Letter

Emergent infrared conformal dynamics: Applications to strongly interacting quantum states

Jeff Maki
Pitaevskii BEC Center, CNR-INO and Dipartimento di Fisica, Università di Trento, I-38123 Trento, Italy

Fei Zhou
Department of Physics and Astronomy, University of British Columbia, 6224 Agricultural Road, Vancouver, British Columbia, Canada, V6T 1Z1

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When the interactions are scale invariant, the quantum dynamics of a quantum gas are strongly constrained by a resultant conformal symmetry. In this Letter we study the expansion dynamics of strongly interacting quantum systems in a shallow harmonic trap in one and three spatial dimensions while interparticle interactions break the scale symmetry explicitly. Our main finding is that in one dimension the dynamics can be strongly constrained by an emergent infrared conformal dynamics (EIRCD) which significantly reduces entropy production, as opposed to three dimensions where there is no EIRCD in the strong coupling limit. We investigate the possibility and signatures of EIRCD in terms of the damping rate of the large amplitude oscillations of the gas, as well as the work done following a two-quench protocol. We find that the damping and the work done are constrained by the EIRCD, and become vanishingly small in the infrared limit when the final harmonic trap frequency is small. Our analysis is based on a close connection between the renormalization group equation flow and expansion dynamics in real space, and as such can be readily applied to a wide range of strongly interacting systems, like one-dimensional (1D) quantum gases and the three-dimensional (3D) unitary Fermi gas.

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Introduction. There have been intensive efforts to understand scale symmetry in strongly interacting quantum gases [1–21]. One remarkable consequence is that the expansion of strongly interacting quantum gases in free space and harmonic traps can exhibit unique features which we denote as *conformal dynamics*; the unitary evolution can be equivalent to a time-dependent *dilation* of the many-body wave function [4,6] up to a time-dependent gauge transformation. Normally, the appearance of conformal dynamics requires that the interactions be *fully* scale invariant. This requires fine-tuning the atomic interactions using Feshbach [22] or confinement-induced resonances [23].

One important parameter to characterize conformal dynamics and its breaking is the (thermodynamic) entropy production rate. Conformal symmetry implies that there exists a class of expansion or contraction dynamics that are isentropic. In contrast, the expansion of generically interacting gases is usually accompanied with finite entropy production and is irreversible [19].

The amount of entropy production for a nearly scale-invariant system depends on how these interactions rescale as one approaches long wavelengths or long timescales, i.e., the infrared (IR) limit. If the effects of breaking scale symmetry are IR irrelevant, entropy production is restricted to short timescales, while the long time dynamics can still be isentropic and conformal. Such *emergent infrared conformal dynamics* (EIRCD) are not associated with the exact symmetry of the microscopic interactions, rather it is an IR symmetry which only appears in the long wavelength dynamics. However, if the scale-breaking interactions are IR relevant, one expects an appreciable entropy production rate at long times.

The study of entropy production is important as it can play a pivotal role in thermalization. Zero entropy production, which only occurs in conformal dynamics, is a sufficient, but not necessary, condition for the absence of thermalization. For instance, one-dimensional (1D) quantum gases are integrable in free space [24–26] and cannot thermalize, yet they can have finite entropy production for generic interactions [27]. Recent evidence seems to suggests this integrable system thermalizes towards the standard Gibbs ensemble [28,29] in the presence of a harmonic potential when one includes a finite diffusion term which is present in the gradient expansion unless the interactions are scale symmetric, as conformal symmetry forbids entropy production. It has yet to be understood how the vanishingly small entropy production rate appears when the scale symmetry is broken explicitly and there are EIRCD. In this Letter we investigate the general possibility of EIRCD in the expansion dynamics of strongly interacting d-dimensional quantum gases and how EIRCD appears in the dynamics. As described in Fig. 1, we consider a strongly interacting d-dimensional gas in a time-dependent harmonic trap with frequency $\omega(t)$. At $t = 0$, we quench the harmonic trapping potential from ω_1 to $\omega_2 \ll \omega_1$, allowing the gas to expand. After a hold time t_h we then quench back to ω_1. We show that for strongly interacting gases in one dimension, there are EIRCD in the IR limit, when $\omega_2/\omega_1 \ll 1$, even if the microscopic interactions break scale symmetry, i.e., the long-time dynamics are fully dictated by *the infrared stable scale symmetric interactions* and are isentropic. This is in comparison to a strongly interacting three-dimensional (3D) quantum gas, which does not exhibit EIRCD, and there is a finite entropy production rate in the long-time dynamics. The main foci of

