PHYSICAL REVIEW A VOLUME 11, NUMBER 6 JUNE 1975

Cooperative interaction of atoms with a radiation field: Evolution of the field density matrix

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(Received 19 June 1974; revised manuscript received 6 January 1975)

The evolution of the radiation field in a model for interacting radiation and matter in cooperative-atom states is studied via the density matrix. The exact treatment possible for small numbers of cooperating atoms reveals effects not apparent from approximation treatments for long interaction times and/or strong interactions. The superradiance concept needs modification under these conditions, and the photon statistics can be greatly modified by the interaction. The representation for the density matrix uses eigensolutions of the model obtained earlier.

I. INTRODUCTION

There has been much interest recently in cooperative effects of atoms interacting with electromagnetic radiation. Dicke[1] first described a simplified model of a single-mode radiation field interacting with N two-level atoms which are, among themselves, noninteracting except through the field. The model is representable by the Hamiltonian

$$\mathcal{H}=\omega a^{\dagger}a+\omega_0 S_3+K(a^{\dagger}S_-+aS_+) \tag{1.1}$$

in the rotating-wave approximation or by a circularly polarized field. The first term is the free-field energy, the second the free-atom energy in a static field, and the third the interaction with a radiation field transverse to the static field. $a^{\dagger}$ and a are the photon creation and annihilation operators, respectively. The representation for the atoms is in terms of the collective pseudo-angular-momentum of the atoms. S_3 is the component parallel to the static field. S_- and S_+ are raising and lowering operators for the atomic system. The model can also be cast in different (equivalent) forms.[2,3] In the formulation of Walls and Barakat[3] it has application to nonlinear optics.

Dicke used a perturbative approach in discussing the model. Considerable work has been done recently on determination of the eigenstates of this system.[4-6] The eigenstates can be written as a mixture of states, each having the same noninteracting energy if $\omega=\omega_0$:

$$|\psi_{nl}\rangle=\sum_{j=1}^{C+1} W_{nl}^{(j)}\epsilon_j|n+j-1\rangle ,$$

where C is the number of atoms in the cooperative state; $W_{nl}^{(j)}$ is the jth component of the lth eigenvector; ϵ_j represents the state of excitation of the atoms—$j=1$ corresponds to all atoms excited and $j=C+1$ to all atoms in the ground state; $|m\rangle$ is the state of the field in which exactly m photons are present. n is a parameter which can be loosely characterized as the minimum number of photons it is possible to find in the state $|\psi_{nl}\rangle$, except that n can range over negative values as large as $-C$. For negative n, $W_{nl}^{(j)}\equiv 0$ for $j=1,2,\ldots,|n|$. There are $C+1$ eigenstates for $n\geq 0$ and $C-|n|+1$ eigenstates for $n<0$.

The dynamics of this and related systems have also been studied using various approaches.[7] Bonifacio and Preparata[8] treat the case of spontaneous emission for macroscopically large numbers of atoms under resonance conditions. Weiss[9] considers the nonresonant case for an asymptotically large number of atoms in a WKB (Wentzel-Kramers-Brillouin) approximation. Jaynes and Cummings[10] considered the exact treatment of the single-particle case in a photon-number representation for the field. The behavior of the field has also been considered in a diagonal coherent-state representation by Chandra and Prakash[2] and Mallory[11] for a single atom in a weak-coupling limit.

In this work the evolution of the density operator is studied using a photon-number representation for the field and an expansion in terms of the eigenstates for which exact calculations can be made using a high-speed digital computer for a small number of atoms.

The motivation for this work is that in present laser-related optical devices, which may involve resonant cavities, effective interaction times can be long enough and coupling between field and atoms strong enough that perturbation and other approximation techniques may be insufficient. For example, it is shown that the statistics of the field distribution can be modified greatly, and that Dicke's superradiance effects[1] may also be changed.

In Sec. II we discuss the formalism of the evolution and some of its characteristics. In Sec. III we discuss the general results. In Sec. IV the more important conclusions are summarized. The notation of Ref. 5 is used here, except that C here is $2s$ in Ref. 5. C is called the cooperation number in this work, as it is the number of cooperating atoms. (The cooperation number of Dicke is s.)

