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An order-parameter field theory for turbulent fluctuations

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With the use of the Herivel-Lin variational principle to describe incompressible inviscid fluids, a fluctuation Hamiltonian is derived in terms of a complex field $\Psi(\vec{r},t)$ representing the fluid. The source-free part of the field current describes the velocity field. The field interaction is found to be of current-current type, i.e., $(\Psi^* \operatorname{grad} \Psi - \text{c.c.})^2$ instead of the commonly used $|\Psi|^4$ type. The pressure, together with a complete energy-momentum tensor of the Ψ field, is introduced, depending on the boundary conditions. The latter imply a long-range interaction near the onset of turbulence. Viscosity is taken into account by proper extension of the equation of motion of $\Psi(\vec{r},t)$, which turns out to be of the Landau type. This equation is solved exactly for laminar plane shear flow. The eddy interaction in k space for fully developed turbulence is given together with a model Hamiltonian for the effects of viscosity. Finally the transient behavior of the Ψ-field amplitude of a fixed spatial mode near the onset of turbulence is compared with experiments.

I. INTRODUCTION

There has been considerable progress in the last few years in our understanding of the singular behavior of systems near continuous phase transitions. The decisive step was a proper treatment of the order-parameter fluctuations.[1]

Another, even older problem is the understanding of the nature of turbulence, its onset in a laminar flow as well as its fully developed phase. In 1944 Landau[2] postulated a description of the transition to turbulence in analogy to his theory of phase transitions. As the turbulent "order parameter" he took the velocity field amplitude A of an instability of the laminar flow. The Reynold's number Re characterizes the external conditions; thus $A \propto (\mathrm{Re} - \mathrm{Re}_c)^{1/2}$. With increasing Re a second instability appears, etc. A whole sequence of transitions finally leads to fully developed turbulence.

There is another correspondence between turbulence and phase transitions: Kolmogoroff[3] in 1941, von Weizsäcker,[4] and Heisenberg[5] in 1948 developed a scaling theory of the highly turbulent state. The scaling idea is also essential in order to understand critical fluctuations near continuous phase transitions.[1,6,7]

If the onset of turbulence is indeed similar to the onset of special ordering at a phase transition the question arises, what is the order parameter? Landau suggested an equation for the time average of the amplitude squared of an instability. The corresponding equation for the amplitude A itself reads

$$\frac{d}{dt}A = \gamma_1 A - \tfrac{1}{2}\alpha A^3 . \tag{1}$$

Let us take for granted that all turbulent behavior is contained in the Navier-Stokes equation[8]

$$\partial_t \vec{v} + (\vec{v} \cdot \operatorname{grad})\vec{v} = -\operatorname{grad}(P/\rho) + \nu \Delta \vec{v} . \tag{2}$$

How can Eq. (1) be derived from Eq. (2)? The latter shows a second-order interaction term, while there is a third-order one in the former equation.

Equation (1) is a deterministic equation for the mean amplitude. Since we are aware at present of the eminent role played by the fluctuations, we want to take them into account. In this paper I would like to suggest how this might be achieved.

Fluctuations, at least in equilibrium phase transitions, are governed by a "Hamiltonian," sometimes also called (restricted) "free energy." This determines the distribution of fluctuations as well as the equation of motion. For fluids only this latter one is given, namely, the Navier-Stokes equation.

In order to find an appropriate Hamiltonian, I will look first (Sec. II) for a Lagrangian, which implies just the fluid equations via Hamilton's principle of least action. It is known[9] that there is no such Lagrangian for the velocity field in the Eulerian description. In particular the kinetic term cannot be derived from any kind of Lagrangian. Instead, Herivel and Lin[10-12] gave a variational principle with constraints in terms of certain "potential functions," describing *inviscid* fluids.

The Herivel-Lin principle with constraints is the starting point of this paper. A complex "order parameter" field $\Psi(\vec{r}, t)$ is introduced (Sec. III) together with a Lagrangian for this field. The field equation of motion is derivable from a least-action principle *without* any constraints. The current of the Ψ field determines the physical velocity field. The defining relation depends on the boundary conditions, i.e., on the actual flow pattern. The pressure P is also defined as a certain observable of the Ψ field (Sec. IV). For the observables $\vec{v}$ and P the Navier-Stokes equation for inviscid fluids holds.

I then derive the Ψ-field Hamiltonian (Sec. V) together with the whole energy-momentum tensor (Sec. VI). The Hamiltonian contains second- and fourth-order terms in Ψ. In contrast to the usual Landau free energy, there is no $|\Psi|^4$ coupling but instead a $(\Psi^* \operatorname{grad}\Psi - \text{c.c.})^2$ interaction of the fluctuations.

Viscosity is taken into account (Sec. VII) by extending the Ψ-field equation of motion. The form of the observable $\vec{v}$ as a functional of Ψ is unchanged. Approximate equations of motion can be represented by a Hamiltonian, which besides the kinetic terms already mentioned includes a second-order term proportional to viscosity ν (Sec. VIII).

To illustrate the use of the Ψ field (Secs. IX – XI) (i) an exact solution of its equation of motion is derived in the example of steady shear flow between two parallel planes. (ii) The turbulent interaction in the fully developed phase is given as the *transverse current* squared, and (iii) the transient behavior of the Ψ amplitude is compared with Eq. (1), suggested by Landau, as well as with experiments.