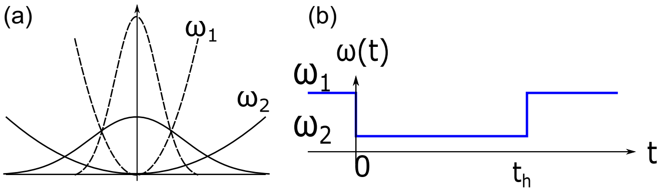

FIG. 1. (a) Schematic of the experimental protocol with (b) a time-dependent frequency. (a) A scale-invariant gas is prepared in thermal equilibrium in a harmonic trap with frequency ω_1. It is then released to a much broader trap with frequency ω_2 causing the gas to expand and oscillate. After a hold time t_h the trap is quenched back to ω_1. (b) The time dependence of the trapping potential.

this work are the following: (i) the origins of EIRCD; (ii) characterizing the entropy production rate in EIRCD; and (iii) an experimental protocol for detecting EIRCD using a two-quench protocol.

(i) Origins of EIRCD: We discuss the possibility of EIRCD using a model Hamiltonian for a quantum gas (either spin-1/2 fermions or bosons) with zero-ranged interactions in d dimensions and inside a time-dependent harmonic trap of frequency $\omega(t)$: $\mathcal{H}_{\omega(t)} = \mathcal{H} + \omega^2(t)C$, where $C = \int d^d r \frac{r^2}{2}\psi^\dagger(\mathbf{r})\psi(\mathbf{r})$ [30]. We also define the Hamiltonian $\mathcal{H}$:

$$\mathcal{H} = \int d^d\mathbf{r}\,\frac{1}{2}\psi^\dagger(\mathbf{r})\left(-\frac{1}{2}\nabla^2\right)\psi(\mathbf{r}) + \int d^d\mathbf{r}\, g(\Lambda)\psi^\dagger(\mathbf{r})\psi^\dagger(\mathbf{r})\psi(\mathbf{r})\psi(\mathbf{r}), \quad (1)$$

where $\psi^\dagger(\mathbf{r})$ is the creation operator and $g(\Lambda)$ is the coupling constant which depends on the ultraviolet (UV) scale of the theory Λ. (We suppress the spin indices for spin-1/2 fermions and set $\hbar = m = 1$.)

Since the dynamics are related to the behavior of scale-breaking interactions in the IR limit, it is beneficial to recall how Eq. (1) behaves under a scale transformation. To emphasize the scaling, we express the Hamiltonian in units of Λ^2 and we rewrite the interaction as $g(\Lambda)\Lambda^{d-2} = c_d\tilde{g}(\Lambda)$ where $\tilde{g} = \tilde{g}^* + \delta\tilde{g}$ and c_d is a nonzero positive constant that depends on the dimension d. The scale-invariant part of the interaction is $\tilde{g}^*$ while $\delta\tilde{g}$ breaks scale symmetry. Physically, g^* corresponds to the interaction at the Feshbach or confinement-induced resonance, while a finite $\delta\tilde{g}$ is related to the detuning from the resonance. The value of $\tilde{g}^*$ can be obtained via the renormalization group equations (RGEs) [31]. Apart from the trivial noninteracting fixed point $\tilde{g}^* = 0$, there is a strongly interacting fixed point $\tilde{g}^* = (2-d)$ when $d = 1, 3$.

We expand the dimensionless Hamiltonian as $\mathcal{H} = \mathcal{H}^* + \delta\mathcal{H}_I$, where $\mathcal{H}^*$ is the scale-invariant part while $\delta\mathcal{H}_I$ breaks the scale symmetry and is proportional to $\delta\tilde{g}(\Lambda)$. Thus under a scale transformation $\delta\mathcal{H}_I$ transforms nontrivially

$$\Lambda \to \frac{\Lambda}{\lambda}, \quad \delta H_I \to \frac{1}{\lambda^\eta}\delta H_I, \quad (2)$$

where λ is a scaling factor larger than unity, i.e., $\lambda > 1$, as this generates a flow towards the IR limit. The scaling dimension of δH_I, $\eta = \eta(d, \tilde{g}^*)$, is universal; it only depends on the spatial dimension d and $\tilde{g}^*$. It can be obtained from solving the RGEs around $\tilde{g}^*$.

