Quantum statistics of light after one-photon interaction with matter*

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The time development of the normally ordered generating functional of a light beam interacting linearly with a collection of two-level atoms is determined exactly. Rigorous conclusions on changes of the statistical properties of the beam are drawn. It is shown that the field after the interaction appears to be the superposition of a thermal field and of a field statistically similar to the initial one.

I. INTRODUCTION

The quantum statistical description of light fields after one-photon interaction with a resonant medium has recently been extensively investigated.[1-3] Most of the previous analyses have been devoted to studying the first and second moments of the photon number distribution after the interaction. Shen has shown that, when all atoms of the medium are in the ground state, the nature of radiation remains the same after the interaction, although the field amplitude decreases exponentially with time.[1] Chandra and Prakash have derived the linear term in the time development of the second moment of the photon number distribution and they have drawn conclusions about the changes in statistical properties of the incident light.[2] Finally, Loudon has treated essentially the same problem, but he has obtained solutions correct to all orders in time for the first and second moments of the photon number distribution.[3]

In the present work we derive the exact time development of the normally ordered generating functional of a field interacting linearly with a collection of two-level atoms and we draw rigorous conclusions on the changes of the statistical properties of the field after the interaction. We show that this field appears to be the superposition of two fields: One is statistically similar to the field before the interaction and the other is a thermal field. These conclusions confirm the accepted view that stimulated emission maintains the coherence properties of stimulating light, whereas spontaneous emission generates incoherent light.[4,5]

II. EQUATION OF MOTION FOR DENSITY OPERATOR OF FIELD

Let us consider the interaction of light with an ensemble of two-level atoms in thermal equilibrium. These atoms have an electric dipole transition between atomic states $|1\rangle$ and $|2\rangle$ with frequency separation ω, which coincides with the photon frequency of the kth mode. Following Shen, the interaction Hamiltonian between the system of photons and the ensemble of two-level atoms is given by[1]

$$\hat{H}_{\rm int} = \sum_i \left[\chi\, \hat{c}_{2i}^\dagger \hat{c}_{1i} \hat{\vec{E}}_k^{(-)}(\vec{r}_i) + \chi^* \hat{c}_{2i} \hat{c}_{1i}^\dagger \hat{\vec{E}}_k^{(+)}(\vec{r}_i)\right],$$

where

$$\hat{\vec{E}}_k^{(+)}(\vec{r}_i) = [\hat{\vec{E}}_k^{(-)}(\vec{r}_i)]^\dagger = i(2\pi\hbar\omega)^{1/2}\vec{u}_k(\vec{r}_i)\hat{a}$$

is the positive-frequency part of the kth-mode electric field at the position of ith atom. Here $\hat{c}_{1i}$, $\hat{c}_{2i}$, $\hat{c}_{1i}^\dagger$, and $\hat{c}_{2i}^\dagger$ are annihilation and creation operators for the ith atom in the states $|1\rangle$ and $|2\rangle$, respectively; $\hat{a}$ and $\hat{a}^\dagger$ are photon annihilation and creation operators; χ is the matrix element for the transition.

The equation of motion for the density operator of the total system is

$$i\hbar \frac{\partial}{\partial t}\hat{\rho}_{\rm tot} = [\hat{\mathcal{H}}_{\rm int}, \hat{\rho}_{\rm tot}],$$

where $\hat{\mathcal{H}}_{\rm int}$ is the interaction Hamiltonian in the interaction picture. By iteration, for small Δt, we have

$$\hat{\rho}_{\rm tot}(t+\Delta t) = \hat{\rho}_{\rm tot}(t) - \frac{i}{\hbar}\int_t^{t+\Delta t} dt_1 [\hat{\mathcal{H}}_{\rm int}(t_1), \hat{\rho}_{\rm tot}(t)] - \frac{1}{\hbar^2}\int_t^{t+\Delta t} dt_2 \int_t^{t_2} dt_1 [\hat{\mathcal{H}}_{\rm int}(t_2), [\hat{\mathcal{H}}_{\rm int}(t_1), \hat{\rho}_{\rm tot}(t)]]. \quad (1)$$