II. HERIVEL-LIN VARIATIONAL PRINCIPLE

Consider an incompressible isentropic fluid which we assume in the beginning to be inviscid. We are looking for a variational principle which describes the velocity field in the Eulerian description.

Such principles have been introduced among others by Herivel[10] (see also Serrin[11]). His results can be used only if one improves them using an idea of Lin,[11,12] who showed how to take proper care of the peculiarity of the $\vec{v}$ field, namely that it actually arises from the flow of matter. To complete the historical background, one should mention earlier work by Clebsch,[13] who first gave the potential representation of $\vec{v}$, and by Bateman.[14] Another branch is connected with the equations of superfluid flow[15,16] and goes back to Eckart.[17] Quite recently Seliger and Whitham[18] gave an extended discussion of variational principles in continuum mechanics, which summarizes best the starting point for the present attempt to find a Hamiltonian for turbulent fluctuations. The Herivel-Lin variational principle has been applied also to relativistic perfect fluids.[19,20]

A reasonable guess of a Lagrangian density is kinetic energy $\frac{1}{2}\rho\vec{v}^2$ minus potential energy ρu. Since the density ρ as well as the temperature T are constant, the internal energy density u is also constant, $u=u_0$, and may be omitted. Incompressibility, $\operatorname{div}\vec{v}=0$, is taken into account by introducing a Lagrange parameter field ϕ:

$$\delta \int (\tfrac{1}{2}\vec{v}^2 - \phi \operatorname{div}\vec{v})\, dV = 0.$$

Variation with respect to $\vec{v}$ yields $\vec{v} = -\operatorname{grad}\phi$. This does not allow for vortices, although even in laminar shear flow $\operatorname{rot}\vec{v} \neq 0$.

This long-standing difficulty has been overcome by Lin.[11,12] His argument is as follows: The velocity fields admissible for variation must be further restricted to take into account the fact that $\vec{v}(\vec{r}, t)$ is generated by the flow of matter. Thus if a "label" is attached to each part of matter at some arbitrary but fixed time t_0, this label will be a streaming invariant expressing "conservation of identity."

The simplest label, taken by Lin, is the position vector $\vec{a}$. Conservation of identity then means, there exists a function $\vec{\alpha}(\vec{r}, t)$, which fulfills the initial condition $\vec{\alpha}(\vec{a}, t_0) = \vec{a}$ and satisfies $\partial_t \vec{\alpha} + (\vec{v} \cdot \operatorname{grad})\vec{\alpha} = 0$. Of course, another labeling at t_0 would do the same job, thus introducing a large amount of arbitrariness. The only essential condition is the *existence* of a certain field $\vec{\alpha}(\vec{r}, t)$ (besides $\vec{v}$, ϕ) which remains constant under the operation $D_t = (\partial_t + \vec{v} \cdot \operatorname{grad})$.

With reference to the Clebsch representation of the fluid velocity ($\vec{v} = \operatorname{grad}\chi + \lambda \operatorname{grad}\mu$) Lin as well as others later used a one-component field α instead of the vector label field $\vec{\alpha}$. A more formal argument refers to Pfaff's theorem; see e.g. Ref. 18.

I think that $\alpha(\vec{r}, t)$ nevertheless can be interpreted as a label field: As the one-dimensional continuum has the same power as the three-dimensional one,[21,22] one can use a one-dimensional manifold as the set of labels for the fluid at t_0. In using it one must, on the other hand, be aware of benefits and drawbacks in comparison with the more natural three-dimensional label set. The obvious benefit is the reduction of the number of fields one deals with from three to one, vector to scalar. A drawback is that restrictions in the initially chosen label function $\alpha(\vec{r}, t_0)$ have to be observed.

If one insists, for example, on a *unique* labeling, one inevitably must give up the continuity of α as a function of $\vec{r}$. This is a consequence of the fact (e.g., proven by Kamke[21]) that a *continuous* image function from a three- to a one-dimensional set cannot be one to one.

Since we even want to differentiate α, we give up unique labeling. The question remains, whether further more severe restrictions for admissible initial label functions $\alpha(\vec{r}, t_0)$ have to be considered.

If we add the conservation of identity as another

$\vec{v}$-field constraint by means of a further Lagrange-multiplier field $\gamma(\vec{r}, t)$, we get the Herivel-Lin variational principle with constraints for perfect incompressible fluids:

$$\delta \int_{t_1}^{t_2} dt \int_V dV \left[\tfrac{1}{2}\vec{v}^2 - \phi \operatorname{div} \vec{v} - \gamma(\partial_t \alpha + \vec{v} \cdot \operatorname{grad}\alpha)\right] = 0. \tag{3}$$

Especially by variation with respect to $\vec{v}$ one gets as one of the equations of motion

$$\vec{v} = -\operatorname{grad}\phi + \gamma \operatorname{grad}\alpha.$$

III. INTRODUCTION OF A COMPLEX ORDER PARAMETER FIELD

Varying Eq. (3) with respect to α together with the constraint of conserved identity leads to

$$D_t \gamma = 0 , \quad D_t \alpha = 0.$$

Looking at these equations of motion, we see that γ and α are indistinguishable. This suggests that one change the Herivel-Lin principle. First, we symmetrize it: Integrate the γ term by parts and get an equivalent extremum principle; take $\frac{1}{2}$ of the sum of both and redefine the auxiliary fields $\frac{1}{2}\gamma \Rightarrow \gamma$, $\phi - \frac{1}{2}\alpha\gamma \Rightarrow \phi$; integrate the ϕ term by parts.

