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Phase-space description of the thermal relaxation of a $(2J+1)$-level system

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The master equation describing the relaxation of a $(2J+1)$-level system is analyzed in the diagonal representation for the density operator using the coherent atomic states as a basis. The c-number quasiprobability functions corresponding to the density operator are found to satisfy a second-order partial differential equation on the surface of the Bloch sphere. This equation is solved in the steady-state limit, and a few moments of interest of the atomic operators are calculated in terms of classical-like integrals in the phase space of the atomic variables. In the high-temperature approximation the partial differential equation is solved exactly for all time by a simple eigenfunction expansion procedure. For the special case of a two-level system an exact solution is also available for arbitrary values of the reservoir temperature.

I. INTRODUCTION

The transformation of operator equations into c-number differential equations has been the subject of numerous investigations in quantum optics.[1] A variety of mapping techniques[2] have been developed to deal with problems involving Bose-Einstein operators which have led to general rules of correspondence, one of the most common being, for example, the antinormal ordering procedure. In particular, it is well known that certain master equations describing the evolution of electromagnetic field operators[3] can be mapped into c-number Fokker-Planck equations evolving in the phase space of the complex eigenvalues of the field destruction operator. In terms of the c-number representation, expectation values as well as multitime averages of quantum observables have been reduced to classical-like integrals[1] which bear a striking resemblance to the results of the classical theory of Markoff processes.

Recently a continuous basis representation has been introduced by Arecchi *et al.*[4] (ACGT representation) to describe collections of two-level atoms. The close formal similarity between the ACGT representation and the Glauber-Sudarshan coherent-state representation[5] motivated an earlier investigation,[6] in which a superradiant master equation derived independently by Agarwal[7] and by Bonifacio *et al.*[8] has been mapped into a Fokker-Planck equation evolving on the surface of the Bloch sphere.

The mapping procedure employed in Ref. 6 is reminiscent of the early attempts in which the Glauber P representation was used to arrive at a phase-space description of the operator equations of motion for electromagnetic field observables. More recently[9] certain rules of correspondence have been developed for quantum-mechanical angular momentum operators. The procedure employed in this work follows closely the method used in Ref. 6, except that here we apply our newly developed rules of correspondence, which greatly facilitate the calculational procedure.

We consider the master equation describing the evolution of the density operator for an arbitrary $(2J+1)$-level system interacting with a thermal reservoir in the Markoff approximation, and construct the differential equation for the quasiprobability function associated with the density operator in the ACGT representation.

While the exact solution of the time-dependent phase-space equation appears to be a difficult task, we have been able to construct the time-independent steady-state solution,[10] and from this we have calculated some of the relevant expectation values for the atomic operators. As expected, our results coincide with those derived from the usual assumption of canonical equilibrium.

In the special case of very low reservoir temperatures, the original master equation is formally identical to the superradiant master equation discussed in Refs. 7 and 8. Hence the results of Ref. 6 can be used to describe the low-temperature limit.

We investigate also the limit of very high reservoir temperatures; an exact time-dependent solution is obtained in this case for the c-number partial differential equation using eigenfunction expansion techniques.

Finally, we specialize the master equation for a two-level system including the elastic contributions. We obtain here also an exact solution to the associated c-number equation for arbitrary initial conditions, in terms of spherical harmonic functions.

II. DERIVATION OF THE c-NUMBER DIFFERENTIAL EQUATION

The time evolution of an arbitrary $(2J+1)$-level system in weak contact with a large heat reservoir can be described in terms of the Markovian master equation (see H. Haken in Ref. 1)

$$\frac{dW}{dt}=\nu([J^-W,J^+]+[J^-,WJ^+]) +\delta([J^+W,J^-]+[J^+,WJ^-]) -\tfrac{1}{2}\eta([WJ^+J^-,J^-J^+]+[J^-J^+,J^+J^-W]), \tag{2.1}$$

where W is the time-dependent density operator for the $(2J+1)$-level system, ν and δ are the atomic transition rates, and η is a parameter which accounts for phase-relaxation effects. The operators $J^{\pm}$ are the usual atomic raising and lowering operators which, together with J_3, satisfy the angular momentum algebra

$$[J^+,J^-]=2J_3,\quad [J_3,J^{\pm}]=\pm J^{\pm}. \tag{2.2}$$

In the first part of the paper we confine our attention to purely inelastic effects (i.e., we set $\eta=0$).

We observe that the evolution of the density operator represented by Eq. (2.1) is such that the total-angular-momentum operator J^2 is conserved. This property of the master equation makes it convenient to consider a continuous-basis representation in the angular momentum subspace of a given value of J.

