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Inverse Faraday effect in plasmas

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The Kubo response-function formalism is utilized to obtain expressions for the magnetization created in a plasma by a circularly polarized electromagnetic wave. This effect, the inverse Faraday effect, has previously been studied only in the dipole approximation. The present treatment includes plasma polarization effects and allows consideration of the thermal motion and relativistic effects associated with high-powered laser radiation. A physical interpretation of the results is presented in which the angular momentum stored in the plasma is shown to be the basis of the effect.

I. INTRODUCTION

The inverse Faraday effect is the production of a uniform magnetization in a medium in which a circularly polarized electromagnetic wave propagates. This effect and its relation to the Faraday effect (the rotation of a plane polarized wave propagating along the direction of an external magnetic field) was first discussed, in general, by Pershan[1] using free-energy considerations. The magnitude of the effect in solids has been calculated[2] and observed.[3] In plasmas, the effect has been interpreted in terms of a one-electron dipole approximation calculation[4] based on the fact that the rotating electric field associated with the circularly polarized radiation field drives the electrons into circular orbits[5] (the effect of ions can be neglected to a first approximation). The radius and velocity of the electrons describing the circular motion are proportional to the electric field of the electromagnetic wave. Hence, a magnetization proportional to the radiation intensity is obtained. For high radiation-field intensities, relativistic effects are expected and these effects have been calculated[6] in essentially the same approximation as in Ref. 4. Experiments tending to confirm the existence of the inverse Faraday effect in plasmas have been performed.[7]

In this paper, a general formulation of the inverse Faraday effect is given using the Kubo response-function formalism. Calculations of these response functions in the linear theory of the interaction of weak radiation fields with plasmas has in the past considerably simplified the understanding of relaxation phenomena.[8-10] However, as was mentioned above, the effect to be calculated here is proportional to the intensity of the radiation field and, hence, is of second order in the field amplitude. This means second-order response functions must be considered. The theory of these nonlinear response functions has formed the basis of much of the recent interest in response theory. For example, the interaction of strong electromagnetic fields in dilute gases has been treated using the Liouville technique to study the nonlinear response.[11] In addition, in certain cases such as the scattering of radiation by plasmas[12] or solids[13] it has been possible to relate these second-order processes to linear response functions. In these cases, the theory takes a particularly simple form. It will be shown here that the inverse Faraday effect can also be related to the functions that describe the linear response of the plasma, and hence, that a simple physical interpretation of the theory is possible.

In Sec. II, the physical basis of the inverse Faraday effect in terms of energy and angular-momentum considerations will be presented. The formulation of the theory in terms of second-order response functions and the calculation of these functions in terms of linear-response theory then follows in Sec. III. The first-principles calculation presented here permits the explanation of certain discrepancies among expressions which have appeared in the literature[4,6,7] and is discussed in Sec. IV. In addition, since the effect will be described in terms of dielectric functions and conductivities, this permits, for example, the inclusion of collective effects, thermal motion, ion contributions and specifically magnetic effects. Finally, in Sec. IV, the effect of absorption on the inverse Faraday effect is discussed.

II. INVERSE FARADAY EFFECT

As was mentioned in Sec. I, the physical basis of the inverse Faraday effect in plasmas can be seen by considering a textbook problem,[5] the trajectory of a particle of charge e and rest mass m_0 in the field of a circularly polarized wave of angular frequency ω and amplitude E_0. Choosing a frame of reference in which the particle has zero average velocity and position, the solution for a left circularly polarized (positive helicity) wave propagating in the $\vec{z}$ direction is circular motion with radius

$$r=|e|E_0/m_0\omega^2\gamma, \tag{1}$$

where $\gamma^2 = 1 + (eE_0/m_0c\omega)^2$ is the relativistic factor, and with transverse momentum

$$p_\perp = |e| E_0/\omega , \tag{2}$$

where $\vec{p} = \vec{P} - (e/c)\vec{A} = -(e/c)\vec{A}$ in the chosen frame of reference. The motion is such that the particle velocity is at all times perpendicular to the electron field and parallel to the magnetic field of the circularly polarized wave and thus, no work is done in maintaining the rotary motion. Therefore, in interaction with the radiation, the charged particle has acquired angular momentum $L_Z = e^2E_0^2/m_0\omega^2\gamma$ and hence, N such particles will have a magnetization per unit volume V given by

$$M_Z = -\frac{|e|}{2m_0c\gamma} L_Z = -\frac{|e|}{2m_0c}\frac{\omega_p{}^2}{4\pi}\frac{E_0^2}{\omega^3\gamma^2}, \tag{3}$$

where $\omega_p^2 = 4\pi Ne^2/Vm_0$ is the plasma frequency.

The creation of this magnetization by the circularly polarized wave is the inverse Faraday effect. However, in ordinary laboratory plasmas the orbit radius of electrons may attain macroscopic dimensions[4] and the neglect of electron interactions and collective effects in the derivation of Eq. (3) may be unjustified. To include these effects, in the next section the response-function formalism will be used to calculate directly the magnetization of the plasma. For this purpose it will be convenient to use another, but equivalent, physical interpretation of the effect, related to the energy considerations presented by Pershan.[1] This interpretation is based on the fact that the interaction between the circularly polarized radiation and the material system results in energy (and angular momentum) being transferred from the radiation field to the material system. That is, if we consider a rotationally invariant system of particles in its ground state in the absence of a radiation field, the angular momentum of this system is zero. By switching on a circularly polarized radiation field, the angular momentum (and hence, magnetization) acquired by the systems of particles will be equal to the angular momentum lost by the radiation field. In terms of the simple one-particle calculation presented above, the total energy of the particle in circular motion (kinetic + potential) is $U_p = \omega L_Z$. Similarly, a circularly polarized electromagnetic wave has energy[14]

$$U_R = \omega L_Z . \tag{4}$$

Thus, when energy is transferred from the field to the particle, angular momentum is transferred also.