II. DENSITY MATRIX EVOLUTION

We are interested in statistical aspects of the temporal evolution of the field. We will use the reduced-field-density operator for the field state specification. We work in the Schrödinger picture in which the density operator ρ is time dependent and evolves as[12]

$$\rho(t) = e^{-i\mathcal{H}t}\rho(0)e^{i\mathcal{H}t} . \qquad (2.1)$$

For our purposes, which include numerical work, the photon-number representation for the field is useful, i.e.,

$$\rho = \sum_{j,k} \sum_{n,m=0}^{\infty} \epsilon_j \,|n\rangle\langle m|\, \tilde{\epsilon}_k \rho_{jknm} ,$$

where $|n\rangle$ is the pure n-photon state. The reduced-field-density matrix is then

$$P_{mn} = \sum_{q\,r} \tilde{\epsilon}_q \rho \epsilon_r .$$

Summations over atomic states extend from 1 to $C+1$, or over the appropriate reduced range as discussed in Sec. I. [The reduced range is easily established by remembering that $W_{ml}^{(j)} \equiv 0$ for certain values of j and l if $m<0$; those values being $j=1,2,\ldots,|m|$ for all l, and $l=C+m+2, C+m+3, \ldots, C+1$ for all j.] This representation has been obtained by Tavis.[13] The results for the reduced-field matrix elements are

$$P_{mn}(t) = \sum_{jk\,r} V_{m-r+1}^{(jr)} V_{n-r+1}^{(kr)*} \rho_{jk(m+j-r)(n+k-r)}(0) , \qquad (2.2)$$

where

$$V_m^{(jp)} \equiv \sum_{l} e^{-itE_{ml}} W_{ml}^{(j)} W_{ml}^{(p)} . \qquad (2.3)$$

E_{ml} is the eigenvalue for state $|\psi_{ml}\rangle$.

In the case where the field and atom states are both initially diagonal,

$$\rho = \sum_{j} \sum_{m=0}^{\infty} \rho_{jm} \epsilon_j \tilde{\epsilon}_j |m\rangle\langle m| ,$$

Eq. (2.2) reduces to

$$P_m(t) = \sum_{jp} |V_{m-p+1}^{(jp)}|^2 \rho_{j(m+j-p)}(0) . \qquad (2.4)$$

Jaynes and Cummings[10] point out that a diagonal initial matrix remains forever diagonal. This remains true for the cooperative-atom case, as is apparent in the reduction of Eq. (2.2) to Eq. (2.4). The remainder of this paper will consider Eq. (2.4) with the additional condition that no initial atom-field correlations exist, i.e.,

$$\rho_{jn}(0) = A_j(0)P_n(0) .$$

Then

$$P_m(t) = \sum_{jp} |V_{m-p+1}^{(jp)}|^2 A_j(0) P_{(m+j-p)}(0) . \qquad (2.5)$$

Some further manipulation of this last equation is informative. From Eq. (2.3),

$$|V_n^{(jp)}|^2 = \sum_{ll'} e^{-it(E_{nl}-E_{nl'})} W_{nl}^{(j)} W_{nl'}^{(j)} W_{nl}^{(p)} W_{nl'}^{(p)} .$$

By interchanging l and l' and rewriting the sum over l and l' as one-half the original plus one-half of the interchanged expression, and then separating out the terms for which $l=l'$, we find

$$|V_n^{(jp)}|^2 = \sum_{l} (W_{nl}^{(j)} W_{nl}^{(p)})^2 + \sum_{l} \sum_{l'\neq l} W_{nl}^{(j)} W_{nl'}^{(j)} W_{nl}^{(p)} W_{nl'}^{(p)} \cos(E_{nl}-E_{nl'})t . \qquad (2.6)$$