We are only interested in the light field and we can study the reduced density operator for the field alone. This reduced density operator, at time $t=0$, is given by

$$\hat{\rho}_f(t) = {\rm tr}_{\rm at}\{\hat{\rho}_{\rm tot}(t)\},$$

where ${\rm tr}_{\rm at}$ denotes the trace operation over the atomic variables. We assume that at time $t=0$, when the interaction is switched on, the radiation field and the atoms are decoupled and

$$\hat{\rho}_{tot}(0)=\hat{\rho}_f(0)\otimes\prod_i \hat{\rho}_i(0),$$

where $\hat{\rho}_i(0)$ is the thermal equilibrium density operator for the ith atom. If we assume also that the thermal equilibrium of the atomic system is hardly disturbed by the photon field, Eq. (1) leads to write the appropriate coarse-grained derivative. Thus the following equation of motion for the light field is obtained:

$$\frac{\partial}{\partial t}\hat{\rho}_f=-\beta n_1(\hat{a}^\dagger\hat{a}\hat{\rho}_f-2\hat{a}\hat{\rho}_f\hat{a}^\dagger+\hat{\rho}_f\hat{a}^\dagger\hat{a}) -\beta n_2(\hat{a}\hat{a}^\dagger\hat{\rho}_f-2\hat{a}^\dagger\hat{\rho}_f\hat{a}+\hat{\rho}_f\hat{a}\hat{a}^\dagger), \quad (2)$$

where

$$\beta=(4\hbar)^{-1}\sum_i \omega|\chi|^2|\vec{u}_k(\vec{r}_i)|^2 g(\omega).$$

Here n_1 and n_2 are the thermal populations for the two atomic states and $g(\omega)$ is the line-shape function.

We emphasize that Eq. (2) can describe the small signal gain of the laser amplifier also. If we want to study other operating conditions of the laser, the atomic system must be treated in higher-order perturbation theory and Eq. (2) must be replaced by a more complicated one.[6] For the purpose of the present paper we can confine ourselves to a discussion of Eq. (2).

III. FIELD AFTER ONE-PHOTON INTERACTION

We notice that Eq. (2) can be solved formally with iteration methods; as shown in the following, this treatment is simple and very convenient to study the statistical properties of fields interacting with matter. This solution of Eq. (2) is

$$\hat{\rho}_f(t)=\sum_{n=0}^{\infty}\frac{1}{n!}t^n\prod_{j=1}^{4}\prod_{i=1}^{n}\prod_{i'=1}^{n}\prod_{i''=1}^{n}\left[\frac{1}{2\pi}\int_0^{2\pi}d\theta_i^{(j)}\right] \times\hat{l}_{n+1-i'}\hat{\rho}_0\hat{h}_{i''}, \quad (3)$$

where $\hat{\rho}_0=\hat{\rho}_f(0)$ is the density operator of the field at time $t=0$, when the interaction is switched on. The sets of operators $\hat{l}_j$ and $\hat{h}_j$ are given by

$$\hat{l}_j(\hat{a},\hat{a}^\dagger)=-i\beta n_1\hat{a}^\dagger\hat{a}\exp[i\theta_j^{(1)}]-i\beta n_2\hat{a}\hat{a}^\dagger\exp[i\theta_j^{(1)}]+i\exp[i\theta_j^{(2)}]+(2\beta n_1)^{1/2}\hat{a}\exp[i\theta_j^{(3)}]+(2\beta n_2)^{1/2}\hat{a}^\dagger\exp[i\theta_j^{(4)}], \quad (4a)$$

$$\hat{h}_j(\hat{a},\hat{a}^\dagger)=-i\exp[-i\theta_j^{(1)}]+i\beta n_1\hat{a}^\dagger\hat{a}\exp[-i\theta_j^{(2)}] +i\beta n_2\hat{a}\hat{a}^\dagger\exp[-i\theta_j^{(2)}]+(2\beta n_1)^{1/2}\hat{a}^\dagger\exp[-i\theta_j^{(3)}]+(2\beta n_2)^{1/2}\hat{a}\exp[-i\theta_j^{(4)}]. \quad (4b)$$

