Fluctuation-dissipation theorems for classical processes

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For certain types of classical processes a fluctuation-dissipation theorem (FDT) holds. The validity of such a theorem is preserved in each order of mass renormalization in the perturbation scheme recently developed by Martin, Siggia, and Rose (MSR). As a consequence, whenever the response function can be expressed in terms of the two-point correlation function via a FDT, the perturbation scheme of MSR simplifies considerably. Especially, it reduces to the scheme constructed by Kawasaki for random processes obeying detailed balance which are (i) linearly damped and (ii) linearly driven by Gaussian white noise.

I. INTRODUCTION

Nonlinear classical processes have recently met with increasing interest. Such processes occur, e.g., in the laser, in turbulent liquids, and in near-critical systems. Except for a few fortunate cases, however, the corresponding nonlinear field equations cannot be solved analytically or even numerically. In general, one has to resort to perturbation expansions for the physically relevant correlation functions of the field considered. Moreover, it was only quite recently that a formalism was developed which allows one to carry out such a perturbation expansion, including renormalization effects, in a systematic manner.[1]

Although the formalism of Martin, Siggia, and Rose (MSR) in many ways resembles Schwinger's, Feynman's, and Matsubara's perturbation theories for quantum systems in equilibrium, it is usually more difficult to handle than its quantum counterpart for the following reason. Both in the quantum-mechanical and the classical cases one has to calculate simultaneously the two-point correlation function and the response function of the field characterizing the linear response of the field to an external force. For quantum systems with detailed balance these two functions are linearly related to one another by the fluctuation-dissipation theorem (FDT). It is precisely this theorem which has rendered the calculation of correlation functions for quantum systems in thermal equilibrium a conceptually and technically simple affair, at least for such cases where the renormalized perturbation theory needs to be pushed to low orders only. In the classical case, however, a FDT does not, in general, exist. Therefore, the formalism of MSR must usually be worked out to generate expressions for both the two-point correlation function and the response function. There are, however, classical processes for which a FDT does indeed hold. Three classes of such processes will be discussed here. Two of these are random processes characterized by detailed balance and by the linear appearance of δ-correlated Gaussian forces in the stochastic field equation. The additional condition of the field equation not containing reversible terms defines the first class (class A).[2,3] The second class (class B) requires the irreversible terms in the field equation to be linear in the field. Finally, class C is constituted by systems in thermal equilibrium for which the field equation can be derived from a Hamiltonian.[3]

For classical processes of all three classes, the two "Dyson equations" of MSR for the two-point correlation function and the response function are compatible with the respective FDTs in each order of mass renormalization. Therefore one of these equations can be replaced by the FDT. The perturbative construction of the correlation function and the response function then becomes just as easy a task as it is for quantum systems in thermal equilibrium.

For class B the formalism of MSR is equivalent to the diagrammatic perturbation scheme of Kawasaki.[4] It is interesting to note that processes of this class have Gaussians as stationary probability distributions for the field.

In Sec. II we present the FDTs mentioned. Sections III and IV explain the formalism of MSR for stochastic field equations needed for treating the classes A and B. Sections V–VII demonstrate the FDTs pertaining to the three classes to be preserved in each order of mass renormalization. Section VI also shows the equivalence of the perturbation schemes of MSR and Kawasaki for class B.

II. FLUCTUATION-DISSIPATION THEOREMS

We first consider a random process characterized by the Langevin equations

$$\frac{d\psi_i(t)}{dt} = F_i(\psi) + f_i(t). \tag{2.1}$$

The label i of the random variables ψ_i may be discrete or continuous. Without loss of generality we assume the ψ_i to be even or odd under time reversal,

$$\psi_i \to \tilde{\psi}_i = \epsilon_i \psi_i, \quad \epsilon_i = \pm 1, \quad \text{for } t \to -t. \tag{2.2}$$

We will, for brevity, call the set $\psi = \{\psi_i\}$ a random field. The drift vector $F_i(\psi)$ may be a nonlinear functional of the field and represents a nonrandom force on $\psi_i(t)$. $f_i(t)$ is a stochastic force assumed to be δ correlated, Gaussian, and independent of ψ. Especially,

$$\langle f_i(t) f_j(t') \rangle = 2D_{ij}\,\delta(t - t'). \tag{2.3}$$

The symmetric diffusion matrix D_{ij} is assumed positive.

It is well known[5] that the Langevin equations (2.1) are equivalent to the following Fokker-Planck equation for the probability density $P(\psi, t)$ of the random field:

$$\frac{dP(\psi, t)}{dt} = L(\psi) P(\psi, t), \tag{2.4}$$

$$L(\psi) = -\frac{\partial}{\partial \psi_{\bar{i}}} F_{\bar{i}}(\psi) + \frac{\partial^2}{\partial \psi_{\bar{i}} \partial \psi_{\bar{j}}} D_{\bar{i}\bar{j}}.$$

Here and later on barred indices are to be summed (or integrated) over.

Graham[3] has shown that the random processes characterized by Eqs. (2.1)–(2.3) obey detailed balance if the drift vector F_i and the diffusion matrix D_{ij} fulfill the following "potential conditions":

$$\begin{aligned} &D_{ij} = \epsilon_i \epsilon_j D_{ij}, \\ &\frac{\partial}{\partial \psi_j} D^{-1}_{i\bar{i}} F^{\mathrm{irr}}_{\bar{i}}(\psi) = \frac{\partial}{\partial \psi_i} D^{-1}_{j\bar{i}} F^{\mathrm{irr}}_{\bar{i}}(\psi), \\ &\frac{\partial}{\partial \psi_{\bar{i}}} F^{\mathrm{rev}}_{\bar{i}}(\psi) + F^{\mathrm{rev}}_{\bar{i}}(\psi) D^{-1}_{\bar{i}\bar{i}} F^{\mathrm{irr}}_{\bar{i}}(\psi) = 0, \end{aligned} \tag{2.5}$$

where the reversible and irreversible parts of the drift are defined as

$$\begin{aligned} F^{\mathrm{rev}}_i(\psi) &= \tfrac{1}{2}[F_i(\psi) - \epsilon_i F_i(\tilde{\psi})], \\ F^{\mathrm{irr}}_i(\psi) &= \tfrac{1}{2}[F_i(\psi) + \epsilon_i F_i(\tilde{\psi})]. \end{aligned} \tag{2.6}$$

If detailed balance holds, the stationary probability density reads[3]

$$P_{\mathrm{st}}(\psi) = \mathfrak{N} e^{-\phi} \tag{2.7}$$

where $\mathfrak{N}$ is a normalization constant and ϕ a "potential" to be determined by its gradient in ψ space,

$$\frac{\partial \phi}{\partial \psi_i} = -\frac{\partial}{\partial \psi_i} \ln P_{\mathrm{st}} = -D^{-1}_{i\bar{j}} F^{\mathrm{irr}}_{\bar{j}}. \tag{2.8}$$

We will be interested in the stationary two-point correlation function

$$\begin{aligned} C_{ij}(t - t') &= \langle \psi_i(t) \psi_j(t') \rangle - \langle \psi_i \rangle \langle \psi_j \rangle \\ &= \int d\psi\, \psi_i\, e^{L(t-t')} \psi_j P_{\mathrm{st}} \\ &\quad - \left(\int d\psi\, \psi_i P_{\mathrm{st}} \right) \left(\int d\psi\, \psi_j P_{\mathrm{st}} \right) \end{aligned} \tag{2.9}$$

and the response function $R_{ij}(t - t')$ describing the linear response of the variable ψ_i at time t to an infinitesimal external force $\delta F_i(t)$ switched on at some instant t_0 as

$$\begin{aligned} \delta \langle \psi_i(t) \rangle &= \langle \psi_i(t) \rangle - \langle \psi_i \rangle \\ &= \int_{t_0}^{t} dt'\, R_{i\bar{j}}(t - t') \delta F_{\bar{j}}(t'). \end{aligned} \tag{2.10}$$

Let us calculate the response function $R_{ij}(t - t')$ for the stationary case. To this end we observe that the external force $\delta F_i(t)$ enters linearly on the right-hand side in the Langevin equations (2.1) and gives rise to an additional term

$$\delta L(t) = -\delta F_{\bar{j}}(t)\, \partial / \partial \psi_{\bar{j}} \tag{2.11}$$

in the Fokker-Planck differential operator. Following the switch-on of $\delta F_i(t)$ at time t_0, the probability density $P(\psi, t)$ deviates from the stationary distribution P_{st}. It may then be formally expanded in powers of $\delta L(t)$ as[6]

$$\begin{aligned} P(\psi, t) &= \left(\exp \int_{t_0}^{t} dt' \left[L + \delta L(t') \right] \right)_+ P_{\mathrm{st}} \\ &= \left(e^{L(t-t_0)} + \int_{t_0}^{t} dt' e^{L(t-t')} \delta L(t') e^{L(t'-t_0)} + \cdots \right) P_{\mathrm{st}} \\ &= P_{\mathrm{st}} + \int_{t_0}^{t} dt'\, e^{L(t-t')} \delta L(t') P_{\mathrm{st}} + \cdots. \end{aligned} \tag{2.12}$$

The linear response $\delta \langle \psi_i(t) \rangle$ therefore reads as shown in Eq. (2.10) with

$$R_{ij}(t - t') = \Theta(t - t') \int d\psi\, \psi_i e^{L(t-t')} \left(-\frac{\partial}{\partial \psi_j} \right) P_{\mathrm{st}}. \tag{2.13}$$

By introducing the operator

$$\hat{\psi}_i = -\partial / \partial \psi_i, \tag{2.14}$$

the response function R_{ij} can be written in the form of a correlation function

$$\begin{aligned} R_{ij}(t - t') &= \Theta(t - t') \langle \psi_i(t) \hat{\psi}_j(t') \rangle \\ &= \langle (\psi_i(t) \hat{\psi}_j(t'))_+ \rangle. \end{aligned} \tag{2.15}$$

It is interesting to note that the operators $\hat{\psi}_i$ are conjugate to the field ψ_i in the sense of the commutation relations

$$[\psi_i, \hat{\psi}_j] = \delta_{ij} \quad \text{for equal times.} \tag{2.16}$$

By inserting Eq. (2.8) in Eq. (2.13), we get the following equivalent expression for R_{ij}:

$$R_{ij}(t-t') = -\Theta(t-t')\langle \psi_i(t) F_{\bar{k}}^{\mathrm{irr}}(\psi(t'))\rangle D^{-1}_{\bar{k}j}. \quad (2.17)$$

This is sometimes referred to as a "generalized fluctuation-dissipation theorem." Nonetheless, it is of little use, as it stands, since it does not relate the response function R_{ij} to the two-point correlation function C_{ij}. It is easy to see, however, that there are two classes of random processes for which (2.17) indeed implies a FDT.

