Transition probabilities in a strong external field*

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The exact Green's function for a harmonic oscillator interacting with an oscillatory external electric field in the dipole approximation is used to evaluate the transition probabilities in closed form. The case of a particle in crossed electric and magnetic fields is also considered.

We shall investigate the behavior of a charged particle in a strong, time-dependent external field. In Sec. I, we consider a charged oscillator in a harmonically varying but spatially constant electric field; in Sec. II, an electron in a constant magnetic field and an orthogonal, harmonically varying, and spatially constant electric field. We shall use the exact Green's function for these systems in order to study these systems.

I. FORMALISM

The unitary time-displacement operator $U(t)$ satisfies

$$i\partial U/\partial t = H(t)U(t) \ . \tag{1}$$

The Green's function[1] for the system is given by

$$G(x,x';t) = \langle x|U(t)|x'\rangle \ . \tag{2}$$

If the "Heisenberg" operators

$$\begin{aligned} x_+^{\rm op}(t) &= U^\dagger x^{\rm op} U = f_+(x^{\rm op}, p^{\rm op}, t) \ , \\ x_-^{\rm op}(t) &= U x^{\rm op} U^\dagger = f_-(x^{\rm op}, p^{\rm op}, t) \end{aligned} \tag{3}$$

can be found as explicit functions of $x^{\rm op}$, $p^{\rm op}$, and t, then we can find a set of partial differential equations for G:

$$\begin{aligned} xG &= \langle x|x^{\rm op}U|x'\rangle = \langle x|Ux_+^{\rm op}(t)|x'\rangle \\ &= \langle x|Uf_+(x^{\rm op}, p^{\rm op}, t)|x'\rangle \\ &= f_+(x', -(1/i)\partial/\partial x', t)G \ , \end{aligned} \tag{4}$$

$$\begin{aligned} x'G &= \langle x|Ux^{\rm op}|x'\rangle = \langle x|x_-^{\rm op}U|x'\rangle \\ &= f_-(x, (1/i)\partial/\partial x, t)G \ , \end{aligned} \tag{5}$$

and

$$i\partial G/\partial t = H(x, (1/i)\partial/\partial x, t)G \ . \tag{6}$$

These partial differential equations can be solved for many simple systems and the G obtained explicitly when subjected to the boundary condition

$$G(x,x';0) = \delta(x-x') \ . \tag{7}$$

It is clear that we can also obtain $G(p,p';t)$,

$$G(p,p';t) = \langle p|U|p'\rangle \ , \tag{8}$$

by similar considerations, i.e.,

$$\begin{aligned} pG &= g_+(p', (1/i)\partial/\partial p', t)G \ , \\ p'G &= g_-(p, -(1/i)\partial/\partial p, t)G \ , \\ i\partial G/\partial t &= H(-(1/i)\partial/\partial p, p, t)G \ , \\ G(p,p';0) &= \delta(p-p') \ . \end{aligned} \tag{9}$$

Linear systems, i.e., those for which the differential equations for $x_\pm^{\rm op}$ or $p_\pm^{\rm op}$ are linear, can always be solved explicitly.

II. HARMONIC OSCILLATOR

We first consider the case of a harmonic oscillator[2] interacting with an oscillatory external electric field in the dipole approximation. Since the motion normal to the field is unaffected, this is a one-dimensional problem with Hamiltonian

$$H = \tfrac{1}{2}[(p^{\rm op})^2 + \omega^2(x^{\rm op})^2] - eEx^{\rm op}\cos\Omega t \ . \tag{10}$$

We shall treat the case $\Omega=\omega$ separately even though, as we shall see, the Green's function and other quantities are given correctly as $\Omega\to\omega$.