Since the oscillatory terms are symmetric in l and l', we need only sum over $l'>l$ and double the result. Inserting $|V_n^{(jp)}|^2$ into Eq. (2.5) we obtain

$$\begin{aligned} P_r(t) = {} & \sum_{jpl} (W_{(r+j-p)l}^{(j)} W_{(r+j-p)l}^{(p)})^2 P_{(r+j-p)}(0) A_j \\ & + 2 \sum_{jpl} \sum_{l'>l} W_{(r+j-p)l}^{(j)} W_{(r+j-p)l'}^{(j)} W_{(r+j-p)l}^{(p)} W_{(r+j-p)l'}^{(p)} P_{(r+j-p)}(0) A_j \cos(E_{(r+j-p)l} - E_{(r+j-p)l'})t . \end{aligned} \qquad (2.7)$$

Thus the solution is a constant plus a sum of oscillatory terms of frequency equal to the differences in eigenvalues. Scharf[6] has stated that if all the eigenvalues were evenly spaced, a regular oscillation of physical quantities would result. This can be seen in Eq. (2.7). If $E_{nl}-E_{nl'}$ were the same for all n and l, the equation would be a cosine Fourier analysis of an exactly periodic function.

The unevenness of spacing causes a modulation of this basic periodicity if the eigenvalues are not too different. Physical quantities such as expected photon number, variance, entropy (defined by $S \equiv -k\mathrm{Tr}\rho\ln\rho$) will then also display this modulation, but in a more complicated form, since the period of P_r may depend on r. Consider the initially diagonal correlationless distribution of Eq. (2.5) or Eq. (2.7). Note from the summation limits that $P_m(t)$ can only be influenced by $P_n(0)$ where $m - C \leqslant n \leqslant m + C$, even if $A_j = 0$ for all j. (If the matter state is in some pure j state, e.g., $j = 1$, the excited state, the range of the initial photon distribution contribution will be even more restricted.) Hence until the cooperation number C becomes of the same order as the initial mean photon number, we would expect no shifts in the gross shape of the photon distribution, but only a sharing of the probability flux over a range of $n \approx 2C$. This implies, for example, that no shift from a chaotic distribution toward a Poisson distribution characteristic of a coherent state could possibly occur until the above condition is satisfied.

For the distribution of Eq. (2.5) the time behavior of the distribution does not depend on ω nor on the interaction strength K and the off-resonance parameter $\Delta\omega$ separately (where $\Delta\omega \equiv \omega - \omega_0$), in any significant way, but only on $K/\Delta\omega$. This is due to certain commutation properties of the constituent parts of the Hamiltonian, Eq. (1.1). We can write

$$\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2 ,$$

where

$$\mathcal{H}_1 = \omega(a^\dagger a + s_3), \quad \mathcal{H}_2 = -\Delta\omega s_3 + K(a^\dagger s_- + a s_+).$$

It can be easily shown that $[\mathcal{H}_1, \mathcal{H}_2] = 0$.[14] Assume

$$\rho(0) = \rho_A(0)\rho_F(0).$$

Then

$$\rho(t) = e^{-i\mathcal{H}_2 t} e^{-i\mathcal{H}_1 t} \rho_A(0)\rho_F(0) e^{+i\mathcal{H}_1 t} e^{+i\mathcal{H}_2 t} .$$

If $\rho_A(0)$ and $\rho_F(0)$ are diagonal, it can be shown that

$$[\rho_A, e^{i\mathcal{H}_1 t}] = [\rho_F, e^{i\mathcal{H}_1 t}] = 0 .$$

Hence

$$\rho(t) = e^{-i\mathcal{H}_2 t} \rho(0) e^{+i\mathcal{H}_2 t} , \tag{2.8}$$

and if we let $\Delta\omega' = \Delta\omega a$, $K' = Ka$, and $t' = t/a$, we have $\mathcal{H}_2 t$, and hence $\rho(t)$ unchanged. Thus, changing K and $\Delta\omega$ while maintaining $K/\Delta\omega$ unchanged merely scales the time. We do not use a normalized time, since doing so makes it awkward to consider both the cases $K \to 0$ and $\Delta\omega \to 0$. This scaling has appeared in some special cases in earlier work,[2, 9, 15, 16] but to the author's knowledge it has not been recognized that it is a general property of the model even in the nonresonant case. The scaling property can be used to reduce the parameters which must be considered in numerical calculations.

It can also be shown that $\rho(t)$, and hence $P_m(t)$, is independent of the sign of K if $\rho(0)$ is diagonal.

