 0. \tag{4.10}$$

We observe that the coefficients of A_1' and A_1 are the stationary processes belonging to $\mathfrak{D}_0$ because of the property (4.6). Therefore, invoking the theory in Sec. II, we again look for a solution in the form (2.30),

$$A_1(x,\omega)=\exp\left(\int_0^x \lambda_A(T^a\omega)\,da\right), \tag{4.11}$$

where the stationary process $\lambda_A(T^x\omega)$ satisfies the differential equation

$$\lambda_A'+\lambda_A^2-\frac{\epsilon_2'}{\epsilon_2}\lambda_A+\frac{k^2}{4}\left[\frac{2i}{k}\left(\frac{\epsilon_2'}{\epsilon_2}\epsilon_0-\epsilon_0'\right)+\epsilon_0^2-|\epsilon_2|^2\right]=0, \tag{4.12}$$

which is an equation in the space $\mathfrak{D}_0$. Putting

$$\lambda_A=\tfrac{1}{2}ik\epsilon_0+\epsilon_2\nu, \tag{4.13}$$

we note that the random process $\nu(x,\omega)$ belongs to $\mathfrak{D}_{2k}$, namely,

$$D^a\nu(x,\omega)=e^{2ika}\nu(x,\omega). \tag{4.14}$$

By the transformation (4.13), (4.12) turns into the equation for ν,

$$\nu'+\epsilon_2\nu^2+ik\epsilon_0\nu-\tfrac{1}{4}k^2\epsilon_2^*=0, \tag{4.15}$$

which is an equation in $\mathfrak{D}_{2k}$. Once it is solved, λ_A and A_1 are in turn obtained. Since

$$A_{-1}=-\frac{1}{\epsilon_2}\left(\epsilon_0A_1+\frac{2i}{k}A_1'\right)=-\frac{2i}{k}\nu A_1, \tag{4.16}$$

the solution (4.8) is now written

$$\psi(x,\omega)=\exp\left(ikx+\int_0^x\lambda_A(T^a\omega)\,da\right)-\frac{2i}{k}\nu(x,\omega)\exp\left(-ikx+\int_0^x\lambda_A(T^a\omega)\,da\right), \tag{4.17}$$

$$=\exp\left(\int_0^x(ik+\lambda_A(T^a\omega)\,da\right)\times\left\{1-\frac{2i}{k}\nu(x,\omega)e^{-2ikx}\right\}. \tag{4.18}$$

By (4.14) the term in the curly brackets is a stationary process belonging to $\mathfrak{D}_0$ so that the approximate expression (4.18) also has the form (2.27) (see Ref. 16). The two traveling waves in (4.17) do not look like the complex conjugate of the other; however, they can be transformed as follows. Using (4.13), (4.15), and the relation (C3), namely,

$$|\nu(x,\omega)|^2=\tfrac{1}{4}k^2,\quad \mathrm{Re}(\epsilon_2\nu)\neq 0, \tag{4.19}$$

which follows from the conservation law (Appendix C), we can prove the equality,

$$\nu'(x,\omega)/\nu(x,\omega)=\lambda_A^*(T^x\omega)-\lambda_A(T^x\omega). \tag{4.20}$$

Therefore, (4.17) can be recast into the form

$$\psi(x,\omega)=\exp\left(ikx+\int_0^x\lambda_A(T^a\omega)\,da\right)-\frac{2i}{k}\nu(0,\omega)\exp\left(-ikx+\int_0^x\lambda_A^*(T^a\omega)\,da\right), \tag{4.21}$$

which consists of two traveling waves in the complex-conjugate forms; the individual traveling wave can never be a solution, however. In view of (4.19) we can put

$$-(2i/k)\nu(0,\omega)=e^{-2i\phi(\omega)}, \tag{4.22}$$

$\phi(\omega)$ being a real random variable. Multiplying (4.21) by a constant $\frac{1}{2}e^{i\phi}$, we also have the solution in the real form,

$$\psi(x,\omega)=\exp\left(\int_0^x\alpha_A(T^a\omega)\,da\right)\times\cos\left(kx+\int_0^x\beta_A(T^a\omega)\,da+\phi(\omega)\right), \tag{4.23}$$

where we have put

$$\lambda_A(T^x\omega)=\alpha_A(T^x\omega)+i\beta_A(T^x\omega). \tag{4.24}$$

Thus we notice that our solution (4.21) or (4.23) is a totally standing wave which is randomly modulated.

B. Approximate solution for Gaussian random medium

We now solve the equation (4.15) for ν when ϵ is a Gaussian process generated by the Brownian-motion process as given by (A22). Since ν is a slowly-varying narrow-band process in $\mathfrak{D}_{2k}$ generated by the processes ϵ_0, ϵ_2, and ϵ_2^*, it is to be expanded in terms of slowly-varying bases of $\mathfrak{D}_{2k}$ made up of the Wiener-Hermite differentials associated with $dB_0(s)$, $dB_2(s)$, and $dB_2^*(s)$. Some lowest-degree bases of $\mathfrak{D}_{2k}$ meeting such requirements are[17]

$$e^{-isx}\,dB_2^*(s), \tag{4.25}$$

$$e^{i(s-s')x}\,\hat{h}^{(2)}[dB_0(s),\ dB_2^*(s')], \tag{4.26}$$

$$e^{-i(s+s'-s'')x}\,\hat{h}^{(3)}[dB_2^*(s),dB_2^*(s'),dB_2(s'')], \tag{4.27}$$

$$e^{i(s+s'-s'')x}\,\hat{h}^{(3)}[dB_0(s),dB_0(s'),dB_2^*(s'')], \tag{4.28}$$

where (4.27) and (4.28) are third-degree bases. We will see that the first term in the expansion is already a fairly good approximation, but to show the method of calculation we expand ν in terms of the first two bases,

$$\nu(x)=\int_{-k/2}^{k/2}F_1(s)\,e^{-isx}\,dB_2^*(s)+\int\int_{-k/2}^{k/2}F_2(s,s')e^{i(s-s')x}\,dB_0(s)\,dB_2^*(s'). \tag{4.29}$$