We define the continuous-basis representation in the subspace of angular momentum J in terms of the states $|\theta,\varphi\rangle$ defined by[4]

$$|\theta,\varphi\rangle=\sum_{m=-J}^{J}|J,m\rangle\binom{2J}{m+J}^{1/2}(\sin\tfrac{1}{2}\theta)^{J+m} \times(\cos\tfrac{1}{2}\theta)^{J-m}e^{-i(J+m)\varphi}. \tag{2.3}$$

The variables θ and φ correspond to the polar and the azimuthal angle on the Bloch sphere, respectively, and the states $|J,m\rangle$ ($|m|\le J$) are the eigenstates of J_3 and J^2. The states defined by Eq. (2.3) are normalized to unity, but are not orthogonal to one another, i.e.,

$$|\langle\theta,\varphi|\theta',\varphi'\rangle|^2=(\cos\tfrac{1}{2}\Theta)^{4J}, \tag{2.4}$$

where $\cos\Theta=\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\varphi-\varphi')$. The set of $|\theta,\varphi\rangle$ states, however, is over-complete in the subspace of the angular momentum J in the sense that the $(2J+1)$-dimensional identity operator can be resolved as

$$1=\frac{2J+1}{4\pi}\int d\Omega\,|\theta,\varphi\rangle\langle\theta,\varphi|=\sum_{m=-J}^{J}|J,m\rangle\langle J,m|, \tag{2.5}$$

where $d\Omega=\sin\theta\,d\theta\,d\varphi$ is the differential element of solid angle. As shown by Arecchi *et al.*, the continuous-basis representation (2.3) can be used to represent arbitrary operators as a continuous superposition of diagonal projectors $|\theta,\varphi\rangle\langle\theta,\varphi|$. In particular, one can introduce the quasiprobability function $P(\theta,\varphi,t)$ as

$$W(t)=\int d\Omega\,P(\theta,\varphi,t)|\theta,\varphi\rangle\langle\theta,\varphi|. \tag{2.6}$$

The function P represents, loosely speaking, a weighting function for the $|\theta,\varphi\rangle$ states on the surface of the Bloch sphere for a given density operator $W(t)$.

For future reference we shall adopt the notation

$$W(t)=\int d\Omega\,P(\Omega,t)\Lambda(\Omega), \tag{2.7}$$

where Ω stands for the independent variables θ and φ, and where $\Lambda(\Omega)$ is the projector operator

$$\Lambda(\Omega)\equiv|\Omega\rangle\langle\Omega| =\sum_{p,q=0}^{2J}\binom{2J}{p}^{1/2}\binom{2J}{q}^{1/2}e^{-i(p-q)\varphi}(\sin\tfrac{1}{2}\theta)^{p+q} \times(\cos\tfrac{1}{2}\theta)^{4J-(p-q)}|J,p\rangle\langle J,q|. \tag{2.8}$$

Next, we replace $W(t)$ in the master equation (2.1) with Eq. (2.7) and notice, as shown in the Appendix, that this results in the following terms on the right-hand side of the equation:

$$\begin{aligned}
J^-WJ^+&=\int d\Omega\,P(\Omega,t)\mathfrak{D}_{J^-}\mathfrak{D}^*_{J^-}\Lambda(\Omega),\\
J^+J^-W&=\int d\Omega\,P(\Omega,t)\mathfrak{D}_{J^-}\mathfrak{D}_{J^+}\Lambda(\Omega),\\
WJ^+J^-&=(J^+J^-W)^+,\\
J^+WJ^-&=\int d\Omega\,P(\Omega,t)\mathfrak{D}_{J^+}\mathfrak{D}^*_{J^-}\Lambda(\Omega),\\
J^-J^+W&=\int d\Omega\,P(\Omega,t)\mathfrak{D}_{J^+}\mathfrak{D}_{J^-}\Lambda(\Omega),\\
WJ^-J^+&=(J^-J^+W)^+.
\end{aligned} \tag{2.9}$$

The $\mathfrak{D}$ operators in the integrals of Eqs. (2.9) are all represented in terms of c-number differential forms as shown in the Appendix. When Eqs. (A15)–(A20) are used in Eqs. (2.9), the master equation takes the form

$$\int d\Omega \frac{\partial}{\partial t} P(\Omega,t)\Lambda(\Omega) = \nu \int d\Omega\, P(\Omega,t)\left[\left(-2J\sin\theta - \frac{\sin\theta}{1+\cos\theta}\right)\frac{\partial}{\partial\theta} + (1-\cos\theta)\frac{\partial^2}{\partial\theta^2} - \frac{\cos\theta}{1+\cos\theta}\frac{\partial^2}{\partial\varphi^2}\right]\Lambda(\Omega)$$
$$+\delta\int d\Omega\, P(\Omega,t)\left[\left(2J\sin\theta + \frac{\sin\theta}{1-\cos\theta}\right)\frac{\partial}{\partial\theta} + (1+\cos\theta)\frac{\partial^2}{\partial\theta^2} + \frac{\cos\theta}{1-\cos\theta}\frac{\partial^2}{\partial\varphi^2}\right]\Lambda(\Omega). \tag{2.10}$$