III. GENERAL RESULTS

We consider the problem of the evolution of a thermal field placed in contact with the assembly of cooperating atoms and allowed to interact for an arbitrary time without losses. The problem is, of course, idealized but exact results can be obtained for a small number of atoms. The initial field distribution is given by

$$P_n = m_0^{-1}(1 + 1/m_0)^{-(n+1)} . \tag{3.1}$$

For typical thermal fields the mean photon number per mode, m_0, is much less than unity.[17] We consider this situation. We also consider a case in which there is no initial field. Thus there is initially only spontaneous emission.

For the single-atom case, analytic eigensolutions for the model[5] yield for an initially excited matter state (i.e., $A_1 = 1$, $A_2 = 0$), using Eq. (2.7),

$$\begin{aligned} P_r &= \tfrac{1}{2}(Q_{r+} + Q_{(r-1)-}) + \tfrac{1}{2} Q_{r-} \cos(2K\eta_r t) \\ &\quad - \tfrac{1}{2} Q_{(r-1)-} \cos(2K\eta_{r-1} t) . \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} &Q_{n\pm} \equiv [1 \pm (\alpha/\eta_n)^2] P_n(0), \quad \eta_r \equiv (\alpha^2 + r + 1)^{1/2}, \\ &\alpha \equiv \Delta\omega/(2K), \quad \text{and } P_{-1} \equiv 0 . \end{aligned} \tag{3.3}$$

For an initially chaotic distribution with mean photon number $m_0 \ll 1$, these probabilities oscillate essentially sinusoidally with a small modulation. Analytic expressions for the mean photon number, its variance, etc., can be written from Eq. (3.2), but the general results do not simplify enough to be very instructive. A process analogous to spontaneous emission corresponding to $P_j(0) = \delta_{j0}$ can be obtained from Eq. (3.2). In this case $P_j(t) = 0$ for $j \neq 0$, 1, and thus $P_1(t) = 1 - P_0(t)$. This is a binomial distribution for one trial. Therefore $\bar{n}(t) = P_1(t) = [2(1 + \alpha^2)]^{-1}[1 - \cos 2K\eta_0 t]$ and $\sigma^2(t) = \bar{n}(t)[1 - \bar{n}(t)]$. This variance is smaller than the variance of either a chaotic distribution or of the Poisson distribution associated with the coherent state.[18] (This variance is characteristic of a fermion system. It will occur, however, for *any* system which has zero probability for occupation by more than one particle. Here, there can be no more than one photon because of insufficient energy.)

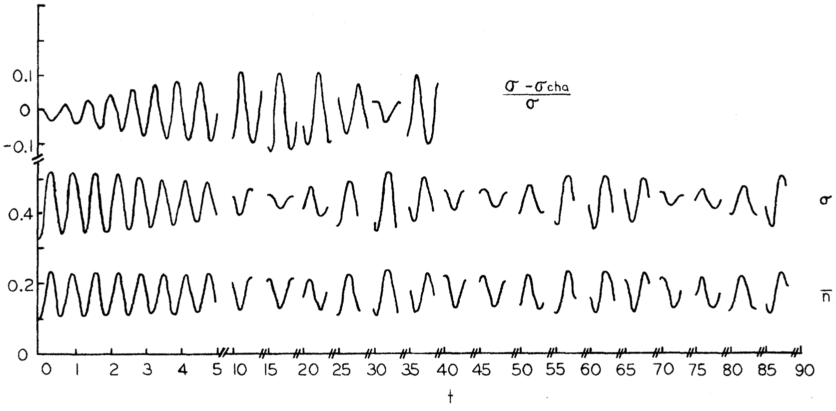

FIG. 1. Field behavior. Atoms initially excited, $C=3$, $m_0=0.1$, $K=1$, $\Delta\omega=10$. Note the time axis breaks for longer times. Time scale is uniform except for breaks.

A computer program was written which calculates the distribution from Eq. (2.5). The details of this program were described recently.[19] Using these results the mean photon number, $\bar{n}$, standard deviation $\sigma=[\overline{n^2}-\bar{n}^2]^{1/2}$, etc., were then calculated in a straightforward manner.

