PHYSICAL REVIEW A VOLUME 11, NUMBER 1 JANUARY 1975

Brownian motion of elastically deformable bodies

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(Received 15 November 1973; revised manuscript received 1 July 1974)

Extensions of the Case and Ornstein-Uhlenbeck treatments of rigid-body Brownian motion to an elastically deformable body in d dimensions are given. The motion of Brownian deformations in a fluid is considered to be locally damped by a retarding friction. To describe the expected fluctuations, velocity and displacement space-time correlation functions of a body are obtained from the relaxation-function theorem. A general equipartition theorem is obtained without postulating certain fluctuating sources. However, it is possible to exhibit those fluctuating sources which would yield our correlation functions, and thus a generalized Einstein relation between fluctuations and retarding dissipations is obtained, which is proved to be necessary and sufficient. Sum rules and both Green's and scattering functions are given for strings, elastically deformable rods, and viscoelastic gels. Curves for the ensemble-average-square displacements and their integrations over the entire length are given.

I. INTRODUCTION

In order to describe the approach toward Brownian equilibrium of a string or an elastic rod, van Lear and Uhlenbeck[1] used a Langevin equation and postulated (a) the equipartition of energy among vibrational modes, and (b) a local damping with a constant frictional coefficient. As a result, they verified Zeeman and Houdijk's experiment[2] and formula[3] for a Brownian wire suspended in a gas in the long-time limit. The present unified treatment of Brownian deformations of elastic bodies in d dimensions is motivated partly by light-scattering experiments,[4] model calculations,[5] stochastic solid mechanics,[6] and partly by the recent success of Case[7] in his treatment of the usual rigid-body Brownian motion with the relaxation-function theorem.

The method we will use is based on (i) a natural extension of van Lear and Uhlenbeck's expansion in terms of our eigenvectors of a d-dimension stress-force operator and (ii) the relaxation-function theorem and techniques of Case.[7] A well-known fact[8] is that the relaxation-function theorem requires no detailed structure of the Hamiltonian H_0 of the body and the fluid medium except the mere existence of an equilibrium canonical distribution,

$$\rho(H_0)=\frac{\exp(-H_0/k_BT)}{\int \exp(-H_0/k_BT)\,d\Gamma}\,, \qquad (1.1)$$

which is used throughout the article for the ensemble average $\langle\cdots\rangle_0$. Consequently we have proved (a) as well as extended (a) to include a more general retarding friction than (b). Based on the relaxation-function theorem, we verified the displacement space-correlation function obtained by Harris and Hearst[9] as one of our special cases in Sec. IV. Furthermore, we have rigorously derived a formula for Tanaka, Hocker, and Benedek's experiment[10] of light scattering from a viscoelastic gel.

II. MODELS AND FIELD CORRELATION FUNCTIONS

Brownian motion of an elastically deformable body in a fluid is realistic, but complicated, as many degrees of intramolecular freedom are involved. Moreover, Brownian motions of the center of mass and of the rotational orientations are generally coupled to a locally Brownian deformation. We consider only the latter effect by rigidly supporting a body at a portion of its boundary. Furthermore, since the probing radiation has a relatively longer wavelength than intramolecular spacings, continuum models of elastic bodies are used. We use, for example, a string as the continuum model of beads and springs. The latter is used by Rouse[11] and Zimm[12] in polymer dynamics. Also, a rod model is used by Harris and Hearst,[9] which includes the worm-like coil model[13] of the macromolecule. Moreover, membranes, plates, and a three-dimensional sponge soaked with a viscous liquid are often considered.

Our theory begins with the Newtonian equation of motion:

$$\rho\ddot{u}_j-\frac{\partial\sigma_{j\alpha}}{\partial x_\alpha}+\int_{-\infty}^{\infty}dt_0\int d\vec{x}_0\,\beta_{j\alpha}(t-t_0,\vec{x}|\vec{x}_0)\times\dot{u}_\alpha(t_0,\vec{x}_0)=0. \qquad (2.1)$$

The jth component of displacement vector $\vec{u}(t,\vec{x})$ of an element of a body from its equilibrium location is restored linearly by an internal stress force

$$\frac{\partial\sigma_{j\alpha}}{\partial x_\alpha}=\left((\kappa+\mu 3^{-1})\frac{\partial^2}{\partial x_j\,\partial x_\beta}+\delta_{j,\beta}\mu\nabla^2\right)u_\beta. \qquad (2.2)$$

According to the linear theory of strain,[14] we have made no distinction in (2.2) between the coordinates $\vec{x}$ of points in the body before and after the deformation. However, three general restrictions will be placed on the externally retarding tensor function $\beta_{jk}(t, \vec{x}|\vec{x}_0)$ in the following three paragraphs denoted by (i), (ii), (iii).

(i) Isotropically local restriction:

$$\beta_{jk}(t, \vec{x}|\vec{x}_0) = \delta_{jk}\rho f(t)\delta(\vec{x}-\vec{x}_0). \tag{2.3}$$

Then $\vec{u}$ can be decomposed into transverse $\vec{u}_t$ ($\mathrm{div}\vec{u}_t=0$) and a longitudinal $\vec{u}_l$ ($\mathrm{curl}\vec{u}_l=0$) components. Thus, we can formally simplify (2.1) and write

$$\rho\ddot{\psi}+\rho L_x\psi+\rho\int_{-\infty}^{\infty}dt_0\, f(t-t_0)\dot{\psi}(t_0,\vec{x})=0, \tag{2.4}$$

where $\psi(t,\vec{x})$, a scalar field, denotes the component of $\vec{u}$ which is either perpendicular or parallel to a propagating vector of elastic waves of the body. The linear operator L_x can be identified explicitly for the continuum models mentioned previously as follows:

$$d=3,\quad L_x=-C_t^2\nabla^2,\quad -C_l^2\nabla^2,$$

$$C_t^2\equiv\mu/\rho,\quad C_l^2\equiv(\kappa+\mu 3^{-1})/\rho; \tag{2.5}$$

$$d=2,\quad L_x=-(\tau/\rho h)\nabla^2,\quad Eh^2[12\rho(1-\sigma^2)]^{-1}\nabla^4; \tag{2.6}$$

$$d=1,\quad L_x=-(\tau/\rho)\partial^2/\partial x^2,\quad (EI/\rho)\partial^4/\partial x^4. \tag{2.7}$$

Moreover, we identify C_t and C_l as the transverse and longitudinal velocities of elastic sound, $C_l \gtrsim (\frac{4}{3})^{1/2}C_t$; and respectively μ and κ as the shear and bulk modulus; h, the thickness; E, the adiabatic or isothermal Young's modulus, $3[(3\kappa)^{-1}+\mu^{-1}]^{-1}$. The adiabatic or isothermal Poisson ratio σ of the transverse compression to the longitudinal extension is defined, $\frac{1}{2}\geq\sigma\equiv(3\kappa-2\mu)/2(3\kappa+\mu)>-1$. For rubber, $\mu\ll\kappa$, $\sigma\cong\frac{1}{2}$; for usual materials, $\sigma\geq 0$. For a rod of radius r and mass M per unit length, the cross-sectional moment of inertia $I=\frac{1}{4}\pi r^2 M$.