In three dimensions or $d = 3$, there is a scale-invariant attractive interaction $\tilde{g}^* = -1$, which represents a Feshbach resonance with an infinite scattering length $a_{3D} = +\infty$ [31]. The scaling dimension for $\delta\mathcal{H}_I$ is $\eta = -1 < 0$. The scale-invariant fixed point Hamiltonian $\mathcal{H}^*$ is IR unstable under a scale transformation; any small but finite $\delta\mathcal{H}_I$ will be amplified during the course of rescaling towards the IR limit [16]. In one dimension, the scale-invariant interactions are repulsive, $\tilde{g}^* = +1$, and represent infinite strength contact interactions, i.e. the 1D scattering length $a_{1D} = 0$. The associated scaling dimension is $\eta = +1 > 0$, and the effect of δH_I diminishes as the running scale Λ is lowered; $\mathcal{H}^*$ is IR stable. Similar arguments apply to the noninteracting fixed point. In three dimensions the noninteracting fixed point is IR stable, while it is unstable in one dimension. In two dimensions, there is only a single noninteracting fixed point which is also stable. This can be seen by from the RG and noting $\eta = 0$ in two dimensions.

In the expansion dynamics of strongly interacting nearly scale-invariant gases, the particle density $n(t)$ depends on time t and can dramatically differ from their initial values at $t = 0$. We quantify the effects of δH_I on the dynamics by evaluating it at the energy scale $n^{2/d}(t)$, or equivalently at a momentum scale set by the instantaneous Fermi momentum: $k_F(t) \propto n^{1/d}(t)$. It is then natural to define a time-dependent running momentum scale at which we evaluate $\delta\mathcal{H}_I$ that is proportional to the ratio of the density at time t to the initial density $k_F(t) \approx [n(t)/n(0)]^{1/d}k_F(0)$. In this way we define the time-dependent rescaling factor $\lambda(t) \sim [n(t)/n(0)]^{1/d}$.

From this point of view, expansion or contraction dynamics can probe the renormalization group flow of the symmetry breaking term $\delta\mathcal{H}_I$ [32]. To observe substantial renormalization effects during the dynamics, it is thus imperative to consider the far-away-from-equilibrium expansion (contraction) dynamics, so that the rescaling factor $\lambda(t)$ can be much larger (smaller) than unity. This is in contrast to the perturbative linear response regime where $\lambda(t)$ always remains close to 1.

One can then directly study exactly conformal dynamics (CD) and the possibility of EIRCD by considering a thermally equilibrated quantum gas in an isotropic d-dimensional harmonic trap that is quenched from from a frequency ω_1 to ω_2. Provided $\omega_2 \ll \omega_1$, one can then generate an appreciable dynamic renormalization group flow towards the IR limit.

When the interactions are exactly scale symmetric, there are CD. In this case the dynamics of the density matrix are completely self similar with a time-dependent rescaling factor [4,11,16,19]

$$\lambda^2(t) = \cos^2(\omega_2 t) + \frac{1}{\tilde{\omega}_2^2}\sin^2(\omega_2 t), \quad \tilde{\omega}_2 = \frac{\omega_2}{\omega_1}. \quad (3)$$

Equation (3) is a periodic function with period $T_2 = \pi/\omega_2$. It takes on values $\lambda(t) \in [1, \lambda_{\max}]$, with a maximum value $\lambda_{\max} = 1/\tilde{\omega}_2$ and a minimum value $\lambda_{\min} = 1$.

