Note on Kroll and Watson's low-frequency result for the multiphoton bremsstrahlung process

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Through a detailed analysis of the second Born amplitude, an alternative derivation of Kroll and Watson's low-frequency result for the multiphoton bremsstrahlung process is given.

After unsuccessful attempts by many authors[1] to relate the multiphoton bremsstrahlung cross section to the elastic-scattering cross section, it was first shown by Kroll and Watson[2] that this relation for a low-frequency electromagnetic (em) field is (in units where $\hbar = c = 1$)

$$\frac{d\sigma^{(N)}}{d\Omega} = \frac{k_f}{k_i} J_N^2\left(\frac{e\vec{a}\cdot\vec{Q}}{m\omega}\right)\frac{d\sigma_{\rm el}(\epsilon,\vec{Q})}{d\Omega}, \tag{1}$$

where k_i and k_f are the initial and the final momenta of the electron, a is the amplitude of the vector potential of the em field of frequency ω, $d\sigma_{\rm el}/d\Omega$ is the differential cross section for elastic scattering to be evaluated at the momentum transfer $\vec{Q}$ $(=\vec{k}_f - \vec{k}_i)$, and at the incident energy ϵ

$$\epsilon = \frac{1}{2m}\left(\vec{k}_i - \frac{\hat{a}mN\omega}{\hat{a}\cdot\vec{Q}}\right)^2 . \tag{2}$$

The intention of this note is to give an alternate derivation of (1) by analyzing in detail the second Born amplitude for this process.

The second Born amplitude is (apart from a numerical constant)

$$T_{B2} = \int_{-\infty}^{\infty} dt \langle X_{k_f}(r,t)| V(r) \int d^3r' \int_{-\infty}^{t} dt' \times G(r,r';t,t')V(r')|X_{k_i}(r',t')\rangle , \tag{3}$$

where

$$X_k(r,t) = e^{-i\vec{k}\cdot\vec{r}} \exp\left[\frac{-i}{2m}\left(k^2 t - \frac{2\vec{k}\cdot\vec{a}e}{\omega}\sin\omega t\right)\right]$$

and

$$G(r,r';t,t') = \int d^3k\, X_k(r,t)X_k^*(r',t') .$$

Using the Bessel-function expansion

$$e^{iy\sin\phi} = \sum_{n=-\infty}^{\infty} J_n(y)e^{in\phi}, \tag{4}$$

one can write (3), for an N-photon bremsstrahlung process, as

$$T_{B2}^{(N)} = 2\pi i \delta\left(\frac{k_f^2 - k_i^2}{2m} - N\omega\right) \times \sum_{n=-\infty}^{\infty} \int d^3k\; V(|\vec{k}_f - \vec{k}|)V(|\vec{k} - \vec{k}_i|) \times \frac{J_n[e\vec{a}\cdot(\vec{k}-\vec{k}_i)/m\omega]J_{N-n}[e\vec{a}\cdot(\vec{k}_f-\vec{k})/m\omega]}{(k^2 - k_i^2)/2m - n\omega}. \tag{5}$$

In the following the low-frequency limit of this amplitude will be studied.

For any given potential, if we do a term-by-term analysis of this amplitude, it is found that as a function of the argument of the Bessel function and its order, the dominant contribution comes from those terms where the argument is equal to its order, i.e., when

$$n = e\vec{a}\cdot(\vec{k}-\vec{k}_i)/m\omega \tag{6}$$

and

$$N - n = e\vec{a}\cdot(\vec{k}_f - \vec{k})/m\omega . \tag{7}$$

From (6) and (7) we get

$$n = N\hat{a}\cdot(\vec{k}-\vec{k}_i)/\hat{a}\cdot\vec{Q} , \tag{8}$$

where $\hat{a}$ is the unit vector in the direction of $\vec{\alpha}$ and $\vec{Q}$ is the momentum transfer.

Using (8) and the addition theorem

$$\sum_{n=-\infty}^{\infty} J_n(u)J_{N-n}(v) = J_N(u+v), \tag{9}$$

Eq. (5) can be written in the form (see noted added in proof)

$$T_{B2}^{(N)} = 2\pi i\delta\left(\frac{k_f^2 - k_i^2}{2m} - N\omega\right)J_N\left(\frac{e\vec{a}\cdot\vec{Q}}{m\omega}\right) \times \int d^3K\, \frac{V(|\vec{K}_f - \vec{K}|)V(|\vec{K}-\vec{K}_i|)}{K^2/2m - K_i^2/2m}, \tag{10}$$

where

$$\vec{K}_i = \vec{k}_i - N\omega m\hat{a}/\hat{a}\cdot\vec{Q} ,$$

$$\vec{K}_f = \vec{k}_f - N\omega m\hat{a}/\hat{a}\cdot\vec{Q} .$$

To demonstrate that the matrix element in (10) corresponds to elastic scattering, we need to show that

$$|\vec{K}_f| = |\vec{K}_i| . \tag{11}$$

This can easily be verified by using the energy-conservation δ function in (10).

We have thus shown that in the low-frequency limit the second Born amplitude can be written as a product of a Bessel function and an elastic-scattering amplitude. By following the above analysis the third and higher Born amplitudes can also be written in the product form. Thus the sum of all these amplitudes constitutes Eq. (1).

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Note added in proof. If we use the following integral representation in Eq. (5):

$$J_n(x)J_{m-n}(y) = \frac{(-1)^m}{2\pi}\int_0^{2\pi} e^{-in\phi} \times J_m\big((x^2+y^2+2xy\cos\phi)^{1/2}\big) \times\left(\frac{xe^{i\phi}+y}{xe^{-i\phi}+y}\right)^{m/2} d\phi ,$$

then because of large variation in n, we can approximate it as

$$J_n(x)J_{m-n}(y) \approx (-1)^m J_m(x+y)\delta[n - mx/(x+y)] .$$

This then leads to Eq. (10).

[1]See, e.g., V. Kasyanov and A. Starostin, Zh. Eksp. Teor. Fiz. 48, 295 (1965) [Sov. Phys.—JETP 21, 193 (1965)].

[2]N. M. Kroll and K. M. Watson, Phys. Rev. A 8, 804 (1973).