PHYSICAL REVIEW A VOLUME 11, NUMBER 5 MAY 1975

Scaling theory of hydrodynamic turbulence

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(Received 6 January 1975)

A phenomenological scaling theory for incompressible fluid turbulence in the limit of infinite Reynolds number is proposed. The local vorticity and local dissipation are taken as scaling variables with scaling dimensions $2/3 - \zeta/2$ and $\mu/2$, respectively. The 1941 Kolmogorov theory corresponds to $\mu = \zeta = 0$. Experimentally, ζ is small and $\mu \approx 1/2$. This choice of scaling variables gives immediate and simple predictions about measured or readily measurable scaling exponents. An additional dimensionality-dependent scaling relation, $\mu = d - 8/3 + 2\zeta$, is proposed and supported by a physically plausible argument. This relation, which is consistent with experiment, suggests that the 1941 Kolmogorov theory is exact for $2 < d < 8/3$ and has small corrections for $d = 3$. Dynamical reasons for this behavior are suggested. The relation of scaling behavior to intermittency of the dissipation is briefly discussed.

I. INTRODUCTION

Incompressible fluid turbulence in the limit of very small viscosity (very large Reynolds number) exhibits scaling properties reminiscent of critical-point fluctuations.[1] The behavior at small scales is universal and isotropic, and independent of the details of the large-scale motions. Fluctuations over a very large range of scales are important, and it is the insulation of the small scales from the direct influence of the large scales which is the essential physical feature of the problem. Important qualitative aspects of the problem were already recognized in 1941 by Kolmogorov,[2] who proposed his famous $k^{-5/3}$ energy spectrum using dimensional arguments. We begin by a restatement of the 1941 Kolmogorov idea in a language appropriate to the generalization we propose.

We work in $\vec{r}$ space, and take as our basic random variable

$$\psi(x) = \partial u/\partial x \ , \tag{1}$$

where u is one Cartesian component of the velocity field and x is one Cartesian component of the position. For isotropic turbulence the statistical properties of $\psi(x)$ will be essentially the same as those of the local vorticity. The basic correlation function of interest is

$$G(r, \lambda) = \langle \psi(x)\psi(x+r)\rangle \ , \tag{2}$$

where λ is the kinematic viscosity. An essential assumption is that the effects of viscosity are felt only at small scales. Energy is put into the fluid at an average rate $\bar{\epsilon}$ per unit volume at scales of order L, cascaded to small scales, and eventually dissipated on scales of order ξ. Characteristic times decrease with decreasing scale size; so the small scales are in a steady state. The insulation of large from small scales is assumed to be good enough that the energy input $\bar{\epsilon}$ is independent of the viscosity λ.

For convenience assume that the Reynolds number is varied by keeping $\bar{\epsilon}$ and L fixed and varying λ. There are two natural scaling assumptions. The first concerns the inertial subrange

$$\xi \ll r \ll L \ . \tag{3}$$

In this range correlation functions are assumed to have their zero-viscosity form, and this is taken as a power law

$$G(r, 0) \sim r^{-(2-\eta)} \ . \tag{4}$$

The second scaling assumption is that the dissipation length

$$\xi = \xi_0 \lambda^{\nu} \tag{5}$$

varies as a power of the viscosity. The scaling exponents η and ν are to be determined from experiment or dynamical theory. The reader familiar with critical phenomena will be aided in forming an analogy by the choice of notation.

The exponents η and ν are not, however, independent. From the mechanism of energy dissipation, we have[2]

$$\bar{\epsilon} = 15\lambda\langle\psi^2(0)\rangle \ . \tag{6}$$

We define the dissipation length ξ through

$$G(0, \lambda) \sim G(\xi, 0) \sim \xi^{-(2-\eta)} \ , \tag{7}$$

which implies that $(2-\eta)\nu = 1$. The power-law divergence of $G(r, 0)$ at small r is rounded off at r of the order of ξ in order to give the dynamically constrained value of mean square vorticity. Equations (4), (5), and (7) are all satisfied in the 1941 Kolmogorov theory with $\eta = \frac{2}{3}$ and $\nu = \frac{3}{4}$. We will assume that they remain satisfied when deviations from the 1941 theory are included. Suppose that $\eta = \frac{2}{3} + \zeta$, where ζ is a small correction. The corresponding exponent for the energy spectrum will be $\frac{5}{3}+\zeta$, and the dissipation-length exponent will be