From

$$i\partial x_+^{\rm op}/dt = [x_+^{\rm op}, H(x_+^{\rm op}, p_+^{\rm op}, t)] \ , \tag{11}$$

$$x_+^{\rm op} = \left(x^{\rm op} - \frac{eE}{\omega^2-\Omega^2}\right)\cos\omega t + \frac{p^{\rm op}}{\omega}\sin\omega t + \frac{eE}{\omega^2-\Omega^2}\cos\Omega t \ , \tag{12}$$

and using

$$Ux_+^{\rm op}U^\dagger = x^{\rm op} \ , \quad Ux^{\rm op}U^\dagger = x_-^{\rm op} \ , \tag{13}$$

we obtain

$$x_-^{\rm op} = x^{\rm op}\cos\omega t - \frac{p^{\rm op}}{\omega}\sin\omega t - \frac{\Omega}{\omega}\frac{eE}{\omega^2-\Omega^2}\sin\omega t\sin\Omega t + \frac{eE}{\omega^2-\Omega^2}(1-\cos\Omega t\cos\omega t) \ ; \tag{14}$$

for $\Omega=\omega$

$$x_+^{\rm op} = x^{\rm op}\cos\omega t + (p^{\rm op}/\omega)\sin\omega t + (eE/2\omega)t\sin\omega t \ , \tag{15}$$

$$x_-^{op} = x^{op}\cos\omega t - (p^{op}/\omega)\sin\omega t + (eE/2\omega^2)\sin^2\omega t \ . \tag{16}$$

After substitution into the partial differential equations, the Green's function is found to be of the form

$$G(x, x'; t) = \left(\frac{\omega}{2\pi i \sin\omega t}\right)^{1/2} g(t)\exp\left(\frac{i\omega}{2\sin\omega t}\right) \times [x^2\cos\omega t - 2xx' + x'^2\cos\omega t + \alpha(t)x + \beta(t)x'] \ , \tag{17}$$

where

$$|g(t)| = 1$$

and

$$\alpha_{NR} = -[2eE/(\omega^2 - \Omega^2)][(\Omega/\omega)\sin\omega t \sin\Omega t - (1 - \cos\Omega t\cos\omega t)] \ ,$$
$$\beta_{NR} = [2eE/(\omega^2 - \Omega^2)](\cos\Omega t - \cos\omega t) \ , \tag{18}$$

or for $\Omega = \omega$

$$\alpha_R = (eE/\omega^2)\sin^2\omega t \ ,$$
$$\beta_R = (eE/\omega)t\sin\omega t \ . \tag{19}$$

(NR means nonresonant and R means resonant.) It is to be noted that

$$\lim_{\Omega\to\omega} \alpha_{NR} = \alpha_R, \quad \lim_{\Omega\to\omega} \beta_{NR} = \beta_R \ .$$

The transition amplitude from a state n to a state m is given, as usual, by

$$a_{mn}(t) = \int_{-\infty}^{\infty} dx\, dx'\, \psi_m^*(x) G(x, x', t)\psi_n(x') \tag{20}$$

with

$$\psi_n(x) = (\omega/\pi)^{1/4}[1/(2^n n!)^{1/2}]e^{-\omega x^2/2} h_n(\omega^{1/2}x), \tag{21}$$

which can be readily evaluated from the generating function for Hermite polynomials[3]:

$$e^{-s^2 + 2(\omega)^{1/2}sx} = \sum_{n=0}^{\infty} \frac{h_n(\omega^{1/2}x)}{n!} s^n \ . \tag{22}$$

If we define

$$A_{ss'}(t) = e^{-(s^2+s'^2)}\int_{-\infty}^{\infty} dx\, dx'\, e^{2\omega^{1/2}(sx+s'x')} \times e^{-\omega(x^2+x'^2)/2} G(x, x'; t) = \sum_{m,n=0} s^m s'^n b_{mn}(t) \ , \tag{23}$$

then

$$a_{mn} = \left(\frac{\omega}{\pi}\right)^{1/2}\left(\frac{n!\,m!}{2^n 2^m}\right)^{1/2} b_{mn} \ ,$$
$$A_{ss'} = g(t)\left(\frac{\pi}{\omega}\right)^{1/2} e^{-i\omega t/2} \times \exp[-(\omega/16\sin^2\omega t)(\alpha^2 + 2\alpha\beta e^{-i\omega t} + \beta^2)] \times \exp[2e^{-i\omega t}(ss' + s\lambda + s'\lambda')] \ , \tag{24}$$

where

$$\lambda = (i\omega^{1/2}/4\sin\omega t)(\alpha e^{i\omega t} + \beta) \ ,$$
$$\lambda' = (i\omega^{1/2}/4\sin\omega t)(\beta e^{i\omega t} + \alpha) \ , \tag{25}$$