First of all, we notice that, by means of the orthogonality of the Wiener-Hermite differentials, $\langle\epsilon_2\nu\rangle$ becomes

$$\langle\epsilon_2\nu\rangle=\int_{-k/2}^{k/2} G(2k+s)F_1(s)\,ds\equiv\mu. \tag{4.30}$$

Substituting (4.29) into (4.15) and using (A8), we obtain a set of the function equations for F's. The first two lowest-degree equations are

$$-isF_1(s)+2\int_{-k/2}^{k/2} G(2k+s')F_1(s')\,ds'\,F_1(s)+ik\int_{-k/2}^{k/2} G(s')F_2(-s',s)\,ds'-\tfrac{1}{4}k^2G^*(2k+s)=0, \tag{4.31}$$

$$i(s-s')F_2(s,s')+2\int_{-k/2}^{k/2} G(2k+s'')F_1(s'')\,ds''\,F_2(s,s')+2\int_{-k/2}^{k/2} G(2k+s'')F_2(s,s'')\,ds''\,F_1(s')+ikG(s)F_1(s')=0. \tag{4.32}$$

Neglecting the third term in (4.32) compared to the fourth, we obtain

$$F_2(s,s')\cong -ikG(s)F_1(s')/[i(s-s')+2\mu]. \tag{4.33}$$

Substituting this in (4.31) yields

$$F_1(s)=\tfrac{1}{4}(ik^2)G^*(2k+s)\big/[s+2i\mu-N(s)],\quad N(s)=k^2\int_{-k/2}^{k/2}\frac{|G(s')|^2\,ds'}{s+s'+2i\mu}. \tag{4.34}$$

Then (4.30) becomes

$$\mu=\frac{ik^2}{4}\int_{-k/2}^{k/2}\frac{|G(2k+s)|^2\,ds}{s+2i\mu-N(s)}, \tag{4.35}$$

which is a dispersion relation for determining μ. Since μ is of the order of σ^2, the right-hand integral can be approximated by a principal-value integral for small σ. Depending on the assumption $\mathrm{Re}\mu\gtrless 0$ in calculating the integral, we obtain

$$\mu\cong\frac{ik^2}{4}\,\mathrm{P}\int_{-k/2}^{k/2}\frac{|G(2k+s)|^2}{s}\,ds\pm\frac{k^2\pi}{4}|G(2k)|^2,\quad \cong\frac{ik^3}{4}|G(2k)|^{2\prime}\pm\frac{k^2\pi}{4}|G(2k)|^2, \tag{4.36}$$

which is consistent with the assumption; here $|G(2k)|^{2\prime}$ denotes the gradient of the power spectrum at $2k$ and P indicates the principal value of the integral which is evaluated approximately assuming $|G|^2$ is a slowly-varying function. Upon calculating $\langle\lambda_A\rangle$ using (4.13), we have $\langle\epsilon_0\rangle=0$. However, in view of the fact that $\langle\epsilon_2\nu\rangle$ given by (4.36) is of the second order in σ, we need to make a correction to ϵ_0,

$$\epsilon_0\to\epsilon_0-\tfrac{2}{3}|\epsilon_1|^2+\tfrac{1}{8}|\epsilon_2|^2+\tfrac{1}{3}|\epsilon_3|^2+\cdots, \tag{4.37}$$

on evaluating the average, these terms being taken from the first bracket in (B4). Then we obtain

$$\lambda_0\equiv\langle\lambda_A\rangle=\mu+ik\int_{-k/2}^{k/2}[-\tfrac{1}{3}|G(k+s)|^2+\tfrac{1}{16}|G(2k+s)|^2+\tfrac{1}{6}|G(3k+s)|^2]\,ds,\quad \cong\mu+ik^2[-\tfrac{1}{3}|G(k)|^2+\tfrac{1}{16}|G(2k)|^2+\tfrac{1}{6}|G(3k)|^2]. \tag{4.38}$$

Using these expressions for λ_0, F_1, and F_2, we obtain an approximate solution for ν by (4.29) and in turn λ_A by (4.13), which we can write[18]

$$\lambda_A=\lambda_0+\int_{-k/2}^{k/2}e^{isx}\left(\frac{ik}{2}G(s)+\int_{-k/2}^{k/2}F_2(s,s')G(2k+s')\,ds'\right)dB_0(s)+\int\int_{-k/2}^{k/2}e^{i(s-s')x}G(2k+s)F_1(s')\hat{h}^{(2)}[dB_2(s),dB_2^*(s')]. \tag{4.39}$$

In the approximate evaluation of various parameters in the next section, however, we neglect F_2 term which usually gives the higher-order effect and also drop $N(s)$ from the denominator of (4.34) which is due to F_2 (see Appendix D).

Using (4.39) in (4.21) or (4.23), we obtain a representation for ψ, by means of which we can examine the characteristics of the wave solution. Since $|G(2k)|^2\neq 0$, we have $\mathrm{Re}(\epsilon_2\nu)\neq 0$ by (4.30) and (4.36). Therefore, the solution (4.21) or (4.23) represents a totally standing wave without energy flow (Appendix C). Furthermore, depending on the sign of $\mathrm{Re}\mu$, we obtain two modes of solution whose envelope increases or decreases in the manner of $e^{\pm\alpha_0 x}$,

$$\alpha_0=|\mathrm{Re}\mu|=|\langle\alpha_A\rangle|. \tag{4.40}$$

Because of these features of the solution, we conclude that the wave in the random medium with $|G(2k)|^2\neq 0$ is the cutoff mode. When $|G(2k)|^2=0$, however, we need to investigate higher-order terms to determine whether or not the mode is cut off.

C. Average and variance of log-amplitude and phase

We calculate several statistical parameters characterizing the wave solution (4.23) or (4.21), using the approximate expression (4.39) without F_2 term. We call

$$|A| = \exp\left(\int_0^x \alpha_A(T^a\omega)\,da\right) \tag{4.41}$$

the amplitude or the envelope and denote it by the symbol $|A|$. Also we call

$$\Theta = kx + \int_0^x \beta_A(T^a\omega)\,da + \phi(\omega) \tag{4.42}$$

the phase (we discard ϕ in the following since we are interested in Θ as a function of x). We interpret the amplitude by the envelope and the phase by the undulation (or the nodes) of the real wave form (4.23), and not by the absolute value or the argument of a complex wave. However, $|A|$ and Θ can be measured by means of $|1/T|$ and $\arg T$ of the complex transmission coefficient [see Sec. IV E].