$$\nu=(\tfrac{4}{3}-\zeta)^{-1} . \quad (8)$$

The essential difficulty with the 1941 Kolmogorov theory is not, however, directly associated with the "order-parameter" correlation function $G(r,\lambda)$. The dimensional arguments of the 1941 theory are extremely restrictive for correlations involving other dynamical variables. Consider for example the variable

$$\epsilon(x)=\lambda\psi^2(x) , \quad (9)$$

which is a one-dimensional counterpart of the local dissipation rate. The correlation function

$$E(r,\lambda)=\langle\epsilon(x)\epsilon(x+r)\rangle \quad (10)$$

has the property that

$$E(0,\lambda)=\lambda^2\langle\psi^4(0)\rangle\sim\bar{\epsilon}^2\langle\psi^4(0)\rangle/\langle\psi^2(0)\rangle^2 . \quad (11)$$

The variation of $E(0,\lambda)$ with Reynolds number is the same as that of the kurtosis of the local velocity derivative. It is natural to assume that this has the power-law form

$$E(0,\lambda)\sim\lambda^{-\alpha} . \quad (12)$$

The dimensional arguments of the 1941 theory require $\alpha=0$. To see this note that $E(0,\lambda)\bar{\epsilon}^{-2}$ is a dimensionless quantity. The only dimensionless parameter on which it can depend is the Reynolds number

$$R=\bar{\epsilon}^{1/3}L^{4/3}\lambda^{-1} .$$

In the 1941 theory, the basic assumption is that there is no dependence on the external length scale L. Without such dependence, however, there can be no dependence on Reynolds number of dimensionless quantities defined at a single spatial point.

The assumption that $\alpha=0$ does not seem reasonable, and does not work experimentally. In Sec. II we consider $\alpha\neq0$, but make the assumption that $\epsilon(x)$ is a scaling variable. This assumption is remarkably restrictive and leads to several immediate and simple predictions concerning the properties of readily measurable correlation functions. In Sec. III we consider how scaling exponents might depend on spatial dimensionality. The 1941 theory plays the role of a mean-field theory with dimensionality-independent exponents. We propose a d-dependent hyperscaling relation which is consistent with experiment for $d=3$. A slightly modified analogy with critical phenomena suggests that the hyperscaling is valid for $d>\frac{8}{3}$, and that the 1941 theory is valid for $d<\frac{8}{3}$. In Sec. IV we discuss briefly some possible reasons for the conclusions of Secs. II and III. Finally, in Sec. V we turn to the question of intermittency as reflected in more complicated statistical properties of $\epsilon(x)$ and $\psi(x)$.

II. LOCAL DISSIPATION AS A SCALING VARIABLE

In 1962 Kolmogorov proposed a modification of his 1941 theory to take into account the fluctuations in dissipation.[3] This 1962 proposal emphasized the intermittency of the dissipation as reflected in the probability distribution of the average dissipation

$$\epsilon_r=\frac{1}{r}\int_0^r\epsilon(x)dx . \quad (13)$$

In Sec. V we will consider briefly the higher moments of ϵ_r, but for now we restrict our attention to $\langle\epsilon_r^2\rangle$, noting that

$$\frac{1}{2}\frac{d^2}{dr^2}(r^2\langle\epsilon_r^2\rangle)=E(r,\lambda). \quad (14)$$

[Equation (14) is directly analogous to the familiar relation between mean-square displacement and the velocity correlation function in Brownian-motion theory.]