Next, a complete and orthonormal (CON) set of deformation eigenvectors $\{X_n(\vec{x})\}$ of the linear, self-adjoint, stress-force operator L_x is introduced, in order to span the Hilbert space, as follows:

$$L_x X_n(\vec{x})=\lambda_n X_n(\vec{x}), \tag{2.8}$$

$$\sum_n X_n(\vec{x})X_n^*(\vec{x})=\delta(\vec{x}-\vec{x}_0), \tag{2.9}$$

$$(X_n, X_m)\equiv\int d\vec{x}\, X_n^*(\vec{x})X_n(\vec{x}). \tag{2.10}$$

The elasticity stability condition requires the eigenvalues λ_n's to be positive definite. If we expand (2.4) in the Hilbert space,

$$\psi(t,\vec{x})=\sum_n Y_n(t)X_n(\vec{x}). \tag{2.11}$$

Then, the expansion amplitude is given by

$$Y_n\equiv(X_n,\psi) \tag{2.12}$$

and governed by

$$\ddot{Y}_n+\lambda_n Y_n+\int_{-\infty}^{\infty}dt_0\, f(t-t_0)\dot{Y}_n(t_0)=0, \tag{2.13}$$

which is obtained by taking the inner product of (2.4) with $X_n(\vec{x})$. In this form, Eq. (2.13) is a generalization of van Lear and Uhlenbeck's equation for the retarding friction $\rho f(t)$. However, we do not postulate fluctuating forces as they did. In this form, Eq. (2.13) is also a mild generalization of Case's equation for a Brownian rigid body. Case's equation might be thought of as one component of the set of equations associated with a zero eigenvalue. This zero eigenvalue corresponds to zero deformation and is therefore not included in our CON set of deformation eigenvectors.

(ii) Causal restriction:

$$f(t)=0,\quad t<0, \tag{2.14}$$

where f is real, positive definite for $t\geq 0$.

Then a Hermitian Green's function associated with Eq. (2.4) is also causal,[15] namely

$$G(t,\vec{x}|\vec{x}_0)=0,\quad t<0, \tag{2.15}$$

satisfies

$$\ddot{G}+L_x G+\int_{-\infty}^{\infty}dt_0\, f(t-t_0)\dot{G}(t_0,\vec{x}|\vec{x}_0)=\delta(t)\delta(\vec{x}-\vec{x}_0). \tag{2.16}$$

It is convenient but not necessary to decompose

$$G(t,\vec{x}|\vec{x}_0)=\sum_n K_n(t)X_n^*(\vec{x}_0)X_n(\vec{x}) \tag{2.17}$$

in terms of a set of petit Green's functions $\{K_n(t)\}$ governed by

$$\ddot{K}_n+\lambda_n K_n+\int_{-\infty}^{\infty}dt_0\, f(t-t_0)\dot{K}_n(t_0)=\delta(t), \tag{2.18}$$

and satisfied by the causality condition

$$K_n(t)=0,\quad t<0. \tag{2.19}$$

Our convention for the Fourier transform with respect to time will always be the following:

$$f_\omega\equiv\int_{-\infty}^{\infty}e^{i\omega t}f(t)\,dt,\quad f_\omega^*=f_{-\omega}. \tag{2.20}$$

Then transforming the petit Green's function gives

$$K_{n,\omega}=[-i\omega(-i\omega+f_\omega)+\lambda_n]^{-1},\qquad K_{n,\omega}^*=K_{n,-\omega}, \tag{2.21}$$

while transforming the system Green's function yields

$$G_\omega(\vec{x}|\vec{x}_0)=\sum_n K_{n,\omega}X_n^*(\vec{x}_0)X_n(\vec{x}),$$
$$G_\omega^*(\vec{x}|\vec{x}_0)=G_{-\omega}(\vec{x}_0|\vec{x}). \quad (2.22)$$

Moreover, since the transformed amplitude equation (2.13) can be written as

$$Y_{n,\omega}K_{n,\omega}^{-1}=0,$$

then by $ax\delta(x)=0$ we obtain the singular solution:

$$Y_{n,\omega}=a_{n,\omega}\delta(K_{n,\omega}^{-1})$$
$$=\sum_{l=1}^{2}a_{n,\omega_l}|\partial K_{n,\omega_l}^{-1}/\partial\omega_l|^{-1}\delta(\omega-\omega_l). \quad (2.23)$$

In order to be able to continue the Fourier inversion analytically into a complex $\omega=\omega_1+i\omega_2$

$$Y_n(t)=(2\pi)^{-1}\int_C \exp(-i\omega_1 t+\omega_2 t)\,Y_{n,\omega}\,d\omega, \quad (2.24)$$

we must impose the third restriction.

(iii) Dissipative restriction:

$$K_{n,\omega}^{-1}\equiv -i\omega(-i\omega+f\omega)+\lambda_n\neq 0, \quad \mathrm{Im}\omega>0. \quad (2.25)$$

Namely, the zeros of $K_{n,\omega}^{-1}$, which are singularities of $Y_{n,\omega}$, must lie at the lower half of the complex ω plane, where $\mathrm{Im}\omega\leq 0$. In case a singularity lies on the real axis of ω, the contour C must be chosen to be parallel to and just above the real axis of ω, in order to satisfy the causality condition (2.19) by closing the upper semicircle for $t<0$. Then by using the causality condition, the zero-frequency Green's function equals the time-integrated Green's function,

$$G_0\equiv\lim_{\omega\to 0}G_\omega=\lim_{\omega\to 0}\int_{-\infty}^{\infty}Ge^{i\omega t}\,dt$$
$$=\int_{-\infty}^{\infty}G\,dt=\int_{0-\epsilon}^{\infty}G\,dt\equiv\int_{|0|}^{\infty}G\,dt,$$

and in terms of the zero-frequency limit of the bilinear decomposition

$$G_0\equiv\lim_{\omega\to 0}G_\omega=\lim_{\omega\to 0}\sum_n K_{n,\omega}X_n^*X_n$$
$$=\sum_n\lambda_n^{-1}X_n^*X_n=\int_{|0|}^{\infty}G\,dt \quad (2.26)$$

equals, in turn, to the spectral decomposition of the steady-state Green's function

$$L_xG_0(\vec{x}|\vec{x}_0)=\delta(\vec{x}-\vec{x}_0);\quad G_0=\sum_n\lambda_n^{-1}X_n^*X_n, \quad (2.27)$$

because $L_xX_n=\lambda_nX_n$, which cancels λ_n^{-1} and gives $\sum_n X_n^*X_n=\delta(\vec{x}-\vec{x}_0)$. Mercer's theorem[17] guarantees the convergence of bilinear formula (2.27) since all λ_n's have the same sign and, in fact, are real, positive definite according to the stability condition.