Let us now introduce the normally ordered generating functional of the field,

$$C(\xi)=\mathrm{tr}_f\{\exp(\xi\hat{a}^\dagger)\exp(-\xi^*\hat{a})\hat{\rho}\}, \quad (5)$$

where tr_f denotes the trace operation over the field variables. It is well known that the functional $C(\xi)$ completely characterizes the density operator $\hat{\rho}$ of the field.[7,8] When a field is described by its generating functional $C(\xi)$ the modal densities,

$$\mathrm{tr}_f\{\hat{a}^{\dagger n}\hat{a}^m\hat{\rho}\}=\langle\hat{a}^{\dagger n}\hat{a}^m\rangle,$$

are given by

$$\langle\hat{a}^{\dagger n}\hat{a}^m\rangle=(-1)^m\frac{\partial^{n+m}}{\partial\xi^n\partial\xi^{*m}}C(\xi)\bigg|_{\xi=\xi^*=0}. \quad (6)$$

Moreover, the generating functional $C_{tot}(\xi)$ of a field which is obtained by the superposition of two independent fields is given by

$$C_{tot}(\xi)=C_1(\xi)C_2(\xi),$$

where $C_1(\xi)$ and $C_2(\xi)$ are the generating functionals for the component fields.[7]

For the following we need some properties for boson operators. When λ is a c number and $\hat{f}(\hat{a},\hat{a}^\dagger)$ is a function that can be expanded in a power series of $\hat{a}$ and $\hat{a}^\dagger$, then[9]

$$\hat{f}(\hat{a},\hat{a}^\dagger)\exp(\lambda\hat{a}^\dagger)=\exp(\lambda\hat{a}^\dagger)\hat{f}(\hat{a}+\lambda,\hat{a}^\dagger), \quad (7a)$$

$$\exp(\lambda\hat{a}^\dagger\hat{a})\hat{f}(\hat{a},\hat{a}^\dagger)\exp(-\lambda\hat{a}^\dagger\hat{a}) =\hat{f}[\hat{a}\exp(-\lambda),\hat{a}^\dagger\exp(+\lambda)]. \quad (7b)$$

We also have

$$[\hat{a},\hat{f}]=\frac{\partial}{\partial\hat{a}^\dagger}\hat{f},\quad[\hat{a}^\dagger,\hat{f}]=-\frac{\partial}{\partial\hat{a}}\hat{f}. \quad (8)$$

From Eqs. (3) and (5) the generating functional of the field interacting linearly with the atomic system may be written as

$$C_f(\xi,t)=\sum_{n=0}^{\infty}\frac{1}{n!}t^n\mathrm{tr}_f\left\{\prod_{j=1}^{4}\prod_{i=1}^{n}\prod_{i'=1}^{n}\prod_{i''=1}^{n}\left[\frac{1}{2\pi}\int d\theta_i^{(j)}\right]\hat{h}_{i''}\exp(i\omega\hat{a}^\dagger\hat{a}t)\exp(\xi\hat{a}^\dagger)\exp(-\xi^*\hat{a})\exp(-i\omega\hat{a}^\dagger\hat{a}t)\hat{l}_{n+1-i'}\hat{\rho}_0\right\},$$

since the reduced density operator $\hat{\rho}_f(t)$ for the field has been expressed in the interaction picture. By using Eqs. (7) the generating functional becomes