In the method of solution given in Ref. 5 the initial conditions are chosen such that in the observer's frame of reference the particle starts out with the L_Z of Eq. (3) and since the motion is such that no work is done, these energy considerations are not evident. However, if the initial conditions are arbitrary, as in the case of a thermal distribution of initial positions and velocities, the same solution can be found by adiabatically switching on the external field and the final angular momentum at the end of the switching procedure can be expressed in terms of the work done during the adiabatic switching. Since the process is adiabatic, no energy is added to the system, radiation plus particles, it is simply redistributed.

More precisely, when the radiation field is switched on currents are induced in the system. The work done by the fields on these induced currents during the switching process can be interpreted in terms of the energy stored or propagating in the medium (Ref. 14, p. 197). This is the microscopic point of view in which the total electric and magnetic fields are E and B and the energy-density terms which appear in Poynting's theorem are

$$(E^2 + B^2)/8\pi + \int \vec{E}\cdot(\vec{j} + \langle\vec{j}\rangle)\, dt ,$$

where $\vec{j}$ is the true external current and $\langle\vec{j}\rangle$ is the induced current, sometimes written $\partial\vec{P}/\partial t + \vec{\nabla}\times\vec{M}$, and this last term is the energy density stored in the medium. Alternatively, and perhaps, more familiarly, this energy is usually included in the quantities $\vec{D}$ and $\vec{H}$ and the total electromagnetic energy density, written $(\vec{E}\cdot\vec{D} + \vec{B}\cdot\vec{H})/8\pi$. In Sec. III we shall adopt the microscopic point of view and calculate the angular momentum (or, more exactly, the magnetization per unit volume) supplied by the radiation field when it does work on the induced currents during the adiabatic switching process. The basic fields to be used are, therefore, $\vec{E}$ and $\vec{B}$.

III. RESPONSE-FUNCTION CALCULATION

In this section, a microscopic theory of the magnetization created in a plasma by an external circularly polarized electromagnetic wave will be presented using the relaxation formalism. This formalism as will be seen, permits general results to be derived which do not depend on the usual dipole or one electron approximation discussed in Sec. II.

The external electric and magnetic fields, $\vec{E}^0$ and and $\vec{B}^0$, are created by an external charge and current density ρ_0 and $\vec{j}_0$ and satisfy the relevant Maxwell equations[15]

$$\begin{aligned} ic\vec{k}\times\vec{B}^0(k,\omega) &= -i\omega\vec{E}^0(k,\omega) + 4\pi\vec{j}_0(k,\omega), \\ ic\vec{k}\times\vec{E}^0(k,\omega) &= +i\omega\vec{B}^0(k,\omega), \end{aligned} \tag{5}$$

where the Fourier transform,

$$\vec{E}(k,\omega)=\int d^3x\int dt\, e^{-i\vec{k}\cdot\vec{x}}e^{i\omega t}\,\vec{E}(x,t),$$

has been taken. These external fields induce local charges and currents $\langle\rho\rangle, \langle\vec{j}\rangle$, which create fields $\langle\vec{E}\rangle, \langle\vec{B}\rangle$ that combine with the external fields to produce the total local fields, $\vec{E}$ and $\vec{B}$, in the plasma. Thus, we have

$$\begin{aligned}\vec{E}(k,\omega)&=\vec{E}^0(k,\omega)+\langle\vec{E}(k,\omega)\rangle,\\ \vec{B}(k,\omega)&=\vec{B}^0(k,\omega)+\langle\vec{B}(k,\omega)\rangle,\end{aligned}\tag{6}$$

and the corresponding Maxwell equations

$$\begin{aligned}ic\vec{k}\times\vec{B}(k,\omega)&=-i\omega\vec{E}(k,\omega)+4\pi[\vec{j}_0(k,\omega)+\langle\vec{j}(k,\omega)\rangle],\\ ic\vec{k}\times\vec{E}(k,\omega)&=i\omega\vec{B}(k,\omega).\end{aligned}\tag{7}$$

The problem is thus to calculate the induced magnetization at time t due to the external fields $\vec{E}^0$ and $\vec{B}^0$ which are switched on adiabatically in the plasma starting at $t=-\infty$. Due to the presence of spatial and time dispersion, the fields at r, t depend on the fields at all r' and previous times t'. Thus, we divide $\vec{E}^0$ for $t>0$ into two parts[15]

$$\vec{E}^0=\vec{E}^{(01)}+\vec{E}^{(02)},$$

where $\vec{E}^{(01)}$ depends on the switching procedure for $t<0$, and $\vec{E}^{(02)}$ is the external fields after the external sources are turned off at time $t=0$.

For a uniform time-independent system, these external fields are linearly related (neglecting third-order field effects[16]) to the total local fields in the plasma by $\vec{\epsilon}$, the dielectric tensor of the medium as follows:

$$\vec{E}^{(01)}(x,t)=\int d^3x\int_{-\infty}^{0}dt'\,\vec{\epsilon}(x-x',t-t')\vec{E}^{(1)}(x',t'),$$

$$\vec{E}^{(02)}(x,t)=\int d^3x\int_{0}^{t}dt'\,\vec{\epsilon}(x-x',t-t')\vec{E}^{(2)}(x',t').$$