The first of these classes is made up by purely irreversible random processes (class A), that is for

$$F_i(\psi) = F_i^{\mathrm{irr}}(\psi), \quad F_i^{\mathrm{rev}}(\psi) = 0. \quad (2.18)$$

In this case Eq. (2.17) can be rewritten, upon using the time-reversal properties of ψ and F and the Langevin equations (2.1),

$$\begin{aligned} R_{ij}(t-t') &= -\epsilon_i\epsilon_{\bar{k}}\langle F_{\bar{k}}^{\mathrm{irr}}(t)\psi_i(t')\rangle D^{-1}_{\bar{k}j}\Theta(t-t') \\ &= -\epsilon_i\epsilon_{\bar{k}}\left\langle\left(\frac{d\psi_{\bar{k}}(t)}{dt} - f_{\bar{k}}(t)\right)\psi_i(t')\right\rangle D^{-1}_{\bar{k}j}\Theta(t-t') \\ &= -\epsilon_i\epsilon_{\bar{k}}\Theta(t-t')\frac{\partial}{\partial t}C_{\bar{k}i}(t-t')D^{-1}_{\bar{k}j} \\ &\quad + \epsilon_i\epsilon_{\bar{k}}\Theta(t-t')\langle f_{\bar{k}}(t)\psi_i(t')\rangle D^{-1}_{\bar{k}j}. \end{aligned}$$

The second term vanishes, for $t<t'$ because of the step function and for $t>t'$ since the action of the δ-correlated force f_k on ψ_i is a retarded one. We therefore arrive at the FDT[2,3]

$$R_{ij}(t-t') = -\Theta(t-t')\frac{\partial}{\partial t}C_{i\bar{k}}(t-t')D^{-1}_{\bar{k}j}$$

or (2.19)

$$\frac{\partial}{\partial t}C_{ij}(t-t') = -R_{i\bar{k}}(t-t')D_{\bar{k}j} + D_{i\bar{k}}R_{j\bar{k}}(t'-t)$$

for purely irreversible random processes with detailed balance. The second version of the FDT in (2.19) follows from the first one with $C_{ij}(t) = \epsilon_i\epsilon_j C_{ij}(-t)$.

Let us return to the expression (2.17) and observe that it obviously leads to a FDT for yet another class of random processes (class B). If the irreversible drift $F_i^{\mathrm{irr}}(\psi)$ is linear in the field,

$$F_i^{\mathrm{irr}}(\psi) = U_{i\bar{k}}\psi_{\bar{k}}, \quad (2.20)$$

Eq. (2.17) immediately becomes a FDT,

$$R_{ij}(t-t') = -\Theta(t-t')C_{i\bar{l}}(t-t')U_{\bar{k}\,\bar{l}}D^{-1}_{\bar{k}j}. \quad (2.21)$$

Note that the potential conditions (2.5) then imply the symmetry

$$D^{-1}_{j\bar{k}}U_{\bar{k}l} = D^{-1}_{l\bar{k}}U_{\bar{k}j} \quad (2.22)$$

and, according to Eq. (2.8), the stationary probability density $P_{\mathrm{st}}(\psi)$ to be Gaussian,

$$P_{\mathrm{st}}(\psi) = \mathfrak{N}\exp[-\tfrac{1}{2}\psi_{\bar{j}}C^{-1}_{\bar{j}\bar{k}}(0)\psi_{\bar{k}}] \quad (2.23)$$

with the stationary second moments

$$C_{ij}(0) = \langle\psi_i(0)\psi_j(0)\rangle = -U^{-1}_{i\bar{k}}D_{\bar{k}j}. \quad (2.24)$$

By using Eq. (2.24) we may write the FDT (2.21) as

$$R_{ij}(t-t') = \Theta(t-t')C_{i\bar{k}}(t-t')C^{-1}_{\bar{k}j}(0)$$

or (2.25)

$$C_{ij}(t-t') = R_{i\bar{k}}(t-t')C_{\bar{k}j}(0) + C_{i\bar{k}}(0)R_{j\bar{k}}(t'-t)$$

for random processes with linear damping and detailed balance. Random processes of class B have been considered by Kawasaki.[4]

Let us finally turn to class C, that is to conservative systems in thermal equilibrium. For such systems the set of $2n$ variables ψ_i contains n pairs of conjugate coordinates and momenta q_ν, p_ν whose equations of motion are

$$dq_\nu/dt = \partial H/\partial p_\nu, \quad dp_\nu/dt = -\partial H/\partial q_\nu. \quad (2.26)$$

The Hamiltonian H also determines the equilibrium phase space density as

$$P_{\mathrm{st}}(q,p) = \mathfrak{N}e^{-\beta H(q,p)}. \quad (2.27)$$

For our purposes it is slightly more convenient to use the variables ψ_i in the following arrangement:

$$\{\psi_1\psi_2\cdots\psi_n\psi_{n+1}\psi_{n+2}\cdots\psi_{2n}\} = \{q_1\cdots q_n p_1\cdots p_n\}, \quad (2.28)$$

that is

$$\psi_i = \begin{cases} q_\nu & \text{for } i=\nu\leq n \\ p_\nu & \text{for } i=n+\nu>n. \end{cases}$$

Then Hamilton's equations (2.26) can be written in the more compact form

$$\frac{d\psi_i}{dt} = D_{i\bar{j}}\frac{\partial H}{\partial\psi_{\bar{j}}}, \quad D_{ij} = -D_{ji} = \delta_{i+n,j} - \delta_{i-n,j}. \quad (2.29)$$

The matrix D_{ij} obeys

$$D_{i\bar{j}}D_{\bar{j}k} = -\delta_{ik}. \quad (2.30)$$

We now add time-dependent external forces $\delta F_i(t)$ to the equations of motion (2.29)

$$\frac{d\psi_i}{dt} = D_{i\bar{j}}\frac{\partial H}{\partial\psi_{\bar{j}}} + \delta F_i(t) = D_{i\bar{j}}\frac{\partial}{\partial\psi_{\bar{j}}}[H+\delta H(t)] \quad (2.31)$$

with

$$\delta H(t) = -\psi_{\bar{i}}D_{\bar{i}\,\bar{j}}\delta F_{\bar{j}}(t).$$

To determine the linear response $\delta\langle\psi_i(t)\rangle$, we note that the phase-space density P deviates, due to the external perturbation, from the canonical

form (2.27) according to Liouville's equation. It may be written as

$$P(t)=\left(\exp\int_{t_0}^{t}dt'\,[L+\delta L(t')]\right)_{+}P_{\rm st} \tag{2.32}$$

with the Liouvillian

$$L=\frac{\partial H}{\partial q_{\bar\nu}}\frac{\partial}{\partial p_{\bar\nu}}-\frac{\partial H}{\partial p_{\bar\nu}}\frac{\partial}{\partial q_{\bar\nu}},$$

$$\delta L(t)=\frac{\partial \delta H(t)}{\partial q_{\bar\nu}}\frac{\partial}{\partial p_{\bar\nu}}-\frac{\partial \delta H(t)}{\partial p_{\bar\nu}}\frac{\partial}{\partial q_{\bar\nu}}=-\delta F_{\bar i}(t)\frac{\partial}{\partial \psi_{\bar i}}.$$

By expanding $P(t)$ in terms of the perturbation $\delta L(t)$, we obtain for the response function

$$R_{ij}(t)=\Theta(t)\int d\psi\,\psi_i\,e^{Lt}\left(-\frac{\partial}{\partial\psi_j}\right)P_{\rm st}. \tag{2.33}$$

After using

$$-\frac{\partial}{\partial\psi_i}P_{\rm st}=\beta\frac{\partial H}{\partial\psi_i}P_{\rm st}$$

and Hamilton's equations in the form (2.29), we arrive at the well-known FDT

$$R_{ij}(t)=-\beta\Theta(t)\frac{\partial}{\partial t}C_{i\bar k}(t)D_{\bar k j}$$

or (2.34)

$$\frac{d}{dt}C_{ij}(t)=R_{i\bar k}(t)D_{\bar k j}/\beta-(1/\beta)D_{\bar k i}R_{j\bar k}(-t)$$

for conservative systems in thermal equilibrium. Note the similarity to the FDT (2.19) for class A. Although this similarity is purely formal, its use will prove fruitful below.

III. THE SELF-CONSISTENT PERTURBATION SCHEME OF MSR FOR RANDOM PROCESSES

This section reviews the formalism developed by Martin *et al.*[1] as extended to random processes. We start from the field equation[7]

$$\begin{aligned}\frac{\partial}{\partial t_1}\psi(1)=&U_1(1)+U_2(1\bar2)\psi(\bar2)+U_2'(1\bar2)f(\bar2)\\&+U_3(1\bar2\bar3)\psi(\bar2)\psi(\bar3)+U_3'(1\bar2\bar3)\psi(\bar2)f(\bar3)\\&+U_3''(1\bar2\bar3)f(\bar2)f(\bar3).\end{aligned} \tag{3.1}$$

Here the index $1=\{i_1,t_1\}$ represents the time and all discrete and continuous variables. Barred indices are to be summed or integrated over. The coefficients $U_n(1\cdots n)$ are assumed known functions of their arguments. Especially, the U_n are instantaneous and stationary

$$U_n(1\cdots n)\sim\delta(t_1-t_2)\delta(t_1-t_3)\cdots\delta(t_1-t_n). \tag{3.2}$$

The field equation (3.1) differs from that considered in Ref. 1 by the appearance of the stochastic force field f. For the sake of generality, we have allowed f to enter the field equation nonlinearly and do not, for the time being, make any specific assumption concerning the statistical properties of f, i.e., the behavior of the correlation function $\langle f(1)f(2)\cdots f(n)\rangle$. As with MSR, we introduce a field $\hat\psi(1)$ conjugate to $\psi(1)$ in the sense

$$\begin{aligned}&[\psi(i_1,t),\ \hat\psi(i_2,t)]=\delta_{i_1i_2},\\&[\hat\psi(i_1,t),\hat\psi(i_2,t)]=[\psi(i_1,t),\psi(i_2,t)]=0,\\&[\hat\psi(1),f(2)]=[\psi(1),f(2)]=0.\end{aligned} \tag{3.3}$$