$$C_f(\eta, t) = \mathrm{tr}_f\left\{\exp(\eta\hat{a}^\dagger)\exp(-\eta^*\hat{a})\sum_{n=0}^{\infty}\frac{1}{n!}t^n\prod_{j=1}^{4}\prod_{i=1}^{n}\prod_{i'=1}^{n}\prod_{i''=1}^{n}\left[\frac{1}{2\pi}\int d\theta_i^{(j)}\right]\hat{\mathfrak{h}}_{i'}\hat{l}_{i'}\hat{\rho}_0\right\},$$

where

$$\eta = \xi\exp(+i\omega t) \tag{9}$$

and

$$\hat{\mathfrak{h}}_j(\hat{a}, \hat{a}^\dagger) = \hat{h}_j(\hat{a}+\eta, \hat{a}^\dagger+\eta^*) = -i\exp[-i\theta_j^{(1)}] + i\beta n_1(\hat{a}^\dagger+\eta^*)(\hat{a}+\eta)\exp[-i\theta_j^{(2)}] + i\beta n_2(\hat{a}+\eta)(\hat{a}^\dagger+\eta^*)\exp[-i\theta_j^{(2)}]$$
$$+(2\beta n_1)^{1/2}(\hat{a}^\dagger+\eta^*)\exp[-i\theta_j^{(3)}] + (2\beta n_2)^{1/2}(\hat{a}+\eta)\exp[-i\theta_j^{(4)}]. \tag{10}$$

When we set

$$\hat{L}_j = -i\beta n_1\exp[i\theta_j^{(1)}]\left(\hat{a} - \frac{\partial}{\partial\hat{a}^\dagger}\right)\left(\hat{a}^\dagger + \frac{\partial}{\partial\hat{a}}\right) - i\beta n_2\exp[i\theta_j^{(1)}]\left(\hat{a}^\dagger + \frac{\partial}{\partial\hat{a}}\right)\left(\hat{a} - \frac{\partial}{\partial\hat{a}^\dagger}\right)$$
$$+i\exp[i\theta_j^{(2)}] + (2\beta n_1)^{1/2}\exp[i\theta_j^{(3)}]\left(\hat{a} - \frac{\partial}{\partial\hat{a}^\dagger}\right) + (2\beta n_2)^{1/2}\exp[i\theta_j^{(4)}]\left(\hat{a}^\dagger + \frac{\partial}{\partial\hat{a}}\right),$$

by means of Eqs. (8) we can write

$$C_f(\eta, t) = \mathrm{tr}_f\left\{\exp(\eta\hat{a}^\dagger)\exp(-\eta^*\hat{a})\sum_{n=0}^{\infty}\frac{1}{n!}t^n\prod_{j=1}^{4}\prod_{i=1}^{n}\left[\frac{1}{2\pi}\int d\theta_i^{(j)}\right]\hat{\mathfrak{h}}_i\hat{L}_i\hat{\rho}_0\right\},$$

since, for a c number, it is $\hat{L}_j c = c\hat{l}_j$.

The generating functional of the field can be expressed, therefore, in the following compact form:

$$C_f(\eta, t) = \mathrm{tr}_f\{\exp(\eta\hat{a}^\dagger)\exp(-\eta^*\hat{a})\exp[t\hat{K}(\eta)]\hat{\rho}_0\}, \tag{11}$$

where

$$\hat{K}(\eta) = \prod_{j=1}^{4}\left[\frac{1}{2\pi}\int d\theta_i^{(j)}\right]\hat{\mathfrak{h}}_i\hat{L}_i.$$

By performing the integrations over $\theta^{(j)}$ in the above expression we obtain

$$\hat{K}(\eta) = -\beta n_1\left(\hat{a} - \frac{\partial}{\partial\hat{a}^\dagger}\right)\left(\hat{a}^\dagger + \frac{\partial}{\partial\hat{a}}\right) - \beta n_2\left(\hat{a}^\dagger + \frac{\partial}{\partial\hat{a}}\right)\left(\hat{a} - \frac{\partial}{\partial\hat{a}^\dagger}\right) - \beta n_1(\hat{a}^\dagger+\eta^*)(\hat{a}+\eta)$$
$$-\beta n_2(\hat{a}+\eta)(\hat{a}^\dagger+\eta^*) + 2\beta n_1(\hat{a}^\dagger+\eta^*)\left(\hat{a} - \frac{\partial}{\partial\hat{a}^\dagger}\right) + 2\beta n_2(\hat{a}+\eta)\left(\hat{a}^\dagger + \frac{\partial}{\partial\hat{a}}\right).$$