Thus, $\vec{E}^{(1)}$ is the total field in the medium for $t\leq 0$ and $\vec{E}^{(2)}$ the total field after the sources are turned off. Since the dielectric function is causal (i.e., is zero for negative values of its time argument), these fields can be Fourier transformed and satisfy Eq. (5) which becomes for $t>0$, at which time the sources have been turned off:

$$\begin{aligned}c^2\vec{k}\times[\vec{k}\times\vec{\epsilon}(k,\omega)\vec{E}^{(2)}(k,\omega)]+\omega^2\vec{\epsilon}(k,\omega)\vec{E}^{(2)}(k,\omega)\\ =-c^2\vec{k}\times[\vec{k}\times\vec{E}^{(01)}(k,\omega)]-\omega^2\vec{E}^{(01)}(k,\omega).\end{aligned}\tag{8}$$

The dielectric tensor can be written in terms of a transverse part ϵ_T and a longitudinal part ϵ_L as follows:

$$\epsilon_{ij}=(k_ik_j/k^2)\epsilon_L+(\delta_{ij}-k_ik_j/k^2)\epsilon_T,\tag{9}$$

and the corresponding transverse local fields satisfy

$$E_T^{(2)}(k,\omega)=-E_T^{(01)}(k,\omega)/\epsilon_T(k,\omega).\tag{10}$$

In the following, we shall calculate the magnetization produced in a plasma by turning on the field $\vec{E}^{(01)}$ adiabatically from $t=-\infty$ to $t=0$ and express the result for $t>0$ in terms of the total local field in the medium $\vec{E}^{(2)}(k,\omega)$. The field $\vec{E}^{(01)}$ is turned on adiabatically since the independent systems of charged particles and external photons which are present at $t=-\infty$ are to be brought into contact in such a manner that the total energy of the systems remains constant (i.e., a temperature T can be defined). At $t=0$, a single system of particles and field exists with some of the energy and angular momentum of the external field now stored in the plasma and the rest present in terms of local fields.

If a dissipative process is also included, there is also a certain quantity of field energy dissipated during the adiabatic switching procedure. This energy is rejected to an external heat bath, and must not be considered to be stored in the plasma. If it were, a unique temperature could not be defined. The object, therefore, of the following calculation is to obtain the difference in magnetization of the plasma between $t=-\infty$ when the external sources begin to be switched on and $t=0$ when they are switched off. After $t=0$ we observe the relaxation of the magnetization. Note that the time $t=0$ is arbitrary since after the systems are fully in contact if the sources are not turned off, all the energy supplied to the system is dissipated to the heat bath in a steady state process.

Since, as was stated in the Introduction, we are considering a process proportional to the intensity of the incident field, the methods which have been developed for linear response[8,17] must be carried to second order. Thus, we calculate, to second order

$$\langle M_Z(t)\rangle=\mathrm{Tr}\rho(t)M_Z,\tag{11}$$

where M_Z is the Z component of the magnetization operator and $\rho(t)$ is the density operator for the system perturbed by a circularly polarized electromagnetic wave propagating along the Z axis. In the following, we shall use a wave with negative helicity (right-hand circular polarization), which for the electron gas, will yield a magnetic field in the positive Z direction.

Positive helicity produces the same magnetic field, but in the negative Z direction. Thus, the general case can be obtained by an appropriate combination of the two helicities[7]. The magnetization operator is taken to be

$$\vec{M}_Z=\frac{1}{V}\int\frac{d\vec{r}}{2c}\{\vec{r}\times\vec{J}(r,t)\}_Z,\tag{12}$$

where

$$\vec{J}(r,t)=\sum_i \frac{e_i}{2m_i}\sum_\alpha \left[\vec{P}^i_\alpha-\frac{e_i}{c}\vec{A}(r,t),\,\delta(r-r^i_\alpha(t))\right]_+ \tag{13}$$

is the symmetrized current density operator; e_i, m_i are the charge and mass of the ith species, $\vec{P}^i_\alpha$ is the momentum operator for the αth particle of the ith species; and $\vec{A}(r,t)$ is the vector potential in the Coulomb gauge $\vec{k}\cdot\vec{A}(k,\omega)=0$, which is related to the transverse fields by

$$\vec{E}_T=-(1/c)\dot{\vec{A}},\quad \vec{B}=\vec{\nabla}\times\vec{A}. \tag{14}$$

The average in Eq. (11) is calculated using a density operator obtained from the solution to the Liouville equation

$$i\,d\rho(t)/dt=[H,\rho(t)], \tag{15}$$

with the initial condition $\rho(-\infty)=\rho_0$. In the above, ρ_0 is the equilibrium density operator and H is the total Hamiltonian of the system

$$H=H_P+H_R+H_I+H_I^0. \tag{16}$$

Here, H_P is the plasma particle Hamiltonian, H_R is that for the internal radiation field, and H_I is the matter-field interaction which, in the absence of the external field, can be written in the Coulomb gauge

$$H_I=-\frac{1}{c}\int d\vec{r}\,\vec{J}(r,t)\cdot\vec{A}(r,t) -\frac{e^2}{2mc^2}\int d\vec{r}\,n(r,t)(\vec{A}(r,t))^2, \tag{17}$$

where $n(r,t)$ is the particle-density operator. The term H_I^0 is the extra term added to the Hamiltonian due to the interaction with an external perturbation $\vec{A}^{(01)}$.