The statistical properties of $\hat\psi$ are determined by requiring

$$\langle\hat\psi(1)Q\rangle=0, \tag{3.4}$$

where Q is an arbitrary functional of ψ, $\hat\psi$, and f. Then $\hat\psi(1)$ must obey the equation of motion

$$\begin{aligned}-\frac{\partial}{\partial t_1}\hat\psi(1)=&U_2(\bar21)\hat\psi(\bar2)+2U_3(\bar2\bar31)\hat\psi(\bar2)\psi(\bar3)\\&+U_3'(\bar21\bar3)\hat\psi(\bar2)f(\bar3).\end{aligned} \tag{3.5}$$

In order to write Eqs. (3.1) and (3.5) in a compact form, we introduce the spinors

$$\Psi(1)=\begin{pmatrix}\psi(1)\\\hat\psi(1)\end{pmatrix},\qquad \Phi(1)=\begin{pmatrix}\psi(1)\\\hat\psi(1)\\f(1)\end{pmatrix} \tag{3.6}$$

and extend our summation convention such that a barred spinor index, e.g., $\bar1$, implies a sum over all spinor components. An underlined index, e.g., $\underline1$, is allowed to refer to the components $\psi,\hat\psi$ of Ψ only, whereas all other indices occurring in the following spinor equations may refer to all components $\psi,\hat\psi,f$ of Φ. The field equations (3.1) and (3.5) may then be combined to the spinor equation

$$\begin{aligned}-\frac{\partial}{\partial t_1}i\sigma(\underline{1\bar2})\Phi(\bar2)=&\gamma_1(\underline1)+\gamma_2(\underline{1\bar2})\Phi(\bar2)\\&+\tfrac12\gamma_3(\underline{1\bar2\bar3})\Phi(\bar2)\Phi(\bar3),\end{aligned} \tag{3.7}$$

with appropriately chosen coefficients $\gamma_n(1\cdots n)$ and the matrix

$$i\sigma(\underline{12})=\begin{pmatrix}0&\delta(12)&0\\-\delta(12)&0&0\end{pmatrix}. \tag{3.8}$$

We remark that the coefficients γ_n can be chosen symmetric in their arguments $1\cdots n$ with the restriction that the underlined index must not refer to f.[8] Note also that only those components of the "bare vertex" $\gamma(\underline{123})$ do not vanish for which one and only one index refers to $\hat\psi$.

We now turn to the calculation of the expectation values $\langle\psi(1)\rangle$, $\langle\psi(1)\psi(2)\rangle$, and $\langle(\psi(1)\hat\psi(2))_+\rangle$. To this end, we introduce the generating functional

$$S(\Lambda)=\langle(\exp[\Lambda(\bar{1})\Phi(\bar{1})])_+\rangle$$
$$=\langle(\exp[\eta(\bar{1})\psi(\bar{1})+\hat{\eta}(\bar{1})\hat{\psi}(\bar{1})+\lambda(\bar{1})f(\bar{1})])_+\rangle \quad (3.9)$$

and the cumulants

$$\langle\langle\Phi(1)\Phi(2)\cdots\Phi(n)\rangle\rangle=\frac{\delta^n \ln S(\Lambda)}{\delta\Lambda(1)\delta\Lambda(2)\cdots\delta\Lambda(n)} \quad (3.10)$$

with the auxiliary parameters

$$\Lambda(1)=\begin{pmatrix}\eta(1)\\ \hat{\eta}(1)\\ \lambda(1)\end{pmatrix}$$

which will, eventually, be set equal to zero. The double brackets indicate that the average is to be performed for $\Lambda\neq 0$, e.g.,

$$\langle\langle\Psi(1)\rangle\rangle=\frac{\langle(\Psi(1)\exp[\Lambda(\bar{1})\Phi(\bar{1})])_+\rangle}{S(\Lambda)}. \quad (3.11)$$

Note that for $\Lambda=0$ the time-ordered two-point cumulants

$$\langle\langle\Psi(1)\Psi(2)\rangle\rangle|_{\Lambda=0}=\langle(\Psi(1)\Psi(2))_+\rangle-\langle\Psi(1)\rangle\langle\Psi(2)\rangle \quad (3.12)$$

have the desired correlation functions $C(12)$ and $R(12)$ as their matrix elements. By using the definition (3.9) of S and the field equation (3.7), we can generate an infinite hierarchy of equations of motion for the cumulants $\langle\langle\Phi(1)\cdots\Phi(n)\rangle\rangle$ of all orders. The first of these equations reads

$$G_{(0)}^{-1}(\underline{1}\bar{2})\langle\langle\Phi(\bar{2})\rangle\rangle=\gamma_1(\underline{1})+\Lambda(\underline{1})+\tfrac{1}{2}\gamma_3(\underline{1}\bar{2}\bar{3})[\langle\langle\Phi(\bar{2})\Phi(\bar{3})\rangle\rangle+\langle\langle\Phi(\bar{2})\rangle\rangle\langle\langle\Phi(\bar{3})\rangle\rangle] \quad (3.13)$$

with

$$G_{(0)}^{-1}(\underline{12})=-\frac{\partial}{\partial t_1}i\sigma(\underline{12})-\gamma_2(\underline{12}). \quad (3.14)$$

The whole hierarchy can be obtained from Eq. (3.13) by differentiating with respect to Λ. For instance, by differentiating just once, we obtain

$$G_{(0)}^{-1}(\underline{1}\bar{2})\langle\langle\Phi(\bar{2})\Phi(3)\rangle\rangle=\delta(\underline{13})+\tfrac{1}{2}\gamma_3(\underline{1}\bar{2}\bar{3})\times[2\langle\langle\Phi(\bar{2})\rangle\rangle\langle\langle\Phi(\bar{3})\Phi(3)\rangle\rangle+\langle\langle\Phi(\bar{2})\Phi(\bar{3})\Phi(3)\rangle\rangle]. \quad (3.15)$$

In order to replace the infinite hierarchy by a finite set of equations in a convenient manner, we introduce the Legendre transform of S as

$$L(\langle\langle\Psi\rangle\rangle,\langle\langle f\rangle\rangle)=\ln S-\Lambda(\bar{1})\langle\langle\Phi(\bar{1})\rangle\rangle \quad (3.16)$$

and, as its derivatives, the renormalized vertices

$$\Gamma_n(12\cdots n)=\frac{\delta^n L}{\delta\langle\langle\Phi(1)\rangle\rangle\delta\langle\langle\Phi(2)\rangle\rangle\cdots\delta\langle\langle\Phi(n)\rangle\rangle}. \quad (3.17)$$

Whenever necessary in the following, we shall write the vertices Γ_n in a more explicit form by displaying the spinor components as, e.g.,

$$\Gamma_2(\Psi(1),f(2))=\frac{\delta^2 L}{\delta\langle\langle\Psi(1)\rangle\rangle\delta\langle\langle f(2)\rangle\rangle}$$

or, even more explicitly, e.g.,

$$\Gamma_2(\psi(1),f(2))=\frac{\delta^2 L}{\delta\langle\langle\psi(1)\rangle\rangle\delta\langle\langle f(2)\rangle\rangle},$$

$$\Gamma_2(\hat{\psi}(1),f(2))=\frac{\delta^2 L}{\delta\langle\langle\hat{\psi}(1)\rangle\rangle\delta\langle\langle f(2)\rangle\rangle}.$$

By using the prescription (3.17), we easily find for the first few vertices

$$\Gamma_1(\psi(1))=-\eta(1),\quad \Gamma_1(\hat{\psi}(1))=-\hat{\eta}(1),\quad \Gamma_1(f(1))=-\lambda(1), \quad (3.18)$$

$$\langle\langle\Phi(1)\Phi(\bar{2})\rangle\rangle\Gamma_2(\bar{2}3)=-\delta(13), \quad (3.19)$$

$$\langle\langle\Phi(1)\Phi(2)\Phi(3)\rangle\rangle=\langle\langle\Phi(1)\Phi(\bar{1})\rangle\rangle\langle\langle\Phi(2)\Phi(\bar{2})\rangle\rangle\times\langle\langle\Phi(3)\Phi(\bar{3})\rangle\rangle\Gamma_3(\bar{1}\bar{2}\bar{3}). \quad (3.20)$$

We may now rewrite the equation of motion (3.15) for the two-point function with the help of Eq. (3.20):

$$G_{(0)}^{-1}(\underline{1}\bar{2})\langle\langle\Phi(\bar{2})\Phi(3)\rangle\rangle=\delta(\underline{13})+\Sigma(\underline{1}\bar{2})\langle\langle\Phi(\bar{2})\Phi(3)\rangle\rangle \quad (3.21)$$

with the "self-energy"

$$\Sigma(\underline{12})=\gamma_3(\underline{12}\bar{3})\langle\langle\Phi(\bar{3})\rangle\rangle+\tfrac{1}{2}\gamma_3(\underline{1}\bar{2}\bar{3})\langle\langle\Phi(\bar{2})\Phi(\bar{4})\rangle\rangle\langle\langle\Phi(\bar{3})\Phi(\bar{5})\rangle\rangle\Gamma_3(\bar{4}\bar{5}2)$$
$$\equiv\Sigma^{\mathrm{HF}}(\underline{12})+\Sigma^{c}(\underline{12}). \quad (3.22)$$

Finally we get an equation for the three-point vertex by combining Eqs. (3.21) and (3.19):

$$G_{(0)}^{-1}(\underline{12})=-\Gamma_2(\underline{12})+\Sigma(\underline{12}) \quad (3.23)$$

and differentiating this with respect to $\langle\!\langle\Phi(3)\rangle\!\rangle$,

$$\Gamma_3(\underline{123})=\frac{\delta\Sigma(\underline{12})}{\delta\langle\!\langle\Phi(3)\rangle\!\rangle}=\gamma_3(\underline{123})+\frac{\delta\Sigma^{c}(\underline{12})}{\delta\langle\!\langle\Phi(3)\rangle\!\rangle}. \quad (3.24)$$