We now examine the effects of $\delta\mathcal{H}_I$ on the aforementioned CD. At leading order in $\delta\mathcal{H}_I$, the scale-breaking perturbation becomes effectively time dependent during the dynamics. We can capture the time dependence by rescaling $\delta\mathcal{H}_I(t)$ by $\lambda(t)$ defined in Eq. (3). In three dimensions, $\eta = -1$ and $\delta\mathcal{H}_I(t)$ is appreciable over the entire period and becomes more relevant when $\lambda(t)$ approaches $\lambda_{\max}$, i.e., when $t \simeq T_2/2$. In one dimension, $\eta = +1$ and $\delta\mathcal{H}_I(t)$ is only appreciable for small time windows $\delta t \sim 1/\omega_1$ near $t = 0$ or $t = T_2$ when the scaling factor $\lambda(t) \sim 1$ is at a minimum and the gas is most dense; $\delta\mathcal{H}_I$ is strongly suppressed around $t = T_2/2$ when the gas is most dilute, $\lambda(t) \approx \lambda_{\max}$. Accordingly, there are EIRCD in one dimension, provided $\tilde{\omega}_2$ is sufficiently small, while in three dimensions EIRCD are absent.

(ii) EIRCD and entropy production: When the interactions break scale symmetry, the oscillatory conformal dynamics in Eq. (3) will become damped, a signal of thermodynamic entropy production $\lambda^2(t) \propto e^{-\Gamma_I(t)t}$, where $\Gamma_I(t)$ is the instantaneous damping rate. We quantify the extent of the breaking of scale symmetry and the onset of EIRCD using the the dissipation rate of the oscillatory conformal dynamics averaged over one period, $\Gamma(\tilde{\omega}_2) = \frac{1}{T_2}\int_0^{T_2}\Gamma_I(t)dt$. When $\tilde{\omega}_2 \to 0$, we can expand $\Gamma(\tilde{\omega}_2)$ in terms of $\tilde{\omega}_2$:

$$\Gamma(\tilde{\omega}_2) = \Gamma_0 + \Gamma_1\tilde{\omega}_2 + \Gamma_2\tilde{\omega}_2^2 + \cdots;$$
$$\Gamma_{0,1,2} = \omega_1^2\tau_F\gamma_{0,1,2}\left[\left(\frac{a_{sc}^2}{\tau_F}\right)^{2-d}, T\tau_F, \omega_1\tau_F\right], \tag{4}$$

where $\tau_F \propto n(0)^{-2/d}$ is a Fermi time defined in terms of the initial density at the center of the trap. The general structure of the dimensionless functions $\gamma_{0,1,2}$ are highly complex and we will focus on the limit $(a_{\rm sc}^2/\tau_F)^{2-d} \ll 1$, with $a_{\rm sc}$ being the d-dimensional scattering length $a_{\rm sc} = a_{\rm 3D}$ for $d = 3$ and $a_{\rm sc} = a_{\rm 1D}$ for $d = 1$.

When $\mathcal{H}$ is fine-tuned to have scale-invariant interactions, $\mathcal{H} = \mathcal{H}^*$, $\Gamma = 0$ for all values of $\tilde{\omega}_2$ and arbitrary temperatures T [16,19]. This is a signature of CD. For generically interacting systems Γ will be finite and will be proportional to $(a_{\rm sc}^2/\tau_F)^{2-d}$ near the strongly interacting fixed point.

The leading behavior of Γ for finite $\delta\mathcal{H}_I(t)$ in the IR limit depends on the scaling dimension η defined in Eq. (2). When $\eta = -1$, as in three dimensions, the symmetry-breaking action of $\delta\mathcal{H}_I(t)$ is relevant as the gas expands. Entropy production occurs approximately uniformly over the entire period T_2 leading to an ω_f-independent average dissipation rate, i.e., a finite Γ_0. Thus $\Gamma \to \omega_0^2\tau_F\gamma_0$ is approximately a constant as $\tilde{\omega}_2 \to 0$, as in generic strongly interacting systems. This signifies finite dissipation in the IR limit and the absence of EIRCD. Instead when $\eta = 1$, as in one dimension, $\delta\mathcal{H}_I(t)$ becomes strongly suppressed in the bulk of the period when $\lambda(t) \gg 1$. Dissipation due to $\delta\mathcal{H}_I(t)$ is then confined to a small time window of $\delta t \sim \pi/\omega_1 \ll T_2$ when $\lambda(t) \approx 1$, i.e., $t = 0, T_2, 2T_2$, and so on. The averaged dissipation rate therefore is inversely proportional to T_2 or proportional to ω_2. So we expect that $\gamma_0 = 0$ while γ_1 remains finite. This scaling implies that if one approaches the IR limit, $\tilde{\omega}_2 \to 0$, Γ becomes vanishingly small indicating an unexpected dynamical phase with EIRCD. Such nearly dissipationless dynamics can be directly studied in experiments of strongly interacting 1D quantum gases.