If we integrate the right-hand side of Eq. (2.10) by parts, and note that the sum of the surface terms vanishes identically, we finally arrive at the following partial differential equation for the quasiprobability function $Q(\Omega,t) \equiv \sin\theta\, P(\Omega,t)$:

$$\frac{\partial}{\partial t}Q(\Omega,t) = \frac{\partial}{\partial\theta}\left[\left(2J(\nu-\delta)\sin\theta + \nu\frac{\sin\theta}{1+\cos\theta} - \delta\frac{\sin\theta}{1-\cos\theta}\right)Q(\Omega,t)\right] + \frac{\partial^2}{\partial\theta^2}\{[\nu(1-\cos\theta)+\delta(1+\cos\theta)]\,Q(\Omega,t)\}$$
$$-\frac{\partial^2}{\partial\varphi^2}\left[\left(\nu\frac{\cos\theta}{1+\cos\theta} - \delta\frac{\cos\theta}{1-\cos\theta}\right)Q(\Omega,t)\right]. \tag{2.11}$$

Equation (2.11) is the main result of this section. The steady-state solution of Eq. (2.11) and the special case of very large reservoir temperatures will be discussed in Secs. III and IV.

III. STEADY-STATE SOLUTION OF THE PHASE-SPACE EQUATION

We look for a solution of the phase-space differential equation (2.11) corresponding to the steady-state condition of thermal equilibrium.[11] We first write Eq. (2.11) in the form of a continuity equation,

$$\frac{\partial Q}{\partial t} = -\frac{\partial}{\partial q_i}(A_iQ) + \frac{\partial^2}{\partial q_i\,\partial q_j}(D_{ij}Q) \equiv -\frac{\partial I_i}{\partial q_i}, \tag{3.1}$$

where the summation convention has been used for repeated indices, and where the probability current I_i has been defined as

$$I_i = A_iQ - \frac{\partial}{\partial q_j}(D_{ij}Q). \tag{3.2}$$

The indices i and j range over the values 1 and 2 and the variables q_i and q_j stand for θ and φ, for $i,j = 1$ or 2, respectively. In the steady state it is not necessary that the probability current I_i vanish. In this case, however, we verify that the detailed-balance condition is satisfied, i.e., the current density vanishes when the steady-state condition is reached. The detailed-balance condition requires that

$$A_iQ - \frac{\partial}{\partial q_j}(D_{ij}Q) = 0, \tag{3.3}$$

or

$$D_{ij}\frac{\partial Q}{\partial q_j} = \left(A_i - \frac{\partial D_{ij}}{\partial q_j}\right)Q. \tag{3.4}$$

Following Lax,[1] we write Eq. (3.4) in the form

$$\frac{\partial \ln Q}{\partial q_k} = (D^{-1})_{ki}\left(A_i - \frac{\partial D_{ij}}{\partial q_j}\right) \equiv U_k, \tag{3.5}$$

where $(D^{-1})_{ki}$ is the (k,i) element of the inverse diffusion matrix.[12] Since Q, in general, is a continuous differentiable function of its arguments, we must have

$$\frac{\partial^2 \ln Q}{\partial q_k\,\partial q_l} = \frac{\partial^2 \ln Q}{\partial q_l\,\partial q_k},$$

so that the condition for detailed balance can be written in the form

$$\frac{\partial U_k}{\partial q_l} = \frac{\partial U_l}{\partial q_k}. \tag{3.6}$$

From Eqs. (3.5), (3.1), and (2.11) it is seen that $U_\varphi = 0$ and $U_\theta \equiv U_\theta(\theta)$; hence Eq. (3.6) is satisfied identically.

Under the condition of detailed balance the steady-state solution $Q(\theta,\varphi)$ must satisfy the pair of equations (3.3) (for $i = 1,2$), which have the explicit form

$$\left(2J(\nu-\delta)\sin\theta + \frac{\nu\sin\theta}{1+\cos\theta} - \frac{\delta\sin\theta}{1-\cos\theta}\right)Q(\theta,\varphi) + \frac{\partial}{\partial\theta}\left[\Big(\nu(1-\cos\theta)+\delta(1+\cos\theta)\Big)Q(\theta,\varphi)\right] = 0, \tag{3.3a}$$

$$\frac{\partial}{\partial\varphi}\left[\left(\nu\frac{\cos\theta}{1+\cos\theta} - \delta\frac{\cos\theta}{1-\cos\theta}\right)Q(\theta,\varphi)\right] = 0. \tag{3.3b}$$

From Eq. (3.3b) we see what we might have expected, namely, that the steady-state solution $Q(\theta,\varphi)$ is actually independent of φ. Equation (3.3a) can be integrated at once with the additional requirement that the normalization condition

$$\int_0^\pi d\theta \int_0^{2\pi} d\varphi\, Q(\theta,\varphi) = 1,$$

be satisfied. The result is

$$Q(\theta)=[(2J+1)/2\pi](\nu-\delta)[(2\delta)^{-(2J+1)}-(2\nu)^{-(2J+1)}]^{-1}\times\sin\theta[(\nu+\delta)-(\nu-\delta)\cos\theta]^{-2(J+1)}. \tag{3.7}$$