which leads to[4]

$$a_{mn} = g(t)e^{-i\omega t/2}\exp\left(-\frac{\omega}{16\sin^2\omega t}(\alpha^2 + 2\alpha\beta e^{-i\omega t} + \beta^2)\right)\left(\frac{2^m}{2^n}\frac{n!}{m!}\right)^{1/2}\lambda^{m-n}L_n^{m-n}(-2e^{-i\omega t}\lambda\lambda')$$
$$= g(t)e^{-i\omega t/2}\exp\left(-\frac{\omega}{16\sin^2\omega t}(\alpha^2 + 2\alpha\beta e^{-i\omega t} + \beta^2)\right)\left(\frac{2^n}{2^m}\frac{m!}{n!}\right)^{1/2}\lambda'^{\,n-m}L_m^{n-m}(-2e^{-i\omega t}\lambda\lambda') \ , \tag{26}$$

where L_m^{n-m} and L_n^{m-n} are associated Laguerre polynomials.

If we let

$$\tau = -2e^{-i\omega t}\lambda\lambda' = (\omega/8\sin^2\omega t)(\alpha^2 + 2\alpha\beta\cos\omega t + \beta^2) \ , \tag{27}$$

then

$$P_{mn}(t) = |a_{mn}(t)|^2 = e^{-\tau}(m!/n!)\tau^{n-m}L_m^{n-m}(\tau)^2 = e^{-\tau}(n!/m!)\tau^{m-n}L_n^{m-n}(\tau)^2 \ , \tag{28}$$

$$\tau_{NR} = \frac{e^2E^2}{(\omega^2 - \Omega^2)^2}\left(\frac{\Omega^2 - \omega^2}{2\omega}\sin^2\Omega t + (\Omega + \omega)\sin^2\tfrac{1}{2}(\omega - \Omega)t - (\Omega - \omega)\sin^2\tfrac{1}{2}(\omega + \Omega)t\right) \ , \tag{29}$$

$$\tau_R = \frac{e^2E^2}{8\omega^3}(\omega^2t^2 + 2\omega t\sin\omega t\cos\omega t + \sin^2\omega t) \ . \tag{30}$$

There are several points to be noted. First, if $\Omega \neq \omega$ the state n is never permanently depleted although the time taken for P_{mn} to return to zero (from its initial value of zero) may be quite long. On the other hand, if $\Omega = \omega$ then the state is rapidly depleted with all probabilities $\to 0$ as $t \to \infty$, i.e., transitions to all states become infinitesimal but in such a way that the total probability remains one. That this must be so follows from the unitarity of G, but an explicit proof will be given in the Appendix. Second, if Ω is an integral multiple of ω, contrary to usual expectations, there is no resonance with states n away from the initial state. This is the result of assuming a dipole interaction: It seems reasonable to expect that if higher multipoles were taken into account, then resonance would occur with frequency n times the fundamental frequency for the case of a 2^n multipole interaction. Third, the time dependence at resonance does not go like e^{-t} but rather like e^{-t^2}.

III. CROSSED FIELDS

Let us now consider a particle in a magnetic field in the z direction and an oscillatory electric field in the x direction. The Hamiltonian is

$$H=(1/2m)[\vec{\mathrm{p}}^{\,\mathrm{op}}-e(\vec{\mathrm{A}}^{\,\mathrm{op}})^2]-eEx_1^{\mathrm{op}}\cos\Omega t \quad . \tag{31}$$

In this case, we shall obtain $G(\vec{\mathrm{p}},\vec{\mathrm{p}};t)$. We shall use the symmetric form for $\vec{\mathrm{A}}$:

$$\vec{\mathrm{A}}=-\tfrac{1}{2}\vec{\mathrm{x}}\times\vec{\mathrm{B}} \quad .$$

Thus, the behavior in the z direction is that of a free particle and

$$G(\vec{\mathrm{p}},\vec{\mathrm{p}}';t)=\overline{G}(p_1p_2,p_1'p_2';t)G^0(p_3,p_3';t) \quad , \tag{32}$$

and in what follows we shall only be concerned with $\overline{G}$. (We shall drop the bar for simplicity.) We shall only deal with the resonant case: If $\omega=eB/m$ and $\Omega=\omega$ (note that this is double the Larmor frequency),