Were it not for the presence of the stochastic forces, we would, up to now, have just reproduced the procedure of MSR, and Eqs. (3.13), (3.21), (3.22), and (3.24) would constitute a complete set of equations for $\langle\!\langle\Phi(1)\rangle\!\rangle$, $\langle\!\langle\Phi(1)\Phi(2)\rangle\!\rangle$, and $\Gamma_3(123)$. In our case, however, these equations involve but do not determine the "pure f vertex" $\Gamma_3(f(1)f(2)f(3))$. Even worse, Eqs. (3.22) and (3.24) contain the term

$$\frac{\delta\Gamma_3(f(1)f(2)f(3))}{\delta\langle\!\langle f(4)\rangle\!\rangle}=\Gamma_4(f(1)f(2)f(3)f(4)), \quad (3.25)$$

that is the pure f vertex of fourth order which,

of course, can also not be calculated from the equations of motion. This lack of information is due to the absence of an equation of motion for the random force $f(1)$. However, we have assumed the statistical properties of $f(1)$, that is, e.g., the pure f cumulants $\langle\langle f(1)\cdots f(n)\rangle\rangle|_{\Lambda=0}$ to be known. It is natural to expect and indeed proven in the Appendix that all pure f vertices can, for $\Lambda=0$, be expressed in terms of the pure f cumulants. The strategy for calculating the wanted correlation functions $\langle\langle \Psi(1)\Psi(2)\rangle\rangle$ in renormalized perturbation theory is therefore obvious. From Eqs. (3.22) and (3.24) one generates either an expansion of Γ_3 in terms of γ_3 (mass renormalization) or of γ_3 in terms of Γ_3 (vertex renormalization). Once such a series is obtained, one sets $\Lambda=0$ and feeds in the values of the pure f vertices constructed from the known pure f cumulants. In the case of mass renormalization, two nonlinear equations for $\langle\langle\psi(1)\psi(2)\rangle\rangle|_{\Lambda=0}$ and $\langle\langle\psi(1)\hat{\psi}(2)\rangle\rangle|_{\Lambda=0}$ result, whereas in vertex renormalization a complete set of equations for these two functions and the vertices $\Gamma_3(123)\neq\Gamma_3(f(1)f(2)f(3))$ is obtained. This procedure is, in general, somewhat more complicated than for nonrandom processes since the pure f vertices are of zero order in the interaction γ_3. So, in order to get Γ_3 correct to nth order in γ_3, one has to iterate Eqs. (3.22) and (3.24) more than n times.

We will, in the rest of this paper, be concerned with random processes of much lesser generality than those admitted by the field equation (3.1). For random processes obeying a fluctuation-dissipation theorem, all results of this section simplify enormously as we shall see immediately.

IV. THE SPECIAL CASE OF LINEAR GAUSSIAN RANDOM FORCES

We have already referred to a theorem, to be proven in the Appendix, which allows the calculation of the pure f vertices from the pure f cumulants. It is a trivial and quite intuitive consequence of this theorem that for a random force f with Gaussian statistics all pure f vertices of order $n\geq 3$ vanish for $\Lambda=0$ (even for $\eta=0$; $\hat{\eta},\lambda$ arbitrary). It follows that the mass-renormalized expansion of Γ_3 involves no ingredients other than the propagators $\langle\langle\Phi(1)\Phi(2)\rangle\rangle$ and the bare interaction $\gamma_3(\underline{1}23)$. If, moreover, the Gaussian noise $f(1)$ enters the field equation (3.1) linearly, with $U_2'(\underline{1}2)=\delta(\underline{1}2)$ as is necessary for the FDTs of Sec. II to hold, the bare interaction γ_3 has no f index. Then the self-energy is, in mass renormalization, made up by the propagators $\langle\langle\Psi(1)\Psi(2)\rangle\rangle$ and bare interactions and cannot have an f index either. By writing out the "Dyson equation" (3.21) in components, we then have, for $\Lambda=0$,

$$G_{(0)}^{-1}(\hat{\psi}(1)\psi(\bar{2}))\langle\langle\psi(\bar{2})\psi(3)\rangle\rangle=\langle\langle f(1)\psi(3)\rangle\rangle+\Sigma(\hat{\psi}(1)\psi(\bar{2}))\langle\langle\psi(\bar{2})\psi(3)\rangle\rangle+\Sigma(\hat{\psi}(1)\hat{\psi}(\bar{2}))\langle\langle\hat{\psi}(\bar{2})\psi(3)\rangle\rangle, \tag{4.1}$$

$$G_{(0)}^{-1}(\hat{\psi}(1)\psi(\bar{2}))\langle\langle\psi(\bar{2})\hat{\psi}(3)\rangle\rangle=\delta(13)+\Sigma(\hat{\psi}(1)\psi(\bar{2}))\langle\langle\psi(\bar{2})\hat{\psi}(3)\rangle\rangle, \tag{4.2}$$

$$G_{(0)}^{-1}(\hat{\psi}(1)\psi(\bar{2}))\langle\langle\psi(\bar{2})f(3)\rangle\rangle=\langle\langle f(1)f(3)\rangle\rangle+\Sigma(\hat{\psi}(1)\psi(\bar{2}))\langle\langle\psi(\bar{2})f(3)\rangle\rangle. \tag{4.3}$$

The third of these equations is immediately solved by expressing the mixed correlation function $\langle\langle\psi f\rangle\rangle$ in terms of the response function

$$\langle\langle\psi(1)f(3)\rangle\rangle=\langle\langle\psi(1)\hat{\psi}(\bar{2})\rangle\rangle\langle\langle f(\bar{2})f(3)\rangle\rangle. \tag{4.4}$$

This can be inserted into Eq. (4.1), whereupon $\langle\langle\psi f\rangle\rangle$ is completely eliminated. By introducing the matrix

$$2D(12)=\begin{pmatrix}0 & 0\\ 0 & \langle\langle f(1)f(2)\rangle\rangle\end{pmatrix}, \tag{4.5}$$

we may write the Dyson equations for $\langle\langle\Psi(1)\Psi(2)\rangle\rangle$ as

$$G_{(0)}^{-1}(1\bar{2})\langle\langle\Psi(\bar{2})\Psi(3)\rangle\rangle=\delta(12)+[\Sigma(1\bar{2})+2D(1\bar{2})]\langle\langle\Psi(\bar{2})\Psi(3)\rangle\rangle \tag{4.6}$$

and determine the self-energy Σ from the original MSR equations

$$\Sigma(12)=\gamma_3(12\bar{3})\langle\langle\Psi(\bar{3})\rangle\rangle+\Sigma^c(12), \tag{4.7}$$

$$\Sigma^c(12)=\tfrac{1}{2}\gamma_3(1\bar{2}\bar{3})\langle\langle\Psi(\bar{2})\Psi(\bar{4})\rangle\rangle\times\langle\langle\Psi(\bar{3})\Psi(\bar{5})\rangle\rangle\Gamma_3(\bar{4}\bar{5}2), \tag{4.8}$$

$$\Gamma_3(123)=\gamma_3(123)+\delta\Sigma^c(12)/\delta\langle\langle\Psi(3)\rangle\rangle, \tag{4.9}$$

where none of the indices refers to f any longer. When showing, in the next two sections, how Eqs. (4.6) and (4.7) simplify when a FDT holds, we shall use the diagrammatic notation suggested by MSR (see Fig. 1). Instead of writing down the complete set of rules for constructing the diagrams corresponding to the mass-renormalized self-energy in all orders (which we shall not need), we prefer to draw the reader's attention to the following naive but nonetheless important observations. (i) Σ contains a "left" and a "right" corner vertex each of which has one "free" index. (ii) All other indices of all vertices are joined by lines. (iii) Each vertex has one $\hat{\psi}$ index and two ψ indices.

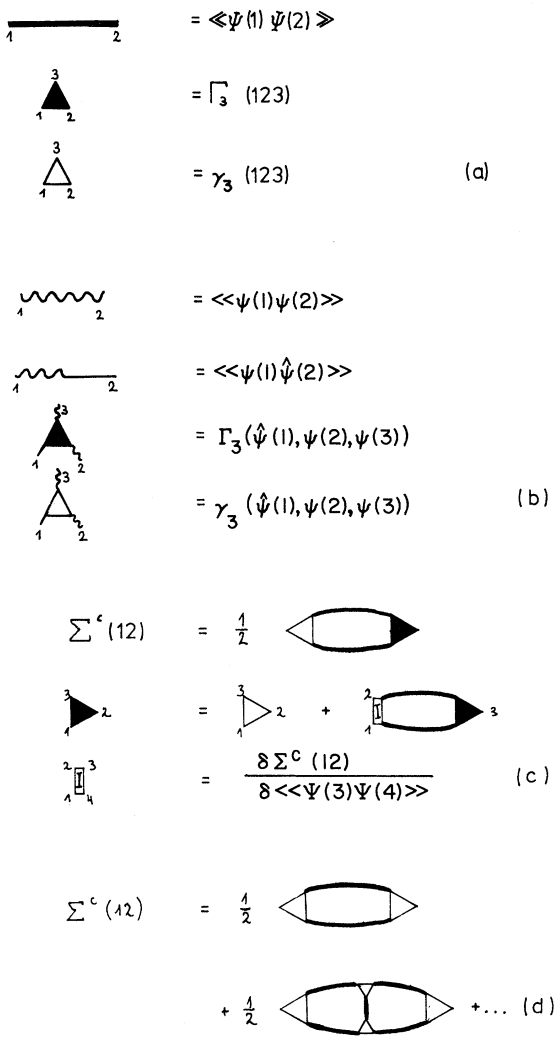

FIG. 1. Symbols for the propagator and the vertices (a) and their components (b). Representation of Eqs. (4.8) and (4.9) (c) and the first terms of the mass-renormalized expansion of Σ^c (d).

(iv) Each line is either $\langle\langle\psi\psi\rangle\rangle$ or $\langle\langle\psi\hat{\psi}\rangle\rangle$. (v) A closed chain of $\langle\langle\psi\hat{\psi}\rangle\rangle$ lines vanishes (e.g., Fig. 2). This follows from $\langle\langle\psi(1)\hat{\psi}(2)\rangle\rangle = 0$ for $t_1 < t_2$. (vi) The $\hat{\psi}$ index of any vertex within a diagram is followed by a $\langle\langle\hat{\psi}\psi\rangle\rangle$ line ending at another vertex whose $\hat{\psi}$ index is again taken up by a $\langle\langle\hat{\psi}\psi\rangle\rangle$ line and so forth. The arising chain of $\langle\langle\hat{\psi}\psi\rangle\rangle$ lines is either closed and thus vanishes or ends a free $\hat{\psi}$ index.