Consequently, we have, with straightforward calculations, that

$$\exp[t\hat{K}(\eta)] = 1 + tN(\hat{D}+A) + \tfrac{1}{2}t^2N^2[(\hat{D}+A)^2 + (\hat{D}+2A)] + (t^3/3!)N^3[(\hat{D}+A)^3 + 3(\hat{D}+A)(\hat{D}+2A) + (\hat{D}+2A) + 2A] + \cdots \tag{12}$$

where, for the sake of brevity, we have used the following notations:

$$\hat{D} = \eta\hat{a}^\dagger - \eta^*\hat{a}, \quad A = -(n_1+n_2)(n_2-n_1)^{-1}\eta\eta^*, \quad N = \beta(n_2-n_1). \tag{13}$$

Since

$$\exp[(e^{Nt}-1)(\hat{D}+A)] = 1 + tN(\hat{D}+A) + \tfrac{1}{2}t^2N^2[(\hat{D}+A)^2 + (\hat{D}+A)] + (t^3/3!)N^3[(\hat{D}+A)^3 + 3(\hat{D}+A)^2 + (\hat{D}+A)] + \cdots,$$

$$\exp[\tfrac{1}{2}(e^{Nt}-1)^2A] = 1 + \tfrac{1}{2}t^2N^2A + \tfrac{1}{2}t^3N^3A + \cdots,$$

Eq. (12) becomes

$$\exp[t\hat{K}(\eta)] = \exp[(e^{Nt}-1)(\hat{D}+A)]\exp[\tfrac{1}{2}(e^{Nt}-1)^2 A].$$

Thus, the generating functional (11) is

$$C_f(\eta, t) = \mathrm{tr}_f\{\exp(\eta\hat{a}^\dagger)\exp(-\eta^*\hat{a})\exp[(e^{Nt}-1)(\eta\hat{a}^\dagger - \eta^*\hat{a})] \times \exp[\tfrac{1}{2}(e^{2Nt}-1)A]\hat{\rho}_0\}.$$

Finally, from the Baker-Hausdorff equality[10]

$$\exp(\hat{X})\exp(\hat{Y}) = \exp(\hat{X} + \hat{Y} + \tfrac{1}{2}[\hat{X}, \hat{Y}]),$$

valid when the operators $\hat{X}$ and $\hat{Y}$ commute with their commutator $[\hat{X}, \hat{Y}]$, the generating functional of the field after the interaction can be written as

$$C_f(\xi, t) = \mathrm{tr}_f[\exp(\exp\{[\beta(n_2-n_1)+i\omega]t\}\xi\hat{a}^\dagger)\exp(-\exp\{[\beta(n_2-n_1)-i\omega]t\}\xi^*\hat{a})\hat{\rho}_0] \times \exp(-n_2(n_2-n_1)^{-1}\{\exp[2\beta(n_2-n_1)t]-1\}\xi\xi^*) . \quad (14)$$

Here we have recalled the notations that have been introduced by Eqs. (9) and (13).