By (4.36), the average of the log amplitude is given as

$$\langle \ln|A|\rangle = \langle \alpha_A\rangle x = \pm\alpha_0 x, \tag{4.43}$$

where

$$\alpha_0 \cong \tfrac{1}{4}(k^2\pi)|G(2k)|^2, \tag{4.44}$$

which we call the average log-amplitude increment (decrement).

We call $\langle\Theta\rangle/x$ the average wave number and $\langle\Theta\rangle/x - k$ the wave-number shift which we write as

$$\Delta k = \langle\beta_A\rangle \cong \tfrac{1}{4}k^3|G(2k)|^{2\prime} - \tfrac{1}{3}k^2|G(k)|^2 + \tfrac{1}{16}k^2|G(2k)|^2 + \tfrac{1}{6}k^2|G(3k)|^2. \tag{4.45}$$

This is independent of the sign in (4.43), but can change its sign depending on the shape of the power spectrum $|G|^2$.

While the average log amplitude increases or decreases linearly with x, the log amplitude has a nonstationary fluctuation about the average value. We show that the variance of the log-amplitude is proportional to $|x|$ for large $|x|$. Let us evaluate the variance

$$d^2 = \langle[\ln|A| - \langle\ln|A|\rangle]^2\rangle = \left\langle\left(\int_0^x (\alpha_A - \langle\alpha_A\rangle)\,da\right)^2\right\rangle, \tag{4.46}$$

where

$$\alpha_A - \langle\alpha_A\rangle = \mathrm{Re}\int\int_{-k/2}^{k/2} e^{i(s-s')x} \times G(2k+s)F_1(s')\hat{h}^{(2)}[dB_2(s), dB_2^*(s')]. \tag{4.47}$$

The asymptotic expression for large x ($x>0$) is calculated to be (Appendix E)

$$d^2 \sim \tfrac{1}{8}\pi k^2|G(2k)|^2 x. \tag{4.48}$$

For negative x, (4.48) is valid if one replaces x by $-x$. Thus we find that the ratio of the standard deviation to the mean, i.e., $d/\langle\ln|A|\rangle$, tends to zero as $x\to\infty$. To put it in another way,

$$\frac{1}{x}\ln|A| \to \frac{1}{x}\langle\ln|A|\rangle = \alpha_0 \quad (x\to\infty) \tag{4.49}$$

holds in the mean-square sense, that is, the log-amplitude increment averaged over the distance x approaches its mathematical expectation as $x\to\infty$. This is due to a law of large numbers.[19] Comparing (4.48) with (4.43), we obtain an interesting relation,

$$d^2 = \tfrac{1}{2}\alpha_0|x|. \tag{4.50}$$

Similarly we evaluate the variance of the phase,

$$\theta^2 = \langle[\Theta - \langle\Theta\rangle]^2\rangle = \left\langle\left(\int_0^x (\beta_A - \langle\beta_A\rangle)\,da\right)^2\right\rangle, \tag{4.51}$$

where

$$\beta_A - \langle\beta_A\rangle = \frac{k}{2}\epsilon_0 + \mathrm{Im}\int\int_{-k/2}^{k/2} e^{i(s-s')x}G(2k+s) \times F_1(s')\hat{h}^{(2)}[dB_2(s), dB_2^*(s')], \tag{4.52}$$

the first and the second terms being orthogonal.[20] θ^2 can be calculated asymptotically (Appendix E),

$$\theta^2 \sim \frac{\pi k^2}{8}[|G(2k)|^2 + 4|G(0)|^2]x = \left(1 + 4\frac{|G(0)|^2}{|G(2k)|^2}\right)d^2, \tag{4.53}$$

which is again proportional to x, and we have $\theta/\langle\Theta\rangle \to 0$ for $x\to\infty$. Thus

$$\frac{1}{x}\Theta \to \frac{1}{x}\langle\Theta\rangle = k + \Delta k \quad (x\to\infty), \tag{4.54}$$

which is due to the ergodic theorem for the stationary process β_A.

We summarize here the expressions of the parameters α_0, Δk, d^2, and θ^2 for the O-U and Z_0 processes whose power spectra are given by (A26) and (A29):

Average log-amplitude increment:

$$\alpha_0 = \frac{\sigma^2}{4}\frac{\kappa k^2}{\kappa^2 + 4k^2} \quad \text{(O-U)}; \tag{4.55}$$

$$\alpha_0 = 2\sigma^2\frac{\kappa k^4}{(\kappa^2+4k^2)^2} \quad (Z_0). \tag{4.56}$$

Average wave-number shift:

$$\Delta k = \frac{\sigma^2 \kappa}{\pi} \left(-\frac{k^4}{(\kappa^2+4k^2)^2} - \frac{k^2}{3(\kappa^2+k^2)} + \frac{k^2}{16(\kappa^2+4k^2)} + \frac{k^2}{6(\kappa^2+9k^2)} \right) \quad \text{(O-U)}; \tag{4.57}$$

$$\Delta k = \frac{\sigma^2 \kappa}{\pi} \left(-\frac{k^4(\kappa^2-4k^2)}{(\kappa^2+4k^2)^3} - \frac{2k^4}{3(\kappa^2+k^2)^2} + \frac{k^4}{2(\kappa^2+4k^2)^2} + \frac{3k^4}{(\kappa^2+9k^2)^2} \right) \quad (Z_0). \tag{4.58}$$

Variance of log-amplitude fluctuation:

$$d^2 = \frac{\sigma^2}{8} \frac{\kappa k^2}{\kappa^2+4k^2} x \quad \text{(O-U)}; \tag{4.59}$$

$$d^2 = \sigma^2 \frac{\kappa k^4}{(\kappa^2+4k^2)^2} x \quad (Z_0). \tag{4.60}$$

Variance of phase fluctuation:

$$\theta^2 = (5+16k^2/\kappa^2)\, d^2 \quad \text{(O-U)}; \tag{4.61}$$

$$\theta^2 = d^2 \quad (Z_0). \tag{4.62}$$

We see that α_0 approaches a constant value in the high-frequency limit $k \to \infty$, showing a saturation characteristic. Clearly, in terms of average wave number, the medium is dispersive. Δk for the O-U process is negative at any k, but Δk for Z_0 process can change its sign. Δk shows the saturation also. θ^2(O-U) is usually larger than $\theta^2(Z_0)$ because for the O-U process $|G(0)|^2$ is the largest value of the spectrum. For the Z_0 process, however, $|G(0)|^2$ becomes zero. This is explained by the fact that $|G(s)|^2, |s| < \frac{1}{2}k$, which corresponds to the slowly changing part of ϵ, mainly affects the phase. The saturation characteristics and the law of large numbers (4.49) or (4.54) were also observed in the case of random slabs.[6]

The above formulas for the statistical parameters and other related properties are experimentally demonstrated by means of the computer simulation of the random media represented by the two processes. The details will be published in a subsequent paper.