It is far from obvious that $\langle\epsilon_r^2\rangle$ or $E(r,\lambda)$ should have an inertial-range power-law behavior, but this part of the 1962 Kolmogorov prediction appears to have considerable experimental support. Measurements of the spectrum of dissipation fluctuations[4-6] indicate that

$$E(r,0)\sim r^{-\mu} , \quad (15)$$

with μ independent of Reynolds number. The most accurate published value[5] is $\mu=0.51\pm0.02$. The dissipation spectrum also appears to cut off at high wave number in essentially the same way as the energy spectrum.[5] This suggests the scaling relation

$$E(0,\lambda)\sim E(\xi,0)\sim\lambda^{-\mu\nu} . \quad (16)$$

The exponent relation

$$\alpha=\mu\nu \quad (17)$$

has, in fact, been previously suggested.[6] Taking $\mu=\frac{1}{2}$ and $\nu=\frac{3}{4}$, Eqs. (12) and (17) state that the kurtosis of the local velocity derivative should increase as the $\frac{3}{8}$ power of the Reynolds number. This is consistent with but not sensitively tested by experiment. Note that the 1941 theory requires $\alpha=\mu=0$.

The essential physical statement in Eq. (16) is that the same characteristic length governs the dissipation fluctuations and the vorticity fluctuations. In analogy to critical phenomena this immediately suggests that we take $\psi(x)$ and $\epsilon(x)$ to be scaling variables with scaling dimensions[7]

$$y_\psi = 1 - \tfrac{1}{2}\eta \text{ and } y_\epsilon = \tfrac{1}{2}\mu \ , \tag{18}$$

respectively. We then consider the cross correlation function

$$C(r,\lambda) = \langle \psi(x)\epsilon(x+r)\rangle \ . \tag{19}$$

This should have the inertial-range form

$$C(r,0) \sim r^{-q} \ , \tag{20}$$

with

$$q = y_\psi + y_\epsilon = \tfrac{2}{3} + \tfrac{1}{2}\mu - \tfrac{1}{2}\zeta \approx 0.90 \ . \tag{21}$$

This inertial-range exponent has not been measured, but should not be difficult to obtain by reprocessing existing data tapes. We further expect the scaling relation

$$C(0,\lambda) \sim C(\xi,0) \sim \lambda^{-q\nu} \ . \tag{22}$$

Note, however, that

$$C(0,\lambda) = \lambda\langle \psi^3(0)\rangle = \lambda^{-1/2}\bar{\epsilon}^{3/2} S(\lambda) \ , \tag{23}$$

where

$$S(\lambda) = \langle \psi^3(0)\rangle / \langle \psi^2(0)\rangle^{3/2} \sim \lambda^{-\theta} \tag{24}$$

is the skewness of the velocity derivative. Combining Eq. (8) with Eqs. (21)–(24), we have

$$\theta = q\nu - \tfrac{1}{2} = \tfrac{1}{2}\mu\nu = \tfrac{1}{2}\alpha \ . \tag{25}$$

Equation (25) states that the skewness varies as the square root of the kurtosis of the velocity derivative. This is again consistent with experiment,[4] but has not been sensitively tested.

To summarize, the observation that $E(r,\lambda)$ has a power-law inertial subrange suggests that the local dissipation should be treated as a scaling variable. This immediately predicts the variation with Reynolds number of the skewness and kurtosis of velocity derivatives, and the inertial range exponent for the correlation between vorticity and local dissipation. These are readily measurable quantities which do not involve the theoretical or experimental difficulties associated with intermittency and higher-order statistics. Their accurate experimental determination would be an important contribution towards an improved basic understanding of turbulence.

III. EFFECTS OF VARIABLE DIMENSIONALITY

To this point the exponents ν and α, or equivalently ζ and μ, are independent. The analogy with critical phenomena suggests that we look for a "hyperscaling" relation between them which depends explicitly on the spatial dimensionality d. In critical phenomena such a relationship is most naturally generated by the invariance of the partition function under a renormalization-group transformation,[7] but it can also be obtained in a more phenomenological context.[8] For turbulence we do not yet have a renormalization-group argument, but we can give a physically plausible fluctuation argument with an interesting if not totally convincing result.