Observation (vi) immediately yields $\Sigma^c(\psi\psi) = 0$, since no diagram to $\Sigma^c(\psi\psi)$ has a free $\hat{\psi}$ index. For $\Sigma^c(\hat{\psi}\psi)$ we conclude that all vertices are connected to the "left $\hat{\psi}$ index" by a chain of $\langle\langle\hat{\psi}\psi\rangle\rangle$ lines. Therefore $\Sigma^c(\hat{\psi}\psi)$ has the structure of a tree as shown in Fig. 3(a). All lines not drawn out there may be imagined to be $\langle\langle\psi\psi\rangle\rangle$ lines joining the ψ indices of internal vertices or $\langle\langle\psi\hat{\psi}\rangle\rangle$ lines joining

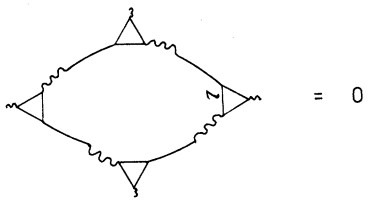

FIG. 2. Example of a vanishing chain of $\langle\langle\psi\hat{\psi}\rangle\rangle$ lines.

the two drawn pieces of the diagram. Obviously, $\Sigma^c(\hat{\psi}(1), \psi(2)) = 0$ for $t_1 < t_2$. Finally, $\Sigma^c(\hat{\psi}\hat{\psi})$ consists of two trees joined together by $\langle\langle\psi\psi\rangle\rangle$ lines [see Fig. 3(b)]. Note that the tree structure of $\Sigma^c(\hat{\psi}\psi)$ and the double-tree structure of $\Sigma^c(\hat{\psi}\hat{\psi})$ are preserved when one has to encounter bare interactions γ_n of higher than third order (Fig. 4).

As a final preparation for the considerations to follow, let us write the Dyson equations (4.6) for the correlation function

$$C(t) \equiv C_{ij}(t) = \langle\langle\psi_i(t)\psi_j(0)\rangle\rangle|_{\Lambda=0} \tag{4.10}$$

and the response function

$$R(t) \equiv R_{ij}(t) = \langle\langle\psi_i(t)\hat{\psi}_j(0)\rangle\rangle|_{\Lambda=0}$$

in the slightly more convenient matrix notation

$$\left(\frac{\partial}{\partial t} - U_2\right)C(t) = 2D\tilde{R}(-t) + \int_{-\infty}^{t} dt'\,\Sigma_{\hat{\psi}\psi}(t-t')C(t') + \int_{-\infty}^{0} dt'\,\Sigma_{\hat{\psi}\hat{\psi}}(t-t')\tilde{R}(-t'), \tag{4.11}$$

$$\left(\frac{\partial}{\partial t} - U_2\right)R(t) = \delta(t) + \int_{0}^{t} dt'\,\Sigma_{\hat{\psi}\psi}(t-t')R(t'). \tag{4.12}$$

The products occurring here have to be understood as matrix products with respect to the indices i. We have used $\langle\langle f(1)f(2)\rangle\rangle|_{\Lambda=0} = \langle f(1)f(2)\rangle = 2D\delta(t_1 - t_2)$ and $\gamma_2(\hat{\psi}(1)\psi(2)) = U_2\delta(t_1 - t_2)$. $\tilde{R}$ is the matrix adjoint of R.

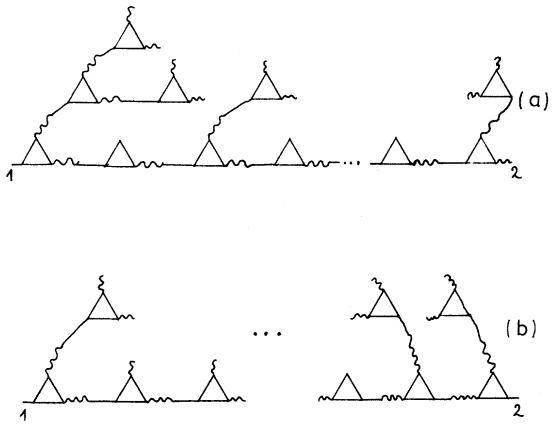

FIG. 3. Tree structure of $\Sigma^c_{\hat{\psi}\psi}$ (a) and double-tree structure of $\Sigma^c_{\hat{\psi}\hat{\psi}}$ (b).

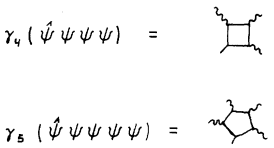

FIG. 4. Bare vertices of higher order.

V. FDT FOR RANDOM PROCESSES OF CLASS A

Recall that random processes of class A obey detailed balance, are driven by Gaussian white noise, and are purely irreversible. Let the drift vector F_i be

$$F_i = F_i^{\mathrm{irr}}$$

$$= \gamma_1(\hat{\psi}_i) + \gamma_2(\hat{\psi}_i\psi_{\bar{j}})\psi_{\bar{j}} + \tfrac{1}{2}\gamma_3(\hat{\psi}_i\psi_{\bar{j}}\psi_{\bar{k}})\psi_{\bar{j}}\psi_{\bar{k}}. \tag{5.1}$$

The bare interactions γ_n are subject to the potential conditions

$$\gamma_n(\hat{\psi}_i\psi_{j_1}\cdots\psi_{j_{n-2}}\psi_{\bar{k}})D_{\bar{k}l} = D_{i\bar{k}}\gamma_n(\psi_{\bar{k}}\psi_{j_1}\cdots\psi_{j_{n-2}}\hat{\psi}_l) \tag{5.2}$$

which are easily derived from Eq. (2.5). For the case $n=3$ we may represent this potential condition and the FDT (2.19)

$$\frac{\partial}{\partial t}C(t) = -R(t)D + D\tilde{R}(-t) \tag{5.3}$$

in graphical form as shown in Fig. 5 where the arrow on the wavy line $C(t_1 - t_2)$ means $\partial/\partial t_1$ (an arrow of opposite direction would mean $\partial/\partial t_2 = -\partial/\partial t_1$).

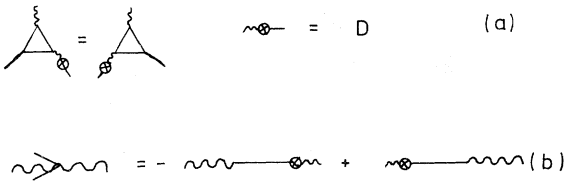

FIG. 5. Graphical representation of the potential condition (5.2) (a) and the FDT (b) for class A, where the arrow on $C(t_1 - t_2)$ means $\partial/\partial t_1$.

We now assert that the Dyson equations (4.11) and (4.12) for C and R are equivalent to one of these equations and the FDT (5.3) where the self-energy Σ may be taken in any order of mass renormalization. To prove this statement we consider Eq. (4.11) for positive times (as for $t<0$ we may invoke the time-reversal properties of C and R) and differentiate with respect to t:

$$\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t} - U_2\right)C(t) = \frac{\partial}{\partial t}\int_{-\infty}^{+\infty} dt'\,[\Sigma_{\hat{\psi}\psi}(t')C(t-t') + \Sigma_{\hat{\psi}\hat{\psi}}(t-t')\tilde{R}(-t')], \qquad t>0.$$

By using the FDT (5.3) we may write this as

$$-\left(\frac{\partial}{\partial t} - U_2\right)R(t)D = \int_{-\infty}^{+\infty} dt'\left(\Sigma_{\hat{\psi}\psi}(t-t')[-R(t')D + D\tilde{R}(-t')] + \frac{\partial\Sigma_{\hat{\psi}\hat{\psi}}(t-t')}{\partial t}\tilde{R}(-t')\right).$$

Since $\Sigma_{\hat{\psi}\psi}(t)$ and $R(t)$ vanish for negative times and $\Sigma^{\mathrm{HF}} \sim \delta(t)$, we obtain

$$\left(\frac{\partial}{\partial t} - U_2\right)R(t)D = \int_0^t dt'\,\Sigma_{\hat{\psi}\psi}(t-t')R(t')D - \int_{-\infty}^0 dt'\left(\Sigma^c_{\hat{\psi}\psi}(t-t')D + \frac{\partial}{\partial t}\Sigma^c_{\hat{\psi}\hat{\psi}}(t-t')\right)\tilde{R}(-t'). \tag{5.4}$$

This is compatible and indeed identical with the equation of motion (4.12) for the response function if and only if

$$\frac{\partial}{\partial t}\Sigma^c_{\hat{\psi}\hat{\psi}}(t) = -\Sigma^c_{\hat{\psi}\psi}(t)D. \tag{5.5}$$

Similarly, the equation of motion for C may be derived from that for R by invoking the FDT (5.3). The exact self-energy obeys, of course, Eq. (5.5) which is nothing else than the FDT expressed in terms of Σ. We have to prove, however, that Eq. (5.5) holds in each order of mass renormalization.

Consider an arbitrary diagram contributing to $\Sigma_{\hat{\psi}\hat{\psi}}$, that is Fig. 3(b), and differentiate with respect to t. The differentiation acts only on the left vertex. Since γ_3 is instantaneous, we can shift $\partial/\partial t$ through the left vertex by using

$$\frac{\partial}{\partial t}[\delta(t-t_1)\delta(t-t_2)] = [\delta(t-t_1)\delta(t-t_2)]\left(\frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2}\right). \tag{5.6}$$

Graphically this is represented by Fig. 6 where we introduce the rule that a diagram with several arrows (time derivatives) means a sum of several such diagrams each of which carries one of the arrows only. By using stationarity and integrating by parts, we then shift the time derivatives to the next vertices as

$$\frac{\partial}{\partial t_1}\langle\!\langle\Psi(t_1)\Psi(t_1')\rangle\!\rangle = -\frac{\partial}{\partial t_1'}\langle\!\langle\Psi(t_1)\Psi(t_1')\rangle\!\rangle$$

$$= +\langle\!\langle\Psi(t_1)\Psi(t_1')\rangle\!\rangle\frac{\partial}{\partial t_1'}. \tag{5.7}$$

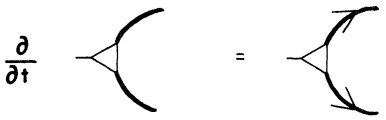

FIG. 6. Representation of Eq. (5.6).

We keep shifting arrows through vertices in this way until all arrows sit on C lines. It is easy to see that out of the potentially many terms represented by such a diagram decorated with arrows, at most a single one survives to give a nonvanishing contribution to $\partial\Sigma_{\hat{\psi}\hat{\psi}}/\partial t$: Recall that, according to (4.16), C lines either connect ψ indices of vertices within the left tree or join the left tree of the diagram considered with the right tree. Every C line within the left tree carries two arrows. The corresponding two terms cancel each other because of

$$\left(\frac{\partial}{\partial t_\nu}+\frac{\partial}{\partial t_\mu}\right)C(t_\nu-t_\mu)=0. \tag{5.8}$$

We may thus omit all arrows on C lines within the left tree. With respect to the C lines joining the two trees of the diagram we must distinguish three cases.