The relaxation function theorem can be derived. Let the system of Hamiltonian H_0 be in equilibrium for $t<t_0$ with an outside force $F(x)$. Then the total Hamiltonian for $t<t_0$ equals

$$H=H_0-\int F\psi\,dx=H_0-\sum_n F_n\,Y_n(t).$$

Let $\langle Y_n(t)\rangle$, without the subscript 0 be averaged with $\rho(H)$ of (1.1) where H_0 is replaced by the above Hamiltonian. Then by means of the multivariable Taylor series expansion of $\rho(H)$ about $\rho(H_0)$, one obtains without using any detail of H_0, the following theorem for the relaxation function:

$$Y_{n,m}^r(t)\equiv\left.\frac{\partial\langle Y_n(t)\rangle}{\partial F_m}\right|_{F=0}$$
$$=(k_BT)^{-1}[\langle Y_n(t)Y_m(t_0)\rangle_0-\langle Y_n(t)\rangle_0\langle Y_m(t_0)\rangle_0]. \quad (2.28)$$

Since the remaining analysis is identical to that of Case,[7] we represent the following results and verify them as follows:

$$\langle\dot{Y}_n(t)\dot{Y}_m(t_0)\rangle_0=\delta_{n,m}(k_BT/2\pi\rho)$$
$$\times\int_C d\omega\exp(-i\omega|t-t_0|)(-i\omega)K_{n,\omega}, \quad (2.29)$$

$$\langle Y_n(t)Y_m(t_0)\rangle_0=\delta_{n,m}(k_BT/2\pi\rho\lambda_n)$$
$$\times\int_C d\omega\exp(-i\omega|t-t_0|)(-i\omega+f_\omega)K_{n,\omega}, \quad (2.30)$$

$$\langle\dot{Y}_n(t)Y_m(t_0)\rangle_0=\delta_{n,m}\,\mathrm{sign}(t-t_0)(k_BT/2\pi\lambda_n\rho)$$
$$\times\int_C d\omega\exp(-i\omega|t-t_0)(-i\omega)$$
$$\times(-i\omega+f_\omega)K_{n,\omega}$$
$$=0,\quad t=t_0, \quad (2.31)$$

where $\mathrm{sign}(t)=\pm 1$ for $t\gtrless 0$. If λ_n were zero for $n=m=0$, then $K_{n,\omega}$ equals $[(-i\omega)(-i\omega+f_\omega)]^{-1}$ by Eq. (2.21) and Eq. (2.29) is reduced to the velocity correlation function of Case[7] and Zwanzig and Bixon[16] for a rigid body in a fluid. Next, one verifies (2.30) by differentiating (2.30) with respect to t and t_0 for $t\gtrless t_0$ and using the following identity

$$i\omega[-i\omega(-i\omega+f_\omega)]\equiv i\omega[K_{n,\omega}^{-1}-\lambda_n].$$

Also the third result (2.31) can be easily verified by differentiating (2.30) with respect to t and using $\partial|t|/\partial t=\mathrm{sign}(t)$. We note that real $x=-x$ if $x=0$ and therefore

$$\left\langle Y(t_0)\frac{\partial Y(t_0)}{\partial t_0}\right\rangle_0=\left\langle Y(-t_0)\frac{\partial Y(-t_0)}{\partial(-t_0)}\right\rangle_0=0, \quad (2.32)$$

which is one of the relaxation conditions required by Case for time-reversal invariance. Next if we define the averaged energies as follows

$$\begin{aligned}\langle \text{K.E.}\rangle &\equiv \tfrac{1}{2}\rho \int d\vec{x}\, \langle \dot{\psi}^2(t,\vec{x})\rangle_0 \\ &= \tfrac{1}{2}\rho \sum_n \langle \dot{Y}_n^2(t)\rangle_0 \equiv \sum_n \langle \text{K.E.}\rangle_n ,\\ \langle \text{P.E.}\rangle &\equiv \tfrac{1}{2} \int d\vec{x}\, \langle \psi(t\vec{x})\rho L_x \psi(t\vec{x})\rangle_0 \\ &= \tfrac{1}{2}\rho \sum_n \lambda_n \langle Y_n^2(t)\rangle_0 \equiv \sum_n \langle \text{P.E.}\rangle_n ,\end{aligned} \tag{2.33}$$

then by applying Case's argument of equal time in the complex ω plane[7] to (2.29) and (2.30) one finds that (2.33) yields the equipartition theorem:

$$\langle \text{K.E.}\rangle_n = \langle \text{P.E.}\rangle_n = \tfrac{1}{2} k_B T. \tag{2.34}$$

Substituting the amplitude correlations (2.29)–(2.31) into our expansion (2.12), we obtain the following main result of elastic space-time correlation functions:

$$\langle \dot{\psi}(t+t_0,\vec{x})\,\dot{\psi}(t_0,\vec{x}_0)\rangle_0 = \frac{k_B T}{\rho}\,\frac{\partial}{\partial |t|}\, G(|t|,\vec{x}|\vec{x}_0), \tag{2.35}$$

$$\langle \dot{\psi}(t+t_0,\vec{x})\psi(t_0,\vec{x}_0)\rangle_0 = \frac{k_B T}{\rho}\,\text{sign}(-t) G(|t|,\vec{x}|\vec{x}_0), \tag{2.36}$$

$$\langle \psi(t+t_0,\vec{x})\psi(t_0,\vec{x}_0)\rangle_0 = \frac{k_B T}{\rho} \int_{|t|}^{\infty} G(t',\vec{x}|\vec{x}_0)\, dt'. \tag{2.37}$$

In principle, the Green's function can be constructed by various methods,[15] although in practice a spectral decomposition of the Green's function given by (2.17) and (2.21) is always available owing to the well-documented[17] eigenfunctions of the linear operator L_x. Some examples will be given in Sec. IV.

III. GENERALIZED EINSTEIN RELATION

As indicated by Case,[7] it is possible to turn the question around and ask as to the stochastic properties of a force, $\rho\tilde{S}(t,\vec{x})$, which, when introduced into Eq. (2.4), will reproduce our elastic space-time correlation functions. The answer to be deduced will be given by the following generalized Einstein fluctuation-dissipation relation:

$$\begin{aligned}\rho^2\langle \tilde{S}\tilde{S}'\rangle_0 = 2k_B T \int \frac{d\omega}{2\pi} \int \frac{d\vec{k}}{(2\pi)^3} \\ \times \exp[i\omega(t-t') - i\vec{k}\cdot(\vec{x}-\vec{x}')]\, 2\,\text{Re}\rho f_\omega .\end{aligned} \tag{3.1}$$

Furthermore, we have written (3.1) in a form that can be generalized to a spatially nonlocal friction kernel $\rho f_{\omega,\vec{k}}$.

In order to reproduce (2.35)–(2.37), we will prove (3.1) to be not only necessary, but also sufficient for a given initial-value problem.