The general behavior of the field quantities is an oscillation about some constant value, reminiscent of the displacement of one of a pair of coupled pendula. This oscillation is also modulated over a longer time period. Since there are no losses in the model, the oscillations continue indefinitely. Figure 1 for three cooperating atoms, initially all excited, is typical for weak coupling and/or far off-resonance, i.e., for $K/\Delta\omega \ll 1$. Both $\bar{n}$ and σ display this pattern. For all initial atomic excitations except all in the ground state the oscillation for $\bar{n}$ is essentially completely above the initial level, as in Fig. 1. The times in Fig. 1 are in units of $1/\omega$ and are relative to the value of K (assuming that $K/\Delta\omega$ remains fixed) as discussed in Sec. II; the values of K and $\Delta\omega$ were chosen for convenience and can also be scaled. For an electric-dipole transition, K is (in units of $\hbar\omega$) of the order[20] of $er_B/(\hbar\omega V)^{1/2}$, where r_B is the Bohr radius and V is the quantization volume. If a quantization volume of 10^{-2} cm^3 (e.g., a cavity of dimensions 1 mm$^2 \times 1$ cm) is assumed, then $K\approx 10^{-10}$, $\Delta\omega$ (in units of ω) is $\approx 10^{-9}$, and the times in Fig. 1 scale upward by 10^{10}. Thus for a radiation wavelength of 6000 Å, $t=1$ in Fig. 1 is about 6 μsec. For $K/\Delta\omega \ll 1$, the oscillation period is almost independent of the number of cooperating atoms for the small numbers calculated. For $K/\Delta\omega \gtrsim 1$, the period increases somewhat initially as C increases from one atom to several. For example, at resonance, the period of oscillation increases by a factor of almost 2 between $C=1$ and $C=5$, but does not change much thereafter. Bonifacio and Preparata[8] calculated oscillation periods for spontaneous emission in the asymptotic limit of large numbers of cooperating atoms for the resonant case. A calculation for 12 cooperating atoms at resonance and for no initial field was made and the resulting oscillation periods compared with asymptotic results calculated[21] from Eq. (59) of Bonifacio and Preparata for (a) the initially excited atomic state [$A_j=\delta_{1j}$ in Eq. (2.5)], and (b) the state $A_j=\delta_{j7}$, corresponding to Dicke's[1] superradiant state. The ratio of the 12-atom result to the asymptotic one is about 1.1 for both the superradiant state and

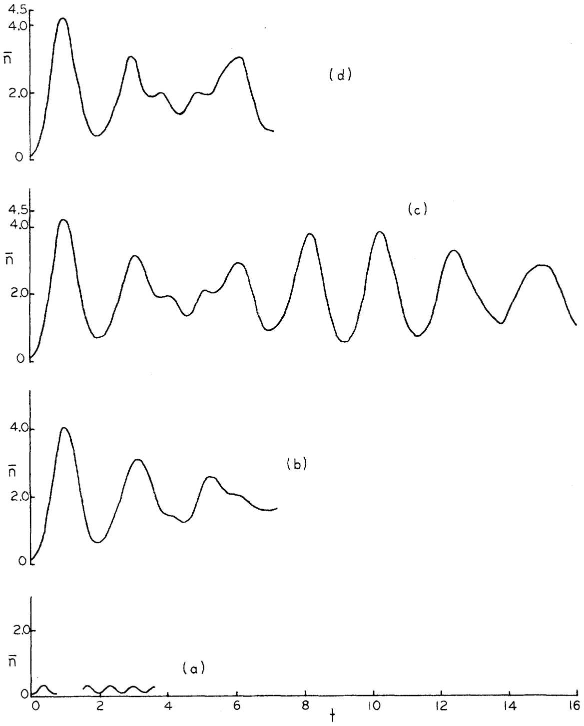

FIG. 2. $\bar{n}$ variation with $\Delta\omega$: $C=5$, $m_0=0.1$, $K=1$. Atoms initially excited. (a) $\Delta\omega=10$, (b) $\Delta\omega=1$, (c) $\Delta\omega=0.1$, (d) $\Delta\omega=0$.