To provide more evidence for this phenomenology, we evaluate the instantaneous dissipation rate to leading order in the scale-breaking interactions using hydrodynamics. We solve the Navier-Stokes equation [33] for a d-dimensional gas

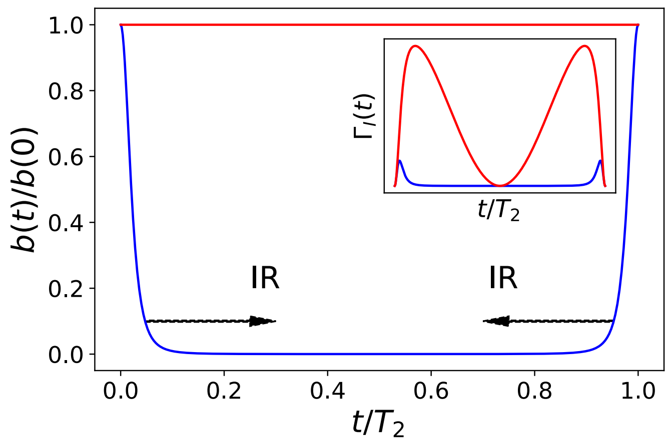

FIG. 2. The time-dependent damping coefficient $b(t)$, Eq. (6), as a function of time for one period of the conformal dynamics in both one (blue) and three dimensions (red). In one dimension the dissipation is only appreciable near $t = 0, T_2$, while in three dimensions there is appreciable dissipation over the entire period. The arrows points towards the IR limit. The inset shows the instantaneous damping rate $\Gamma_I(t)$ over one period.

in a harmonic trapping potential

$$n(\partial_t + \mathbf{v}\cdot\nabla_{\mathbf{r}})\mathbf{v} = -\nabla_{\mathbf{r}}P - n\omega_2^2\mathbf{r}_i + \nabla_{\mathbf{r}}(\zeta\nabla_{\mathbf{r}}\cdot\mathbf{v}). \tag{5}$$

n, $\mathbf{v}$, and P are the local density, velocity, and pressure, respectively. The last term in Eq. (5) describes dissipation and defines the bulk viscosity, ζ [34]. Equation (5) can be solved using a scaling ansatz for the density $n(\mathbf{r}, t) = \lambda^{-d}(t)n[\mathbf{r}/\lambda(t), 0]$ and for the velocity field $\mathbf{v}(\mathbf{r}, t) = \mathbf{r}\dot{\lambda}(t)/\lambda(t)$. For scale-invariant interactions the solution of Eq. (5) for $\lambda(t)$ is also Eq. (3). This ansatz can be obtained by applying an SO(2, 1) conformal field theory to the dynamics at the scale-invariant fixed point [19,20]. For finite $\delta\mathcal{H}_I$, we can still use the scaling ansatz, but the oscillations in $\lambda(t)$ will now be damped due to the dissipation.

In this hydrodynamic framework the instantaneous dissipation rate of the monopole oscillations is defined as $\Gamma_I(t) = |\frac{1}{E}\frac{d\langle E_k\rangle(t)}{dt}|$ where E is the total energy per particle and $\langle E_k\rangle(t)$ is the kinetic energy of the macroscopic flow per particle. $\Gamma_I(t)$ is related to the thermodynamic entropy production rate since $\partial_t S(t) = T^{-1}d\langle E_k\rangle/dt$. From Eq. (5) the change in the kinetic energy is itself proportional to the bulk viscosity as $d\langle E_k\rangle/dt = -d\int\frac{d^d\mathbf{r}}{N}\zeta(\mathbf{r}, t)(\frac{\dot{\lambda}(t)}{\lambda(t)})^2$. Near the scale-invariant fixed point the bulk viscosity possesses a simple scaling form [35–37] which allows one to write

$$\Gamma_I(t) = b(t)\dot{\lambda}^2(t), \quad b(t) = \frac{e_d}{\omega_1\lambda^2(t)}\left(\frac{k_F a_{\rm sc}}{\lambda(t)}\right)^{2\eta}. \tag{6}$$

In Eq. (6), $\eta = 2 - d$ for $d = 1, 3$ and e_d is a time-independent constant that depends on the equation of state of the initial gas. k_F is the Fermi wave number defined at the center of the trap ($k_F = \sqrt{2/\tau_F} \propto n(0)^{1/d}$).