Equation (3.7) can be used to calculate the steady-state expectation values of the operators of interest. In general, the expectation values of normal ordered products of the form $J^{+l}J_3^nJ^{-l'}$ are easier to calculate if we use the diagonal representation (2.6) and the disentangling theorem of Arecchi *et al.*[4] The diagonal representation (2.6) leads to

$$\langle J^{+l}J_3^nJ^{-l'}\rangle=\int d\Omega\, P(\theta,\varphi,t)\langle\theta,\varphi|J^{+l}J_3^nJ^{-l'}|\theta,\varphi\rangle, \tag{3.8}$$

while the disentangling theorem allows the calculation of the diagonal matrix elements $\langle\theta,\varphi|\ldots|\theta,\varphi\rangle$ by direct differentiation of the generating function

$$\chi(\alpha,\beta,\gamma)=[e^{\beta/2}(\sin\tfrac{1}{2}\theta)^2+e^{-\beta/2}(\alpha e^{i\varphi}\sin\tfrac{1}{2}\theta+\cos\tfrac{1}{2}\theta)\times(\gamma e^{-i\varphi}\sin\tfrac{1}{2}\theta+\cos\tfrac{1}{2}\theta)]^{2J} \tag{3.9}$$

as follows

$$\langle\theta,\varphi|J^{+l}J_3^nJ^{-l'}|\theta,\varphi\rangle=\left(\frac{\partial}{\partial\alpha}\right)^l\left(\frac{\partial}{\partial\beta}\right)^n\left(\frac{\partial}{\partial\gamma}\right)^{l'}\chi(\alpha,\beta,\gamma)\bigg|_{\alpha=\beta=\gamma=0}. \tag{3.10}$$

In particular, we have

$$\langle\theta,\varphi|J_3|\theta,\varphi\rangle=-J\cos\theta, \tag{3.11}$$

$$\langle\theta,\varphi|J_3^2|\theta,\varphi\rangle=\tfrac{1}{2}J[(2J-1)\cos^2\theta+1]. \tag{3.12}$$

It is now a simple matter to calculate the steady-state average of J_3. The result of the integration (3.8) is

$$\langle J_3\rangle=\frac{2J+1}{1-\sigma}\,\frac{(1/\sigma)^{2J}-1}{(1/\sigma)^{2J+1}-1}-J\,\frac{1+\sigma}{1-\sigma}, \tag{3.13}$$

where $\sigma\equiv\delta/\nu=e^{(-\hbar\omega/kT)}$. We can write Eq. (3.13) in a more familiar form as

$$\langle J_3\rangle=-\frac{2J+1}{2}\coth\left(\frac{2J+1}{2}\frac{\hbar\omega}{kT}\right)+\frac{1}{2}\coth\left(\frac{1}{2}\frac{\hbar\omega}{kT}\right), \tag{3.14}$$

which is identical to the expectation value of the operator J_3 for a $(2J+1)$-level system in canonical equilibrium with a reservoir at temperature T.

In a similar way we can calculate the second moment of J_3 from Eqs. (3.8) and (3.12). The result is

$$\langle J_3^2\rangle=J(J+1)+\frac{1}{2}\coth^2\left(\frac{1}{2}\frac{\hbar\omega}{kT}\right)-\frac{2J+1}{2}\coth\left(\frac{1}{2}\frac{\hbar\omega}{kT}\right)\coth\left(\frac{2J+1}{2}\frac{\hbar\omega}{kT}\right). \tag{3.15}$$

IV. HIGH-TEMPERATURE LIMIT

As we mentioned before, the exact solution of the phase-space equation (2.11) is quite complicated for arbitrary values of the atomic transition rates ν and δ. Two special cases which lend themselves to exact solutions correspond to the limiting values $T=0$ and $T\to\infty$ for the reservoir temperature.

In the former case, the differential equation, written in terms of the density function

$$Q(\theta,t)=\int_0^{2\pi}d\varphi\, Q(\theta,\varphi,t),$$

is formally identical to the superradiant diffusion equation that has been discussed in Ref. 6. In this limit, the time-dependent solution can be reduced to quadratures. For a discussion of the method of solution we refer the reader to Sec. VII of Ref. 6.

In the limit of a very high reservoir temperature, the solution of Eq. (2.11) can be obtained by eigenfunction expansion techniques. We observe that for $\nu=\delta$ Eq. (2.11) becomes

$$\frac{\partial P(\theta,\varphi,t)}{\partial t}=2\nu\frac{\cos\theta}{\sin\theta}\frac{\partial P(\theta,\varphi,t)}{\partial\theta}+2\nu\frac{\partial^2P(\theta,\varphi,t)}{\partial\theta^2}+2\nu\frac{\cos^2\theta}{\sin^2\theta}\frac{\partial^2P(\theta,\varphi,t)}{\partial\varphi^2}. \tag{4.1}$$