(i) The C line in question neither starts nor ends at the left nor at the right corner vertex. Then the neighborhood of the C line looks like Fig. 7(a). But then there are other diagrams, occurring in the same order of mass renormalization, identical to the one considered except that the part shown in Fig. 7(a) is replaced by Fig. 7(b) or by Fig. 7(c). In fact, the three diagrams referred to here are one and the same diagram with respect to the spinor propagator $\langle\!\langle\Psi(1)\Psi(2)\rangle\!\rangle$. We now use the FDT [Fig. 5(b)] to rewrite Figs. 7(a)–7(c) as Figs. 7(d)–7(f). Finally, with the help of the potential condition (5.2) on Fig. 5(a), we transform the two terms in Fig. 7(d) into Fig. 7(g). We now see that the first term in Fig. 7(g) [or (a)] is canceled by the second term in Fig. 7(f) [or (c)] and the second term in Fig. 7(g) by the first one in Fig. 7(e). The terms qualified as "one other term" in Figs. 7(e) and 7(f) cancel in a similar way with contributions from other diagrams obtained from the same spinor diagram by filling in ψ and $\hat{\psi}$ indices differently. Consequently, no diagram to $\Sigma_{\hat{\psi}\hat{\psi}}$ can contribute to $(\partial/\partial t)\Sigma_{\hat{\psi}\hat{\psi}}$ other than those for which a junction of the left and right trees by a C line involves either the left or the right corner vertex.

(ii) The C line in question starts at the left corner vertex. Then its neighborhood looks like Fig. 8(a). The second equality holds since a chain of $\langle\!\langle\psi\hat{\psi}\rangle\!\rangle$ lines from the left corner vertex to the

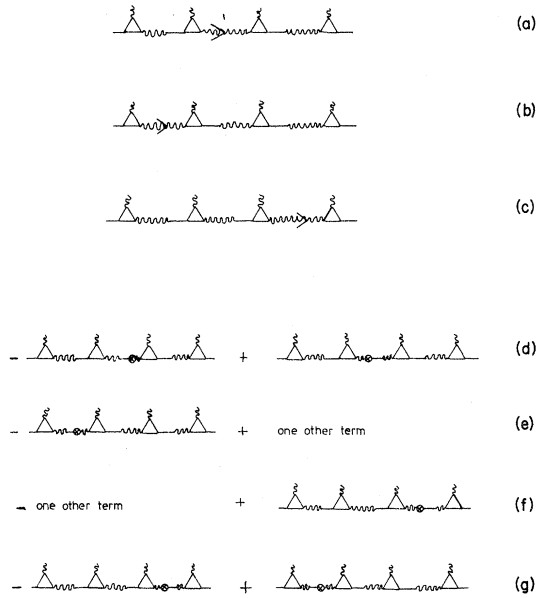

FIG. 7. (a), (b), (c) Parts of diagrams for $(\partial/\partial t)\Sigma^c_{\hat{\psi}\hat{\psi}}(t-t')$ with C-line junctions between left and right tree in the interior. (d), (e), (f) same as (a), (b), (c), respectively, after using the FDT. (g) same as (a) and (d) after using the potential conditions.

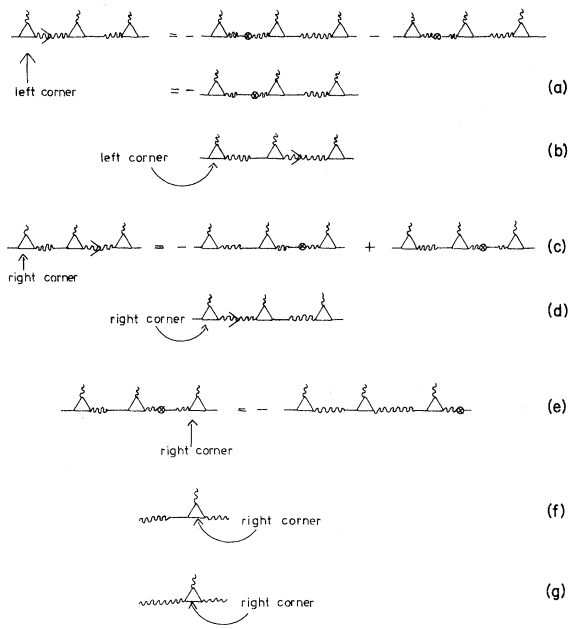

FIG. 8. (a) Part of a diagram for $(\partial/\partial t)\Sigma^c_{\hat{\psi}\hat{\psi}}(t-t')$ with C-line junction between left and right tree at the left corner vertex reexpressed with the help of the FDT and by using $t>t'$. (b) Diagram to $(\partial/\partial t)\Sigma_{\hat{\psi}\hat{\psi}}(t-t')$ cancelling the one in (a). (c) Similar to (a) but with C-line junction at the right corner vertex. (d) Diagram cancelling the second term on the right-hand side in (c). (e) First term on the right-hand side in (c) reexpressed using the potential conditions. (f) Right corner of an arbitrary diagram to $\Sigma^c_{\hat{\psi}\psi}$. (g) Right corner of a diagram to $\Sigma^c_{\hat{\psi}\hat{\psi}}$.

right corner vertex gives no contribution for $t>0$. The remaining term cancels against a contribution from Fig. 8(b) in a manner described for case (i) (a "one other term" is involved). We conclude that only those diagrams to $\Sigma_{\hat{\psi}\hat{\psi}}$ contribute to $(\partial/\partial t)\Sigma_{\hat{\psi}\hat{\psi}}$ which have a C line junction between the left and right tree ending at the right corner vertex. This case is left for discussion.

(iii) The C line in question ends at the right corner vertex. Then its neighborhood looks like Fig. 8(c). The second term there cancels with a contribution from Fig. 8(d) in a manner described for case (i) (again, a "one other term" is involved), whereas the first one may be transformed by using the potential condition (5.2) or Fig. 5(a) so that we get, effectively, Fig. 8(e). This, however, is the neighborhood of the right corner of a diagram to $\Sigma_{\hat{\psi}\psi}$, multiplied from the right by $-D$.

We thus arrive at the following conclusion. The set of diagrams pertaining to $\Sigma_{\hat{\psi}\hat{\psi}}(t)$ in a given order of mass renormalization leads, upon differentiation with respect to the time t, to a set of diagrams of the same order pertaining to $-\Sigma_{\hat{\psi}\psi}(t)D$. To complete our proof, we are left with the task of showing that $(\partial/\partial t)\Sigma_{\hat{\psi}\hat{\psi}}$ gives all terms of $-\Sigma_{\hat{\psi}\psi}D$ in nth order of mass renormalization. This is easy, however, since if in an arbitrary diagram pertaining to $\Sigma_{\hat{\psi}\psi}(t)$, that is in the tree drawn in Fig. 3(a), we replace the element of Fig. 8(f) by that of Fig. 8(g), we get precisely that diagram pertaining to $\Sigma_{\hat{\psi}\hat{\psi}}(t)$ which regenerates the starting contribution to $\Sigma_{\hat{\psi}\psi}$, apart from the factor $-D$, upon being differentiated with respect to t. Hence there is a one-to-one correspondence between the diagrams pertaining to $(\partial/\partial t)\Sigma_{\hat{\psi}\hat{\psi}}$ and $-\Sigma_{\hat{\psi}\psi}D$. We have thus demonstrated the FDT (5.5) to hold in each order of mass renormalization.

The self-consistent perturbation theory obtained from Eqs. (4.6)–(4.9) by iterating (4.9) now simplifies considerably. One needs to use only one of the functions C and R and has to determine only $\Sigma_{\hat{\psi}\hat{\psi}}$ or $\Sigma_{\hat{\psi}\psi}$. Note that Eqs. (4.6)–(4.9) uniquely determine C and R.

Let us conclude by noting that the validity of the above considerations is by no means restricted to the case of cubic interactions. Eventually present vertices $\gamma_n(1\cdots n)$ with $n>3$ would be represented pictorially by nth-order polygons and obey the potential condition (5.2) which may be drawn as in Fig. 9 for $n=4$. None of the steps in the above considerations depends in any way on the number $n-1$ of ψ indices of the bare vertex γ_n.

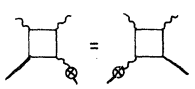

FIG. 9. Class-A potential condition for the bare vertex.

VI. FDT FOR RANDOM PROCESSES OF CLASS B

We assert that for random processes with detailed balance and linear damping which are driven by Gaussian white noise the Dyson equations (4.18) and (4.19) for C and R are equivalent to one of these equations and the FDT (2.25),

$$C(t)=R(t)C(0)+C(0)\tilde{R}(-t) \tag{6.1}$$

with the self-energy Σ taken in any order of mass renormalization. Moreover, the equation of motion (4.19) for R and the FDT (6.1), together with the mass-renormalized expansion for $\Sigma_{\hat{\psi}\psi}$ according to Eqs. (3.22) and (3.24) are equivalent to the graphical perturbation scheme developed by Kawasaki.[4]

In order to prove the above statements, we first use the last of the potential conditions (2.5) and the expression (2.20) for the linear damping to obtain for the cubic vertex $\gamma_3(\hat{\psi}_k\psi_l\psi_m)$, which now represents a reversible interaction,

$$\gamma_3(\hat{\psi}_i\psi_{\bar{l}}\psi_{\bar{m}})C_{\bar{l}j}(0)C_{\bar{m}k}(0)+\gamma_3(\hat{\psi}_j\psi_{\bar{l}}\psi_{\bar{m}})C_{\bar{l}i}(0)C_{\bar{m}k}(0)+\gamma_3(\hat{\psi}_k\psi_{\bar{l}}\psi_{\bar{m}})C_{\bar{l}i}(0)C_{\bar{m}j}(0)=0. \tag{6.2}$$

For later use we provide ourselves with the pictorial representations of the FDT (6.1) and the symmetry (6.2) shown in Fig. 10.