$$\begin{aligned}p_{1+}^{\mathrm{op}}=&-\tfrac{1}{4}m\omega x_2^{\mathrm{op}}(1-\cos\omega t)-\tfrac{1}{4}m\omega x_1^{\mathrm{op}}\sin\omega t\\&+\tfrac{1}{2}p_2^{\mathrm{op}}\sin\omega t+\tfrac{1}{2}p_1^{\mathrm{op}}(1+\cos\omega t)\\&+(eE/4\omega)(3\sin\omega t+\omega t\cos\omega t) \quad ,\end{aligned} \tag{33}$$

$$\begin{aligned}p_{2+}^{\mathrm{op}}=&\tfrac{1}{4}m\omega x_1^{\mathrm{op}}(1-\cos\omega t)-\tfrac{1}{4}m\omega x_2^{\mathrm{op}}\sin\omega t\\&-\tfrac{1}{2}p_1^{\mathrm{op}}\sin\omega t+\tfrac{1}{2}p_2^{\mathrm{op}}(1+\cos\omega t)-\tfrac{1}{4}eEt\sin\omega t \quad ,\end{aligned} \tag{34}$$

$$\begin{aligned}p_{1-}^{\mathrm{op}}=&\tfrac{1}{4}m\omega x_1^{\mathrm{op}}\sin\omega t-\tfrac{1}{4}m\omega x_2^{\mathrm{op}}(1-\cos\omega t)\\&+\tfrac{1}{2}p_1^{\mathrm{op}}(1+\cos\omega t)-\tfrac{1}{2}p_2^{\mathrm{op}}\sin\omega t\\&-(eE/4\omega)(\omega t+2\sin\omega t+\sin\omega t\cos\omega t) \quad ,\end{aligned} \tag{35}$$

$$\begin{aligned}p_{2-}^{\mathrm{op}}=&\tfrac{1}{4}m\omega x_2^{\mathrm{op}}\sin\omega t+\tfrac{1}{4}m\omega x_1^{\mathrm{op}}(1-\cos\omega t)\\&+\tfrac{1}{2}p_1^{\mathrm{op}}\sin\omega t+\tfrac{1}{2}p_2^{\mathrm{op}}(1+\cos\omega t)\\&-(eE/4\omega)\sin^2\omega t \quad .\end{aligned} \tag{36}$$

Inserting the above to obtain the partial differential equations and solving them, we finally obtain for G

$$\begin{aligned}G(p,p';t)=&\frac{g(t)}{i\pi m\omega\sin(\omega t/2)}\exp\left(\frac{i}{m\omega\sin(\omega t/2)}\right)\{\cos(\omega t/2)[(p_1-p_1')^2+(p_2-p_2')^2]+2(p_1'p_2-p_1p_2')\sin(\omega t/2)\\&-(eE/2\omega)[(p_1-p_1')\cos(\omega t/2)(\omega t+3\sin\omega t)+p_2'\sin(\omega t/2)(\omega t-3\sin\omega t)-p_2\sin(\omega t/2+\sin\tfrac{1}{2}\omega t)]\} \quad ,\end{aligned} \tag{37}$$

with $g(t)$ a complicated phase factor. For $t\to 0$, G is so chosen that it goes to $\delta(p_1-p_1')\delta(p_2-p_2')$.

Physically, G is the probability amplitude that a particle entering the crossed fields at $t=0$ with momentum $(p_1', p_2', 0)$ has momentum $(p_1, p_2, 0)$ at time t. As can be seen, the wave packet spreads as time progresses. But let us consider the times $t_n=2\pi n/\omega=nT$, i.e., integral numbers of Larmor periods. Under these circumstances $\sin(\omega t_n/2)=0$ and we must take the appropriate limit. Then

$$G(p,p';t_n)=\tilde{g}(t_n)\delta(p_2-p_2')\delta(p_1-p_1'-eEt_n) \quad , \tag{38}$$

where $\tilde{g}(t_n)$ is a phase factor. Thus, the packet collapses and becomes sharp in momentum again *but with the x component of momentum increased by just the amount expected classically*. Hence, a particle in crossed electric and magnetic fields behaves in a manner very similar to that of a classical particle, and this clearly shows why the electrons in a cyclotron emerge with relatively well-defined energy.