We look for an elementary solution of the form

$$P(\theta,\varphi,t)=e^{-2\nu\lambda t}f(\theta)g(\varphi), \tag{4.2}$$

where λ is an undetermined parameter. The φ-dependent part of the solution is

$$g(\varphi)=e^{im\varphi},\quad m=0,\ \pm1,\ldots, \tag{4.3}$$

which ensures single valuedness with respect to rotations around the polar axis of the Bloch sphere. The θ-dependent part of the solution satisfies the differential equation

$$\frac{d^2f}{d\theta^2}+\frac{\cos\theta}{\sin\theta}\frac{df}{d\theta}+\left(\lambda'-\frac{m^2}{\sin^2\theta}\right)f=0, \tag{4.4}$$

where $\lambda'=\lambda+m^2$. The only regular solutions of Eq. (4.4) are the associated Legendre functions $P_n^m(\cos\theta)$ corresponding to the eigenvalues $\lambda'=n(n+1)$ $(n=0,1,\ldots,|m|\le n)$. It follows that the arbitrary solution of Eq. (4.1) can be expressed as a linear superposition of spherical harmonics $Y_n^m(\theta,\varphi)$,

$$P(\theta,\varphi,t)=\sum_{n=0}^{\infty}\sum_{m=-n}^{n}C_n^m e^{-2\nu[n(n+1)-m^2]t}\,Y_n^m(\theta,\varphi), \tag{4.5}$$

where

$$C_n^m=\int_0^{\pi}d\theta\,\sin\theta\int_0^{2\pi}d\varphi\, Y_n^{m*}(\theta,\varphi)P(\theta,\varphi,0), \tag{4.6}$$

and $P(\theta, \varphi, 0)$ is the initial density function of the atomic system.

V. PHASE-SPACE DIFFERENTIAL EQUATION FOR A TWO-LEVEL SYSTEM

In this section we analyze the behavior of the phase-space density function for a two-level system interacting with a thermal reservoir. Unlike the case of the arbitrary multilevel system discussed in Sec. II, it is possible to find an exact solution for the density function, even including the elastic contributions (the η terms) in the master equation. For $J=\frac{1}{2}$, Eq. (2.1) can be written in the form

$$\frac{\partial W}{\partial t}=\nu(2J^-WJ^+-J^+J^-W-WJ^+J^-)+\delta(2J^+WJ^--J^-J^+W-WJ^-J^+)-\tfrac{1}{4}\eta W+\eta J_3WJ_3 . \quad (5.1)$$

Using the $\mathfrak{D}$-operator techniques mentioned in Sec. II and discussed further in the Appendix, one can derive the differential equation for the density function $Q(\theta,\varphi,t)\equiv\sin\theta P(\theta,\varphi,t)$,

$$\frac{\partial Q}{\partial t}=\frac{\partial}{\partial\theta}\left\{\left[\nu\left(\sin\theta+\frac{1-\cos\theta}{\sin\theta}\right)-\delta\left(\sin\theta+\frac{1+\cos\theta}{\sin\theta}\right)+\frac{1}{4}\eta\sin\theta\cos\theta\right]Q\right\}+\frac{\partial^2}{\partial\theta^2}\left[\left(\nu(1-\cos\theta)+\delta(1+\cos\theta)+\frac{1}{4}\eta\sin^2\theta\right)Q\right]+\frac{\partial^2}{\partial\varphi^2}\left[\left(-\nu\frac{\cos\theta}{1+\cos\theta}+\delta\frac{\cos\theta}{1-\cos\theta}+\frac{1}{4}\eta\right)Q\right]. \quad (5.2)$$

To construct a solution for Eq. (5.2), it is convenient to impose the ansatz

$$Q(\theta,\varphi,t)=\sum_{l=0}^{1}\sum_{m=-l}^{l}Q_{lm}(t)\sin\theta P_l^m(\cos\theta)e^{-im\varphi}, \quad (5.3)$$

where $P_l^m(\cos\theta)$ are the associated Legendre functions. We will have solved the problem if we can construct and solve the equations of motion for the time-dependent expansion coefficients $Q_{lm}(t)$. The same ansatz was used in an attempt to solve the phase-space equation (2.11). The resultant set of first-order linear differential equations for $Q_{l,m}(t)$ was found to be unmanageably complicated. For the case of a two-level system, one is concerned with only four expansion coefficients; furthermore, only two of the four differential equations are coupled to one another (this number grows if one considers multilevel systems). The procedure for arriving at the differential equations in question is fairly simple, but algebraically involved. Here, we merely quote the final result

$$\begin{aligned}\dot{Q}_{0,0}(t)&=0,\\ \dot{Q}_{1,-1}(t)&=-(\nu+\delta+\tfrac{1}{2}\eta)Q_{1,-1},\\ \dot{Q}_{1,0}(t)&=6(\nu-\delta)Q_{0,0}-2(\nu+\delta)Q_{1,0},\\ \dot{Q}_{1,1}(t)&=-(\nu+\delta+\tfrac{1}{2}\eta)Q_{1,1}.\end{aligned} \quad (5.4)$$