the initially excited state. This approximate unity undoubtedly reflects the fact that the differences in adjacent eigenvalues which determine the oscillations via Eq. (2.7) are already close to their asymptotic values at $C = 12$. The oscillation amplitude (and hence the time-averaged $\bar{n}$) increases with increasing number of atoms. This is discussed further below. The modulation of this oscillation does not decrease significantly as the number of cooperating atoms increases from one to 12, the largest number considered. Figure 2 shows the behavior of $\bar{n}$ for $C = 5$ as $K/\Delta\omega$ increases from values much less than one through unity to infinity at resonance for a fixed value of K. The behavior shown is typical for other values of C in the range considered. In the regime $K/\Delta\omega \ll 1$, the period of the oscillation does not depend greatly on K if $\Delta\omega$ is constant. If K is held fixed and $\Delta\omega$ varied in the same range of $K/\Delta\omega$, then the oscillation period varies approximately inversely with $\Delta\omega$.

When $K/\Delta\omega \ll 1$, the amplitude of the oscillations increases rapidly as $K/\Delta\omega$ increases, going approximately as $(K/\Delta\omega)^2$. For $K/\Delta\omega \gtrsim 1$, the amplitudes do not depend greatly on this parameter.

Note that the modulation of the basic oscillation becomes more rapid as resonance is approached.

The statistics of the radiation, in general, do not maintain the chaotic nature of the initial distribution, unlike the limiting case of weak coupling for the single particle.[2, 11] Figure 1 shows the standard deviation σ in photon number and its departure from the value for chaotic distribution with the same (instantaneous) $\bar{n}$, i.e.,

$$\sigma_{\text{cha}}(t) = [\bar{n}(t)(\bar{n}(t)+1)]^{1/2},$$

where $\bar{n}(t)$ is calculated from the model interaction at time t. The standard deviation oscillates about the chaotic value, being less or greater than that value about equally. For this case of three cooperating atoms the departure reaches a maximum of about 10% of the chaotic value. This is about three times as great as the single-particle case for the same ratio $K/\Delta\omega$. This essentially linear dependence continues through six cooperating atoms. It is clear that as the number of cooperating atoms increases, the value of the coupling required to destroy the chaotic nature of the distribution decreases.

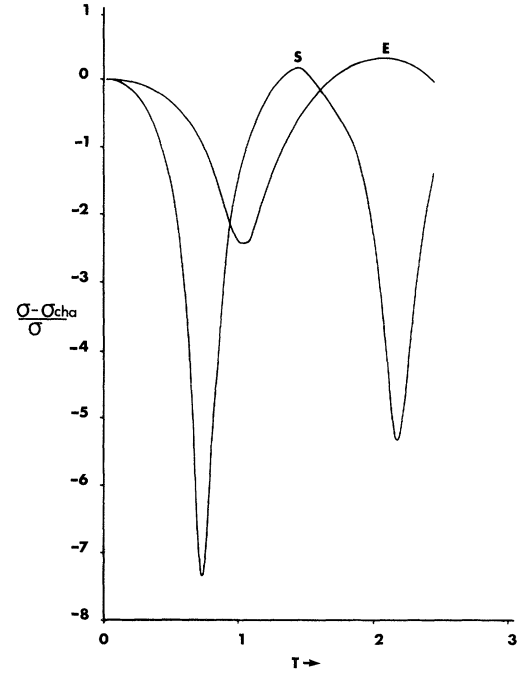

FIG. 3. Standard deviation of the photon number at resonance. $C = 6$, $K = 1$. Chaotic initial field with $m_0 = 0.1$. E is the initially excited atomic state; S is the superradiant state.

As the field comes into resonance with the cooperating atoms, the changes in σ increase markedly. Figure 3 illustrates this behavior. While σ still oscillates about the chaotic value, it is several times less than the chaotic value on the average over the oscillation cycle. Figure 3 also shows the behavior for an initial atomic state which is a coherent mixture of three excited and three ground-state atoms ($C = 6$, $j = 4$). This is a superradiant state as defined by Dicke.[1] Note that the decreased σ is more pronounced than for the initially excited state, and that the initial rate of decrease is greater. This greater reduction in σ for the superradiant state is apparent for off-resonance cases also. Thus under certain conditions the mean photon number in this model is more precisely defined than for a thermal field. It can also be better defined than for a coherent state, represented by a Poisson distribution with $\sigma = \bar{n}^{1/2}$.