Noting that there is an explicit change in the current operator when the external field is applied, so that the total current operator in the presence of this external field is

$$\vec{J}'=\vec{J}-\frac{e^2}{mc}\int d\vec{r}\,n(r,t)\vec{A}^{(01)}(r,t), \tag{18}$$

the perturbation H_I^0 becomes

$$H_I^0=-\frac{1}{c}\int d\vec{r}\,\vec{J}(r,t)\cdot\vec{A}^{(01)}(r,t) +\frac{e^2}{2mc^2}\int d\vec{r}\,n(r,t)\{\vec{A}^{(01)}(r,t)\}^2. \tag{19}$$

Including the explicit change in the magnetization operator, we must now calculate

$$\langle M_Z'(t)\rangle=\langle M_Z(t)\rangle -\frac{e^2}{2mc^2}\frac{1}{V}\int d\vec{r}\{\vec{r}\times\vec{A}^{(01)}(r,t)\langle n(r,t)\rangle\}_Z. \tag{20}$$

Solving Eq. (15) to second order in the external field $\vec{A}^{(01)}$, we obtain, in the usual manner, the gauge invariant expression

$$\begin{aligned}\langle M_Z'(t)\rangle=&\frac{i}{c}\int_0^\infty d\tau\int d\vec{r}\langle[M_Z(\tau),J_\alpha(r)]\rangle_0 A_\alpha^{(01)}(r,t-\tau)-\frac{e^2}{mc^2}\frac{1}{2V}\int_v d\vec{r}(\vec{r}\times\vec{A}^{(01)}(r,t))_Z\langle n\rangle_0\\ &-\frac{ie^2}{2mc^2}\int_0^\infty d\tau\int d\vec{r}\langle[M_Z(\tau),n(r)]\rangle_0\{A^{(01)}(r,t-\tau)\}^2\\ &+\left(\frac{i}{c}\right)^2\int_0^\infty d\tau\int d\vec{r}\int_0^\infty d\tau'\int d\vec{r}'\langle[[M_Z(\tau),J_\alpha(r)],J_\beta(r',-\tau')]\rangle_0 A_\alpha^{(01)}(r,t-\tau)A_\beta^{(01)}(r',t-\tau-\tau')\\ &-\frac{ie^2}{mc^2}\frac{1}{2Vc}\int_0^\infty d\tau\int d\vec{r}\int_0^\infty d\tau'\int d\vec{r}'(\vec{r}'\times\vec{A}^{(01)}(r',t))_Z\langle[n(r',\tau),J_\alpha(r)]\rangle_0 A_\alpha^0(r,t-\tau),\end{aligned} \tag{21}$$

where $\langle\cdots\rangle_0$ is the average taken with respect to the equilibrium density operator (the summation convention has been used with respect to the indices α,β of the spatial components).

Since, as can be easily seen, the magnetization operator is simply related to the total angular momentum operator, it commutes with the unperturbed Hamiltonian of the system. Therefore, the well-known equal time commutation rules can be used to compute the commutators in Eq. (21). Thus, using

$$[M_Z,J_\alpha]=(ie/2mc)(1/V)\{\epsilon_{Z\alpha\beta}J_\beta+(\vec{r}\times\vec{\nabla})_Z J_\alpha\},\quad [M_Z,n(r)]=-(ie/2mc)(1/V)(\vec{r}\times\vec{\nabla})_Z n(r), \tag{22}$$

where $\epsilon_{Z\alpha\beta}$ is the well-known unit antisymmetric tensor, we find that the first three terms of Eq. (21) vanish for a spatially homogeneous isotropic system. We now explicitly write the external field in terms of a monochromatic right circularly polarized vector potential (negative helicity)

$$\vec{A}^{(01)}(r,t)=(A^{(01)}\cos(\vec{k}\vec{r}-\omega t),\,A^{(01)}\sin(\vec{k}\vec{r}-\omega t),\,0)\,e^{\eta t}, \tag{23}$$

where $e^{\eta t}$ (with the limit $\eta\to 0$ implied at the end) is the adiabatic switching factor which will also maintain the causality of the response and insure the convergence of the Fourier transforms. The remaining two terms in Eq. (21) can now be written

$$\begin{aligned}\langle M_Z'(t)\rangle=&\frac{-|e|}{2mc^3}\frac{1}{4V}\int_0^\infty d\tau\int_{-\infty}^\infty d\tau'\int_V d\vec{r}\int_V d\vec{r}'\left(\frac{-1}{i}\right)\\&\times\{\pi^{+-}(r-r',\tau')|A^{(01)}|^2e^{-ik(r-r')}e^{i\omega\tau'}e^{-\eta\tau'}-\pi^{-+}(r-r',\tau')|A^{(01)}|^2e^{ik(r-r')}e^{-i\omega\tau'}e^{-\eta\tau'}\}e^{2\eta(t-\tau)}\\&+\frac{|e|}{mc^3}\frac{1}{4V}\int_{-\infty}^{+\infty}d\tau\int_V d\vec{r}\int_V d\vec{r}'(\vec{r}\times\vec{A}^{(01)}(r,t))_Z\{\pi^+(r-r',\tau)A_-^{(01)}(r',t-\tau)\\&+\pi^-(r-r',\tau)A_+^{(01)}(r',t-\tau)\},\end{aligned} \tag{24}$$

where we have introduced the circular operators

$$A_\pm^0(r,t)=A_x^0(r,t)\pm iA_y^0(r,t)=A^0e^{\pm ikr\mp i\omega t+\eta t},\quad J_\pm(r,t)=J_x(r,t)\pm iJ_y(r,t), \tag{25}$$

and where

$$\pi^{+-}(r-r',t)=i\theta(t)\langle[J_+(r,t),J_-(r',0)]\rangle_0,\quad \pi^\pm(r-r',t)=i\theta(t)\langle[en(r,t),J_\pm(r',0)]\rangle_0. \tag{26}$$

Note that $\pi^{++}=\pi^{--}=0$ for a spatially homogeneous time-reversal-invariant system. Here $\theta(t)$ is the Heaviside step function, zero for negative times. The retarded commutator functions in Eq. (26) have been extensively studied in linear-response theory[8] and have been calculated explicitly for classical and quantum plasmas.[9,10] We have thus reduced the second-order calculations implied in Eq. (21) to the study of integrals over linear-response functions. This will greatly facilitate the physical interpretation of the results.