The solution of the system of equations (5.4) is

$$\begin{aligned}Q_{0,0}(t)&=Q_{0,0},\\ Q_{1,-1}(t)&=Q_{1,-1}(0)e^{-(\nu+\delta+1/2\eta)t},\\ Q_{1,1}(t)&=Q_{1,1}(0)e^{-(\nu+\delta+1/2\eta)t},\\ Q_{1,0}(t)&=Q_{1,0}(0)e^{-2(\nu+\delta)t}+[(3\nu-\delta)/(\nu+\delta)]Q_{0,0}\\ &\quad\times(1-e^{-2(\nu+\delta)t}).\end{aligned} \quad (5.5)$$

From the normalization condition for $Q(\theta,\varphi,t)$ we find that

$$Q_{0,0}(t)=1/4\pi, \quad (5.6)$$

whereas the remaining three constants can be determined from the initial condition as

$$\frac{4\pi}{2l+1}\frac{(l+m)!}{(l-m)!}Q_{l,m}(0)=\int_0^{2\pi}d\varphi\int_0^{\pi}d\theta\, e^{im\varphi}\times P_l^m(\cos\theta)Q(\theta,\varphi,0). \quad (5.7)$$

VI. CONCLUSIONS

The continuous-basis representation of the coherent atomic states provides an elegant description of the evolution of an arbitrary $(2J+1)$-level system interacting with a reservoir at temperature T. The master equation governing the time dependence of the atomic-density operator has been mapped into a c-number equation evolving on the surface of the so-called Bloch sphere, neglecting the elastic contributions. Unfortunately, the exact solution of this differential equation appears to be quite a difficult mathematical problem. We have been able, however, to derive the steady-state distribution for the atomic system, and some of the operator expectation values of interest. They have been calculated using the integral representation (3.8) in the phase-space of the atomic variables and have proved to be identical, as expected, to the canonical expectation values.

In the limit of large reservoir temperatures, we have obtained an exact time-dependent solution in terms of a linear superposition of spherical harmonics. Finally, we have derived the complete phase-space equation for a relaxing two-level system, including the elastic terms of the master equation, and found its exact solution in terms of associated Legendre functions. The method of solution employed in this final calculation could also be used to derive a time-dependent solution for arbitrary values of J, but it would require numerical methods to calculate the time dependence of the expansion coefficients.

A number of useful rules, referred to as the $\mathfrak{D}$-operator calculus, have been collected in the Appendix to facilitate the derivation of the phase-space equation discussed in this paper and for future applications involving angular momentum operators.

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APPENDIX: $\mathfrak{D}$-OPERATOR CALCULUS

We develop the explicit form of the $\mathfrak{D}$ operators which are useful in the mapping of operator equations into c-number differential form. A more complete description of the method can be found in Narducci *et al.*[9] We consider the diagonal coherent-state projector

$$\Lambda(\Omega) \equiv |\Omega\rangle\langle\Omega| = \sum_{p,q=0}^{2J} \Gamma_{p,q}(\Omega), \tag{A1}$$

where

$$\Gamma_{p,q}(\Omega) = |J,p\rangle\langle J,q| \binom{2J}{p}^{1/2} \binom{2J}{q}^{1/2} (\sin\tfrac{1}{2}\theta)^{p+q} (\cos\tfrac{1}{2}\theta)^{4J-(p+q)} e^{-i(p-q)\varphi}. \tag{A2}$$

In view of the applications discussed in Secs. II and V, we wish to establish the existence of a differential operator $\mathfrak{D}(\Omega)$ acting on the angular variables θ and φ, such that the following identity holds:

$$B_n \cdots B_1 |\Omega\rangle\langle\Omega| A_1 \cdots A_m \equiv \mathfrak{D}(\Omega)|\Omega\rangle\langle\Omega|. \tag{A3}$$

In Eq. (A3) the operators B_i and A_j are assumed to be angular-momentum operators acting on the states $|J,p\rangle$ and $\langle J,q|$.

It is useful, in what follows, to consider the first-order derivatives

$$\frac{\partial\Lambda}{\partial\theta} = -2J(\tan\tfrac{1}{2}\theta)\,\Lambda(\Omega) + \frac{1}{\sin\theta}\sum_{p,q}(p+q)\Gamma_{p,q}(\Omega), \tag{A4}$$

$$\frac{\partial\Lambda}{\partial\varphi} = -i\sum_{p,q}(p-q)\Gamma_{p,q}(\Omega). \tag{A5}$$

The operators

$$\Sigma_p(\Omega) \equiv \sum_{p,q} p\Gamma_{p,q}(\Omega), \tag{A6}$$

$$\Sigma_q(\Omega) \equiv \sum_{p,q} q\Gamma_{p,q}(\Omega) = \Sigma_p(\Omega)^+, \tag{A7}$$

can be expressed in terms of $\Lambda(\Omega)$ and its derivatives using Eqs. (A4) and (A5). One finds at once

$$\Sigma_p(\Omega) = \frac{1}{2}\sin\theta\frac{\partial\Lambda}{\partial\theta} + J(1-\cos\theta)\Lambda + \frac{i}{2}\frac{\partial\Lambda}{\partial\varphi}$$
$$= \Sigma_q(\Omega)^+. \tag{A8}$$