Let us examine the initially-superradiant-state evolution further. Using perturbation theory, Dicke found that states in which only part of the cooperating atoms were excited exhibited enhanced spontaneous emission, but that states in which all atoms were excited had normal spontaneous emission rates, i.e., rates proportional to the number of atoms, C. The enhanced rates were proportional to factors as large as C^2.

If the model is examined for times long enough for depletion of the excited states to occur (a case in which perturbation theory is invalid), the situation changes somewhat. Figure 4 compares the behavior of the initially excited and superradiant states over one cycle of oscillation. (The graphs shown are for an initially chaotic distribution with

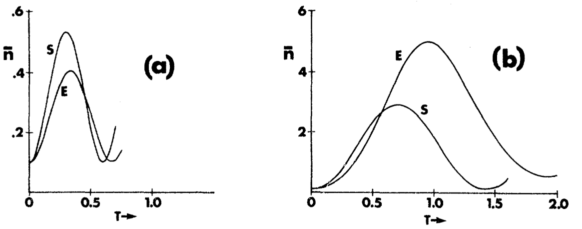

FIG. 4. Field for first cycle of oscillation; comparison of initially excited (E) and superradiant (S) states for off-resonance (a) and resonance (b); $C=6$, $j=4$ for superradiant state, $K=1$, $\Delta\omega=10$ in (a).

mean photon number $m_0=0.1$; the results discussed below are essentially unchanged if there is no initial field.) For appreciable off-resonance as illustrated in Fig. 4(a), not only the initial rise, but also the peak value of the mean photon number $\bar{n}$, is greater for the superradiant state. However, near resonance [see Fig. 4(b)], although $\bar{n}$ for the superradiant state rises faster initially (yielding the higher spontaneous emission found by Dicke), $\bar{n}$ for the initially excited state eventually exceeds it and reaches a higher peak value. This is due to the greater energy stored in the atom-field system in the initially excited case. For the field sufficiently off-resonance with the atoms as in (a) not enough energy is transferred to the field to allow this extra-energy-storage factor to dominate. Thus, if the effective interaction times are long enough, the superradiant state defined by Dicke loses its unique characteristic of generating higher fields.

Figure 5 shows how the value of $\bar{n}$ at the first oscillation peak depends on the cooperation number C in both resonant and nonresonant cases.

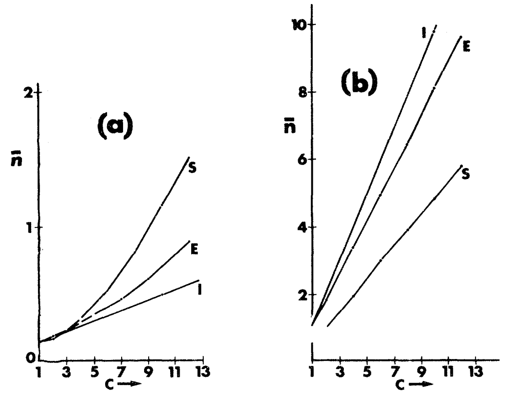

FIG. 5. Peak values of field. Comparison of initially excited (E), superradiant (S), and incoherent (I), as function of number of cooperating atoms for off-resonance (a) and resonance (b); $j=\frac{1}{2}C+1$ for superradiant state, $K=1$, $\Delta\omega=10$ in (a).

For comparison, a graph of the equation

$$n_I=(\bar{n}_1-m_0)C+m_0,$$

representing the photon number for C atoms acting incoherently, is shown. Here $\bar{n}_1$ is the value of $\bar{n}$ at the first peak for a single atom, and m_0 is the mean photon number of the initial chaotic distribution, in this case 0.1. For near-resonance conditions, the peak field obtained with incoherent radiation always exceeds the coherent radiation peak for either the initially excited or superradiant states. Far off-resonance, both these states have peak radiation values exceeding the peak incoherent field. As is clear from Fig. 5, this behavior should hold for arbitrarily large C. For a quantization volume of 10^{-2} cm^3, the oscillation times are of the order of μsec, and so the concept of superradiance might need to be modified with some metastable atomic states.

