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Letter

Driven-dissipative Bose-Einstein condensation and the upper critical dimension

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Driving and dissipation can stabilize Bose-Einstein condensates. Using Keldysh field theory, we analyze this phenomenon for Markovian systems that can comprise on-site two-particle driving, on-site single-particle and two-particle loss, as well as edge-correlated pumping. Above the upper critical dimension, mean-field theory shows that pumping and two-particle driving induce condensation right at the boundary between the stable and unstable regions of the noninteracting theory. With nonzero two-particle driving, the condensate is gapped. This picture is consistent with the recent observation that, without symmetry constraints beyond invariance under single-particle basis transformations, all gapped quadratic bosonic Liouvillians belong to the same phase. For systems below the upper critical dimension, the edge-correlated pumping penalizes high-momentum fluctuations, rendering the theory renormalizable. We perform the one-loop renormalization group analysis, finding a condensation transition inside the unstable region of the noninteracting theory. Interestingly, its critical behavior is determined by a Wilson-Fisher-like fixed point with universal correlation-length exponent $\nu = 0.6$ in three dimensions.

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Introduction. Quantum phase transitions [1–3] in driven-dissipative systems have recently received a surge of attention. Instead of nonanalytic changes in zero-temperature states, phase transitions in open many-body systems are characterized by nonanalyticities in the nonequilibrium steady state which can arise due to a competition between Hamiltonian terms and environment couplings.

Promising experimental platforms to study such phenomena are cold atoms in optical cavities [4,5], lattices and tweezers [6–8], trapped ions [9–11], Rydberg atoms [12–16], superconducting circuits [17–19], and polaritons in circuit-QED or semiconductor-microcavity systems [20–29].

Markovian open quantum systems evolve according to a Lindblad master equation $\partial_t\hat{\varrho} = \mathcal{L}(\hat{\varrho})$ for the density operator $\hat{\varrho}$ with the Liouvillian

$$\mathcal{L}(\hat{\varrho}) = -\mathrm{i}[\hat{H},\hat{\varrho}] + \sum_\alpha \left(\hat{L}_\alpha\hat{\varrho}\hat{L}_\alpha^\dagger - \frac{1}{2}\{\hat{L}_\alpha^\dagger\hat{L}_\alpha,\hat{\varrho}\}\right), \tag{1}$$

where Lindblad operators $\hat{L}_\alpha$ capture the coupling to the environment [30–33]. While exact solutions are rare [34–46], the long-distance physics of typical Markovian many-body systems can be analyzed with Keldysh field theory and renormalization group (RG) techniques [47–49]. Some examples can be found in Refs. [50–55].

This work provides a field-theoretical analysis of driven-dissipative Bose-Einstein condensation (BEC) [24,50,51,56–64] above and below the upper critical dimension. Specifically, we consider bosons on a d-dimensional cubic lattice, comprising the kinetic energy and an on-site two-particle driving term in the Hamiltonian as well as dissipators for on-site single-particle and two-particle loss and an edge-correlated pumping process,

$$\mathcal{L} = -\mathrm{i}[\hat{H},\cdot] + \frac{\gamma_p}{2}\sum_{\langle i,j\rangle}\mathcal{D}[\hat{a}_i^\dagger + \hat{a}_j^\dagger] + 2d\gamma_l\sum_i\mathcal{D}[\hat{a}_i] + \tilde{u}\sum_i\mathcal{D}\big[\hat{a}_i^2\big], \quad \text{with} \tag{2a}$$

$$\hat{H} = -\tilde{J}\sum_{\langle i,j\rangle}\hat{a}_i^\dagger\hat{a}_j + \sum_i\left(d\tilde{J}\,\hat{a}_i^\dagger\hat{a}_i + G\,\hat{a}_i^2\right) + \text{H.c.}. \tag{2b}$$

Here, $\mathcal{D}[\hat{L}_\alpha](\hat{\varrho}) := \hat{L}_\alpha\hat{\varrho}\hat{L}_\alpha^\dagger - \frac{1}{2}\{\hat{L}_\alpha^\dagger\hat{L}_\alpha,\hat{\varrho}\}$ is the dissipator for Lindblad operator $\hat{L}_\alpha$, $\hat{a}_i$ is the bosonic annihilation operator on site i such that $[\hat{a}_i,\hat{a}_j^\dagger] = \delta_{i,j}$ and $[\hat{a}_i,\hat{a}_j] = 0$, and the sum $\sum_{\langle i,j\rangle}$ runs over all lattice edges. The chemical potential has been set to the minimum of the free-boson band. We could add additional on-site particle pumping terms as well as edge-correlated loss. As long as the latter is weaker than the edge-correlated pumping, these would not change the physics qualitatively.