A. Necessary condition

Replacing the linear and deterministic equation (2.4) with a stochastic Langevin equation,

$$\rho\ddot{\psi} + \rho L_x \psi + \rho \int_{-\infty}^{\infty} dt_0\, f(t-t_0)\dot{\psi}(t_0,\vec{x}) = \rho\tilde{S}(t,\vec{x}), \tag{3.2}$$

gives the following definition of fluctuations, denoted always with a tilde:

$$\psi = \psi - \langle\psi\rangle_0 + \langle\psi\rangle_0 \equiv \tilde{\psi} + \langle\psi\rangle_0, \quad \langle\tilde{\psi}\rangle_0 \equiv 0. \tag{3.3}$$

Comparing (3.2) with the Green's function equation (2.16), we know

$$\psi(t,\vec{x}) = \int_{-\infty}^{\infty} dt_0 \int d\vec{x}_0\, G(t-t_0,\vec{x}|\vec{x}_0)\tilde{S}(t_0,\vec{x}_0). \tag{3.4}$$

satisfies (3.2) by a direct substitution. In order to make $\langle\psi\rangle_0$ satisfy the deterministic equation, we require

$$\langle\tilde{S}\rangle_0 = 0. \tag{3.5}$$

Expanding

$$\tilde{S}(t,\vec{x}) \equiv \sum_n \tilde{F}_n(t) X_n(\vec{x}) \tag{3.6}$$

gives a Langevin equation of each displacement amplitude

$$\ddot{Y}_n + \lambda_n Y_n + \int_{-\infty}^{\infty} dt_0\, f(t-t_0)\,\dot{Y}_n(t_0) = \tilde{F}_n(t), \tag{3.7}$$

which by using the definitions of petit Green's function (2.18), (2.19) yields

$$\begin{aligned}Y_n(t) &= \int_{-\infty}^{t} dt_0 K_n(t-t_0)\,\tilde{F}_n(t_0) \\ &= \int_{-\infty}^{\infty} dt_0 K_n(t-t_0)\,\tilde{F}_n(t_0).\end{aligned} \tag{3.8}$$

In terms of the Fourier convolution theorem, this becomes

$$Y_{n,\omega} = K_{n,\omega}\tilde{F}_{n,\omega}, \tag{3.9}$$

and therefore produces

$$\langle \tilde{F}_{n,\omega}\tilde{F}_{n',\omega'}\rangle_0 = K_{n,\omega}^{-1}K_{n',\omega'}^{-1}\langle Y_{n,\omega}Y_{n',\omega'}\rangle_0. \tag{3.10}$$

Transforming the elastic amplitude correlation (2.30) gives

$$\begin{aligned}\langle Y_{n,\omega}Y_{n',\omega'}\rangle_0 = \delta_{n,n'}(k_B T/\rho\lambda_n)\, 2\pi\delta(\omega+\omega') \\ \times[(-i\omega+f_\omega)K_{n,\omega} + (-i\omega'+f_{\omega'})K_{n',\omega'}],\end{aligned} \tag{3.11}$$

which when substituting into Eq. (3.10) yields finally

$$\langle \tilde{F}_{n,\omega} \tilde{F}_{n',\omega'} \rangle_0 = \delta_{n,n'}(k_B T/\rho\lambda_n) 2\pi\delta(\omega+\omega') \times [(-i\omega + f_\omega) K^{-1}_{n',\omega'} + (-i\omega' + f_{\omega'}) K^{-1}_{n,\omega}] = \delta_{n,n'}(k_B T/\rho) 2\pi\delta(\omega+\omega')[f_\omega + f^*_\omega]. \tag{3.12}$$

Here use is only made of the definition (2.21) of $K_{n,\omega}$. One can transform the result (3.12) back to time space,

$$\langle \tilde{F}_n(t)\tilde{F}_{n'}(t')\rangle_0 = \delta_{n,n'}(k_B T/\rho) f(t-t'), \quad t \geq t', \tag{3.13}$$

provided that one has normalized $f(t)$ at the level of (3.1) such that

$$f_\omega \to 0, \quad \omega \to \infty. \tag{3.14}$$

Therefore, substituting (3.12) back into the expansion (3.6) of $\tilde{S}(t,x)$, we obtain (3.1), the generalized Einstein fluctuation-dissipation relation for an elastic body. Furthermore, substituting (3.13) back into (3.6), we show that the following form of Einstein fluctuation-dissipation theorem is a necessary consequence of our elastic space-time correlation function:

$$\langle \tilde{S}(t,x)\tilde{S}(0,x_0)\rangle_0 = (k_B T/\rho) f(t)\delta(x-x_0), \quad t \geq 0. \tag{3.15}$$

B. Sufficient condition

Given some fixed initial conditions,

$$\psi(0,x) = \psi_0(x), \quad \dot{\psi}(0,x) = \dot{\psi}_0(x), \tag{3.16}$$

one can solve the initial-value problem using the Langevin equation and then prove the Einstein relation (3.15) to be sufficient to reproduce our elastic space-time correlation functions, either in the long-time limit or averaging (3.16) over all possible equilibrium values. We will use the standard Laplace transform formula

$$Y_s \equiv L_s Y(t) \equiv \int_0^\infty \exp(-st) Y(t)\, dt, \tag{3.17}$$

and the Fourier-Mellin inversion formula

$$Y(t) \equiv L_t^{-1} Y_s \equiv (2\pi i)^{-1} \int_B ds\, e^{st}\, Y_s \tag{3.18}$$

along the Bronwich contour B, parallel and to the right of the coordinate axis of the imaginary part of s. We will always carry out our equilibrium ensemble average, $\langle\cdots\rangle_0$, in two steps, with the partial ensemble average associated with the fixed initial condition denoted by angular brackets with no subscript ($\langle\cdots\rangle$), and finally over all possible initial values at the end of our calculation. Applying (3.17) to (3.2) gives the transformed amplitude equation

$$Y_s = K_s[v_0 + (s+f_s)Y_0 + \tilde{F}_s] = \langle Y_s \rangle + K_s \tilde{F}_s. \tag{3.19}$$

In (3.2) we used $f(t-t_0)=0$ for $t>t_0$ to cut off the upper integration limit at t, and set $\tilde{s}(t)=0$ for $t<0$ to chop off the lower limit at 0 so that we can use the Laplace convolution theorem. Also we have expanded (3.2) and (3.16) according to (2.11), and used the fact $\langle \tilde{F}_s\rangle = 0$. Furthermore, the mode index n from the amplitude $Y_n(t)$ was suppressed and abbreviated

$$Y(0) \equiv Y_0, \quad \dot{Y}(0) \equiv v_0, \qquad K_s \equiv (s^2 + sf_s + \lambda)^{-1}. \tag{3.20}$$

Rewriting (3.19) by use of our definition of tilde (3.3) as

$$K_s \tilde{F}_s = Y_s - \langle Y_s \rangle \equiv \tilde{Y}_s \equiv L_s \tilde{Y}(t), \tag{3.21}$$

one can calculate

$$K_{s_1}K_{s_2}\langle \tilde{F}_{s_1}\tilde{F}_{s_2}\rangle = L_{s_1}L_{s_2}\langle \tilde{Y}(t_1)\tilde{Y}(t_2)\rangle \tag{3.22a}$$

by knowing the double-Laplace-transformed generalized Einstein fluctuation-dissipation relation

$$\langle \tilde{F}_{s_1}\tilde{F}_{s_2}\rangle = (k_B T/\rho)(s_1+s_2)^{-1}(f_{s_1}+f_{s_2}). \tag{3.22b}$$