Consider now the special case of Eq. (A3) corresponding to $B_1 = J^+$, with the remaining B and A operators equal to the identity operator. From the algebraic property of J^+ we find

$$J^+|\Omega\rangle\langle\Omega| = (\cot\tfrac{1}{2}\theta)e^{i\varphi}\Sigma_p(\Omega), \tag{A9}$$

whence it follows that

$$J^+|\Omega\rangle\langle\Omega| = e^{i\varphi}\left(J\sin\theta + (\cos\tfrac{1}{2}\theta)^2\frac{\partial}{\partial\theta} + \frac{i}{2}\cot\tfrac{1}{2}\theta\frac{\partial}{\partial\varphi}\right)\Lambda(\Omega)$$
$$\equiv \mathfrak{D}_{J^+}(\Omega)\Lambda(\Omega). \tag{A10}$$

A similar calculation leads to the explicit representation of the differential operators $\mathfrak{D}_{J^-}$ and $\mathfrak{D}_{J_3}$, namely,

$$J^-|\Omega\rangle\langle\Omega| = e^{-i\varphi}\left(J\sin\theta - (\sin\tfrac{1}{2}\theta)^2\frac{\partial}{\partial\theta} - \frac{i}{2}\tan\tfrac{1}{2}\theta\frac{\partial}{\partial\varphi}\right)\Lambda(\Omega)$$
$$\equiv \mathfrak{D}_{J^-}(\Omega)\Lambda(\Omega), \tag{A11}$$

and

$$J_3|\Omega\rangle\langle\Omega| = \left(-J\cos\theta + \frac{1}{2}\sin\theta\frac{\partial}{\partial\theta} + \frac{i}{2}\frac{\partial}{\partial\varphi}\right)\Lambda(\Omega)$$
$$\equiv \mathfrak{D}_{J_3}(\Omega)\Lambda(\Omega). \tag{A12}$$

The differential operators corresponding to $\Lambda(\Omega)J^\pm$ and $\Lambda(\Omega)J_3$ follow from the identities

$$\Lambda(\Omega)J^\pm = [J^\mp\Lambda(\Omega)]^+ = \mathfrak{D}^*_{J^\mp}(\Omega)\Lambda(\Omega), \tag{A13}$$

$$\Lambda(\Omega)J_3 = [J_3\Lambda(\Omega)]^+ = \mathfrak{D}^*_{J_3}(\Omega)\Lambda(\Omega), \tag{A14}$$

which result from the Hermitian character of the projector $\Lambda(\Omega)$. The operators $\mathfrak{D}^*_{J^\pm}$, $\mathfrak{D}^*_{J_3}$ are the complex-conjugate differential forms of the $\mathfrak{D}$ operators defined by Eqs. (A10), (A11), and (A12).

More general expressions of the form given by Eq. (A3) can be constructed by repeated applications of the elementary $\mathfrak{D}$ operators given above. Since this calculation involves a substantial amount of algebraic manipulations, we list here the $\mathfrak{D}$ operators of interest for the calculations performed in Secs. II and IV.

$$J^+J^-\Lambda(\Omega) = \left(J^2\sin^2\theta + \frac{1}{2}J(1-\cos\theta)^2\right)\Lambda + \left(J\sin\theta\cos\theta + \frac{1}{4}\sin\theta\,(2-\cos\theta)\right)\frac{\partial\Lambda}{\partial\theta} + i\left(J\cos\theta + \frac{1}{2}\right)\frac{\partial\Lambda}{\partial\varphi}$$
$$-\frac{i}{2}\sin\theta\frac{\partial^2\Lambda}{\partial\theta\partial\varphi} + \frac{1}{4}\frac{\partial^2\Lambda}{\partial\varphi^2} - \frac{1}{4}\sin^2\theta\frac{\partial^2\Lambda}{\partial\theta^2}$$
$$\equiv \mathfrak{D}_{J^-}\mathfrak{D}_{J^+}\Lambda(\Omega), \tag{A15}$$

$$J^-J^+\Lambda(\Omega) = \left(J^2\sin^2\theta + \frac{1}{2}J(1+\cos\theta)^2\right)\Lambda + \left(J\sin\theta\cos\theta - \frac{1}{4}\sin\theta(2+\cos\theta)\right)\frac{\partial\Lambda}{\partial\theta} + i\left(J\cos\theta - \frac{1}{2}\right)\frac{\partial\Lambda}{\partial\varphi}$$
$$-\frac{i}{2}\sin\theta\frac{\partial^2\Lambda}{\partial\theta\partial\varphi} + \frac{1}{4}\frac{\partial^2\Lambda}{\partial\varphi^2} - \frac{1}{4}\sin^2\theta\frac{\partial^2\Lambda}{\partial\theta^2}$$
$$\equiv \mathfrak{D}_{J^+}\mathfrak{D}_{J^-}\Lambda(\Omega), \tag{A16}$$