D. Average wave

In some papers, the average of the wave solution or the Green's function was studied.[5] In order to compare with those results, we calculate the average using the expression (4.21) or (4.23). Here we evaluate the average of one of the two complex-conjugate waves forming a cutoff mode. As we are interested in the behavior of the average wave for large x, we again discard $\nu(0,\omega)$ or $\phi(\omega)$ in the calculation. For simplicity we approximate $\lambda_A \cong \lambda_0 + \frac{1}{2} ik\epsilon_0$, neglecting higher-order terms. We first note the relation

$$\left\langle \exp\left(\frac{ik}{2} \int_0^x \epsilon_0(T^a\omega)\, da \right) \right\rangle = \exp\left\{ -\frac{1}{2} \left\langle \left(\frac{k}{2} \int_0^x \epsilon_0(T^a\omega)\, da \right)^2 \right\rangle \right\} \sim \exp(-\tfrac{1}{4}\pi k^2 |G(0)|^2 x) \tag{4.63}$$

holds for large x where $|G(0)|^2 \neq 0$. This becomes a constant when $|G(0)|^2 = 0$. Then, using (4.36), we obtain

$$\left\langle \exp\left(ikx + \lambda_0 x + \frac{ik}{2} \int_0^x \epsilon_0(T^a\omega)\, da \right) \right\rangle \sim \exp[i(k+\Delta k)x + \tfrac{1}{4}k^2\pi(\pm|G(2k)|^2 - |G(0)|^2)x] \tag{4.64}$$

for large x. We find that when $|G(0)|^2 > |G(2k)|^2$, even the amplitude of the increasing mode (+ sign) decays exponentially with increasing x. Thus, in the case of O-U process, the average amplitude of the increasing mode becomes

$$\exp\{-[\sigma^2 k^4/\kappa(\kappa^2+4k^2)]x\}. \tag{4.65}$$

Physically this implies that the fluctuating phase mostly determined by ϵ_0 is so strong that the increasing amplitude determined by ϵ_2 is overcancelled by the fluctuating phase. Equation (4.65) agrees with the result from the other source.[5] However, we have seen that when $|G(0)|^2 < |G(2k)|^2$, as in the case of Z_0 process with $|G(0)|^2 = 0$, the average amplitude of the increasing mode does not decay with increasing x. In this connection we notice that the exponential decay towards infinity ($x \to \pm\infty$) is not always a sufficient boundary condition for determining the average Green's function in "free" space.

E. Transmission coefficient

The cutoff mode never transports the energy through the infinite medium. If the medium is of finite thickness, however, the energy is transferred by the leakage like the tunnel effect though a cutoff microwave wave guide. Then, for a medium thick enough, the energy transfer decreases exponentially with increasing thickness.

We treat this problem as a boundary-value problem using the two independent solutions of the cutoff modes. Let the increasing and decreasing modes be $\psi_1(x)$ and $\psi_2(x)$, respectively. Then the wave in the medium can be expressed as $\psi(x)$

$=a\psi_1(x)+b\psi_2(x)$. Matching the boundary conditions at $x=0$ and $x=L$, where L denotes the thickness, we obtain the complex transmission coefficient,

$$T=\frac{2ik\begin{vmatrix}\psi_1(0) & \psi_2(0)\\ \psi_1'(0) & \psi_2'(0)\end{vmatrix}}{\begin{vmatrix}\psi_1'(L)-ik\psi_1(L) & \psi_2'(L)-ik\psi_2(L)\\ \psi_1'(0)+ik\psi_1(0) & \psi_2'(0)+ik\psi_2(0)\end{vmatrix}}. \tag{4.66}$$

The numerator is the Wronskian which is a constant independent of L. For large L, $\psi_2(L)$ and $\psi_2'(L)$ are negligible, so that the asymptotic expression of $1/T$ becomes

$$\frac{1}{T}\sim\mathrm{const}\times\left(\psi_1(L)+\frac{i}{k}\,\psi_1'(L)\right). \tag{4.67}$$

We evaluate this using the solution (4.21). Using the fact that ν is a zeroth-order quantity in σ, and neglecting the higher-order terms, we have

$$\frac{1}{T}\sim\mathrm{const}\times\exp\left(-ikL+\int_0^L\lambda_A^*(T^x\omega)\,dx\right), \tag{4.68}$$

that is, the asymptotic form for $1/T$ is proportional to the backward-traveling-wave part of (4.21). Hence, $1/|T|$ asymptotically is the same with the amplitude $|A|$ and $\arg T$ equals the negative phase $-\Theta$. The average and the variance of $\ln|T|$ and $\arg T$ are, therefore, given by those of $\ln|A|$ and Θ obtained in (4.3). In view of the property (4.49), we can say that the transmission coefficient $|T|$ decreases with increasing thickness L in the exponential manner

$$|T|\sim e^{-\alpha_0 L} \tag{4.69}$$

almost certainly. Such a characteristic of the transmission coefficient was also obtained in the case of random stack of dielectric slabs[6] and random point scatterers[4]; in the light of the present theory, it can be explained by the white-noise-like spectra of such random media.

Finally we note that, as shown by the present example, some other boundary-value problems in the random medium can be also treated in terms of two independent modes of solution, such as an excitation, or the "free"-space Green's function. They will be studied in a later work.