Second, let us consider now the resultant variational principle as a common principle of least action

$$\delta \int dt \int dV\, L = 0$$

without any constraints but with six independent fields instead of four, namely $\vec{v}, \alpha, \phi, \gamma$. Subsequently this number will be reduced to only two independent components.

The Lagrange density L is defined as

$$(1/\rho)L = \tfrac{1}{2}\vec{v}^2 - u_0 + \partial_t \phi + \vec{v} \cdot \operatorname{grad}\phi + \alpha(\partial_t \gamma + \vec{v} \cdot \operatorname{grad}\gamma) - \gamma(\partial_t \alpha + \vec{v} \cdot \operatorname{grad}\alpha). \tag{4}$$

ρ is the constant mass density, u_0 the constant internal energy per unit mass. $-u_0 + \partial_t \phi$ does not enter the action, of course, but has been added for formal reasons.

Varying with respect to $\vec{v}$ gives

$$\vec{v} = -\operatorname{grad}\phi + \gamma \operatorname{grad}\alpha - \alpha \operatorname{grad}\gamma . \tag{5}$$

The γ-α part looks like the conventional quantum-mechanical current

$$\vec{j} = (1/2i)[\Psi^* \operatorname{grad}\Psi - \Psi \operatorname{grad}\Psi^*] , \tag{6}$$

if we define the complex (i.e., two component) field

$$\Psi = \gamma + i\alpha.$$

Third, $\vec{v}$ and ϕ will now be eliminated in favor of Ψ, which is considered to be the basic field. Varying with respect to ϕ yields incompressibility, $\operatorname{div} \vec{v} = 0$. From Eq. (5),

$$\Delta\phi = \operatorname{div}\vec{j} . \tag{7}$$

By this equation we can solve ϕ in terms of the Ψ field. Thus $\vec{v}$ is the current $\vec{j}$ of the basic complex field Ψ corrected by a term which removes the possible sources of $\vec{j}$:

$$\vec{v} = \vec{j} - \operatorname{grad}\phi(\vec{j}) . \tag{8}$$

The vortices of $\vec{v}$ are those of $\vec{j}$ and read

$$\operatorname{rot}\vec{v} = 2 \operatorname{grad}\gamma \times \operatorname{grad}\alpha . \tag{9}$$

The equation of motion for Ψ is found by variation of L with respect to γ and α:

$$\partial_t \Psi + \vec{v} \cdot \operatorname{grad}\Psi = 0 . \tag{10}$$

$\vec{v}$ is to be considered as a functional of Ψ, Ψ^*, cf. Eq. (8) together with Eq. (7). To solve this latter equation we explicitly have to introduce the special geometry of the flow problem considered, namely, the boundary values of ϕ. Consider the Green's function $G(\vec{r}, \vec{r}\,')$ in the given geometry,

$$\Delta' G(\vec{r}, \vec{r}\,') = \delta(\vec{r} - \vec{r}\,').$$

Then

$$\phi(\vec{r}) = \int_V G(\vec{r}, \vec{r}\,') \operatorname{div}'\vec{j}(\vec{r}\,')\, dV(\vec{r}\,') + \oint_{F_V} d\vec{F}(\vec{r}\,') \cdot [\phi(\vec{r}\,') \operatorname{grad}' G(\vec{r}, \vec{r}\,') - G(\vec{r}, \vec{r}\,') \operatorname{grad}'\phi(\vec{r}\,')] = \phi_j + \phi_b . \tag{11}$$

The second term, ϕ_b, is solely determined by the boundary; the first one is effected by fluctuations of Ψ via $\vec{j}$. As the boundary values are considered already to be fixed by the laminar flow, ϕ_b is of zero order and ϕ_j of second order in the fluctuating field which is superimposed on the basic laminar flow profile. Equation (11) induces a subdivision of $\vec{v}$ in a zero- and a second-order contribution,

$$\vec{v} = \vec{v}_b + \vec{v}_j , \quad \vec{v}_b = -\operatorname{grad}\phi_b , \quad \vec{v}_j = \vec{j} - \operatorname{grad}\phi_j . \tag{12}$$

The order parameter equation now looks like a typical Landau equation.

$$\partial_t \Psi + \vec{v}_b \cdot \operatorname{grad}\Psi + \vec{v}_j(\Psi^*, \Psi) \cdot \operatorname{grad}\Psi = 0 . \tag{13}$$

It will be supplemented later (see Sec. VII) by viscous terms, mainly by $\nu\Delta\Psi$. The competition of this viscous term with the first-order kinetic term plays the role of the conventional Landau parameter, $a \propto T - T_c$, or γ_1 of Eq. (1). The interaction of the fluctuations is not, as usual, $(\Psi^*\Psi)\Psi$ but rather of current type $(\Psi^* \operatorname{grad}\Psi - \text{c.c.})\Psi$. Note the long-ranging nonlocality introduced by the "Coulomb" character of Green's function G. Note furthermore its dependence on the geometrical parameters of the flow.

IV. PRESSURE; EULER'S EQUATION

To sum up: The fluid is described by a complex field $\Psi = \gamma + i\alpha$. Its equation of motion in the case of an incompressible *perfect* fluid is Eq. (13). Formally Ψ is a modified "square root" of $\vec{v}$; physically it is connected with the movement of labels. From Ψ one obtains the velocity field $\vec{v}$ using Eqs. (12), (11), and (6).