The response function $\pi^{\alpha\beta}$ will later be described in terms of the external conductivity of the system. It is related to the retarded commutator functions π^α by the continuity equation

$$\partial/\partial t\; en(r,t)=i[H,en(r,t)]=-\vec{\nabla}\cdot\vec{J}(r,t), \tag{27}$$

which yields

$$\frac{\partial}{\partial t}\pi^\alpha(r-r',\tau)=\pi^\alpha(r-r',0)\delta(\tau)-\sum_\beta\nabla_\beta\pi^{\beta\alpha}(r-r',\tau). \tag{28}$$

Using this relation and

$$\pi^\alpha(r-r',0)=(-\omega_p^2/4\pi)\nabla_\alpha'\delta^3(r-r'), \tag{29}$$

which is the sum rule which ensures the gauge invariance of the theory, the second term in Eq. (24) may be integrated by parts in the space and time variables. The result is

$$\begin{aligned}\langle M_Z'(t)\rangle=\frac{|e|}{2mc^3}\Bigg\{&\frac{1}{4V}\int_V d\vec{r}\int_V d\vec{r}'\Bigg[\int_{-\infty}^t e^{2\eta\tau}d\tau\int_{-\infty}^{+\infty}d\tau'\,\pi^{+-}(r-r',\tau')2\sin(-k(r-r')+\omega\tau')e^{-\eta\tau'}\\&-e^{2\eta t}\int_{-\infty}^{+\infty}d\tau\,\pi^{+-}(r-r',\tau)2\,\mathrm{Re}\left(\frac{e^{-ik(r-r')+i(\omega+i\eta)\tau}}{\omega+i\eta}\right)\Bigg]\\&+\mathrm{Re}\left(\frac{\omega_p^2}{4\pi(\omega+i\eta)}\right)e^{2\eta t}\Bigg\}\,|A^{(01)}|^2.\end{aligned} \tag{30}$$

In the above we have used the fact that $\pi^{++}=\pi^{--}=0$ and hence, $\pi^{+-}=\pi^{-+}$ for an isotropic system. Equation (30) contains all the magnetization transferred to the plasma between $t=-\infty$ and t, including that part subsequently rejected to the heat bath which maintains the system at a constant temperature. To see this, we now take the time derivative of Eq. (30), being careful to retain in the rate of magnetization those terms with an infinitesimal rate, proportional to η, which result from the differentiation of the adiabatic switching factor $e^{\eta t}$. These infinitesimal rates acting over an infinite time interval result in that part of the magnetization adiabatically stored in the plasma at $t=0$. The time derivative of Eq. (30) is

$$\frac{d}{dt}\langle M_Z'(t)\rangle = \frac{+|e|}{2mc^3}e^{2\eta t}\left(\frac{1}{2V}\int_V d\vec{r}\int d\vec{r}'\left\{\left(\int_{-\infty}^{+\infty}d\tau'\,\pi^{+-}(r-r',\tau)\sin(-k(r-r')+\omega\tau')e^{-\eta\tau'}\right)_{\eta=0}\right.\right.$$
$$-\eta\left(\int_{-\infty}^{+\infty}d\tau'\,\tau'\,\pi^{+-}(r-r',\tau')\sin(-k(r-r')+\omega\tau')e^{-\eta\tau'}\right)_{\eta=0}$$
$$\left.-2\eta\left[\int_{-\infty}^{+\infty}d\tau\pi^{+-}(r-r',\tau)\,\mathrm{Re}\left(\frac{e^{-ik(r-r')+i(\omega+i\eta)\tau}}{\omega+i\eta}\right)\right]_{\eta=0}\right\}$$
$$\left.+2\eta\left(\frac{\omega_p^2}{4\pi\omega}\right)\right)|A^{(01)}|^2, \tag{31}$$

where Eq. (30) has been expanded to first order in a power series in η to display the linear term which it contains.

If now, for illustrative purposes, the $\eta\to 0$ limit is taken to obtain the explicit dissipation in the system, we obtain

$$\lim_{\eta\to 0}\frac{d\langle M_Z'(t)\rangle}{dt} = \frac{|e|}{2mc^3}\,\tfrac{1}{2}\,\mathrm{Im}\pi^{+-}(k,\omega)\,|A^{(01)}|^2, \tag{32}$$

where the Fourier transform has been performed according to

$$\pi^{\alpha\beta}(k,\omega) = \lim_{\eta\to 0}\int d\vec{r}\int d\tau\,\pi^{\alpha\beta}(r,\tau)\times e^{-ik\cdot r+i(\omega+i\eta)\tau}. \tag{33}$$

This result is related to the usual observation that the dissipation in a plasma is proportional to the imaginary part of the response function. This could have been more simply obtained by direct calculation of the dissipation from Eq. (20), neglecting the adiabatic term

$$\frac{d}{dt}\langle M_Z'(t)\rangle = -\frac{e^2}{2mc^2}\frac{1}{V}\times\int d\vec{r}\left(\vec{r}\times\frac{d}{dt}\vec{A}^{(01)}(r,t)\langle n(r,t)\rangle\right)_z, \tag{34}$$

which using a method, identical to that leading to Eq. (30) results also in Eq. (32). To obtain now the adiabatic contribution to the magnetization we integrate the remaining two terms which are proportional to η. This gives

$$\int_{-\infty}^{0}\frac{d}{dt}\langle M_Z'(t)\rangle\,dt = \langle M_Z'(0)\rangle = \frac{|e|}{2mc^3}\frac{1}{2}\left(\frac{1}{2}\frac{\partial\mathrm{Re}\pi^{+-}(k,\omega)}{\partial\omega} - \frac{\mathrm{Re}\pi^{+-}(k,\omega)}{\omega} + \frac{\omega_p^2}{2\pi\omega}\right)|A^{(01)}|^2. \tag{35}$$