APPENDIX A

Wiener's nonlinear theory of the Brownian-motion process has been investigated from various points of view (for detailed references, see Ref. 9). A multiple stochastic integral with respect to the Brownian-motion process was studied systematically by Itô[21] under the name of the multiple Wiener integral. Concerning the notations and definitions of the multiple Wiener integral we follow the Appendix of Ref. 9. We point out that we can introduce another representation for the multiple Wiener integral convenient for solving differential equations. In the present Appendix we summarize the brief descriptions and formulas concerning the alternative representation, which, however, are made intentionally formal to minimize the expositions; a rigorous argument can be made in much the same way as in Ref. 9.

A. Brownian-motion process

Let $B(x,\omega)$, $-\infty<x<\infty$, be the Brownian-motion process (for simplicity we often delete the probability parameter ω from notations). The differential $dB(x)=B(x+dx)-B(x)$ is a real Gaussian variable with the zero mean having the property

$$\langle dB(x)\,dB(x')\rangle=\delta(x-x')\,dx\,dx' \tag{A1}$$

(dB/dx is the so-called Gaussian white noise). The nth degree orthogonal functional made of the differential $dB(x)$, which we call the nth degree Wiener-Hermite differential, is denoted by $h^{(n)}[dB(x_1),\ldots,dB(x_n)]$ [cf. (A5) below]. The definition and related formulas are summarized in Ref. 9.

B. Fourier transform of differential

We write the formal Fourier transform of the white noise dB/dx as

$$\frac{d\hat{B}(s)}{ds}=\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty}e^{-isx}\,dB(x),\quad -\infty<s<\infty. \tag{A2}$$

This is a complex-valued white noise on the s axis whose real and imaginary parts have independent identical distributions having the properties

$$d\hat{B}^*(s)=d\hat{B}(-s), \tag{A3}$$

$$\langle d\hat{B}(s)\,d\hat{B}^*(s')\rangle=\delta(s-s')\,ds\,ds', \tag{A4}$$

where $*$ denotes the complex conjugate. $d\hat{B}(s)$ is sometimes referred to as a complex Gaussian random measure.[22]

C. Complex Wiener-Hermite differential

To simplify notations in the following, we write $dB(s)$ for $d\hat{B}(s)$ unless there is confusion: we discriminate between $dB(s)$ and $dB(x)$ by their arguments s and x. We define the complex Wiener-Hermite differentials associated with $dB(s)$ by

$$\hat{h}^{(0)}=1,$$

$$\hat{h}^{(1)}[dB(s)]=dB(s),$$

$$\hat{h}^{(2)}[dB(s_1),dB(s_2)]=dB(s_1)dB(s_2)-\delta(s_1+s_2)ds_1ds_2,$$

$$\hat{h}^{(3)}[dB(s_1),dB(s_2),dB(s_3)]=dB(s_1)dB(s_2)dB(s_3)-[\delta(s_1+s_2)ds_1ds_2dB(s_3)+\delta(s_2+s_3)ds_2ds_3dB(s_1)+\delta(s_3+s_1)ds_3ds_1dB(s_2)], \quad (A5)$$

etc., which are obtained from $h^{(n)}[dB(x_1),\ldots,dB(x_n)]$ by replacing $dB(x_i)$ by $dB(s_i)$ and $\delta(x_i-x_j)$ by $\delta(s_i+s_j)$. Formally, $\hat{h}^{(n)}/ds_1\cdots ds_n$ is the Fourier transform of $h^{(n)}/dx_1\cdots dx_n$;

$$\hat{h}^{(n)}[dB(s_1),\ldots,dB(s_n)]=\frac{ds_1\cdots ds_n}{(2\pi)^{n/2}}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}e^{-i(s_1x_1+\cdots+s_nx_n)}\times h^{(n)}[dB(x_1),\ldots,dB(x_n)]. \quad (A6)$$

Orthogonality relation:

$$\langle\hat{h}^{(n)}[dB(s_{i_1}),\ldots,dB(s_{i_n})]\hat{h}^{(m)*}[dB(s_{j_1}),\ldots,dB(s_{j_m})]\rangle=\delta_{nm}\delta^n_{ij}ds_{i_1}\cdots ds_{j_m}, \quad (A7)$$

where δ^n_{ij} equals the sum of all distinct products of n delta functions of the form $\delta(s_{i_\nu}-s_{j_\mu})$, $i\equiv(i_1,\ldots,i_n)$, $j\equiv(j_1,\ldots,j_m)$, all i_ν and i_μ occurring only once in each product.

Recurrence formula:

$$dB(s_1)\hat{h}^{(n)}[dB(s_2),\ldots,dB(s_{n+1})]=\hat{h}^{(n+1)}[dB(s_1),\ldots,dB(s_{n+1})]+\sum_{i=2}^{n+1}\delta(s_1+s_i)\hat{h}^{(n-1)}[dB(s_2),\ldots,dB(s_{i-1})dB(s_{i+1}),\ldots,dB(s_{n+1})]. \quad (A8)$$

Transformation property:

$$D^a dB(s,\omega)=dB(s,T^{-a}\omega)=e^{-isa}dB(s,\omega) \quad (A9)$$

$$D^a\hat{h}^{(n)}[dB(s_1),\ldots,dB(s_n)]=e^{-i(s_1+\cdots+s_n)a}\hat{h}^{(n)}[dB(s_1),\ldots,dB(s_n)]. \quad (A10)$$

Complex Wiener-Hermite differential on the mixed basis:

We can consider the Wiener-Hermite differentials in terms of both $dB(s)$ and $dB^*(s)$, which are interpretable by means of (A3); for instance,

$$\hat{h}^{(3)}[dB(s_1),dB^*(s_2),dB^*(s_3)]=dB(s_1)dB^*(s_2)dB^*(s_3)-[\delta(s_1-s_2)ds_1ds_2dB^*(s_3)+\delta(s_2+s_3)ds_2ds_3dB(s_1)+\delta(s_3-s_1)ds_3ds_1dB^*(s_2)]. \quad (A11)$$

Other related formulas (A7)–(A10) can be likewise interpreted.