The pressure P will be introduced now as another observable derived from the Ψ field. Its proper definition follows from the requirement that the observable $\vec{v}$ satisfies the Euler equation.

We calculate $(\partial_t + \vec{v}\cdot\operatorname{grad})\vec{v}$. Using Eqs. (8), (6), and (10) together with the commutation relation $D_t\partial_\lambda - \partial_\lambda D_t = -v_{\mu|\lambda}\partial_\mu$, we find

$$(\partial_t + \vec{v}\cdot\operatorname{grad})\vec{v} = -\operatorname{grad}(\tfrac{1}{2}\vec{v}^2 + \partial_t\phi + \vec{v}\cdot\operatorname{grad}\phi)\ .$$

This suggests that one define the pressure by

$$(1/\rho)P = \tfrac{1}{2}\vec{v}^2 + \partial_t\phi + \vec{v}\cdot\operatorname{grad}\phi - u_0\ , \qquad (14)$$

where u_0 is the constant internal energy density. Then $\vec{v}$ together with P obeys the Euler equation as a *consequence* of the Ψ-field equation of motion.

If we express L of Eq. (4) in terms of the physical fields (satisfying $D_t\alpha = D_t\gamma = 0$), it is equal to the pressure P. The Lagrangian thus has a direct physical meaning.[23]

V. FLUCTUATION HAMILTONIAN

Now, we can apply the methods of conventional field theory. The Ψ-field Lagrangian L can be rewritten as

$$(1/\rho)L = \partial_t\phi + \alpha\partial_t\gamma - \gamma\partial_t\alpha - \tfrac{1}{2}\vec{v}^2 - u_0\ , \qquad (15)$$

if Eq. (5) is used for $\vec{v}$. Let me remark that the action principle with this form of L,

$$\delta\int_{t_1}^{t_2} dt \int_V dV(\alpha\partial_t\gamma - \gamma\partial_t\alpha - \tfrac{1}{2}\vec{v}^2) = 0\ , \qquad (16)$$

is most convenient in deriving immediately the equation of motion (10), if $\delta\vec{v}$ is expressed by $\delta\alpha$ or $\delta\gamma$ via Eq. (5).

As usual, we construct an energy-momentum tensor:

$$T_{\mu\nu} = \sum_a \chi_{a|\mu}\frac{\partial L}{\partial\chi_{a|\nu}} - \delta_{\mu\nu}L\ .$$

Only the order parameter components have nontrivial canonically conjugate momenta

$$\frac{\partial L}{\partial(\partial\gamma/\partial t)} = \rho\alpha\ , \quad \frac{\partial L}{\partial(\partial\alpha/\partial t)} = -\rho\gamma\ .$$

T_{44} is the energy density H. The calculation yields

$$H = \tfrac{1}{2}\rho\vec{v}^2 + \rho u_0\ . \qquad (17)$$

An analog of the Boltzmann principle might also hold in steady states far from equilibrium, stating that the probability for the occurrence of a certain field $\Psi(\vec{r})$ is determined by the exponential of the total energy necessary to realize it. Although as expected $\mathcal{H}$ is quadratic in $\vec{v}$ the Boltzmann distribution is not of Gaussian form since not $\vec{v}$ but Ψ determines the volume element in function space.

The total energy can be remodeled using Eqs. (12) to separate contributions of different order in Ψ:

$$\mathcal{H} = \int_V H\, dV \equiv \mathcal{H}^{(0)} + \mathcal{H}^{(2)} + \mathcal{H}^{(4)}\ . \qquad (18)$$

The zero-order term

$$\mathcal{H}^{(0)} = \int_V (\tfrac{1}{2}\rho\vec{v}_b^{\,2} + \rho u_0)\, dV$$

is entirely determined by the boundaries, as only ϕ_b enters. The term

$$\mathcal{H}^{(2)} = \rho\int_V \vec{v}_b\cdot\vec{v}_j\, dV = -\rho\oint_F \phi\vec{v}_j\cdot d\vec{F}$$

is formally of second order in Ψ. It can also be reduced to boundary values, as $\operatorname{div}\vec{v}_j = \operatorname{div}\vec{j} - \Delta\phi_j = \Delta\phi_b = 0$. This reduction is not possible if $\vec{v}_b$ comprises the whole laminar profile, which is *not* of gradient form in general. The fourth-order term reads

$$\mathcal{H}^{(4)} = \int_V \tfrac{1}{2}\rho\vec{v}_j^{\,2}\, dV = \tfrac{1}{2}\rho\int_V (\vec{j} - \operatorname{grad}\phi_j)^2\, dV$$
$$= \tfrac{1}{2}\rho\int_V (\vec{j}^2 - \operatorname{grad}^2\phi_j)\, dV - \rho\oint_F \phi_j\vec{v}_j\cdot d\vec{F}\ .$$

The volume parts of $\mathcal{H}^{(2)}$ and $\mathcal{H}^{(4)}$ depend on fluctuations of Ψ. All other contributions to the energy $\mathcal{H}$ are fixed by the boundary conditions.