This now represents the magnetization per unit volume of the plasma at $t=0$ when the external sources are turned off. This expression can be written in terms of more familiar quantities by introducing the external conductivity σ^0, which relates the induced current density to the external field

$$\langle J_\alpha(k,\omega)\rangle = \sigma^0_{\alpha\beta}(k,\omega)E^0_\beta(k,\omega). \tag{36}$$

Using this expression the following relations can be obtained from linear-response theory[7-9]:

$$\pi^{\alpha\beta}(k,\omega) = \omega\sigma^0_{\alpha\beta}(k,\omega) + (\omega_p^2/4\pi)\delta_{\alpha\beta}. \tag{37}$$

This relation can be used in our second-order theory, since nonlinear effects occur only in third order for a spatially homogeneous isotropic medium.[16] Thus, Eq. (35) takes the form

$$\langle M_Z'\rangle = \frac{|e|}{2mc^3}\frac{1}{2}\left[\omega^2\frac{\partial}{\partial\omega}\left(\frac{\mathrm{Re}\sigma^0_T(k,\omega)}{\omega}\right)\right]|A^{(01)}|^2, \tag{38}$$

where σ^0_T is the transverse external conductivity defined by

$$\sigma^0_{\alpha\beta} = \frac{k_\alpha k_\beta}{k^2}\sigma^0_L + \left(\delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2}\right)\sigma^0_T. \tag{39}$$

Only the transverse part of σ^0 occurs since we have used the Coulomb gauge and $\vec{A}$ is then transverse.

Equation (38) represents the complete expression to second order for the magnetization produced in a plasma by an externally applied right circularly polarized monochromatic electromagnetic wave. In plasmas, however, as was pointed out in Eq. (10) the external field at $t=0$ is not the total field and the electrodynamics of the plasma is more properly described in terms of the total local field $\vec{E}^{(2)}$ defined by Eq. (10). The local conductivities and dielectric functions σ,ϵ which are the physical parameters usually used to describe the plasma are related to the external conductivities by the Maxwell equations (4) and (6). Writing only those quantities which occur in the transverse response, we have[15]

$$\epsilon_T(k,\omega)-1=\frac{4\pi\omega\sigma_T(k,\omega)}{\omega^2-k^2c^2}=\frac{4\pi\omega\sigma_T^0(k,\omega)}{\omega^2-k^2c^2-4\pi\omega\sigma_T^0(k,\omega)}, \tag{40}$$

$$\sigma_T^0=\sigma_T/\epsilon_T. \tag{41}$$

The separation of the response into longitudinal and transverse quantities is, of course, equivalent[18] to the usual description in terms of a dielectric function and a magnetic permitivity, but has the advantage of avoiding the problem associated with the division of the current into a time derivative of a polarization and the curl of a magnetization.[19]

Substituting Eq. (41) into Eq. (38) and using the definition of Eq. (10) for the total local field at time $t=0$, we obtain, after some algebra, the magnetization in terms of the total field $\vec{A}^{(2)}$,

$$\langle M_Z'\rangle=\frac{|e|}{2mc^3}\left[\frac{\omega^2}{2}\frac{\partial}{\partial\omega}\mathrm{Re}\left(\frac{\sigma_T}{\omega}\right)+\frac{4\pi c^2k^2\omega\,|\sigma_T|^2}{(\omega^2-k^2c^2)^2}\right]|A^{(2)}|^2. \tag{42}$$

In terms of the basic fields, $\vec{E}^{(2)}$ and $\vec{B}^{(2)}$, we have

$$\langle M_Z'\rangle=\frac{|e|}{2mc}\left[\frac{1}{2}\frac{\partial}{\partial\omega}\mathrm{Re}\left(\frac{\sigma_T(k,\omega)}{\omega}\right)|E|^2+|1-\epsilon_T(k,\omega)|^2\frac{|B|^2}{4\pi\omega}\right],\quad t=0, \tag{43}$$

where we now drop the superscript on the field amplitudes, since for $t\geq 0$ it is understood that the total fields in the plasma are given by Eq. (10).

This expression now represents the magnetization of the plasma at $t=0$. For later times, this magnetization decays with the rate given in Eq. (32) which can be written

$$\frac{d\langle M_Z'\rangle}{dt}=\frac{|e|}{2mc}\frac{\mathrm{Im}\sigma_T(k\omega)}{\omega}|E|^2,\quad t>0. \tag{44}$$

These two expressions, Eqs. (43) and (44), are the complete result for the creation and dissipation of magnetization when circularly polarized electromagnetic waves interact with a plasma.

IV. DISCUSSION OF RESULTS

The interpretation in terms of stored angular momentum can be made more explicit by considering the expression for U_p, the energy density associated with an electromagnetic wave in a plasma, neglecting magnetic field effects[20]

$$U_p=\frac{1}{2}\frac{\partial\sigma}{\partial\omega}(k,\omega)|E|^2+\frac{1}{8\pi}|E|^2. \tag{45}$$

Subtracting out the energy density of the radiation field $(1/8\pi)|E|^2$, to obtain the net energy density stored in the plasma, it can be seen that the first term in Eq. (43) is the corresponding expression for the magnetization stored by the electric field, since for a circularly polarized wave the energy is related by Eq. (4) to the angular momentum. We can therefore interpret the first term in Eq. (43) as the magnetization stored by the electric field in the ordered motion of particles in the medium.