We now take the equation of motion (4.11) for C for $t>0$ (for $t<0$ we may invoke the time-reversal properties of C and R),

$$\left(\frac{\partial}{\partial t}-U_2\right)C(t)=\int_{-\infty}^{t}dt'\,\Sigma_{\hat{\psi}\psi}(t-t')C(t')+\int_{-\infty}^{0}dt'\,\Sigma_{\hat{\psi}\hat{\psi}}(t-t')\tilde{R}(-t'), \tag{6.3}$$

and insert the FDT (6.1)

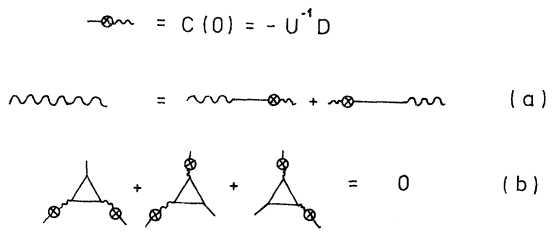

FIG. 10. FDT (a) and potential condition (b) for class B.

$$\left(\frac{\partial}{\partial t} - U_2\right) R(t)C(0) = \int_0^t dt' \, \Sigma_{\hat{\psi}\psi}(t-t')R(t')C(0) + \int_{-\infty}^{0} dt' \left[\Sigma_{\hat{\psi}\psi}(t-t')C(0) + \Sigma_{\hat{\psi}\hat{\psi}}(t-t')\right]\tilde{R}(-t'). \tag{6.4}$$

This is the equation of motion (4.19) for $R(t)$ if and only if the self-energy Σ obeys the FDT

$$\Sigma_{\hat{\psi}\psi}(t)C(0) = -\Sigma_{\hat{\psi}\hat{\psi}}(t) \quad \text{for } t > 0. \tag{6.5}$$

Similarly, one derives (4.11) from (4.12). Our above assertions require as a proof that we demonstrate (6.5) to hold in any order of mass renormalization.

Next, we prove the following auxiliary statement: the integrand of an arbitrary spinor diagram for $\Sigma(t-t')$ with all time variables t_i fixed vanishes when any vertex within the diagram is taken at some instant $t_1 \notin [t', t]$. A vertex within the diagram is connected to three other vertices as shown in Fig. 11(a). Let us assume, without loss of generality, $t_2 \leq t_3 \leq t_4$. In order to see whether the corresponding factor in the integrand indeed vanishes for $t_1 < t_2$, we insert indices. We may omit all terms containing a $\langle\langle \psi(t_1)\hat{\psi}(t_i)\rangle\rangle$ line since these certainly vanish for $t_1 < t_2$. Therefore the element of Fig. 11(a) gives rise to Fig. 11(b). The first equality follows from the FDT [Fig. 8(a)], and the second one from the potential condition [Fig. 8(b)]. Consequently, for $t > t'$, the smallest time argument in the integrand corresponding to an arbitrary Σ diagram is t' which refers to the right corner vertex.[9]

It follows from the above auxiliary statement that in an arbitrary diagram for $\Sigma_{\hat{\psi}\hat{\psi}}$ that is in Fig. 3(b) the right tree degenerates, for $t > t'$, to the right corner vertex. Therefore, the component $\Sigma_{\hat{\psi}\hat{\psi}}$ of an arbitrary spinor diagram for Σ may be pictured as in Fig. 11(c) where the shaded circle means the sum of all left trees obtained by filling in internal indices into the original spinor diagram. Similarly, the component $\Sigma_{\hat{\psi}\psi}(t-t')$ of the original spinor diagram has the structure shown in Fig. 11(d). Now, by inserting the FDT [Fig. 8(a)] into Fig. 11(c), we get Fig. 11(e), where the two equalities arise from using the potential condition [Fig. 10(b)] and then invoking the FDT [Fig. 10(a)]. But this just says that the self-energy FDT (6.5) is preserved in each order of mass renormalization.

Let us note that the stationary functions $C(t)$ and $R(t)$ are already uniquely determined by using one of the Dyson equations (4.11) or (4.12) and the FDT (6.1) for $t > 0$ only, if we complement these equations by the initial condition

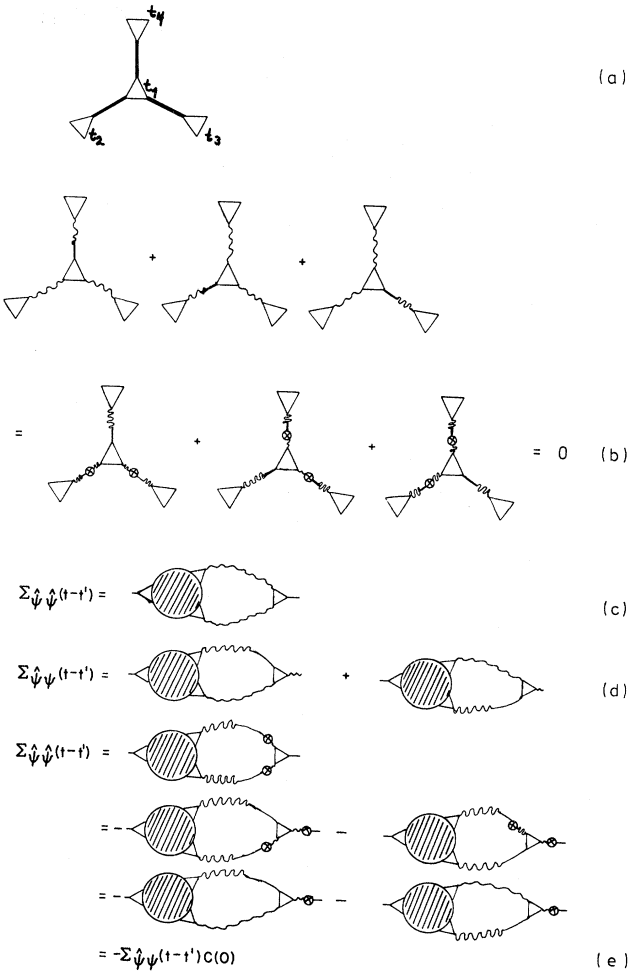

FIG. 11. (a) Part of a self-energy diagram with unspecified indices on the lines. (b) Sum of contributions from (a) when $t_1 < t_2, t_3, t_4$ reexpressed using the FDT and the potential conditions. (c) Structure of all diagrams to $\Sigma^{c}_{\hat{\psi}\hat{\psi}}$ which do not vanish because of the equation in (b). (d) Structure of all diagrams to $\Sigma_{\hat{\psi}\psi}$ which do not vanish because of the equation in (b). (e) By invoking the FDT and the potential conditions $\Sigma^{c}_{\hat{\psi}\hat{\psi}}$ as shown in (c) is transformed into $-\Sigma^{c}_{\hat{\psi}\psi}C(0)$ with $\Sigma_{\hat{\psi}\psi}$ as shown in (d).

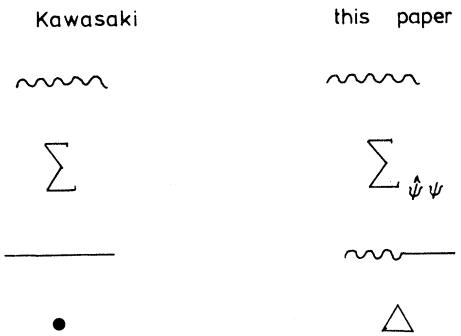

FIG. 12. Correspondence between Kawasaki's (Ref. 2) and our graphs.

$$R_{ij}(0^+)=\delta_{ij}. \tag{6.6}$$

The equivalence of the MSR formalism to Kawasaki's perturbation scheme for random processes of class B is now obvious. The translation rules are listed in Fig. 12.

Kawasaki uses the following rules for constructing his self-energy diagrams. (i) Draw all trees. (ii) Omit all reducible diagrams containing self-energy insertions. (iii) Omit all decorations. Rule (i) is Fig. 3(a), rule (ii) is a consequence of renormalization, and (iii) is the above auxiliary statement.

VII. FDT FOR CONSERVATIVE SYSTEMS IN THERMAL EQUILIBRIUM

Here again, one of the Dyson equations (4.18) for C and (4.19) for R can be replaced by the FDT which now is given by (2.34).

In complete analogy with the considerations of Sec. V for class A, one first writes the FDT in terms of the self-energy Σ. The result is

$$\Sigma_{\hat\psi\psi}(t)\frac{1}{\beta}D-\frac{\partial}{\partial t}\Sigma_{\hat\psi\hat\psi}(t)=0 \quad \text{for } t>0. \tag{7.1}$$

The proof of (7.1) being preserved in all orders of mass renormalization can also be reduced to the one given in Sec. V for class A. To that end we introduce the pictorial representations of D and of the FDT (2.34) shown in Figs. 13(a) and 13(b). Furthermore we provide ourselves with the following properties of the bare interactions $\gamma_n(1\cdots n)$ which follow from the fact that the field equation

$$\frac{d\psi_i}{dt}=\gamma_1(\hat\psi_i)+\gamma_2(\hat\psi_i\psi_{\bar j})\psi_{\bar j}+(1/2!)\gamma_3(\hat\psi_i\psi_{\bar j}\psi_{\bar k})\psi_{\bar j}\psi_{\bar k}$$
$$+(1/3!)\gamma_4(\hat\psi_i\psi_{\bar j}\psi_{\bar k}\psi_{\bar l})\psi_{\bar j}\psi_{\bar k}\psi_{\bar l}+\cdots$$

now derives from a Hamiltonian as

$$\frac{d\psi_i}{dt}=D_{i\bar j}\frac{\partial H}{\partial\psi_{\bar j}}.$$

We obviously have

$$\gamma_3(\hat\psi_i\psi_j\psi_k)=D_{i\bar m}\frac{\partial^3 H}{\partial\psi_{\bar m}\partial\psi_j\partial\psi_k}\bigg|_{\psi=0}, \tag{7.2}$$

$$\gamma_4(\hat\psi_i\psi_j\psi_k\psi_l)=D_{i\bar m}\frac{\partial^4 H}{\partial\psi_{\bar m}\partial\psi_j\partial\psi_k\partial\psi_l}\bigg|_{\psi=0}, \text{ etc.}$$

It follows that

$$(1/\beta)D_{i\bar m}\gamma_3(\psi_{\bar m}\psi_j\hat\psi_k)=\gamma_3(\hat\psi_i\psi_j\psi_{\bar m})(1/\beta)D_{j\bar m},$$

and similarly for $\gamma_4,\gamma_5,\ldots$ [see Fig. 13(c)].

These symmetry conditions have the same appearance as the potential conditions (5.2), Figs. 5(a) and 9. The FDT (7.2) has the same form as that for class A, Fig. 5(b), apart from an over-all sign which does not matter. The proof of the self-energy FDT (7.1) in each order of mass renormalization thus carries over from Sec. V without change.

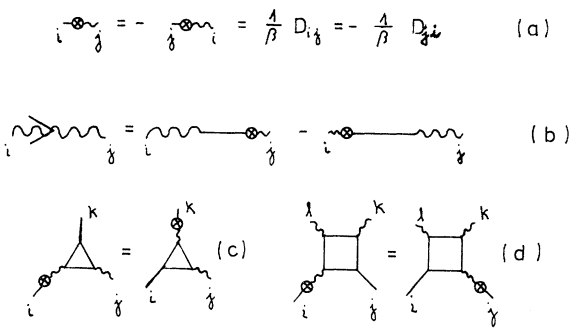

FIG. 13. Representation of the matrix D (a), the FDT (b) and the potential condition for γ_3 (c) and γ_4 (d) for class C.