VI. ENERGY-MOMENTUM TENSOR OF A FLUID

One may ask also for the other components of the energy-momentum tensor $T_{\mu\nu}$. It is convenient to calculate them if one goes at first a step backwards and uses $\vec{v}, \phi, \alpha, \gamma$ as independent fields. Taking into account that $L = P$ for the physical solutions Ψ, one finds the following expressions:

$$T_{ij} = -(\rho v_i v_j + \delta_{ij} P), \quad i,j = 1, 2, 3 , \tag{19a}$$

the well-known stress tensor for a perfect fluid;

$$T_{i4} = -\rho v_i , \quad i = 1, 2, 3 , \tag{19b}$$

the mass current density; and

$$T_{4i} = v_i (P + \rho u_0 + \tfrac{1}{2}\rho \vec{v}^2), \quad i = 1, 2, 3, \tag{19c}$$

the enthalpy current density. T_{44} is the energy density as already discussed.

These formulas coincide with the nonrelativistic limit of the relativistic energy-momentum tensor (cf. Ref. 24). Here we have found them within the framework of a nonrelativistic field theory.

VII. Ψ-FIELD EQUATION FOR VISCOUS FLUIDS

Now the viscous term in the Navier-Stokes equation (2) will be taken into account. This will be achieved by proper extension of the equation of motion (10) for Ψ. In contrast, the representation (5) [as well as (8) together with (6) and (11)] of the velocity field $\vec{v}$ through the current of the basic complex field Ψ shall be retained.

In order to find proper equations of motion for Ψ including viscosity effects, I first express the corresponding term $\nu\Delta\vec{v}$ in Eq. (2) in terms of Ψ. It is

$$\Delta\vec{v} = \Delta(\vec{j} - \text{grad}\phi) = \Delta\vec{j} - \text{grad}\,\text{div}\,\vec{j} = -\text{rot}(\text{rot}\,\vec{j}).$$

The dissipation is determined by the vorticity of $\vec{j}$. For the components we have (with $\gamma_\mu \equiv \partial_\mu \gamma$, etc.)

$$\begin{aligned}\nu\Delta v_\lambda &= 2\nu\partial_\mu(\gamma_\mu \alpha_\lambda - \gamma_\lambda \alpha_\mu) \\ &= 2\nu(\gamma_{\mu\mu}\alpha_\lambda - \alpha_{\mu\mu}\gamma_\lambda + \gamma_\mu \alpha_{\lambda\mu} - \alpha_\mu \gamma_{\lambda\mu}) .\end{aligned} \tag{20}$$

Applying $(\partial_t + \vec{v}\cdot\text{grad})$ to v_λ given by Eq. (5), we can use again the above-mentioned commutation relation $[D_t, \partial_\lambda] = -v_{\mu|\lambda}\partial_\mu$, but we have to give up Eq. (10), which holds for inviscid fluids only. Formally we find

$$\begin{aligned}\partial_t v_\lambda + \vec{v}\cdot\text{grad}\, v_\lambda = &-\partial_\lambda(\tfrac{1}{2}\vec{v}^2 + \partial_t \phi + \vec{v}\cdot\text{grad}\phi) \\ &+ (\alpha_\lambda D_t \gamma - \alpha \partial_\lambda D_t \gamma \\ &- \gamma_\lambda D_t \alpha + \gamma \partial_\lambda D_t \alpha) .\end{aligned} \tag{21}$$

In view of Eq. (20) it is suggestive to add the contributions $\propto \alpha$ and $\propto \gamma$ to the pressure part ∂_λ and to *redefine* the observable "pressure":

$$(1/\rho)P = \tfrac{1}{2}\vec{v}^2 + D_t \phi + \alpha D_t \gamma - \gamma D_t \alpha - u_0 . \tag{22}$$

This is not only compatible with the previous definition (14) (since $D_t\gamma = D_t \alpha = 0$ for inviscid fluids), but the interpretation of L as the pressure of the perfect fluid still holds: $P = L$.

Comparison of Eq. (20) with the nonpressure part of Eq. (21) shows that $\vec{v}$ satisfies the Navier-Stokes equation, if for each $\lambda = 1, 2, 3$,

$$\begin{aligned}\alpha_\lambda D_t \gamma - \gamma_\lambda D_t \alpha = \nu(&\gamma_{\mu\mu}\alpha_\lambda - \alpha_{\mu\mu}\gamma_\lambda \\ &+ \gamma_\mu \alpha_{\lambda\mu} - \alpha_\mu \gamma_{\lambda\mu}) .\end{aligned} \tag{23}$$

There is some freedom to choose $D_t\gamma$ and $D_t\alpha$. If only Eqs. (23) are satisfied, the order parameter field Ψ represents a velocity field, which solves the equations of an incompressible but viscous fluid.

If one sticks to the interpretation of α as a label field that is conserved under D_t also in a viscous fluid, one has $D_t\alpha = 0$; Eqs. (23) then determine $D_t\gamma$. However, it seems reasonable to treat α and γ in a more symmetrical manner. Two possibilities are then suggestive: The first and third terms on the right-hand side determine $D_t\gamma$ or alternatively the first and fourth ones. $D_t\alpha$ is fixed by the corresponding other terms.