Employing the identity

$$\lambda(f_{s_1}+f_{s_2}) \equiv (s_1+f_{s_1})K^{-1}_{s_2} + (s_2+f_{s_2})K^{-1}_{s_2} - (s_1+s_2)[(s_2+f_{s_2})(s_1+f_{s_2})+\lambda],$$

and carrying out one of the double Fourier-Mellin inversions at the singularity $s_1+s_2=0$, one can obtain

$$\langle \tilde{Y}(t_1)\tilde{Y}(t_2)\rangle = (k_B T/\rho\lambda)\{L^{-1}_{t_1-t_2}(s_1+f_{s_1})K_{s_1} + L^{-1}_{t_2-t_1}(s_2+f_{s_2})K_{s_2} - [Y_0^{-2}\langle Y(t_1)\rangle\langle Y(t_2)\rangle|_{v_0=0} + \lambda v_0^{-2}\langle Y(t_1)\rangle\langle Y(t_2)\rangle|_{y_0=0}]\}. \tag{3.23}$$

Because of the damping of the systematic part of the friction, Eq. (3.23) will reproduce the amplitude correlation (2.30) if s is replaced by $-i\omega$ and use is made of the dissipative condition (2.25) in order to combine the first two terms into $L^{-1}_{|t_1-t_2|}(s+f_s)K_s$. Moreover, the left-hand side of (3.23) is decomposed by the definition of tilde into two terms

$$\langle \tilde{Y}(t_0)\tilde{Y}(t_2)\rangle = \langle Y(t_1)Y(t_2)\rangle - \langle Y(t_1)\rangle\langle Y(t_2)\rangle. \tag{3.24}$$

Since as a result of the systematic part of the friction the averaged amplitude decays $\lim\langle Y(t)\rangle = 0$ as $t\to\infty$, then the second term of (3.24) involving the product of averages will vanish as will the remaining terms of the right-hand side of (3.23) when we take the long-time limit: $t_1\to\infty$, $t_2\to\infty$, but

$t_1 - t_2 = t$. Therefore, we finally obtain from (3.23) and (3.24) our equilibrium correlation function (2.30) in the long-time limit. In particular for $t_1 = t_2$, we obtain the equipartition theorem (2.34)

$$\lim_{t_1\to\infty} \rho(\tfrac{1}{2}\lambda)\langle Y^2(t_1)\rangle = \tfrac{1}{2}k_B T.$$

If we had not posed the fixed-initial-value problem, we could average (3.23) also over all possible initial values by means of the equipartition theorem

$$\tfrac{1}{2}\rho\langle v_0^2\rangle_0 = \tfrac{1}{2}k_B T = \rho(\tfrac{1}{2}\lambda)\langle Y_0^2\rangle_0, \tag{3.25}$$

which allow us to cancel terms involving the product of averages on both sides of (3.23) at all times. Clearly, it would then not be necessary to take the long-time limit. This contrast illustrates that the stochastic elements enter the equation of motion through the Langevin source and/or the initial conditions in the stochastic Langevin approach, whereas only the initial conditions are stochastic in the deterministic relaxation approach. Similar proofs for $\langle \dot{Y}(t_1)\dot{Y}(t_2)\rangle$ and $\langle \dot{Y}(t_1)Y(t_2)\rangle$ can be easily constructed, if use is made of the following two equations:

$$L_s \dot{Y}(t) \equiv \dot{Y}_s = Y_0 + sY_s = \langle \dot{Y}_s\rangle + sK_s\tilde{F}_s, \tag{3.26}$$

$$s_1 s_2(f_{s_1} + f_{s_2}) \equiv s_2 K_{s_1}^{-1} + s_1 K_{s_2}^{-1} - (s_1+s_2)(s_1 s_2 + \lambda). \tag{3.27}$$

This completes our development of the approach of equilibrium amplitude correlations from a given initial data by means of the Langevin sources and the systematic retarding friction.

IV. WHITE-NOISE BROWNIAN DEFORMATIONS

An elastic body has been considered to be embedded in the medium characterized by our three conditions, namely isotropically local, causal, and dissipative. Furthermore, the correlation time of fluctuating forces exerted on the body is assumed to be infinitesimally small compared with the decay time of the averaged displacement due to the systematic friction. Since the friction and the force correlation are proportional, both must have a white spectrum, i.e.,

$$\langle \rho\tilde{S}\rho\tilde{S}'\rangle = 2k_B T(\rho f_0)\delta(t-t')\delta(\vec{x}-\vec{x}'). \tag{4.1}$$

Indeed, this is a special case of our generalized Einstein fluctuation-dissipation relation, if we normalize

$$2\int_0^\infty \delta(t)\,dt = \int_{-\infty}^\infty \delta(t)\,dt = 1 \tag{4.2}$$

for the even function $\delta(t) = \delta(-t)$, then causal $f(t) = 0$ for $t < 0$ demands

$$f(t) = 2f_0\delta(t), \quad t \geq 0, \tag{4.3}$$

because

$$f_\omega \equiv \int_{-\infty}^\infty e^{i\omega t} f(t)\,dt$$

$$\equiv \int_0^\infty e^{i\omega t} f(t)\,dt = f_0. \tag{4.4}$$

Using (4.2) and (4.3), our general equation of motion (2.4) reduces to

$$\rho\ddot{\psi} + \rho L_x\psi + \rho f_0\dot{\psi} = 0, \tag{4.5}$$

and therefore is the generalized Langevin equation (3.2):

$$\rho\ddot{\psi} + \rho L_x\psi + \rho f_0\dot{\psi} = \rho\tilde{S}. \tag{4.6}$$

Therefore, the associated Green's equation (2.16) becomes

$$\ddot{G} + L_x G + \rho f_0\dot{G} = \delta(t)\delta(\vec{x}-\vec{x}_0), \tag{4.7}$$

which in terms of the bilinear decomposition (2.17),

$$G(t\vec{x}|\vec{x}_0) = \sum_n K_n(t)X_n^*(\vec{x}_0)X_n(\vec{x}), \quad L_x X_n = \lambda_n X_n, \tag{4.8}$$

gives a petit Green's equation

$$\ddot{K}_n + \lambda_n K_n + \rho f_0\dot{K}_n = \delta(t). \tag{4.9}$$

A Fourier time transform of (4.9) gives the same result as that obtained by substituting (4.4) into (2.21) directly, namely

$$K_{n,\omega} = [-i\omega(-i\omega + f_0) + \lambda_n]^{-1}. \tag{4.10}$$

A simple Fourier inversion gives

$$K_n(t) = (2\pi)^{-1}\int_{-\infty}^\infty d\omega\, e^{-i\omega t}K_{n,\omega}$$

$$= \theta(t)\,e^{-f_0 t/2}\sin(\Omega_n t)\,\Omega_n^{-1}, \tag{4.11}$$

where $\theta(t)$ is the Heaviside function and Ω_n is the vibration frequency of nth mode defined by