ACKNOWLEDGMENT

We thank Paul C. Martin for presenting to us the results of Ref. 1 before publication.

APPENDIX

We here demonstrate that for $\Lambda=\{\eta,\hat\eta,\lambda\}=0$ all pure f vertices $\Gamma_n(f(1)\cdots f(n))|_{\Lambda=0}$ are determined by the pure f cumulants $\langle\!\langle f(1)\cdots f(n)\rangle\!\rangle|_{\Lambda=0}$. This statement has already been used in Sec. III.

Let us start out with $n=3$ and let $\eta=0$ but, for the moment being, $\hat\eta$ and λ arbitrary. Then the following three cumulants vanish:

$$\langle\!\langle\hat\psi(1)\hat\psi(2)\hat\psi(3)\rangle\!\rangle=\langle\!\langle\hat\psi(1)\hat\psi(2)f(3)\rangle\!\rangle$$
$$=\langle\!\langle\hat\psi(1)f(2)f(3)\rangle\!\rangle=0. \tag{A1}$$

As explained in Fig. 16, these identities imply that all third-order vertices without $\hat\psi$ index except for the pure f vertex vanish:

$$\Gamma_3(\psi(1),\psi(2),\psi(3))=\Gamma_3(\psi(1),\psi(2),f(3))$$
$$=\Gamma_3(\psi(1),f(2),f(3))=0. \tag{A2}$$

The argument displayed in Fig. 16 makes use of the general relation (3.20) (Fig. 14) and the graphs for $\langle\!\langle ff\rangle\!\rangle$ and $\langle\!\langle\psi f\rangle\!\rangle$ defined in Fig. 15. As a consequence, we see that only the pure f vertex contributes to the pure f cumulant,

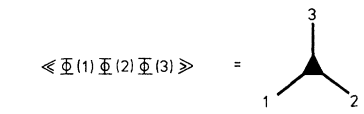

FIG. 14. Graphical version of Eqs. (3.20).

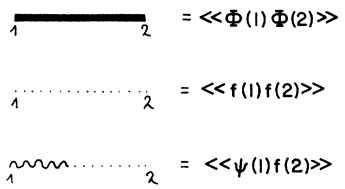

FIG. 15. Symbols for the propagator including stochastic forces.

$$\langle\!\langle f(1)f(2)f(3)\rangle\!\rangle = \langle\!\langle f(1)f(\bar{1})\rangle\!\rangle\langle\!\langle f(2)f(\bar{2})\rangle\!\rangle \times \langle\!\langle f(3)f(\bar{3})\rangle\!\rangle\Gamma_3(f(\bar{1})f(\bar{2})f(\bar{3})). \tag{A3}$$

This proves our initial statement for $n=3$.

Next, we generalize (A2) to all orders:

$$\Gamma_n(\psi(1)\cdots\psi(i)f(i+1)\cdots f(n))=0. \tag{A4}$$

That is, we claim that Γ_n vanishes for $\eta=0$ unless it is a pure f vertex or has at least one $\hat{\psi}$ index. This is most easily verified by induction. Suppose (A4) to hold for some order n. Then, by differentiating with respect to $\hat{\eta}(n+1)$ and $\lambda(n+1)$, we have $(\eta=0)$

$$\begin{aligned}\frac{\delta\Gamma_n(\cdots)}{\delta\hat{\eta}(n+1)} &= \frac{\delta\Gamma_n(\cdots)}{\delta\langle\!\langle\hat{\psi}(\overline{n+1})\hat{\psi}(n+1)\rangle\!\rangle}\langle\!\langle\hat{\psi}(\overline{n+1})\hat{\psi}(n+1)\rangle\!\rangle \\ &+ \frac{\delta\Gamma_n(\cdots)}{\delta\langle\!\langle\psi(\overline{n+1})\hat{\psi}(n+1)\rangle\!\rangle}\langle\!\langle\psi(\overline{n+1})\hat{\psi}(n+1)\rangle\!\rangle \\ &+ \frac{\delta\Gamma_n(\cdots)}{\delta\langle\!\langle f(\overline{n+1})\hat{\psi}(n+1)\rangle\!\rangle}\langle\!\langle f(\overline{n+1})\hat{\psi}(n+1)\rangle\!\rangle \\ &= \frac{\delta\Gamma_n(\cdots)}{\delta\langle\!\langle\psi(\overline{n+1})\hat{\psi}(n+1)\rangle\!\rangle}\langle\!\langle\psi(\overline{n+1})\hat{\psi}(n+1)\rangle\!\rangle.\end{aligned}$$

We conclude

$$\frac{\delta\Gamma_n(\cdots)}{\delta\langle\!\langle\psi(n+1)\rangle\!\rangle} = \Gamma_{n+1}(\psi(1)\cdots\psi(i)f(i+1)\cdots f(n)\psi(n+1)) = 0$$

and similarly

$$\frac{\delta\Gamma_n(\cdots)}{\delta\langle\!\langle f(n+1)\rangle\!\rangle} = \Gamma_{n+1}(\psi(1)\cdots\psi(i)f(i+1)\cdots f(n+1))=0.$$

Since (A4) has already been shown to hold for $n=3$, the induction is complete.

We now phrase the statement at the beginning of this appendix as a *Theorem:* The pure f cumulant $\langle\!\langle f(1)\cdots f(n)\rangle\!\rangle|_{\eta=0}$ can be expressed in terms of the second-order cumulant $\langle\!\langle f(1)f(2)\rangle\!\rangle|_{\eta=0}$ and the pure f vertices $\Gamma_m(f(1)\cdots f(m))$ with $m\leq n$.

Obviously, this theorem allows the calculation of all pure f vertices from the known pure f cumulants. To prove the theorem, we again proceed by induction. For $n=3$ it holds, as shown in (A3). Assuming it to hold for some arbitrary order n, we differentiate the corresponding equation with respect to $\lambda(n+1)$. Thereby the quantities involved go over into $(\eta=0!)$

$$\begin{aligned}&\langle\!\langle f(1)\cdots f(n)\rangle\!\rangle \to \langle\!\langle f(1)\cdots f(n)f(n+1)\rangle\!\rangle, \\ &\Gamma_m(f(1)\cdots f(m)) \to \Gamma_{m+1}(f(1)\cdots f(m)\Phi(\overline{m+1})) \times\langle\!\langle\Phi(\overline{m+1})f(n+1)\rangle\!\rangle \\ &\quad=\Gamma_{m+1}(f(1)\cdots f(m)f(\overline{m+1})) \times\langle\!\langle f(\overline{m+1})f(n+1)\rangle\!\rangle, \\ &\langle\!\langle f(1)f(2)\rangle\!\rangle \to \langle\!\langle f(1)f(2)f(n+1)\rangle\!\rangle \\ &\quad=\langle\!\langle f(1)f(\bar{1})\rangle\!\rangle\langle\!\langle f(2)f(\bar{2})\rangle\!\rangle\langle\!\langle f(n+1)f(\bar{3})\rangle\!\rangle \times\Gamma_3(f(\bar{1})f(\bar{2})f(\bar{3})),\end{aligned}$$

where (A4) has been used. Thus the theorem holds for all orders n.

We finally specialize to Gaussian noise. To display the relation between pure f vertices and pure f cumulants explicitly for this case, we let $\eta=0$ and $\hat{\eta}=0$ but keep, for the time being, λ arbitrary. The generating functional (3.9) then becomes

$$\begin{aligned}S(\eta=0,\hat{\eta}=0,\lambda) &= \langle\exp[\lambda(\bar{1})f(\bar{1})]\rangle \\ &= \exp[\tfrac{1}{2}\lambda(\bar{1})\langle f(\bar{1})f(\bar{2})\rangle\lambda(\bar{2})]\end{aligned} \tag{A5}$$

with some given second moment $\langle f(1)f(2)\rangle$. Without loss of generality, we have assumed $\langle f(1)\rangle=0$.[10] By differentiating $\ln S(\lambda)$ with respect to λ, we obtain the cumulants

$$\begin{aligned}&\langle\!\langle f(1)\rangle\!\rangle = \langle f(1)f(\bar{2})\rangle\lambda(\bar{2}), \\ &\langle\!\langle f(1)f(2)\rangle\!\rangle = \langle f(1)f(2)\rangle, \\ &\langle\!\langle f(1)\cdots f(n)\rangle\!\rangle = 0 \quad \text{for } n\geq 3.\end{aligned}$$

Since all pure f cumulants vanish for $n\geq 3$, the pure f vertices Γ_n for $n\geq 3$ vanish, too. As anticipated in Sec. IV, mass renormalization then contains the bare interactions γ_n only.

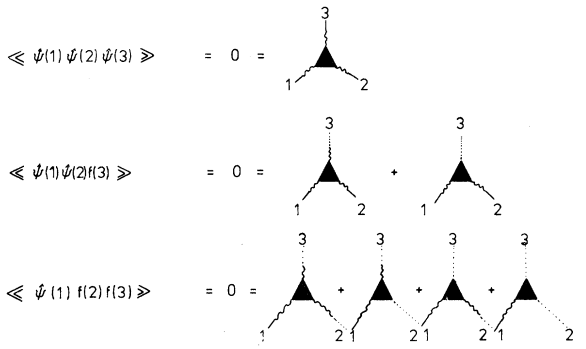

FIG. 16. Vanishing components of the three-point cumulant.

[1]P. C. Martin, E. D. Siggia, and H. A Rose, Phys. Rev. A 8, 423 (1973).
[2]G. S. Agarwal, Z. Phys. 252, 25 (1972).
[3]R. Graham, *Springer Tracts in Modern Physics* (Springer, Berlin, 1973), Vol. 66.
[4]K. Kawasaki, Ann. Phys. (N.Y.) 61, 1 (1970).
[5]R. L. Stratonovich, *Topics in the Theory of Random Noise* (Gordon and Breach, New York, 1963).
[6]$(\cdots)_+$ means the time-ordered product.
[7]Higher-order nonlinearities can be dealt with without further conceptual difficulties.
[8]If bare interactions γ_n with $n>4$ are admitted, some of the following considerations have to be modified. Especially, symmetrization of γ_n leads, in general, to contributions to γ_{n-2} without $\hat{\psi}$ index.
[9]Note that this is Kawasaki's prescription of "decorations" having to be dropped in all diagrams.
[10]A nonvanishing mean value $\langle f \rangle$ can be allotted to the inhomogeneity γ_1 in the field equation.