(i) The following six equations have to be solved simultaneously.

$$\left.\begin{aligned}\alpha_\lambda D_t \gamma &= \nu(\alpha_\lambda \gamma_{\mu\mu} + \alpha_{\lambda\mu}\gamma_\mu) , \\ \gamma_\lambda D_t \alpha &= \nu(\gamma_\lambda \alpha_{\mu\mu} + \gamma_{\lambda\mu}\alpha_\mu) ,\end{aligned}\right\} \quad \lambda = 1, 2, 3 . \tag{24}$$

If α or γ do not depend on certain x_λ, the equations for those λ are satisfied automatically. Generally, from Eqs. (24) the following equations can be deduced:

$$\begin{aligned}D_t\gamma &= \nu\Delta\gamma + \nu\,\text{grad}\gamma\cdot\text{grad}\ln|\text{grad}\alpha| , \\ D_t\alpha &= \nu\Delta\alpha + \nu\,\text{grad}\alpha\cdot\text{grad}\ln|\text{grad}\gamma| .\end{aligned} \tag{25}$$

Among the solutions of Eqs. (25) are, in particular, those that satisfy Eqs. (24).

(ii) The other symmetric choice is

$$\left.\begin{aligned}\alpha_\lambda D_t \gamma &= \nu(\alpha_\lambda \gamma_{\mu\mu} - \alpha_\mu \gamma_{\lambda\mu}) , \\ \gamma_\lambda D_t \alpha &= \nu(\gamma_\lambda \alpha_{\mu\mu} - \gamma_\mu \alpha_{\lambda\mu}) ,\end{aligned}\right\} \quad \lambda = 1, 2, 3 . \tag{26}$$

As before, one can deduce as a consequence of these six equations that

$$\begin{aligned}D_t\gamma &= \nu\left(\gamma_{\mu\mu} - \frac{\alpha_\lambda\alpha_\mu}{\alpha_\nu\alpha_\nu}\gamma_{\lambda\mu}\right) \equiv \nu Q_\alpha \gamma , \\ D_t\alpha &= \nu\left(\alpha_{\mu\mu} - \frac{\gamma_\lambda\gamma_\mu}{\gamma_\nu\gamma_\nu}\alpha_{\lambda\mu}\right) \equiv \nu Q_\gamma \alpha .\end{aligned} \tag{27}$$

Q_α and Q_γ are orthogonal Laplace operators with respect to the α and γ surfaces:

$$Q_\alpha = \left(\delta_{\lambda\mu} - \frac{\alpha_\lambda\alpha_\mu}{\alpha_\nu\alpha_\nu}\right)\partial_\lambda\partial_\mu .$$

In both choices the linear term is $\nu\Delta\Psi$. The mode coupling via nonlinear viscous terms is expected to be of minor importance as compared with the kinetic coupling $\vec{v}_j(\Psi^*, \Psi)\cdot\text{grad}\,\Psi$, which is known to be strong. Thus the following equation might be sufficient to describe a turbulent fluid:

$$\partial_t \Psi + (\vec{v}_b\cdot\text{grad} - \nu\Delta)\Psi + \vec{v}_j(\Psi^*, \Psi)\cdot\text{grad}\Psi = 0 . \tag{28}$$

On the other hand, if Eq. (28) is used, there will be additional terms in the Navier-Stokes equation, which cannot be expressed by $\vec{v}$. These are the last two terms of Eq. (20) with opposite sign.

Remark: If one uses the nonsymmetric-Lagrange density following from Eq. (3), one again comes up with the fundamental equations (23).

VIII. A TURBULENCE-MODEL HAMILTONIAN

It is tempting to propose an additional term of second order in Ψ and $\propto\nu$ in the action principle (16) in order to derive the approximate field equation (28). The easiest guess would be $-\nu|\mathrm{grad}\Psi|^2$. But this leads to

$$D_t\Psi = i\nu\Delta\Psi$$

instead of $\nu\Delta\Psi$. Another guess starts from the idea that the differences between α and γ are responsible for the vorticity [cf. Eq. (9)]. One adds $-\nu[\mathrm{grad}(\gamma-\alpha)]^2$ in Eq. (16) and finds

$$D_t\alpha = D_t\gamma = \nu\Delta(\gamma-\alpha) .$$

Finally one might subtract the first from the second ansatz:

$$D_t\Psi = \nu\Delta\Psi^* . \tag{29}$$

These equations of motion are represented by the Lagrange density

$$(1/\rho)L = \alpha\partial_t\gamma - \gamma\partial_t\alpha - \tfrac{1}{2}\vec{v}^2 + 2\nu\,\mathrm{grad}\gamma\cdot\mathrm{grad}\alpha \tag{30}$$

together with the Hamiltonian density[25]

$$H = \tfrac{1}{2}\rho\vec{v}^2 - 2\rho\nu\,\mathrm{grad}\gamma\cdot\mathrm{grad}\alpha$$
$$= \tfrac{1}{2}\rho\vec{v}^2 + \tfrac{1}{2}\rho\nu i[(\mathrm{grad}\Psi)^2 - (\mathrm{grad}\Psi^*)^2] . \tag{31}$$

The model Hamiltonian (31) is expected to describe a fluid which does not really dissipate energy. The dissipation of one field component $\propto\nu$ is fed back in the other one $\propto(-\nu)$. This might be of minor importance in the universal part of the spectral function in fully developed turbulence.

IX. EXACT Ψ-FIELD SOLUTION FOR LAMINAR SHEAR FLOW

Several questions have still to be answered. What are convenient initial conditions for the label distribution α? What are the proper boundary conditions for Ψ and ϕ? What is the steady-state distribution of Ψ according to the full or approximate equations of motion *including* viscosity? Does such a steady state exist at all?

Especially the last question deserves attention. Namely, if α is something like a label field, it will show permanent motion, even if the velocity field is stationary. To study this I solve the equations of motion (23) for γ and α in the simple case of steady laminar shear flow between infinite parallel plates.