$$\Lambda(\Omega)J^+J^- = \mathfrak{D}^*_{J^-}\mathfrak{D}^*_{J^+}\Lambda(\Omega), \tag{A17}$$

$$\Lambda(\Omega)J^-J^+ = \mathfrak{D}^*_{J^+}\mathfrak{D}^*_{J^-}\Lambda(\Omega), \tag{A18}$$

$$J^-\Lambda(\Omega)J^+ = \left(J^2\sin^2\theta + \frac{1}{2}J(1-\cos\theta)^2\right)\Lambda + \left(-J\sin\theta\,(1-\cos\theta) + \frac{1}{4}\cot\theta\,(1-\cos\theta)^2\right)\frac{\partial\Lambda}{\partial\theta} + \frac{1}{4}(1-\cos\theta)^2\frac{\partial^2\Lambda}{\partial\theta^2}$$
$$+\frac{1}{4}(\tan\tfrac{1}{2}\theta)^2\frac{\partial^2\Lambda}{\partial\varphi^2} \equiv \mathfrak{D}_{J^-}\mathfrak{D}^*_{J^-}\Lambda \equiv \mathfrak{D}^*_{J^-}\mathfrak{D}_{J^-}\Lambda\,, \tag{A19}$$

$$J^+\Lambda J^- = \left(J^2\sin^2\theta + \frac{1}{2}J(1+\cos\theta)^2\right)\Lambda + \left(J\sin\theta\,(1+\cos\theta) + \frac{1}{4}\cot\theta\,(1+\cos\theta)^2\right)\frac{\partial\Lambda}{\partial\theta}$$
$$+\frac{1}{4}(1+\cos\theta)^2\frac{\partial^2\Lambda}{\partial\theta^2} + \frac{1}{4}(\cot\tfrac{1}{2}\theta)^2\frac{\partial^2\Lambda}{\partial\varphi^2}$$
$$\equiv \mathfrak{D}_{J^+}\mathfrak{D}^*_{J^+}\Lambda \equiv \mathfrak{D}^*_{J^+}\mathfrak{D}_{J^+}\Lambda\,. \tag{A20}$$

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[1] J. R. Klauder and E. C. G. Sudarshan, *Fundamentals of Quantum Optics* (Benjamin, New York, 1968); M. Lax, *Statistical Physics, Phase Transitions and Superfluidity, 1966 Brandeis University Summer Institute* (Gordon and Breach, New York, 1968), Vol. II, p. 269; H. Haken, in *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1970), Vol. 25, Sect. 2c.

[2] K. E. Cahill and R. J. Glauber, Phys. Rev. 177, 1857 (1969); 177, 1882 (1969); G. S. Agarwal and E. Wolf, Phys. Rev. D 2, 2161 (1970); 2, 2187 (1970); 2, 2206 (1970); M. Lax and W. H. Louisell, IEEE J. Quantum Electron. QE-3, 47 (1967).

[3] W. H. Louisell and J. H. Marburger, IEEE J. Quantum Electron. QE-3, 348 (1967); G. S. Agarwal, Phys. Rev. A 2, 2038 (1970).

[4] F. T. Arecchi, E. Courtens, R. Gilmore, and H. Thomas, Phys. Rev. A 6, 2211 (1973). See also J. M. Radcliffe, J. Phys. A 4, 313 (1971); C. M. Bowden, Int. J. Quantum Chem. 6, 133 (1972).

[5] R. J. Glauber, Phys. Rev. 131, 2766 (1963); E. C. G. Surdarshan, Phys. Rev. Lett. 10, 277 (1963).

[6] L. M. Narducci, C. M. Bowden, and C. A. Coulter, Nuovo Cimento Lett. 8, 57 (1973); L. M. Narducci, C. A. Coulter, and C. M. Bowden, Phys. Rev. A 9, 829 (1974).

[7] G. S. Agarwal, Phys. Rev. A 4, 1791 (1971).

[8] R. Bonifacio, F. Haake, and P. Schwendimann, Phys. Rev. A 4, 302; 4, 854 (1971).

[9] L. M. Narducci, C. M. Bowden, V. Bluemel, G. P. Carrazana, and R. A. Tuft, Phys. Rev. A (to be published). Different rules have been developed by H. Haken, Z. Phys. 219, 411 (1969). We have decided in favor of the present development because our *c*-number equation contains only a finite number of derivatives for the problem discussed in this paper.

[10] The steady-state aspect of this problem has also been analyzed by G. S. Agarwal [Z. Phys. 258, 409 (1973)] as an application of his general analysis of microre-

versibility for open systems.

[11]R. Graham and H. Haken, Z. Phys. 243, 289 (1971); H. Risken, Z. Phys. 251, 231 (1972).

[12]Here we use the terminology "diffusion matrix" in analogy to the case of a Fokker-Planck equation. In this case our diffusion matrix is nonpositive definite.