It is clear that the field after the interaction with the atomic system appears to be the superposition of two fields whose generating functionals are expressed, respectively, by

$$C_1(\xi, t) = \mathrm{tr}_f[\exp(\exp\{[\beta(n_2-n_1)+i\omega]t\}\xi\hat{a}^\dagger) \times \exp(-\exp\{[\beta(n_2-n_1)-i\omega]t\}\xi^*\hat{a})\hat{\rho}_0]$$

and

$$C_2(\xi, t) = \exp(-n_2(n_2-n_1)^{-1} \times \{\exp[2\beta(n_2-n_1)t]-1\}\xi\xi^*) .$$

We wish first to discuss the statistical properties of the field corresponding to $C_1(\xi, t)$. We notice that the generating functional $C_0(\xi)$ of the field before the interaction with the atomic system is given by

$$C_0(\xi) = \mathrm{tr}_f\{\exp(\xi\hat{a}^\dagger)\exp(-\xi^*\hat{a})\hat{\rho}_0\} .$$

We easily see that $C_0(\xi)$ is formally identical with the functional $C_1(\xi, t)$, provided that the variables ξ and ξ^* are replaced by variables $\xi\exp\{[\beta(n_2-n_1)+i\omega]t\}$ and $\xi^*\exp\{[\beta(n_2-n_1)-i\omega]t\}$. We therefore conclude that these fields are statistically similar. Then, if $\langle\hat{a}^{\dagger n}\hat{a}^m\rangle_0$ and $\langle\hat{a}^{\dagger n}\hat{a}^m\rangle_1$ are the (n, m)th-order normally ordered modal densities obtained from the functionals $C_0(\xi)$ and $C_1(\xi, t)$, respectively, we have, according to Eq. (6), that

$$\langle\hat{a}^{\dagger n}\hat{a}^m\rangle_1 = \exp[(n+m)\beta(n_2-n_1)t] \times \exp[i\omega(n-m)t]\langle\hat{a}^{\dagger n}a^m\rangle_0 .$$

This result shows that the initial field is attenuated by the interaction when $n_1 > n_2$ and amplified when $n_1 < n_2$, which corresponds to an atomic population inversion.

We discuss now the statistical properties of the field corresponding to $C_2(\xi, t)$. This generating functional is a standard Gaussian distribution and therefore describes a thermal field of mean photon number

$$\langle n(t)\rangle = n_2(n_2-n_1)^{-1}\{\exp[2\beta(n_2-n_1)t]-1\} .$$

In fact, the density operator of this field can be written as

$$\hat{\rho}_2(t) = \{1-\exp[-\mu(t)]\}\exp[-\mu(t)\hat{a}^\dagger\hat{a}],$$

with

$$\mu(t) = \ln\{[\langle n(t)\rangle]^{-1}[\langle n(t)\rangle + 1]\} .$$

We point out that, if the field is initially incoherent or if there are no photons present initially, the field is incoherent at all subsequent times.

It may be of interest to discuss the above results also in the P representation.[7] The field before the interaction is described in the P representation by the density operator

$$\hat{\rho}_0 = \int d^2\alpha\, P_0(\alpha)|\alpha\rangle\langle\alpha| .$$

From Eq. (14) we have that the field after the interaction is given by the superposition of two fields. The first is described by the density operator

$$\hat{\rho}_1(t) = \int d^2\alpha\, P_0(\alpha\exp\{[\beta(n_2-n_1)-i\omega]t\})|\alpha\rangle\langle\alpha|$$

and the second by

$$\hat{\rho}_2(t) = [\pi\langle n(t)\rangle]^{-1} \times \int d^2\alpha\, \exp\{-[\langle n(t)\rangle]^{-1}|\alpha|^2\}|\alpha\rangle\langle\alpha| .$$

In $\hat{\rho}_1$ the amplitudes α and α^* of the initial distribution $P_0(\alpha)$ are multiplied by a factor, so the field has changed its statistical properties in a rather trivial way since it is simply a translation of the P distribution in the α space. The second is a thermal field.

To sum up, in the interaction with the atomic system the initial field is amplified or attenuated, but this process generates incoherent light also. Clearly, since the amplified light is obtained by stimulated emission process and incoherent light is generated by spontaneous emission, our results confirm the accepted view[4,5] that stimulated emission maintains the coherence properties of stimulating light, whereas spontaneous emission generates only incoherent light.

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