$$\Omega_n^2 + \tfrac{1}{4}f_0^2 = \lambda_n. \tag{4.12}$$

Clearly, (4.11) satisfies the causal $K_n(t) = 0$, $t < 0$ and can be constructed directly from (4.9). Knowing the petit Green's function (4.11), one can proceed to calculate the elastic space-time correlation functions with various formulas. Either use the solution

$$\psi = \int_{-\infty}^\infty dt_0 \int d\vec{x}_0\, G(t-t_0, \vec{x}|\vec{x}_0)\tilde{S}(t_0, \vec{x}_0) \tag{4.13}$$

via the force correlation (4.1), or use the results (2.35)–(2.37) of the relaxation-function theorem. In terms of the Green's function, one obtains from either method the final results:

$$\langle \dot{\psi}(t+t_0,\vec{x})\dot{\psi}(t_0,\vec{x}_0)\rangle_0 = (k_BT/\rho)\,e^{-f_0|t|/2} \times\sum_n [\cos(\Omega_n|t|) - (f_0/2\Omega_n) \times \sin(\Omega_n|t|)]X_n^*(\vec{x}_0)X_n(\vec{x}), \quad (4.14)$$

$$\langle \psi(t+t_0,\vec{x})\psi(t_0,\vec{x}_0)\rangle_0 = (k_BT/\rho)\,e^{-f_0|t|/2} \times\sum_n \lambda_n^{-1}[\cos(\Omega_n|t|) + (f_0/2\Omega_n)\sin(\Omega_n|t|)] \times X_n^*(\vec{x}_0)X_n(\vec{x}), \quad (4.15)$$

$$\langle \dot{\psi}(t+t_0,\vec{x})\psi(t_0,\vec{x}_0)\rangle_0 = (k_BT/\rho)G(|t|,\vec{x}|\vec{x}_0)\,\mathrm{sign}(-t). \quad (4.16)$$

For equal-time points, the space-time correlations are simplified

$$\langle \dot{\psi}(t,\vec{x})\psi(t,\vec{x}_0)\rangle_0 = 0, \quad (4.17)$$

$$\langle \dot{\psi}(t,\vec{x})\dot{\psi}(t,\vec{x}_0)\rangle_0 = (k_BT/\rho)\delta(\vec{x}-\vec{x}_0), \quad (4.18)$$

$$\langle \psi(t,\vec{x})\psi(t,\vec{x}_0)\rangle_0 = (k_BT/\rho)\sum_n \lambda_n^{-1}X_n^*(\vec{x}_0)X_n(\vec{x}) = (k_BT/\rho)G_0(\vec{x}|\vec{x}_0), \quad (4.19)$$

where G_0 is the zero-frequency limit of G_ω defined generally by (2.26) and therefore (4.19) can be directly obtained from our main result (2.37) by using (2.26). Should a nonwhite friction spectrum ρf_ω be appreciable in reality, the Fourier inversion (4.11), which may be difficult to obtain but in principle exists, gives us different elastic space-time correlation functions with respect to the same set of eigenvectors as that of a white friction spectrum ρf_0.

V. DISCUSSION AND CONCLUSION

The elastic displacement space-time correlation functions given by Eqs. (2.35)–(2.37) are quite general and useful. They are derived from the relaxation-function theorem (2.28), as well as obtained from the generalized Einstein fluctuation-dissipation relation (3.1) via the associated Langevin equation (3.2). According to our theory the intramolecular correlation function

$$\langle|\psi(t,\vec{x})-\psi(0,\vec{x})|^2\rangle_0 = \frac{2k_BT}{\rho}\left(G_0(\vec{x},\vec{x}) - \int_{|t|}^{\infty} dt'\, G(t',\vec{x}|\vec{x}_0)\right) \quad (5.1)$$

is rigorously obtained in terms of the system's Green's functions

$$G(t,\vec{x}|\vec{x}_0) = (2\pi)^{-1}\sum_n \int_C \exp\{-i\omega t\} \times[-i\omega(-i\omega+f_\omega)+\lambda_n]^{-1}\,d\omega \times X_n^*(\vec{x}_0)X_n(\vec{x}), \quad (5.2)$$

and

$$G_0(\vec{x}|\vec{x}) = \sum_n \lambda_n^{-1}X_n^*(\vec{x})X_n(\vec{x}), \quad (5.3)$$

which have been decomposed with respect to the complete and orthonormal set $\{X_n(\vec{x})\}$ of eigenvectors associated with the internal stress-force operator L_x (2.5)–(2.7). Substituting (5.1) into the Pecora–Van Hove scattering function,

$$S(Q,\omega) = \int d\vec{x}\exp[-Q^2(k_BT/\rho)G_0(\vec{x}|\vec{x})] \times\int dx_0\int_{-\infty}^{\infty} dt\,\exp\left(i\omega t + Q^2\frac{k_BT}{\rho} \times\int_{|t|}^{\infty} dt'\, G(t',\vec{x}|\vec{x}_0)\right), \quad (5.4)$$

gives us the differential cross section of radiation scattering from intramolecular fluctuations. More detailed predictions will be given in the forthcoming paper. Here we discuss only our results when the dissipation function ρf_ω has the white spectrum ρf_0 and is reducible to those results in the literature as special cases.

A. Viscoelastic gel, light-scattering experiment

Tanaka, Hocker, and Benedek (THB) used optical (homodyne) mixing spectroscopy to detect the polarized and depolarized light scattering respectively from the longitudinal and transverse displacement thermal fluctuations of a uniform elastic network against an extremely viscous gel liquid held by the network of the size 0.4 cm^3. For a gel of 5% polyacrylamide (prepared from Canalco premixed reagents), THB found: $\rho\sim 1$ g/cm^3; $\mu\sim 10^4$ dyn/$cm^2 \ll \mu$ of rubber; $\kappa\sim 2\mu/3$, i.e., $\sigma\sim 0$ for a sponge; $f\sim 10^{11}$ dyn sec/cm^4; and they derived two decay times τ_f and τ_s for their displacement correlation functions from (2.4) postulating a specific structure of the system's Hamiltonian:

$$\tau_{\text{fast}} \equiv \tfrac{1}{2}\tau_0 \equiv \rho/f \sim 10^{-11}, \qquad \tau_{\text{slow}} \equiv 2/\tau_0\omega_0^2 \equiv f/\rho C^2 q^2 \sim 10^{-3}. \quad (5.5)$$

Here C denotes the elastic sound velocities, either $C_l\sim 10$ m/sec or C_t, respectively given by (2.5) along or perpendicular to the elastic wave propagation vector $\vec{q}$, which is selected out by the light scattering of wave number transfer 10^5 cm^{-1}.