Consider the quantity

$$\langle \psi^4(x)\rangle = \left\langle \int d^d k \, |(\psi^2)_{\vec{k}}|^2 \right\rangle \tag{26}$$

in a d-dimensional fluid. The volume of $\vec{k}$-space contributing to the integral is ξ^{-d}. In the integrand,

$$(\psi^2)_{\vec{k}} = \sum_{\vec{q}} \psi_{\vec{q}} \psi_{\vec{k}-\vec{q}} \ .$$

It is reasonable to suppose that $(\psi^2)_{\vec{k}}$ for $k < \xi^{-1}$ is dominated by vorticity fluctuations $\psi_{\vec{q}}$ with q in the inertial subrange. We thus suppose that the integrand in Eq. (26) can be taken independent of viscosity. This implies that

$$\langle \psi^4(x)\rangle \sim \xi^{-d} \sim \lambda^{-d\nu} \ . \tag{27}$$

Recalling that

$$\langle \epsilon^2(x)\rangle = \lambda^2 \langle \psi^4(x)\rangle \ ,$$

we have

$$\alpha = d\nu - 2 \ . \tag{28}$$

Using the various exponent relations obtained in Sec. II this can be written in the more convenient form

$$\mu = d - \tfrac{8}{3} + 2\zeta \ , \tag{29}$$

which relates the inertial-range exponent of the energy spectrum to the inertial-range exponent of the dissipation fluctuations.

Equation (29) has several interesting features. First, we note that a nonzero value of μ does not require a nonzero value of ζ. This point has been previously emphasized by Kraichnan[9] in a quite different context. We then note that, for $d=3$, the experimental value $\mu = 0.51 \pm 0.02$ implies that $\zeta = 0.09 \pm 0.01$. A small positive value of ζ is frequently stated to be suggested by experiment, but no firm values have been published. Finally, we note that the 1962 Kolmogorov and Obukhov ideas when combined with Eq. (29) give an interesting if somewhat mysterious prediction for μ and ζ. Obukhov[10] suggested that ζ should be determined by replacing $(\bar{\epsilon})^{2/3}$ in the 1941 theory by $\langle \epsilon_r^{2/3}\rangle$. Using Kolmogorov's[3] suggestion that ϵ_r should be a log-normal random variable, this implies $\zeta = \tfrac{1}{9}\mu$. When combined with Eq. (29) for $d=3$ this gives $\mu = \tfrac{3}{7}$ and $\zeta = \tfrac{1}{21}$. These results are quite reasonable, but the physical motivation for Obukhov's suggestion remains unclear.

The dimensional analysis leading to the 1941 Kolmogorov theory leads to dimensionality-independent exponents

$$\alpha_K = 0,\ \nu_K = \tfrac{3}{4},\ \eta_K = \tfrac{2}{3},\ \mu_K = 0,\ \zeta_K = 0\ .$$

In this sense (and so far in no other sense) the 1941 theory can be thought of as a mean field theory for turbulence. The 1941 theory is compatible with the d-dependent scaling relations (28) and (29) only for $d = d^* = \frac{8}{3}$. In critical phenomena mean field theory is valid for $d > d^*$, and the hyperscaling relation $d\nu = 2 - \alpha$ holds for $d < d^*$. For an ordinary critical point $d^* = 4$. We would suggest that for turbulence the situation is reversed, probably due to the universality of large k fluctuations in turbulence as opposed to small k fluctuations in critical phenomena. Thus we expect the 1941 theory to be valid for $d < \frac{8}{3}$, and the hyperscaling relations of Eqs. (28) or (29) for $d > \frac{8}{3}$. In Sec. IV we give some qualitative dynamical arguments in support of this suggestion.

One might hope that the 1941 theory would be valid in two dimensions, but $d = 2$ is a quite special case because of the absence of vortex stretching, which is the basic mechanism of turbulence for $d > 2$. A modified form of the 1941 ideas has been developed for two-dimensional turbulence,[11] and the subject is of considerable interest in its own right. There has been considerable discussion about corrections to the scaling exponents of the modified theory. Using the critical-phenomena analogy the present author has suggested[1] that these corrections should be large. The work presented here makes this earlier suggestion inoperative.