IV. CONCLUSIONS

In this work the evolution of a radiation field in the photon-number representation interacting with C cooperating atoms has been studied.

The density-matrix evolution was found to depend essentially only on the ratio of the coupling constant K to the off-resonance parameter $\Delta\omega$, not on both separately. The general form of the photon-number probabilities, P_n, as a function of time was found to be a time-averaged probability different from the initial value and undamped oscillations (with modulation) about that average.

Field parameters such as the mean photon number and higher moments for the single-particle case can be understood analytically. For cooperating atoms a high-speed digital computer was used to examine the detailed behavior of the field for small numbers (up to 12) of cooperating atoms. Various initial states of the atoms and field were considered and the behavior as a function of the ratio $K/\Delta\omega$ was examined. The general behavior of the mean photon number and its standard deviation is an undamped oscillation about an average which is different from the initial value. The amplitude and period of the oscillation and the displacement of the average depend on $K/\Delta\omega$ in a manner discussed in Sec. III. The basic oscillation is in addition modulated, the degree and quasiperiod of the modulation depending on $K/\Delta\omega$. As resonance is approached, the modulation quasiperiod becomes shorter and the degree of modulation becomes greater. The standard deviation of the photon number for an initially chaotic distribution, characteristic of a thermal source, first becomes less than the chaotic value and then oscillates, but the time average of this oscillation can still be much less than the chaotic value.

For interaction times long enough that perturbation theory is not valid, Dicke's superradiance effects are modified. Fully excited initial states may yield stronger fields than the coherent partially excited "superradiant" states. Even the initially excited state shows an enhanced field over the incoherent (i.e., noncooperating) atomic state. These results are expected to hold for large numbers of cooperating atoms also.

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[1] R. H. Dicke, Phys. Rev. 93, 99 (1954).

[2] N. Chandra and H. Prakash, Phys. Rev. Lett. 22, 1068 (1969).

[3] D. F. Walls and R. Barakat, Phys. Rev. A 1, 446 (1970).

[4] M. Tavis and F. W. Cummings, Phys. Rev. 170, 379 (1968).

[5] W. R. Mallory, Phys. Rev. 188, 1976 (1969).

[6] G. Scharf, Helv. Phys. Acta 43, 806 (1970).

[7] See the review article by S. Stenholm [Phys. Lett. 6C, 1 (1973)] for other references.

[8] R. Bonifacio and G. Preparata, Phys. Rev. A 2, 336 (1970).

[9] R. Weiss, Helv. Phys. Acta 46, 546 (1973).

[10] E. T. Jaynes and F. W. Cummings, Proc. IEEE 51, 89 (1963).

[11] W. R. Mallory, Ph.D. dissertation (Syracuse University, Syracuse, N. Y., 1970); parts of this were published as Ref. 5 above, but the density-matrix work has not been published.

[12] R. C. Tolman, *The Principles of Statistical Mechanics* (Oxford U. P., Oxford, England, 1938), p. 339.

[13] M. Tavis, Ph.D. dissertation (University of California, 1968) (unpublished).

[14] W. H. Louisell, *Radiation and Noise in Quantum Electronics* (McGraw-Hill, New York, 1964), p. 213.

[15] I. R. Senitzky, Phys. Rev. A 3, 421 (1971).

[16] S. Swain, J. Phys. A 5, 1587 (1972).

[17] L. Mandel and E. Wolf, Rev. Mod. Phys. 37, 231 (1965); see Sec. 2.3.

[18] See, e.g., J. R. Klauder and E. C. G. Sudarshan, *Fundamentals of Quantum Optics* (Benjamin, New York, 1968).

[19] W. R. Mallory, Proc. Montana Acad. Sci., 1974 (to be published).

[20] W. Heitler, *The Quantum Theory of Radiation*, 3rd ed. (Oxford U. P., Oxford, England, 1954).

[21] *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. A. Stegun (Dover, New York, 1965), p. 608.