The velocity field has the following special form: $\vec{v} = (u(y), 0, 0)$, with $u(-b)=0$, $u(+b)=u_s$, and $2b$ the distance between the plates. Neglecting viscosity for a moment, Eqs. (10) for $\Psi=\gamma+i\alpha$ are

$$\partial_t\alpha + u(y)\partial_x\alpha = 0, \quad \partial_t\gamma + u(y)\partial_x\gamma = 0.$$

Both are solved by arbitrary functions of $x-u(y)t$. Imagining α as a label distribution, a linear function seems to be a good choice. Therefore I use the ansatz

$$\alpha(x,y,t) = x - u(y)t.$$

At $t=0$ the labeling is $\alpha=x$; i.e., along each flow filament the fluid elements are uniquely characterized by their position. Different filaments are labeled in the *same* manner, independent of y, z. The labels move according to the filament's position with $u(y)$.

We cannot take γ also as linear in the argument $x-ut$, as only *different* α, γ can describe vortices. Therefore we try the simple ansatz $\gamma(y)$, i.e., a constant with respect to $x-ut$.

We use now Eq. (5). Its x component is $u(y) = -\partial_x\phi + \gamma(y)$, thus

$$\phi = x[\gamma(y) - u(y)] + g(y,t).$$

Insert this in the y component equation. It is this equation that actually rules out choices other than $\gamma=\gamma(y)$; i.e., γ follows necessarily once $\alpha(x,y,t)$ has been chosen.

$$0 = -x(\gamma'-u') - \partial_y g + \gamma(-tu') - (x-ut)\gamma' .$$

The coefficient of x has to vanish; therefore $2\gamma'-u'=0$, which yields $\gamma=\frac{1}{2}u(y)+c$. The rest of the y equation gives $g(y,t) = -ctu + f(t)$.

Finally, from the pressure equation we have

$$(1/\rho)P = \tfrac{1}{2}\vec{v}^2 + \partial_t\phi + \vec{v}\cdot\mathrm{grad}\phi - u_0 = d_t f(t) - u_0.$$

This can be satisfied only if P does not depend on space. Because of stationarity P does not depend on t either, so $f=[(1/\rho)P+u_0]t$.

To sum up: the relevant fields for steady laminar shear flow are

$$\alpha = x - u(y)t, \quad \gamma = \tfrac{1}{2}u(y) + c,$$
$$\phi = -\tfrac{1}{2}xu(y) + [(1/\rho)P + u_0]t + c[x - u(y)t], \quad P = P_0. \tag{32}$$

Only $u(y)$ still remains undetermined. To determine it, we have to take care of viscosity, i.e., of the full Eqs. (23). If we use the solution (32), we find: If $\lambda=z$, Eq. (23) is satisfied identically; if $\lambda=y$, the right-hand side adds up to 0, as does the left; only if $\lambda=x$, is there a nontrivial condition, $0 = \nu\gamma''(y)$. Because of viscosity it is $\gamma'' \propto u'' = 0$; i.e., ν determines the linearity of the velocity profile,

$$u(y) = \tfrac{1}{2}u_s(y/b + 1) \ . \tag{33}$$

This completes an exact solution. As expected, it explicitly depends on time t.[26] The solution's vorticity is

$$\begin{aligned}\mathrm{rot}\,\vec{v} &= 2\,\mathrm{grad}\,\gamma \times \mathrm{grad}\,\alpha \\ &= (0,\ 0,\ -u') \\ &= (0,\ 0,\ -u_s/2b).\end{aligned}$$

The Ψ-field current reads

$$\vec{j} = (\tfrac{1}{2}u + c,\ -\tfrac{1}{2}xu' - ctu',\ 0).$$

$$\mathrm{grad}\,\phi = (-\tfrac{1}{2}u + c,\ -\tfrac{1}{2}xu' - ctu',\ 0).$$

As $\mathrm{div}\,\vec{j} = 0$ we conclude that $\phi_j = 0$ and $\phi = \phi_b$.

Once $\alpha(x, y, t)$ had been chosen everything else could be deduced. Another "gauge" of the label field α at $t = 0$ leads to physically equivalent solutions.

X. INTERACTION IN FULLY DEVELOPED TURBULENCE

While at the onset of turbulence the boundaries are decisive—for the form of the instable mode as well as for the laminar background profile—in fully developed turbulence there is a universal region on the length scale in which neither boundaries nor the form of the energy providing laminar background is important. In this universal region turbulence is considered to be homogeneous and isotropic.

To treat this regime we may use $\exp(i\vec{k}\cdot\vec{r})$ as eigenfunctions of the Laplace equation (7). Define

$$\Psi(\vec{r}, t) = \sum_k e^{i\vec{k}\cdot\vec{r}} a_k(t),$$

and find as Fourier representation of the other fields

$$\vec{j}_k = \sum_q \vec{q}\, a^*_{q-k/2} a_{q+k/2} \ ,$$

$$-k^2\phi_k = i\sum_q (\vec{k}\cdot\vec{q})\, a^*_{q-k/2} a_{q+k/2} \ .$$

The velocity $\vec{v}_j(\Psi^*, \Psi)$ of Eq. (12), which is of second order in the field, is just the transverse part of the current:

$$\vec{v}_j(k) = \sum_q [\vec{q} - \vec{k}^0(\vec{k}^0\cdot\vec{q})]\, a^*_{q-k/2} a_{q+k/2} \ .$$