The time dependence of the damping coefficient $b(t)$ over one period of the dynamics is shown in Fig. 2 near the strongly interacting scale-invariant point in one dimension [Fig. 2(a)] and three dimensions [Fig. 2(b)]. In one dimension the damping coefficient is strongly suppressed save for two tiny time windows near $t = 0, T_2$, while in three dimensions it remains close to unity over the entire period. Taking the average of $\Gamma_I(t)$ over the period T_2, we verify the prediction that the leading term in Eq. (4) is γ_0 in three dimensions while it is γ_1 in one dimension.

(iii) Measurement scheme: When the system has EIRCD, the dynamics are nearly reversible as the entropy production is localized to short time windows near $\lambda(t) = 1$. A convenient way to study the reversibility of the dynamics is to examine the work done following a second quench after hold time t_h which returns the system to the original harmonic trap of ω_1, see Fig. 1.

The total work done in this two-quench protocol sensitively depends on the value of t_h and whether there are EIRCD. Generally, the work done is given by

$$W = \frac{(\omega_1^2 - \omega_2^2)}{2}\left(\lambda_C^2(t_h) - 1\right)\langle C\rangle(0),$$
$$\lambda_C^2(t_h) = \frac{\langle C\rangle(t_h)}{\langle C\rangle(0)}. \tag{7}$$

Here we introduce $\lambda_C(t_h)$ as a measure of the actual size of the quantum gas at $t = t_h$, which, in general, differs from $\lambda(t_h)$ defined in Eq. (3), but coincides with $\lambda(t_h)$ for CD. The work done can also be directly related to the size of quantum gases at infinite times [39]. W in this two-quench protocol is positive semi-definite, i.e., $W \geqslant 0$ for all ω_1 and ω_2.

When $\omega_2 \ll \omega_1$, the work done reaches a minimum, $W = 0$, if $\lambda_C^2(t_h) = 1$, i.e., when the dynamics are fully reversible, when $t_h = nT_2$ $n = 1, 2, 3, \ldots$ [38]. If there is entropy production, the amount of work done at any hold time $t_h > 0$ *shall always be larger* than zero, $W(t_h > 0) > 0$, as $\lambda_C(t_h)$ now has to be larger than unity.

The amount of work done after one cycle in this two-quench scheme can be used as an effective probe of EIRCD. In three dimensions the action of $\delta\mathcal{H}_I(t)$ is enhanced in the IR limit, leading to entropy production and conventional thermalization as ω_2 approaches zero. So when $\omega_2 \to 0$, the scaling parameter approaches an equilibrium value $\lambda_C^2(t_h = T_2) \to \lambda_{\text{eq}}^2 = 1/(2\tilde{\omega}_2^2)$ which is much bigger than $\lambda^2(t = T_2) = 1$. This results in $W \propto \tilde{\omega}_2^{-2}$, as $\tilde{\omega}_2$ approaches zero. However, in one dimension near the strong coupling fixed point, dissipation occurs during a short-time window, $\delta t \sim 1/\omega_1$, that is parametrically small compared to T_2. This indicates $\lambda_C^2(t_h = T_2) = 1 + O(\Gamma_1/\omega_1)$ which is very close to unity as $\Gamma_1/\omega_1 \sim \omega_1\tau_F \ll 1$ in the many-body limit.

Another way to visualize this physics is to consider the average power $\tilde{P} = W\omega_2$. Following our discussions on the work, $\tilde{P}$ is proportional to $\tilde{\omega}_2$ in one dimension and becomes vanishingly small in the IR limit, while in three dimensions the average power diverges as $\tilde{P} \propto 1/\tilde{\omega}_2$. We numerically confirm this behavior for the average power and for the work, by solving the Navier-Stokes equation, Eq. (5), using the scaling ansatz [19,20]. The results of the simulation are presented in Fig. 3.