According to our theory, we must construct the Green's function which then gives rigorously the displacement space-time correlation function. To this end, the CON set of eigenfunctions $\{X_n(x)\}$ and eigenvalues $\{\lambda_n\}$ of the stress-force operator L_x given by (2.5) is written as follows:

$$X_n(\vec{x}) = (V)^{-1/2}e^{i\vec{q}_n\cdot\vec{x}}, \quad \vec{q}_n = \vec{n}\,2\pi/l, \quad V = l^3. \quad (5.6)$$

Here $\vec{n}=(n_x, n_y, n_z)$ are integers, and

$$V^{-1}\sum_n \exp[i\vec{q}_n\cdot(\vec{x}-\vec{x}_0)]=\delta(\vec{x}-\vec{x}_0), \qquad (5.7)$$

$$V^{-1}\int d\vec{x}\exp[i(\vec{q}_n-\vec{q}_m)\cdot\vec{x}]=\delta_{n,m}. \qquad (5.8)$$

Operating L_x on the eigenfunction (5.6) yields

$$\lambda_n = C_l^2 q_n^2,\ C_t^2 q_n^2\ . \qquad (5.9)$$

Specifically, THB assumed a constant friction coefficient $f\equiv\rho f_0$; then from (4.8) and (4.11) follows our Green's function.

$$G(t,\vec{x}|\vec{x}_0)=\theta(t)\,e^{-f_0 t/2} \times\sum_n \Omega_n^{-1}\sin(\Omega_n t)X_n^*(\vec{x}_0)X_n(\vec{x}). \qquad (5.10)$$

By definition (4.12) we can rewrite the mode frequency Ω_n in terms of THB notation (5.5):

$$\Omega_n\equiv(\lambda_n-\tfrac{1}{2}f_0^2)^{1/2}=i(\tfrac{1}{2}f_0)(1-4C^2q_n^2/f_0^2)^{1/2} \equiv i(1/\tau_0)(1-\omega_0^2\tau_0^2)^{1/2}, \qquad (5.11)$$

for the overdamped vibration against the extremely viscous gel liquid. Knowing the specific Green's function, we obtain from our unified treatment of an elastic body the following specific result:

$$\langle\psi(t,\vec{x})\psi(0,\vec{x}_0)\rangle_0=\sum_n (k_BT/\rho\lambda_n) \times(A_s e^{i\omega_+|t|}+A_f e^{i\omega_-|t|}) \times X_n^*(\vec{x}_0)X_n(\vec{x}), \qquad (5.12)$$

where we have abbreviated, according to THB,

$$i\omega_\pm\equiv-[1\pm(1-\omega_0^2\tau_0^2)^{1/2}]/\tau_0, \qquad (5.13)$$

and defined analogously

$$A_{\text{short}}\equiv\tfrac{1}{2}[1-(1-\omega_0^2\tau_0^2)^{-1/2}], \qquad (5.14)$$

$$A_{\text{fast}}\equiv\tfrac{1}{2}[1+(1-\omega_0^2\tau_0^2)^{-1/2}]. \qquad (5.15)$$

Thus, we have rigorously derived A_s and A_f which clearly satisfy the conditions imposed by THB, namely

$$A_s+A_f=1,\quad i\omega_+A_s+i\omega_-A_f=0. \qquad (5.16)$$

Given further their approximation

$$\tau_f/\tau_s=\tau_0^2\omega_0^2\sim10^{-8}\ll1, \qquad (5.17)$$

we have

$$A_f\cong1+\tfrac{1}{4}\omega_0^2\tau_0^2\cong1,\quad A_s\cong-\tfrac{1}{4}\omega_0^2\tau_0^2\cong0, \qquad (5.18)$$

and

$$-i\omega_+\cong2/\tau_0\equiv1/\tau_{\text{fast}}, \qquad (5.19)$$

$$-i\omega_-\cong\tfrac{1}{2}\tau_0\omega_0^2\equiv1/\tau_{\text{slow}}. \qquad (5.20)$$

If we express (5.12) also in terms of our expansion amplitude $Y_n(t)$ of $\psi(t,x)$, where our subscript n denotes their wave-vector channel number $\vec{q}_n$ given by (5.6), then we have reproduced the following overdamped result using the THB approximation (5.17):

$$\langle Y_n(t)Y_n(0)\rangle_0=(k_BT/\rho\lambda_n)(A_s e^{i\omega_+|t|}+A_f e^{i\omega_-|t|}) \cong(k_BT/\rho C^2q_n^2)\exp(-|t|/\tau_{\text{slow}}) \qquad (5.21)$$

which has been, apart from the trivial normalization factor $V/(2\pi)^3$, obtained independently by THB and verified with their experimental data obtained at a fixed optical mixing-time interval t and plotted (5.21) against all the channels q^2 axis, which is proportional to τ_s^{-1}.

We note that (5.21) follows also from (2.23) and (2.24) if use is made of our equipartition theorem (3.25). Within the framework of our model equation (2.1), it is natural to suggest that the slightly nonexponential decay, which concerned THB in their polarized light scattering at 90° from the sample at 25°C, is due to a retarding frictional coefficient of ρf_ω. A detailed analysis will be given in a forthcoming article.

B. Polymer dynamics, model calculation

Harris and Hearst (HH) considered an elastic rod of a fixed length by means of a Lagrange multiplier β, which gives the force operator ρL_x in terms of their arc length s:

$$\rho L_s\equiv\alpha\partial^4/\partial s^4-\beta\partial^2/\partial s^2. \qquad (5.22)$$

HH applied a specific white-noise spectrum [defined by their Eq. (46) which is $\frac{1}{3}$ of Eq. (4.1)] to a Langevin equation given by their Eq. (48). Because of the HH white-noise assumption, our generalized Langevin equation (3.7) of the expansion amplitude $Y_n(t)$ includes the HH equation as a special case. Furthermore, by postulating a detailed form of the Hamiltonian H_0 in their canonical distribution Eq. (22), they obtain the following equal-time displacement vector correlation [see HH Eq. (30)]

$$\langle\vec{x}(s)\cdot\vec{x}(s')\rangle=(3k_BT/\alpha)\sum_i\psi_i(s)\psi_i(s')/\mu_i\ . \qquad (5.23)$$

If we identify $\alpha\mu_i/\rho$ and $\psi_i(s)$ in (5.23) respectively to be the definitions of eigenvalue λ_n and eigenfunction $X_n(x)$ of the x component of the stress-force operator ρL_x, the HH correlation result becomes three times the displacement correlation (4.19) of one component. A natural generalization of their calculation of intrinsic viscosity to an externally retarding friction is currently under investigation.

C. String and rod, stochastic solid mechanics

Although van Lear and Uhlenbeck (VU) did not apply Green's function techniques to express final results, they calculated a detailed time course of mean square amplitudes including our (4.19) [see VU Eq. (21)] as the long-time limit, by using the white spectrum (4.1), derived here without using any detail structure of the system's Hamiltonian. However, using the Green's function one can sum up (4.19) by a homogeneous solution representation of Green's function.