The second term in Eq. (43) can be interpreted more easily if we introduce the induced field, using the definition

$$\langle\vec{A}\rangle=(1-\epsilon_T)\vec{A}, \tag{46}$$

a relation also valid to second order in isotropic systems. The second term then becomes $(e/2mc)|\langle\vec{B}\rangle|^2/\omega$, the angular momentum stored in the induced magnetic field associated with the electromagnetic wave. Note it is *not* related to the work done by the magnetic field of the wave, since the electron motion is always parallel to the magnetic vector of the wave and thus, no work is done. It is perhaps more suggestive to write this term as $c\vec{k}\times\langle\vec{B}\rangle\cdot\langle\vec{E}\rangle/\omega^2$, which shows this term to be the magnetization created when the induced electric field does work on the induced vortical currents. This is therefore, a specifically collective effect associated with spatial dispersion in the system and contributes directly to the magnetization. It is always absent in calculations which use the one electron theory or the dipole approximation $(k\to 0)$. In addition, since the classical diamagnetism of a plasma is zero, a quantum calculation of ϵ_T must be performed to correctly include the induced magnetic field.[21] In any case, it will usually be a small correction since the quantum results are always of order k^2 and the intrinsic magnetization of a plasma is also small.[22] In the following, therefore, we consider only the first term in order to compare this formula with previous results.

If we consider a classical electron gas and neglect thermal motion, then for $\omega>\omega_p$ the conductivity, $\sigma_T(k,\omega)=-\omega_p^2/4\pi\omega$ and the first term in Eq. (43) becomes

$$\langle M_Z'\rangle=\frac{|e|}{2mc}\frac{1}{4\pi}\frac{\omega_p^2}{\omega^3}|E|^2. \tag{47}$$

This is the nonrelativistic limit of the one electron calculation given in Eq. (3) and is identical with the result of Ref. 4. However, it should be noted that in the one electron, dipole approximation calculations of Refs. 4 and 5 the field which drives the electrons is the external field. This is of course, correct for dilute plasmas where $\omega_p\ll\omega$, when the external and local fields become identical ($\epsilon_T\sim 1$ in this case). For these plasmas, collective effects are absent and $\vec{E}^{(01)}=\vec{E}^{(2)}$. Some confusion, however, has arisen in the literature over the inclusion of collective effects in Eq. (47). We have shown in the derivation above that collective effects can be included only if a careful distinction is made between the local field $\vec{E}^{(2)}$ and the external driving field $\vec{E}^{(01)}$. When this is done, the correct dipole approximation result, Eq. (47) shows that collective effects are included by replacing the local field by the total field. In addition, it is only in this manner that the relation between the direct and inverse Faraday effects obtained by Pershan[1] and applied to plasmas by Pomeau and Quemada[4] is verified. In Sec. V we will demonstrate this more explicitly in terms of the quasiparticle picture of a plasma.[23]

A similar calculation by Steiger and Woods[6] considers the inverse Faraday effect in terms of the propagation modes available in a plasma for circularly polarized radiation. The Maxwell equations for the induced fields are solved for the induced current density. That is, the fluctuating $\langle \vec{E} \rangle$, in our notation, is considered to be the driving field for the circular motion of the plasma electrons. The assumption is then made that this mode will couple to an external laser field. Using in Eq. (46) the same approximation for σ as used above, the first term in Eq. (43) becomes

$$\langle M_Z^{\prime} \rangle \approx \frac{|e|}{2m_0 c} \frac{1}{4\pi} \frac{\omega_p^2}{\omega^3} \left(\frac{\omega^2 - c^2 k^2}{\omega_p^2} \right)^2 |\langle \vec{E} \rangle|^2 , \qquad (48)$$

in agreement with the result given in Eq. (2.13) of Ref. 6. Note that this result is written in the nonrelativistic limit [it is identical to Eq. (47) but written in terms of the induced field]. To introduce relativistic effects, the authors of Ref. 6 use the identity which relates the induced current to the induced field

$$\langle \vec{J} \rangle = [\omega^2 - k^2 c^2 / 4\pi\omega] \langle \vec{E} \rangle$$

to write a relativistic formula for the induced current. That is, the induced current is assumed to be circular and, thus, can be written $\langle \vec{J} \rangle = Ner\omega = (\omega_p^2/4\pi) m_0 r \omega / e$, where r is the relativistic radius of Eq. (1). When this is done, the factor $(\omega^2 - k^2c^2)/\omega_p^2$, which in this approximation relates the total and induced fields becomes the relativistic factor $1/\gamma$ defined in Eq. (1). Thus, Eq. (48) can be written, in the form of Eq. (3),

$$\langle M_Z^{\prime} \rangle = \frac{|e|}{2m_0 c} \frac{1}{4\pi} \frac{\omega_p^2}{\omega^3} \frac{1}{\gamma^2} |\langle \vec{E} \rangle|^2 , \qquad (49)$$

where $\gamma^2 = 1 + (eE_0/m_0 c\omega)^2 = (1 - \omega^2 r^2/c^2)^{-1}$ using the definition of r in Eq. (1).

One must be careful in interpreting Eq. (49) since it is written in terms of the induced field. As was pointed out in Sec. III the field for $t > 0$ depends on the field $t \leq 0$ and therefore the total field in the plasma is different from the induced field. For this reason, the induced-field intensity cannot be equated with the external laser power.

One can obtain an estimate of relativistic effects by using a relativistic conductivity σ_r in Eq. (43). This is slightly inconsistent since we have used a nonrelativistic Hamiltonian in the derivation. Nevertheless, using the dipole approximation expression[24] $\sigma_r/\omega = -c^2/3a^2\omega^2$, where $a^2 = \kappa TV/4\pi Ne^2$ is the Debye length, in the ultrarelativistic case, when the thermal energy, kT is of the order of $m_0 c^2 \gamma$, we obtain

$$\frac{\omega^2}{2} \frac{\partial}{\partial \omega} \frac{\sigma_r}{\omega} \approx \frac{\omega_p^2}{\gamma\omega} , \qquad (50)$$

which leads to Eq. (3) with the total field replacing the external field.