D. Multiple Wiener integral

The n-tuple Wiener integral with respect to the complex random measure $dB(s)$ can be defined by

$$\hat{I}_n(F)=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}F(s_1,\ldots,s_n)\hat{h}^{(n)}\times[dB(s_1),\ldots,dB(s_n)] \quad (A12)$$

for $F\epsilon L^2(R_n)$. [$L^2(R_n)$ denotes the Hilbert space with the inner product (A16) below, R_n referring to the n-dimensional space.] $\hat{I}_n(F)$ has the properties

$$\hat{I}_n(F)=\hat{I}_n(\tilde{F}), \quad (A13)$$

$$\langle\hat{I}_n(F)\hat{I}_m(G)\rangle=\delta_{nm}n!(\tilde{F},\tilde{G})_n, \quad (A14)$$

where

$$\tilde{F}(s_1,\ldots,s_n)=\frac{1}{n!}\sum_{(i)}F(s_{i_1},\ldots,s_{i_n}). \quad (A15)$$

$(i)=(i_1,\ldots,i_n)$ running over all permutations of $(1,2,\ldots,n)$ and

$$(F,G)_n=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}F(s_1,\ldots,s_n)\times G^*(s_1,\ldots,s_n)ds_1\cdots ds_n. \quad (A16)$$

When $F\epsilon L^2(R_n)$ is a Fourier transform of $f\epsilon L^2(R_n)$,

$$F(s_1,\ldots,s_n)=\frac{1}{(2\pi)^{n/2}}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}e^{i(s_1x_1+\cdots+s_nx_n)}\times f(x_1,\ldots,x_n)dx_1\cdots dx_n, \quad (A17)$$

then we have the equality in the $L^2[\omega]$ sense of the two n-tuple Wiener integrals: $I_n(f)=\hat{I}_n(F)$.

E. Orthogonal development of a functional of Brownian-motion process

A nonlinear functional $\Phi(\omega)$ with finite variance has the orthogonal development in terms of the multiple Wiener integrals (sometimes called the Wiener-Hermite expansion):

$$\Phi(\omega)=\sum_{n=0}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}f_n(x_1,\ldots,x_n)h^{(n)}\times[dB(x_1),\ldots,dB(x_n)], \tag{A18}$$

$$=\sum_{n=0}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}F_n(s_1,\ldots,s_n)\hat{h}^{(n)}\times[dB(s_1),\ldots,dB(s_n)], \tag{A19}$$

where f_n and F_n are related by the Fourier transformation (A17). Because of (A13), f_n and F_n can be regarded as symmetric in their arguments. (A18) and (A19) are different representations for the same functional $\Phi(\omega)$, which we may call x and s representation, respectively, in a manner of quantum theory. A discrete representation is the well-known Cameron-Martin theorem.[21]

F. Stationary process generated by Brownian-motion process

A stationary process derived from $\Phi(\omega)$ by the shift can be given either in x or s representation, using (A10);

$$\Phi(T^x\omega)=\sum_{n=0}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}f_n(x_1-x,\ldots,x_n-x)h^{(n)}\times[dB(x_1),\ldots,dB(x_n)], \tag{A20}$$

$$=\sum_{n=0}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}F_n(s_1,\ldots,s_n)e^{i(s_1+\cdots+s_n)x}\times\hat{h}^{(n)}[dB(s_1),\ldots,dB(s_n)]. \tag{A21}$$

A particular case gives a real stationary Gaussian process

$$\epsilon(T^x\omega)=\int_{-\infty}^{\infty}g(x'-x)\,dB(x')=\int_{-\infty}^{\infty}G(s)e^{isx}\,dB(s), \tag{A22}$$

$$G(s)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}g(x)e^{isx}\,dx,\quad G^*(s)=G(-s), \tag{A23}$$

with the covariance function

$$R(x)=\langle\epsilon(T^x\omega)\epsilon(\omega)\rangle=\int_{-\infty}^{\infty}g(x'-x)g(x')\,dx'=\int_{-\infty}^{\infty}|G(s)|^2e^{isx}\,ds. \tag{A24}$$

O-U (Ornstein-Uhlenbeck) process:

$$g(x)=\sigma\sqrt{2\kappa}\,e^{\kappa x}\quad(x<0),\qquad =0\quad(x>0), \tag{A25}$$

$$G(s)=\sigma\left(\frac{\kappa}{\pi}\right)^{1/2}\frac{1}{\kappa+is},$$

$$|G(s)|^2=\sigma^2\frac{\kappa}{\pi}\,\frac{1}{\kappa^2+s^2}\quad(\sigma,\kappa>0), \tag{A26}$$

$$R(x)=\sigma^2e^{-\kappa|x|}=\sigma^2\frac{\kappa}{\pi}\int_{-\infty}^{\infty}\frac{e^{isx}}{\kappa^2+s^2}\,ds. \tag{A27}$$

Z_0 process:

$$g(x)=\sigma 2\sqrt{\kappa}\,[1+\kappa x]\,e^{\kappa x}\quad(x<0),\qquad =0\quad(x>0), \tag{A28}$$

$$G(s)=\sigma\left(\frac{2\kappa}{\pi}\right)^{1/2}\frac{is}{(\kappa+is)^2},$$

$$|G(s)|^2=\sigma^2\frac{2\kappa}{\pi}\,\frac{s^2}{(\kappa^2+s^2)^2}, \tag{A29}$$

$$R(x)=\sigma^2[1-\kappa|x|]\,e^{-\kappa|x|}. \tag{A30}$$

Z_α process:

$$g(x)=\sigma\left(\frac{\pi}{2}\right)^{1/2}c[2+4\kappa x+(\kappa^2+\alpha^2)x^2]\,e^{\kappa x}\quad(x<0),\qquad =0\quad(x>0), \tag{A31}$$

$$G(s)=\sigma c\frac{s^2-\alpha^2}{(\kappa+is)^3},\quad |G(s)|^2=\sigma^2c^2\frac{(s^2-\alpha^2)^2}{(\kappa^2+s^2)^3}, \tag{A32}$$

$$R(x)=\sigma^2(1+c_1|x|+c_2x^2)e^{-\kappa|x|}, \tag{A33}$$

where c, c_1, and c_2 are functions of κ and α, which can be easily obtained. All these processes are one-sided moving averages: The O-U process is a well-known Gaussian Markov process; the Z_0 process is a double Markov process with zero spectrum at $s=0$, and the Z_α process is a triple Markov process with zero spectrum at $s=\alpha$ $(\alpha\neq 0)$. The corresponding Langevin equations can be readily obtained using the expressions for $G(s)$.