(i) String with rigid ends[15]:

$$(-\tau/\rho)G_0 = x(x_0-1)\theta(x_0-x) + x_0(x-1)\theta(x-x_0). \tag{5.24}$$

(ii) Rod with one end clamped:

$$(6EI/\rho)G_0 = x^2(3x_0-x)\theta(x_0-x) + x_0^2(3x-x_0)\theta(x-x_0). \tag{5.25}$$

(iii) Rod with two ends clamped[17]

$$(6EI/\rho)G_0 = x^2(x_0-1)^2(3x_0-x-2xx_0)\theta(x_0-x) + x_0^2(x-1)^2(3x-x_0-2x_0x)\theta(x-x_0), \tag{5.26}$$

where E is the Young's modulus and I is the cross-sectional moment of inertia, given below Eq. (2.17) for L_x which yields $L_xG_0=\delta$. Brownian deformations at equal time and space are plotted in Fig. 1 for a string and Figs. 2 and 3 for rods; and their integrations over the entire length are respectively the invariant traces (Tr) of Green's functions

$$\mathrm{Tr}G_0 \equiv \int_0^1 dx\, G_0(x,x) = \sum_n \lambda_n^{-1} = (\rho/\tau)(6)^{-1},\ (\rho/EI)(3\times4)^{-1},\ (\rho/EI)(3\times4\times5\times7)^{-1} \tag{5.27}$$

which are proportional to the areas under curves and in turn to the mean-square-ensemble deviations over the entire length l. Equation (5.27) provides us with interesting sum rules over mode frequencies (4.12). Finally we make remarks about (i) how the presence of stochastic forces affect each mode of oscillation, (ii) the ultraviolet catastrophe of equipartition theorem, and (iii) a general nonlocal white noise.

(i) From Figs. 1–3, the equal-time displacement correlation is seen to bear slight resemblance to the shapes of the fundamental or the lowest *undamped* oscillation. Actually for both the string and the rod, the displacement correlations at equal time are sustained by the stochastic noises. In the case of a tuning fork, all "inharmonic" modes decay rapidly in time and only the lowest mode is sustained. On the contrary, due to the "white noises" the displacement correlation of each mode decays exponentially in time at the same rate. Although each mode does share an equal amount of energy, the higher the mode of oscillation the smaller the amplitude of the displacement correlation.

(ii) A realistic source correlation is very sharp in time, but it is not as sharp as $\delta(t)=(2\pi)^{-1} \times \int_{-\infty}^{\infty} d\omega\, 1 \exp(-i\omega t)$ which has a unit amplitude 1 for all ω. Thus it is customary to cut off the amplitude at higher frequencies determined analogous-

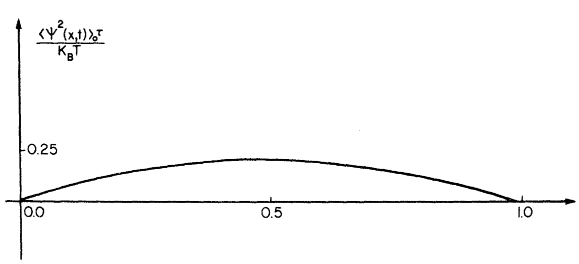

FIG. 1. Under the influence of the "white noise," the displacement correlation function $\langle\psi^2(xt)\rangle_0$, reduced by the tension τ and the thermal energy k_BT, is plotted against a unit length of rigid supports. The total area $\langle\psi^2(xt)\rangle_0\tau/k_BT$ is a universal constant 6^{-1}.

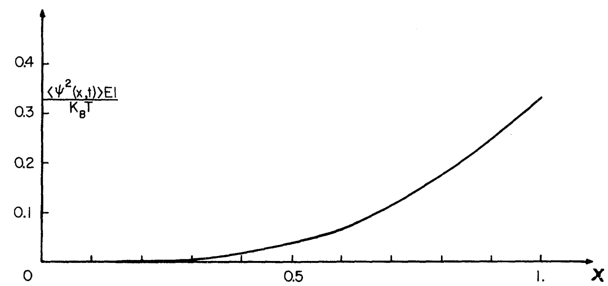

FIG. 2. Under the influence of the "white noise," the displacement correlation function $\langle\psi^2(x,t)\rangle_0$, reduced by the ratio of the flexibility parameter EI and the thermal energy k_BT, is plotted against a unit length rod with one end clamped. The total area $\langle\psi^2(x,t)\rangle_0 EI/k_BT$ is a universal constant, $(3\times4)^{-1}$.

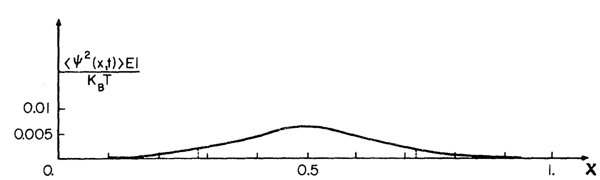

FIG. 3. Under the influence of the "white noise," the displacement correlation function $\langle\psi^2(x,t)\rangle_0$, reduced by the ratio of the flexibility parameter EI and the thermal energy k_BT, is plotted against a unit length rod with two ends clamped. The total area $\langle\psi^2(x,t)\rangle_0 EI/k_BT$ is a universal constant, $(3\times4\times5\times7)^{-1}$.

ly by the Debye's frequency. This is then called the practically "white noises" associated with the cut-off white spectrum. In other words, higher modes of oscillation will not be excited by the realistic "white noises." Therefore, the tail part of the infinite sum over the high-frequency modes will not contribute to the total energy. Then the total energy is finite, which means the string will not be ruptured by the realistic "white noises." It is with this qualification that the white noise is used and in this sense that an equal amount of energy is found for each excitable mode beneath the natural cutoff. A similar qualifying statement should be made for $\delta(x)$ which then explains the δ singularity of Eq. (4.18). There exists at least one case in which (4.1) is kinetically derived.[18] As a matter of fact, the practically white noise present in the Knudsen gas is indeed local in space, for the arrivals of gas particles are always according to a local Poisson distribution.

(iii) A general nonlocal white noise

$$\langle \tilde{S}(\vec{x}, t)\tilde{S}(\vec{x}', t')\rangle = s\delta(t-t')D(\vec{x}-\vec{x}'), \tag{5.28}$$

imposes no mathematical difficulty. One can always expand $\tilde{S}=\sum_n \tilde{F}_n(t)X_n(\vec{x})$ and $D=\sum_{nm} \times D_{nm}X_n^*(\vec{x})X_m(\vec{x}')$ to obtain a mode-mode coupling

$$\langle \tilde{F}_n(t)\tilde{F}_m(t')\rangle = sD_{nm}\delta(t-t'),$$

where $D_{nm}=\delta_{n,m}$ is decoupled for the local white noise. Although the displacement correlation will generally involve a double sum over the mode frequencies Ω_n and Ω_m weighted by the coupling constant D_{nm} and $X_n^*(\vec{x})X_m(\vec{x}')$, the integration of energies at equal time and space over the entire length will reduce, by means of the orthonormality of $X_n^*(\vec{x})X_m(\vec{x}')$, the double sum to a single sum weighted, however, by the diagonal term of the nonlocality coupling constant D_{nn}. Then, we find a partition law of mode energies weighted by D_{nn} for the nth mode.

ACKNOWLEDGMENTS

It is a pleasure to acknowledge discussions with Professor J. J. Hermans, Professor W. Prins, and Professor L. G. Pedersen. This work was supported by the Materials Research Center, University of North Carolina, through Grant No. GH-33632 from the National Science Foundation.

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