Finally, if the absorptive properties of the plasma are taken into account for $t > 0$, the plasma will dissipate the magnetization stored during the adiabatic switching. This dissipation is described by Eq. (44). Since $\mathrm{Im}\sigma_T(k, \omega)$ is the same parameter as that measured in the damping of transverse waves in plasmas,[9,10] Eq. (44) shows explicitly that the decay of the magnetization is proportional to the rate of decay of the angular momentum of the local field in the plasma. Thus, experiments such as those of Ref. 7 in which the rate of decay of the magnetization of the plasma is measured, can give information on the propagation characteristics of transverse circularly polarized waves in plasmas.

V. CONCLUSION

A general response-function formalism has been developed for the analysis of the inverse Faraday effect in plasmas. The formalism allows a simple interpretation in terms of the angular momentum stored or dissipated in the medium. By making a careful distinction between the total local field in the plasma and the external field, a comparison of seemingly different results which occur in the literature was possible. This distinction, which is characteristic of charged systems, has been the source of confusion in the past[20] and must be made whenever an induced quantity is calculated.

The general result of Eq. (43) can be used to include the effects of thermal motion and ions in classical plasmas and thereby improve the classical nonrelativistic calculations. In addition, expressions for the relativistic conductivity are available in the literature[24] and in spite of the nonrelativistic Hamiltonian used in this work an approximation to the relativistic effects expected to occur at high external field strengths can be found. Finally, it would be interesting to use a quantum mechanical response function such as that calculated in Refs. 9 and 10 to find the small diamagnetic and paramagnetic corrections to the effect discussed in Sec. IV.

The distinction between local and external fields is not necessary in neutral gases and Eq. (24) can be used with $\vec{A}^{(01)}$ as the incident total radiation field. The quantity $\sigma_T^0(k,\omega)/\omega$ of Eq. (38) is the susceptibility, $\chi(\omega)$, of the gas which is simply related to the atomic dipole moment correlation function, often calculated in problems related to the spectral line shape of the neutral gas spectrum.[25] It is to be remarked here, however, that it is the real part of this function which determines the inverse Faraday effect while the imaginary part determines the line shape. In addition, since near spectral resonances, $\chi(\omega)$ is strongly frequency dependent, the induced magnetization which is proportional to the derivative should be largest in these regions. Thus, at frequencies near strong spectral lines, this effect may be large enough to be observed. A calculation is presently underway and the results will be published in a future paper.

An estimate of the effect in nonabsorbing gases can be obtained by using the response function for a harmonic oscillator in Eq. (43). For this case we have

$$\mathrm{Re}\chi(\omega)=\frac{\mathrm{Re}\sigma_T(0,\omega)}{\omega}=\frac{1}{4\pi}\frac{\omega_p^2}{\omega_0^2-\omega^2} \tag{51}$$

and hence, for the harmonic oscillator we have

$$\langle M_Z'\rangle_{\mathrm{osc}}=\frac{|e|}{2mc^2}\frac{1}{4\pi}\frac{\omega\omega_p^2}{(\omega_0^2-\omega^2)^2}|E^{(0)}|^2. \tag{52}$$

Thus, a neutral gas, or a collection of noninteracting harmonic oscillators will store angular momentum in the same way as the plasma. This analogy can be made more explicit by expressing the approximate plasma result, Eq. (47), in terms of the external field at time $t=0$, using the dipole approximation relation $\vec{E}_0=(1-\omega_p^2/\omega^2)\vec{E}$. This results in

$$\langle M_Z'\rangle=\frac{|e|}{2mc^2}\frac{1}{4\pi}\frac{\omega\omega_p^2}{(\omega^2-\omega_p^2)^2}|E^{(0)}|^2. \tag{53}$$

Here the plasma collective effects which are usually included in the total field now appear in the fact that the plasma particles react to external fields as a collection of harmonic oscillators with a natural frequency ω_p. This is the usual classical interpretation of a plasma found in most textbooks,[14,20] and is the basis for the quasiparticle approach to plasma dynamics.[23]

Finally, the direct Faraday effect (the rotation of the plane of polarization of a plane polarized wave propagating along the direction of an externally applied magnetic field) can also be understood by the same angular momentum arguments as those given in Sec. II. Thus, the rotation of the plane of polarization can be considered to be the acquisition of angular momentum by the radiation field in interaction with a gyrotropic medium. Since this argument closely parallels the energy consideration in Ref. 1, details can be found in that paper. The symmetry relation between these two effects is also derived in Ref. 1.

The possibility of performing experiments such as those in Ref. 7 where the decay of the magnetization is observed is perhaps the most interesting method of studying the inverse Faraday effect in plasmas. Experiments of this type performed as a function of frequency can give information on the dissipative properties of plasmas. Perhaps, more interesting is the possibility of similar experiments in neutral gases where the magnetization created can be detected by the Zeeman shift of the atomic lines of a noninterfering probe gas. This will yield the real part of the polarizability, while the imaginary part can be obtained from experiments on the rate of decay of the magnetization.

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We mention this here only because of the possiblity of confusion arising from the notation. For the interested reader, the external field, E_0 and the induced field, $\langle E\rangle$ are denoted by $\tilde{E}$ and E in Ref. 18. The notation used in this paper is the same as that in Ref. 9 and, as pointed out there, our conductivities differ by a factor of i from the usual definition.
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