APPENDIX B

Substituting (4.7) into (2.1) gives

$$\sum_{n=-\infty}^{\infty}\left(A_n''+2ikA_n'+[(ink)^2+k^2]A_n+k^2\sum_{m=-\infty}^{\infty}\epsilon_{n-m}A_m\right)e^{inkx}=0. \tag{B1}$$

Neglecting A_n'' and equating each coefficient of e^{inkx} to zero, we have a set of equations for the narrow-band processes A_n:

$$\frac{2in}{k}A_n'-(n^2-1)A_n+\sum_{m=-\infty}^{\infty}\epsilon_{n-m}A_m=0,\qquad n=0,\pm1,\pm2,\ldots. \tag{B2}$$

Since A_1 and A_{-1} are of zeroth order, we treat the equations with $n=\pm 1$ as the dominant equations and the other as perturbations. Noting that the resonance frequency $(n^2-1)k/n$, $|n|\geq 2$, lies outside the band, we neglect A_n' for the approximate evaluation of A_n;

$$A_n \cong \frac{1}{n^2-1}\left[\epsilon_{n-1}A_1+\epsilon_{n+1}A_{-1}\right], \quad n\neq \pm 1. \tag{B3}$$

Substituting this into the dominant equations, we have

$$\frac{2i}{k}A_1'+\left[\epsilon_0-\frac{2}{3}|\epsilon_1|^2+\frac{1}{8}|\epsilon_2|^2+\frac{1}{3}|\epsilon_3|^2+\cdots\right]A_1 +\left[\epsilon_2-\epsilon_1^2+\frac{2}{3}\epsilon_1^*\epsilon_3+\cdots\right]A_{-1}=0, \tag{B4}$$

$$-\frac{2i}{k}A_{-1}'+\left[\epsilon_0-\frac{2}{3}|\epsilon_1|^2+\frac{1}{8}|\epsilon_2|^2+\frac{1}{3}|\epsilon_3|^2+\cdots\right]A_{-1} +\left[\epsilon_2^*-\epsilon_1^{*2}+\frac{2}{3}\epsilon_1\epsilon_3^*+\cdots\right]A_1=0. \tag{B5}$$

We see that the coefficient in the first square bracket belongs to $\mathfrak{D}_0$ while that in the second belongs to $\mathfrak{D}_{\pm 2k}$ and that the effect of $A_n, n\neq\pm 1$, on the dominant equations is of the second order in σ. Hence the lowest-order approximation is tantamount to neglecting $A_n, n\neq\pm 1$, from the beginning.

APPENDIX C

We consider the conservation law following from the equations (4.9) and (4.15). From (4.9), we easily obtain

$$|A_1|^2-|A_{-1}|^2=\text{const.}, \tag{C1}$$

which shows the energy conservation of the approximate solution (4.8). From (4.15) we obtain

$$|\nu|^{2\prime}+(|\nu|^2-\tfrac{1}{4}k^2)(\epsilon_2\nu+\epsilon_2^*\nu^*)=0, \tag{C2}$$

which means

$$|\nu|^2=\tfrac{1}{4}k^2, \quad \mathrm{Re}[\epsilon_2\nu]\neq 0, \tag{C3}$$

$$|\nu|^2=\text{const.}, \quad \mathrm{Re}[\epsilon_2\nu]=0. \tag{C4}$$

These relations can also be understood by the relation following from (4.11), (4.13), and (4.16);

$$|A_1|^2-|A_{-1}|^2=\left[1-\frac{4}{k^2}|\nu|^2\right]\exp\left(2\int_0^x \mathrm{Re}(\epsilon_2\nu)\,da\right). \tag{C5}$$

When $\mathrm{Re}[\epsilon_2\nu]\neq 0$ holds (see Sec. IV B), we have (C3), such that

$$|A_1|^2-|A_{-1}|^2=0, \tag{C6}$$

which means there is no energy flow.

It should be noticed that the conservation law above does hold regardless of ϵ_0: in other words, ϵ_0 contributes only to the phase fluctuation of the solution.

APPENDIX D

The conservation law $|\nu|^2=\frac{1}{4}k^2$ implies that the complex vector of the random process ν lies on the circle with radius $\frac{1}{2}k$ and that only the phase angle of ν varies as a random function. We will check this for the first-order approximation,

$$\nu(x)=\int_{-k/2}^{k/2}F_1(s)e^{-isx}dB_2^*(x),$$
$$F_1(s)=\frac{ik^2}{4}\,\frac{G^*(2k+s)}{s+2i\lambda_0}, \tag{D1}$$

where the F_2 term is neglected. Then

$$|\nu|^2=\int_{-k/2}^{k/2}|F_1(s)|^2ds+\iint_{-k/2}^{k/2}F_1(s)F_1^*(s')\times e^{i(s'-s)x}\hat{h}^{(2)}[dB_2(s),dB_2^*(s')]. \tag{D2}$$

The random part is to be successively cancelled by taking the higher-order terms in the expansion. We calculate the first integral for sufficiently small σ^2:

$$\int_{-k/2}^{k/2}|F_1(s)|^2ds\cong\frac{k^4}{16}\int_{-k/2}^{k/2}\frac{|G(2k+s)|^2ds}{s^2+[\frac{1}{2}k^2\pi|G(2k)|^2]^2}\cong\frac{k^2}{8}, \tag{D3}$$

which is one half the correct value $\frac{1}{4}k^2$, that is, the radius of the complex vector (D1) is about 0.7 of the correct value. The upper limit $\frac{1}{4}k^2$ can be approached by adding the higher-degree terms in the expansion. It is shown that the inclusion of F_2 term as well as the third term based on (4.28) does not improve $|\nu|^2$. It is because those terms are generated by ϵ_0, which never affects the conservation law (see Appendix C). It is, however, shown that the third-degree term based on (4.27) gives rise to the additional constant about $\frac{1}{16}k^2$ to (D2), which makes up three-fourths the correct value $\frac{1}{4}k^2$. Thus the increase in the number of the terms makes the constant part of $|\nu|^2$ closer to $\frac{1}{4}k^2$ and the random part smaller.