In this Letter, we first derive the Keldysh action in the continuum limit. According to the subsequent mean-field treatment, above the upper critical dimension, the condensation transition, as induced by pumping γ_p and/or two-particle driving G, occurs right at the boundary between the stable and unstable regions of the noninteracting theory [65]. The tree-level scaling analysis yields the upper critical dimension $d_c = 4$. Next, we discuss the Gaussian approximation for $d > d_c$, finding that with nonzero two-particle driving G the condensate is gapped, and gapless otherwise. In fact, a transition between two gapped phases inside the stable region of the noninteracting theory can be excluded on general grounds by Proposition 5 of Ref. [66]. Lastly, we carry out the one-loop RG analysis with an ϵ expansion for $d < d_c$, finding a condensation transition inside the unstable region of the noninteracting theory. The nonequilibrium phase diagram is then determined by a Wilson-Fisher-like fixed point [59,67–69]. The results are summarized and compared to BEC in closed systems. The appendices in the Supplemental Material discuss the experimental realization and provide details for the analytical investigations [70].

Keldysh action. Similar to procedures for closed systems [71,72], we can use a Trotter decomposition of the time evolution operator to write the evolved density operator as $\hat{\varrho}(t) = e^{\mathcal{L}t}\hat{\varrho}(0) = (e^{\mathcal{L}t/N_t})^{N_t}\hat{\varrho}(0)$. Then, inserting resolutions of the identity in terms of coherent states between all factors and taking the trace gives the partition function Z as a product of coherent-state matrix elements $\langle\boldsymbol{\psi}'_+|e^{\mathcal{L}\delta t}[|\boldsymbol{\psi}_+\rangle\langle\boldsymbol{\psi}_-|]|\boldsymbol{\psi}'_-\rangle$ with $\hat{a}_i|\boldsymbol{\psi}_\pm\rangle = \psi_{\pm,i}|\boldsymbol{\psi}_\pm\rangle$. Taking the continuous-time limit $N_t \to \infty$, one arrives at [47,49]

$$Z = \int \mathscr{D}[\psi_\pm, \psi_\pm^*]e^{-S} \quad \text{with Keldysh action} \tag{3a}$$

$$S = \int_t [\boldsymbol{\psi}_+^\dagger \partial_t \boldsymbol{\psi}_+ - \boldsymbol{\psi}_-^\dagger \partial_t \boldsymbol{\psi}_- - \mathscr{L}(\boldsymbol{\psi}_\pm, \boldsymbol{\psi}_\pm^*)], \tag{3b}$$

where $\mathscr{L}(\boldsymbol{\psi}_\pm, \boldsymbol{\psi}_\pm^*)$ are Liouvillian matrix elements. Staring with lattice spacing a, we can take the spatial continuum limit by replacing the variables $\psi_{\pm,i}$ with fields $a^{d/2}\psi_\pm(\boldsymbol{x}_i)$ and sums $\sum_i$ by integrals $a^{-d}\int_{\boldsymbol{x}} \equiv a^{-d}\int \mathrm{d}^d x$. For terms that act on lattice edges $\langle i, j\rangle$, we express $\psi_\pm(\boldsymbol{x}_j)$ in terms of $\psi_\pm(\boldsymbol{x}_i)$ and its derivatives up to second order. This step assumes that relevant field configurations are sufficiently smooth, and we will indeed find the gap to close at quasi-momentum $k = 0$, such that long-range fluctuations dominate. We arrive at the Keldysh action

$$\begin{aligned} S = \int_{\boldsymbol{x},t} \Big[&\psi_+^*\partial_t\psi_+ - \psi_-^*\partial_t\psi_- - 2d(\gamma_l\psi_+\psi_-^* + \gamma_p\psi_+^*\psi_-) \\ &+ d(\gamma_l + \gamma_p)(\psi_+^*\psi_+ + \psi_-^*\psi_-) \\ &- \frac{a^2\gamma_p}{4}(2\psi_-\nabla^2\psi_+^* - \psi_+^*\nabla^2\psi_+ - \psi_-^*\nabla^2\psi_-) \\ &- \mathrm{i}a^2\tilde{J}(\psi_+^*\nabla^2\psi_+ - \psi_-^*\nabla^2\psi_-) + \mathrm{i}G(\psi_+^2 - \psi_-^2 + \text{c.c.}) \\ &- \frac{a^d\tilde{u}}{2}(2\psi_+^2\psi_-^{*2} - |\psi_+|^4 - |\psi_-|^4)\Big]. \end{aligned} \tag{4}$$