We should note that the hyperscaling relation $d\nu = 2 - \alpha$ is not above suspicion even in critical phenomena. It works exactly for the two-dimensional Ising model, and for renormalization-group calculations close to four dimensions. For the three-dimensional Ising model, however, the best numerical evidence suggests a small deviation, which can be encompassed by an additional exponent ω^*, sometimes called the anomalous dimension of the vacuum.[8] If such an exponent were to exist for turbulence, the conclusions of Sec. II relating different experimental exponents would be unchanged, but Eqs. (28) and (29) would no longer hold. We would not be at all surprised if Eq. (29) were not numerically accurate in three dimensions, but we would expect the prediction of $\frac{8}{3}$ as a crossover dimension to remain correct. In order to understand better the significance of this crossover we need a deeper physical understanding of why the 1941 theory gives the exponent $\nu = \frac{3}{4}$.

Finally, we note that the idea of a fractional dimension being related to dissipation fluctuations in turbulence has been introduced elsewhere.[12] The connection to the present work is not clear, and would be of some interest to understand.

IV. DYNAMICAL ARGUMENTS

The scaling behavior proposed in Sec. II strongly suggests an analysis in terms of a fixed point of an appropriate probability functional under a renormalization-group transformation. In an earlier paper[1] we made a crude attempt to analyze turbulence from this point of view. Our formal starting point was to look for steady-state solutions to the Fokker-Planck equation first proposed by Edwards.[13] This formal point of view still seems appropriate since it eliminates the unwanted time dependence and allows a simple parameterization of the driving force at large scales. We are left with a linear functional differential equation in which the infinite number of degrees of freedom plays an essential role. We showed how the 1941 Kolmogorov theory could arise if the stirring length L were taken to infinity from the beginning. Our use of the term "renormalization group" in this context was, however, somewhat misleading. If a renormalization-group transformation is thought of as a refined scale transformation at fixed cutoff,[7] then taking the cutoff to infinity first is a degenerate case in which the dynamically crucial distinction between a scale transformation and a renormalization-group transformation is lost. To carry out a renormalization-group transformation for turbulence, we must find a procedure for integrating out the low-wave-number degrees of freedom while keeping the stirring length L finite. To date we have had no success with this problem.

We also suggested that the observed deviations from the 1941 Kolmogorov theory were incompatible with a fixed-point description of turbulence in the limit of vanishing viscosity. The experimental evidence that we used concerned the two-point probability $p(\Delta u)$, where

$$\Delta u = u(x + r) - u(x)\ .$$

When r is an inertial range scale, the 1941 theory predicts that $p(\Delta u)$ has the same shape for all r if Δu is appropriately scaled. Experimentally there are definite deviations from this condition of statistical self-similarity,[14] but our earlier claim that these deviations rule out fixed-point behavior is incorrect.

Fixed-point scaling requires that correlation functions like

$$\langle \psi(x)\psi(x + r)\psi(x + r + \rho)\rangle\ , \tag{30}$$

with r and ρ inertial range scales, have scaling properties determined by the scaling dimension of $\psi(x)$. The scaling behavior of

$$\langle \psi^2(x)\psi(x+r)\rangle \tag{31}$$

need not, however, be simply related to that of $\langle\psi(x)\psi(x+r)\rangle$, as we have already discussed in Sec. II. The experimental evidence concerns correlation functions like Eq. (31) rather than like Eq. (30).

The analogy to critical phenomena suggests[15] that one look for a scaling variable quadratic in the velocity field whose correlation function diverges more weakly with Reynolds number than does the "order-parameter" correlation function $G(r,\lambda)$. This would be analogous to the energy variable as a scaling variable in critical phenomena with the associated specific-heat singularity. As suggested in Sec. II, the local dissipation $\epsilon(x)$ is a reasonable candidate for this role. It remains to be seen if the more detailed statistical properties of $\epsilon(x)$ are compatible with fixed-point scaling. We discuss this point briefly in Sec. V.

The possibility of a quadratic scaling variable is also suggested by the structure of the Navier-Stokes equations. These equations couple the velocity field at a point to terms quadratic in the velocity field at a point. The equation of motion for $G(r,\lambda)$ contains correlation functions like Eq. (31), but does not contain correlation functions like Eq. (30).