APPENDIX E

First we note the asymptotic formula for large x $(x>0)$,

$$\left\langle\left(\int_0^x u(x')\,dx'\right)^2\right\rangle=\int_0^x\int_0^x R(x'-x'')\,dx'\,dx'' \sim x\int_{-\infty}^{\infty}R(x')\,dx'=2\pi P(0)x, \tag{E1}$$

where $u(x)$ is a stationary process with the covariance function $R(x)=\langle u(x)u(0)\rangle$ and $P(0)$ denotes the value at $t=0$ of the power spectrum $P(t)$ which is the Fourier transform of $R(x)$. (E1) is valid for $x<0$ if x is replaced by $|x|$.

We apply (E1) to the stationary process $\alpha_A-\langle\alpha_A\rangle$, whose covariance function is given by, using (4.47),

$$R(x)=\frac{1}{4}\int\int_{-k/2}^{k/2} e^{i(s-s')x}\,|G(2k+s)F_1(s')+G^*(2k+s')F_1^*(s)|^2\,ds\,ds'. \tag{E2}$$

$P(t)$ is obtained from (E2) by changing the variable by $t=s-s'$, $t'=s+s'$, and taking the coefficient of e^{itx}. $P(0)$ is, however, easily obtained,

$$P(0)=\frac{1}{4}\int_{-k/2}^{k/2}|G(2k+s)F_1(s)+G^*(2k+s)F_1^*(s)|^2\,ds. \tag{E3}$$

We evaluate this using the approximate expression

$$F_1(s)\cong\frac{ik^2}{4}\,\frac{G^*(2k+s)}{s+i\tau}\,,\quad \tau=\frac{k^2\pi}{4}|G(2k)|^2. \tag{E4}$$

Then

$$P(0)=\frac{k^4}{4^3}\int_{-k/2}^{k/2}|G(2k+s)|^4\,\frac{4\tau^2}{(s^2+\tau^2)^2}\,ds \cong\frac{k^2}{16}|G(2k)|^2. \tag{E5}$$

Substituting this into (E1) gives (4.48).

We again apply (E1) to $\beta_A-\langle\beta_A\rangle$ and calculate the covariance function, using (4.52),

$$R(x)=\frac{k^2}{4}\int_{-k/2}^{k/2}e^{isx}|G(s)|^2\,ds+\frac{1}{4}\int\int_{-k/2}^{k/2}e^{i(s-s')x}|G(2k+s)F_1(s')-G^*(2k+s')F_1^*(s)|^2\,ds\,ds'. \tag{E6}$$

The value of the power spectrum at the origin is

$$P(0)=\frac{k^2}{4}|G(0)|^2+\frac{1}{4}\int_{-k/2}^{k/2}|G(2k+s)F_1(s)-G^*(2k+s)F_1^*(s)|^2\,ds, \cong\frac{k^2}{4}|G(0)|^2+\frac{k^2}{16}|G(2k)|^2, \tag{E7}$$

which agrees with (E5) when $|G(0)|^2=0$. Substituting this into (E1) gives (4.53).

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[11]N. J. Vilenkin, *Special Functions and the Theory of Group Representations* (American Mathematical Society, Providence, 1968).

[12]When n^2 or ϵ is a periodic function with period L, the Floquet solution has the form (see Ref. 7),

$$\psi(x)=e^{\lambda x}u_\lambda(x),\quad u_\lambda(x+L)=u_\lambda(x),\ \lambda:\ \text{const.},$$

which is derived by means of the invariance under periodic translations.

[13]There are a number of $\epsilon(T^x\omega)$ such that (2.31) yields a solution $\lambda(T^x\omega)$ $(\langle|\lambda|^2\rangle<\infty)$: in fact, one can substitute any (differentiable) stationary process for λ in (2.31) to obtain such ϵ. For arbitrary ϵ, however, one can not always obtain λ with $\langle|\lambda|^2\rangle<\infty$ from (2.31).

[14]In this case $\psi(x,\omega)$, a functional of ϵ, is regarded as a functional of the Brownian-motion process. Such change of the basic probability space may be allowed if the measure P of ϵ is absolutely continuous with respect to the measure of the Brownian-motion process: for instance, it is so for the O-U process.

[15]C. Hayashi, *Nonlinear Oscillations in Physical Systems* (McGraw-Hill, New York, 1964).

[16]We note the transformation property of A_n: Putting

$$A_n(x,\omega)=\exp\left(\int_0^x\lambda_A(T^a\omega)\,da\right)a_n(x,\omega),$$

we write

$$\psi(x,\omega)=\exp\left(\int_0^x[ik+\lambda_A(T^a\omega)]da\right)\sum_n a_n(x,\omega)e^{i(n-1)kx}$$

in the form of (2.27). If $\sum_n a_n e^{i(n-1)kx}$ is regarded as D^a-invariant, the process $a_n(x,\omega)$ belongs to $\mathfrak{D}_{-(n-1)k}$.

[17]A constant, i.e., a zeroth-degree basis, does not belong to $\mathfrak{D}_{2k}$. By (A10), (4.4), and (4.5), the Wiener-Hermite differentials appearing in these bases can be written

$$\hat{h}^{(2)}[dB_0(s),dB_2^*(s')]=dB_0(s)dB_2^*(s'),$$

$$\hat{h}^{(3)}[dB_2^*(s),dB_2^*(s'),dB_2(s'')]=dB_2^*(s)dB_2^*(s')dB_2(s'')-\delta(s-s'')\,ds\,ds''\,dB_2^*(s')-\delta(s'-s'')\,ds'\,ds''\,dB_2^*(s),$$

$$\hat{h}^{(3)}[dB_0(s),dB_0(s'),dB_2^*(s'')] = \hat{h}^{(2)}[dB_0(s),dB_0(s')]dB_2^*(s'').$$

The first and the last equations are due to the factorization property of $\hat{h}^{(n)}$.

[18]Because of the correction (4.37), we have other $\hat{h}^{(2)}$ terms which, however, are shown to give only σ^4-order effect so that they can be neglected in the present approximation.

[19]It is rewritten in a familiar form

$$\lim_{x\to\infty} \frac{1}{x}\int_0^x \alpha_A(T^a\omega)\, da = \langle\alpha_A\rangle,$$

which is an ergodic theorem for the stationary process α_A. Since the measure-preserving transformation T^x is metrically transitive in the case of the Brownian-motion process (Ref. 23), the equality holds for almost all ω.

[20]See Ref. 18.

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