Finally, if one considers the nonresonant case, i.e., $\Omega\neq eB/m$, then we find that once again after an integral number of Larmor periods, we return to sharp values of p_1 and p_2, but in this case

$$G(p,p';t_n)=\tilde{g}'(t_n)\delta(p_2-p_2')\delta(p_1-p_1') \quad ,$$

and there is *no* gain in energy. This is true for any $\Omega\neq eB/m$ including integral multiples of eB/m. For there to be an effect of an integral multiple frequency, the spatial variation of $\vec{\mathrm{E}}$ must be significant; i.e., nonlinear interaction terms must be present.

APPENDIX

We present a proof that

$$\sum_{m=0}^{\infty}P_{mn}(t)=1 \quad \text{for arbitrary } n,t. \tag{A1}$$

We proceed by induction with respect to n:

$$P_{m0}(t)=e^{-\tau}(1/m!)\tau^{m} , \tag{A2}$$

since $L_0^{\alpha}=1$. Hence,

$$\sum_m P_{m0}=e^{-\tau}\sum_{m=0}^{\infty}\frac{1}{m!}\tau^m=1 . \tag{A3}$$

We assume

$$\sum_m P_{m,n-1}=e^{-\tau}\sum_{m=0}^{\infty}\frac{(n-1)!}{m!}\tau^{m+1-n}(L_{n-1}^{m+1-n})^2=1 , \tag{A4}$$

$$\sum_m P_{mn}=e^{-\tau}\sum_{m=0}^{\infty}\frac{n!}{m!}\tau^{m-n}(L_n^{m-n})^2 . \tag{A5}$$

But[4]

$$L_n^{\alpha}(x)=(1/n)[(n+\alpha)L_{n-1}^{\alpha}-xL_{n-1}^{\alpha+1}] , \tag{A6}$$

$$\sum P_{mn}=e^{-\tau}\sum_{m=0}^{\infty}\frac{(n-1)!}{m!}\tau^{m-n}L_n^{m-n}[mL_{n-1}^{m-n}-\tau L_{n-1}^{m-n+1}] ,$$

$$=e^{-\tau}\left(\sum_{m=1}^{\infty}\frac{(n-1)!}{(m-1)!}\tau^{m-n}L_n^{m-n}L_{n-1}^{m-n}-\sum_{m=0}^{\infty}\frac{(n-1)!}{m!}\tau^{m-n+1}L_n^{m-n}L_{n-1}^{m-n+1}\right) , \tag{A7}$$

and[3]

$$L_n^{m-n}=L_n^{m-n+1}-L_{n-1}^{m-n+1} . \tag{A8}$$

Therefore,

$$\sum P_{mn}=e^{-\tau}\left(\sum_{m=1}^{\infty}\frac{(n-1)!}{(m-1)!}\tau^{m-n}L_n^{m-n}L_{n-1}^{m-n}-\sum_{m=0}^{\infty}\frac{(n-1)!}{m!}\tau^{m-n+1}L_n^{m-n+1}L_{n-1}^{m-n+1}+\sum_{m=0}^{\infty}\frac{(n-1)!}{m!}\tau^{m-n+1}(L_{n-1}^{m-n+1})^2\right) \tag{A9}$$

$$=e^{-\tau}\sum_{m=0}^{\infty}\frac{(n-1)!}{m!}\tau^{m-n+1}(L_{n-1}^{m-n+1})^2 \tag{A10}$$

if we change summation variables in the first tensor. Thus,

$$\sum P_{mn}=\sum P_{m,n-1}=1 .$$

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[1]L. F. Landovitz (unpublished). Also, for a very thorough discussion of quantum mechanics in terms of Green's functions cf. J. Schwinger, *Quantum Kinematics and Dynamics* (Benjamin, New York, 1970).

[2]This problem can also be solved by standard techniques. Cf. P. W. Langhoff, S. T. Epstein, and M. Karplus, Rev. Mod. Phys. 44, 602 (1972).

[3]Cf. L. I. Schiff, *Quantum Mechanics* (Benjamin, New York, 1968).

[4]Cf. W. Magnus, F. Oberhettinger, and R. P. Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics* (Springer-Verlag, New York, 1966).