To understand the dependence of scaling exponents on dimensionality we must consider the dynamical mechanisms whose interplay determines the small-scale fluctuations. The $(\vec{v}\cdot\vec{\nabla})\vec{v}$ convective nonlinearity in the Navier-Stokes equations is responsible for the cascade of energy to small scales. The $\vec{\nabla}p$ pressure term, when combined with the incompressibility condition, is strongly nonlocal in ordinary space, and does not contribute to the transfer of energy in $\vec{k}$ space.[2] In the 1941 Kolmogorov theory the external length scale L does not appear in the final results. As we have seen this requires a strong suppression of dissipation fluctuations. This can only come about if the nonlocal pressure term couples separate regions of space strongly enough to prevent any tendency towards independent cascades in different regions with different turbulent intensities. Spatial intermittency and a build up of dissipation fluctuations would indicate that the pressure term is no longer able to compete equally with the local cascade due to the inertial terms. To illustrate this effect in an extreme case consider the Burgers equation in one dimension.[16] Here the pressure term is absent, and the convective terms lead to well defined shock fronts, an extreme case of spatial intermittency.

As has been emphasized by Kraichnan[9] a turbulent steady state is very far from equilibrium with the high-wave-number modes very weakly populated. An essentially one-way cascade in $\vec{k}$ space is set up in which energy flows from the originally populated small-k region to the much larger unpopulated large-k region. As the dimensionality increases the volume of $\vec{k}$ space into which energy can cascade grows. Perhaps the local cascade via the convective terms can no longer be completely balanced by the averaging effect of the pressure terms when the dimensionality is sufficiently high. Intermittency as reflected in deviations from the 1941 theory would be expected at high enough dimensionality. The qualitative ideas leading to Eq. (27) would be expected to be valid only in this high-dimensionality situation.

We then note that the scaling argument leading to Eq. (27) is compatible with the 1941 theory only for $d=\frac{8}{3}$. We have a plausible argument, reinforced by the reasonable numerical results for $d=3$, that the scaling argument should be valid for $d>\frac{8}{3}$. The 1941 theory is assumed valid for $d<\frac{8}{3}$ by analogy to the crossover at $d=4$ in critical phenomena. These arguments must be considered as speculative at the present time, but they do indicate that the results of Secs. II and III are not unreasonable as possible consequences of the Navier-Stokes equations.

V. INTERMITTENCY

Much of the recent literature on very high-Reynolds-number turbulence has focused on the problem of spatial intermittency. The observed random variables have a tendency towards "spottiness" which becomes more pronounced on small scales and at higher Reynolds numbers. A proper statistical measure of this effect is not easily constructed. The most popular choice is the probability distribution of the random variable ϵ_r defined by Eq. (13). It was originally postulated by Kolmogorov[3] that the logarithm of ϵ_r should have a normal distribution. This suggests that ϵ_r should be a product of independently distributed random variables. This idea was developed by Yaglom,[17] and put on a more satisfactory mathematical basis by Novikov.[18] Novikov introduced the ratio variables

$$q_{r,l}=\epsilon_r/\epsilon_l \,, \tag{32}$$

which he proved to have some very interesting properties. In particular, if the scale similarity conditions

$$\langle q^p_{r,l}\rangle=(l/r)^{\mu_p} \tag{33}$$

are satisfied for all integer p and for $r<l$, then $q_{r,\rho}$ is statistically independent of $q_{\rho,l}$ for $r<\rho<l$, and the probability distribution of $q_{r,l}$ is asymp-

totically log-normal for $r \ll l$. The approach to log-normality is sufficiently slow, however, that the scaling exponents μ_p are not equal to the exponents μ_p^* for the asymptotic distribution. In the Yaglom-Novikov picture intermittency is thus a natural consequence of scale similarity, and is not a separate phenomenon of intrinsic interest.

It is reasonable to expect that for $r \ll l$ the fluctuations in ϵ_l are negligible compared to the fluctuations in ϵ_r. Thus the probability distribution of ϵ_r should define the same scaling exponents as defined by Eq. (33). This ergodic hypothesis [Novikov's Eq. (1.6)] is at the heart of Yaglom's original argument for the intermittency of ϵ_r. Recent experiments[19] strongly suggest, however, that this ergodic hypothesis is not valid. Perhaps this is because the random variables ϵ_r and ϵ_l remain strongly correlated even when $r \ll l$. This is not too surprising in light of the slow decay of the correlation function $E(r, 0)$ associated with the small value of μ. Experimentally $\mu_2 = 0.22$, which is much smaller than the value $\mu = 0.51$ determined from $\langle \epsilon_r^2 \rangle$ and discussed extensively in Secs. II and III. The inequality $\mu_2 < \mu$ is consistent with long-range positive correlations between ϵ_r and ϵ_l suppressing the fluctuations in $q_{r,l}$. In fact, it is just this possibility which makes $q_{r,l}$ a bounded random variable and allows Novikov to give proofs relating intermittency and scale similarity.