Before concluding, we want to make two remarks. First, we parametrized this EIRCD using the dissipation rate, Eq. (4). We illustrate that in one dimension the leading contribution to the decay rate is $\Gamma \propto \tilde{\omega}_2$ as $\tilde{\omega}_2 \to 0$ while in three dimensions Γ is approximately constant. This difference controls whether

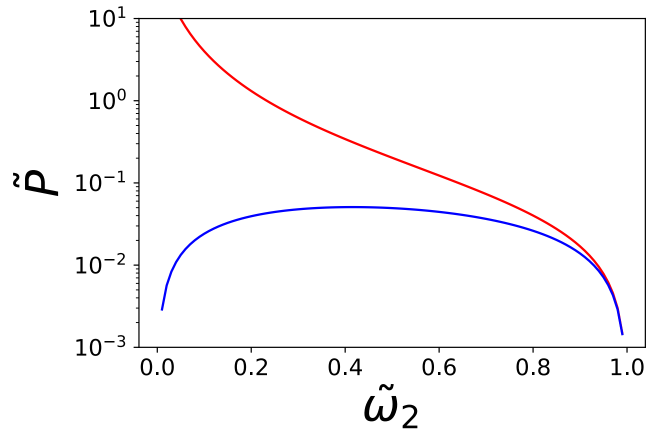

FIG. 3. Dynamic simulation of the average power of the work done following a two-quench protocol: $\tilde{P} = W\omega_2$ as a function of $\tilde{\omega}_2 = \omega_2/\omega_1$ in three dimensions (red) and one dimension (blue) for dimensionless bulk viscosity $\tilde{\zeta} = d^2 \int d^d\mathbf{r}\zeta(\mathbf{r}, 0)/[2N\omega_1\langle C\rangle(0)] = 0.1$. In the IR limit, $\tilde{\omega}_2 \to 0$, the average power vanishes as ω_2 in one dimension, while it diverges as ω_2^{-1} in three dimensions.

there are EIRCD or not. Equation (4) ought to be contrasted to *linearized hydrodynamics* when $\lambda(t) = 1 + \delta\lambda(t)$ with $|\delta\lambda(t)| \ll 1$. This limit is opposite to the deep scaling regime we focuson in this Letter where $\lambda_{\max} \gg 1$. In the IR scaling regime ($\tilde{\omega}_2 \to 0$), the dynamics are highly nonlinear but are self-similar and conformal. The linearized hydrodynamic limit can be realized when $\tilde{\omega}_2 \approx 1$ or $|\omega_2 - \omega_1| \ll \omega_1$; and the damping is given by $\Gamma_{\text{c.m.}} \approx \omega_1^2\tau_R$, for a relaxation time τ_R. Extending our results to the linear response limit naturally reproduces the previous literature on collective modes [40,41].

Second, Eqs. (4) and (6) also imply that the thermalization rate will be parametrically smaller for systems with EIRCD. This means that the thermalization rate for 1D Bose gases in the strongly interacting limit is smaller than in the weakly interacting limit. Some numerical simulations of the Lieb-Liniger model seem to suggest that the thermalization rate is an order of magnitude larger in the weakly interacting limit [29]. We also verified that the phenomenology associated with the EIRCD is consistent with generalized hydrodynamics in the presence of a diffusion term, which provides an accurate description of 1D Bose gases [27,28,42–55]. Similarly, one expects the thermalization rate for strongly interacting 3D Fermi gases to be much larger than weakly interacting systems due to the lack of EIRCD in the strongly interacting limit.

In summary, we investigated the dynamics of strongly interacting quantum gases in quenched harmonic traps in $d = 1, 3$ dimensions. Although we focused on the pragmatic case of strongly interacting gases in $d = 1, 3$, we note that our formalism is quite general and can also apply for weakly interacting gases, two dimensions ($d = 2$), and to other types of scale-breaking interactions, like effective range corrections and three-body interactions.

For the strongly interacting 1D and 3D gas, we found that the expansion dynamics in one dimension generates a renormalization group flow towards the IR limit which renders the breaking of scale invariance irrelevant. Since the strongly interacting fixed point in one dimension is IR stable and robust, EIRCD can be viewed as a dynamical phase that remains to be further studied in experiments examining the scaling dynamics discussed here. It also remains to further explore in a more quantitative manner the relation between the point of view of EIRCD in the proximity of *infrared stable fixed points*, which can also occur in higher spatial dimensions, and the fascinating microscopic dynamics of 1D integrable systems such as the Lieb-Liniger model but in a harmonic trap.

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