Homogeneity and isotropy of the turbulence furthermore means that Ψ represents the eddies without any laminar additions. Therefore the eddy interaction of kinetic type is $\mathcal{H}^{(4)}$ of Eq. (18). Its $\vec{k}$-space density distribution

$$H_k^{(4)} = \tfrac{1}{2}\rho V \vec{v}_j(k)\cdot\vec{v}_j{}^*(k)$$

is a transverse-current–transverse-current interaction.

$$\mathcal{H}^{(4)} = \tfrac{1}{2}\rho V \sum_{k,p,q} [\vec{p}\cdot\vec{q} - (\vec{p}\cdot\vec{k}^0)(\vec{k}^0\cdot\vec{q})]\, a^*_{p-k} a^*_{q+k} a_{p+k} a_{q-k} \ . \tag{34}$$

The interaction matrix element reads

$$\begin{aligned} &U_4(k, p, q) = p_i P_{ij}(\vec{k}^0) q_j = \vec{p}\cdot\vec{q} - (\vec{p}\cdot\vec{k}^0)(\vec{k}^0\cdot\vec{q}) \ , \\ &P_{ij} = P_{ji} = \delta_{ij} - k_i^0 k_j^0 , \ \text{transverse projector.}\end{aligned} \tag{35}$$

Finally the viscous part of the Hamiltonian (31) is written

$$\mathcal{H}^{(\nu)} = \tfrac{1}{2}\rho V \sum_k \nu k^2 i (a_k a_{-k} - a_k^* a_{-k}^*) \ . \tag{36}$$

XI. APPROXIMATE Ψ BEHAVIOR AT THE ONSET OF TURBULENCE

Clearly an exact treatment of the Ψ equations (23) gives the same physical results as exact solutions of the Navier-Stokes equation (2). But if an approximate equation for a turbulence order parameter is needed, in view of the presented results it seems more natural to write down a Landau equation for the amplitude $\psi(t)$ of a spatially fixed Ψ mode, rather than for the velocity amplitude $A(t)$ [cf. Eq. (1)]:

$$\frac{d\psi}{dt} = a\psi - b\psi^3 \ . \tag{37}$$

In the linear regime Eqs. (37) and (1) are equivalent, $a = \gamma_1/2 \propto \mathrm{Re} - \mathrm{Re}_c$. They differ in the nonlinear parts. $b\psi^3$ would correspond to a term $\propto A^2$ in Eq. (1).

Consequences of Eq. (37) are: (i) The steady velocity amplitude A_0 of the new mode which appears at Re_c increases $\propto \psi_0^2 = a/b \propto \mathrm{Re} - \mathrm{Re}_c$, rather than $\propto (\mathrm{Re} - \mathrm{Re}_c)^{1/2}$ according to Landau's suggestion (1).

(ii) The transient behavior of the normalized velocity amplitude $\psi^2/\psi_0^2 = A/A_0$ as a function of normalized time $\tau = \gamma_1 t$ is

$$\psi^2/\psi_0^2 = e^{\sigma\tau}/(1 + e^{\sigma\tau}) \ . \tag{38}$$

Using the Landau equation (1) one derives $\psi^2/\psi_0^2 = e^{\tau}/(3 + e^{2\tau})^{1/2}$ instead of Eq. (38) (see, for example, Ref. 27). Figure 1 shows a comparison of both theoretical functions as well as experimental points, gained by Donnelly and Schwarz[27] from ion current measurements at the Taylor in-

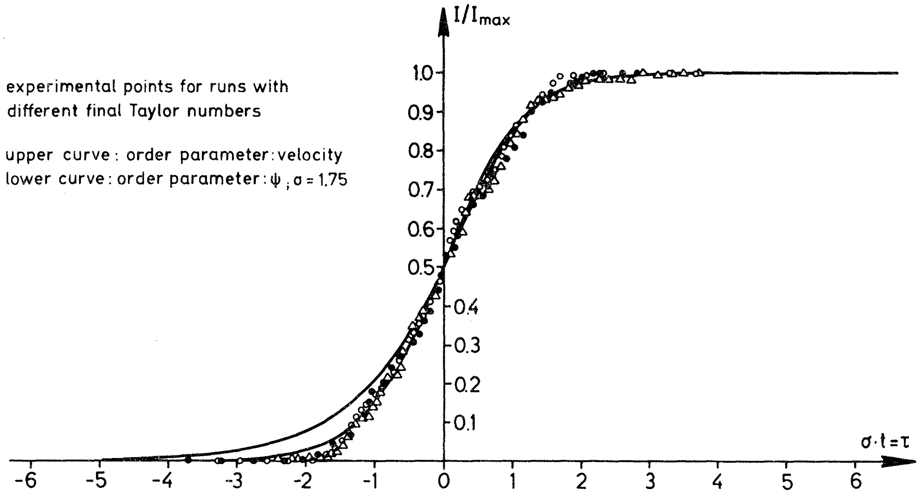

FIG. 1. Normalized ion current ∝ velocity amplitude vs normalized time. The Taylor number of the shear flow between concentric cylinders is suddenly raised above the critical one. Experimental data from Ref. 27.

stability. σ is chosen to be 1.75 instead of 1 in order to have good agreement between both theoretical curves and experiments if $\tau \gtrsim 0$. γ_1 has also been fitted (by Donnelly and Schwarz) to have agreement in this regime.

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