We thus find that the entire basis for understanding the intermittency of the dissipation in terms of the properties of ratio variables is put into question by experiment. Ironically, these same experiments strongly support the elegant work of Novikov on the properties of the ratio variables. The question of the intermittency of the dissipation field requires a thorough reexamination without the use of ratio variables. Perhaps we will again find that the 1962 Kolmogorov idea is essentially correct and is essentially equivalent to the scaling hypothesis of Sec. II. Perhaps we will find that the intermittency of $\epsilon(x)$ requires an intrinsically weaker form of scaling than that associated with a fixed point. The question is open and of considerable interest.

Finally, we consider the effects of intermittency on the energy spectrum $E(k)$. In the 1941 theory $E(k)$ has the form

$$E(k) \sim k^{-(1+\eta)} f(k\xi) , \tag{34}$$

with the shape of the cutoff function $f(x)$ independent of Reynolds number. Physically we think of intermittency as associated with cascades in different regions of space going on to some extent independently. Regions of stronger turbulence will cascade more efficiently than regions of weaker turbulence. This should lead to a cutoff function $f(x)$ whose shape depends on Reynolds number, particularly in the asymptotic large-k region. The simplest way to examine this question is to study the various moments

$$M_n(\lambda) = \int_0^\infty k^n E(k) dk . \tag{35}$$

For $n=2$, Eq. (35) defines the dissipation length and the scaling exponent ν. To study $n=4$ consider the skewness of the velocity derivative as a function of Reynolds number. This is given by[20]

$$S(\lambda) = \tfrac{1}{7}(270)^{1/2} \lambda M_4(\lambda)[M_2(\lambda)]^{-3/2} . \tag{36}$$

If the cutoff function had a universal shape, Eq. (36) would imply that

$$\theta = \tfrac{3}{2}\zeta\nu , \tag{37}$$

where θ is defined by Eq. (24). This is to be compared with the result $\theta = \frac{1}{2}\mu\nu$ of Eq. (25) obtained from the scaling hypothesis of Sec. II. Theoretically we can not exclude the possibility that $\mu = 3\zeta$, which would make Eqs. (25) and (37) agree, but experimentally it is unlikely that 3ζ could be as large as μ. If we accept Eq. (25) we conclude that the skewness grows with Reynolds number faster than predicted by Eq. (34). This is consistent with our qualitative picture of intermittency enhancing the large-k regions of the spectrum. Similar arguments can be given for the intermittency parameter

$$A = M_2(\lambda) M_6(\lambda)[M_4(\lambda)]^{-2}$$

introduced by Wyngaard and Pao.[4] This parameter would be constant if Eq. (34) were correct. Experimentally, it appears to increase weakly with Reynolds number.

Note that we have to be careful in making analogies with the scaling properties expected near a critical point. The basic scaling property is the existence of a single correlation length applicable to both the vorticity and dissipation correlation functions. This is a reasonable and appealing simplifying assumption. The stronger scaling property of Eq. (34) seems, however, not to hold.

The phenomenological theory presented in the present paper allows the possibility for significantly more effective use of existing experimental data and suggests feasible new experiments. Future progress will depend, however, on more refined experiments and on genuine dynamical theory starting from the Navier-Stokes equations.

ACKNOWLEDGMENTS

This work was supported by the National Science Foundation through Grant No. ENG74-08501 and through the Materials Science Center at Cornell

University. The hospitality of the National Center for Atmospheric Research in Boulder, Colorado, where this work was started in the summer of 1974, is appreciated. I would like to thank Paul Martin for suggesting the search for a quadratic scaling variable and Michael Fisher and Robert Kraichnan for constructive skepticism about the ideas presented in Ref. 1.

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