Note that the two-particle driving term in the Hamiltonian breaks the superparticle number conservation such that $[\hat{\varrho}, \sum_i \hat{a}_i^\dagger\hat{a}_i] \neq 0$. Hence, for nonzero G, the system only has the discrete $\mathcal{PT}$ symmetry [73], while it has a continuous U(1) symmetry under

$$\hat{a}_i \mapsto e^{\mathrm{i}\alpha}\hat{a}_i \quad \text{when} \quad G = 0. \tag{5}$$

For further analysis, it is useful to perform the Keldysh rotation [49] from fields $\psi_\pm$ to

$$\psi_c := (\psi_+ + \psi_-)/\sqrt{2}, \quad \psi_q := (\psi_+ - \psi_-)/\sqrt{2}, \tag{6}$$

which leads to the action

$$\begin{aligned} S = \int_{\boldsymbol{x},t} \Big[&\psi_c^*\partial_t\psi_q + \psi_q^*\partial_t\psi_c - t_1(\psi_c^*\psi_q - \psi_q^*\psi_c) \\ &+ t_2\psi_q^*\psi_q + K_1\big(\psi_c^*\nabla^2\psi_q - \psi_q^*\nabla^2\psi_c\big) + 2K_2\psi_q^*\nabla^2\psi_q \\ &- \mathrm{i}J\big(\psi_c^*\nabla^2\psi_q + \psi_q^*\nabla^2\psi_c\big) + 2\mathrm{i}G(\psi_c\psi_q + \psi_c^*\psi_q^*) \\ &+ \frac{u}{2}\big(\psi_c^2\psi_c^*\psi_q^* + \psi_q^2\psi_c^*\psi_q^* - \text{c.c.}\big) + 2u\psi_c\,\psi_c^*\psi_q\,\psi_q^*\Big], \end{aligned} \tag{7}$$

where we have reparametrized the model with

$$t_1 := d(\gamma_l - \gamma_p), \quad t_2 := 2d(\gamma_l + \gamma_p) \geqslant 0, \quad J := a^2\tilde{J},$$
$$K_1 = K_2 := a^2\gamma_p/4 \geqslant 0, \quad \text{and} \quad u := a^d\tilde{u} \geqslant 0. \tag{8}$$

We have introduced two separate parameters K_1 and K_2 which, while they are identical in the original model, will turn out to scale differently in the RG analysis.

Mean-field theory. Let us denote the solution of the saddle-point equations

$$\frac{\delta S}{\delta\psi_c^*} = 0, \quad \frac{\delta S}{\delta\psi_q^*} = 0 \tag{9}$$

by $\bar{\psi}_c$, $\bar{\psi}_q$. Due the conservation of probability, all terms in the action (7) are at least linear in ψ_q or ψ_q^* [47,49]. Hence, the first equation leads to $\bar{\psi}_q = 0$. Then, the second equation yields

$$t_1\bar{\psi}_c + 2\mathrm{i}G\bar{\psi}_c^* + \frac{u}{2}\rho\bar{\psi}_c = 0 \quad \text{with} \quad \rho \equiv |\bar{\psi}_c|^2. \tag{10}$$

Multiplying this by $\bar{\psi}_c$ and $\bar{\psi}_c^*$, respectively, we find

$$\bar{\psi}_c^2 = \frac{-2\mathrm{i}G\rho}{t_1 + \frac{u}{2}\rho} \quad \text{and} \quad \bar{\psi}_c^{*2} = \frac{\mathrm{i}}{2G}\Big(t_1 + \frac{u}{2}\rho\Big)\rho. \tag{11}$$

Now, using that these are complex conjugates of each other leads to the equation

$$\Big[4G^2 - \Big(t_1 + \frac{u}{2}\rho\Big)^2\Big]\rho = 0. \tag{12}$$

Solving for ρ, we arrive at the mean-field solution

$$\bar{\psi}_q = 0, \quad \rho = \begin{cases} 0 & \text{for } t_1 > 2|G|, \\ \frac{2}{u}(2|G| - t_1) & \text{for } t_1 < 2|G|. \end{cases} \tag{13}$$

To obtain $\bar{\psi}_c$, one simply substitutes ρ into Eq. (11).

Now, recalling that [47,49]

$$\begin{aligned} \langle\psi_{c,i}^*(t)\psi_{c,j}(t')\rangle &\equiv \frac{1}{Z}\int \mathscr{D}[\psi_\pm, \psi_\pm^*]\,\psi_{c,i}^*(t)\psi_{c,j}(t')\,e^{-S} \\ &= \mathrm{Tr}(\{\hat{a}_i(t), \hat{a}_j^\dagger(t')\}\,\hat{\varrho}_{\text{ss}}), \end{aligned} \tag{14}$$

where $\hat{\varrho}_{\text{ss}}$ denotes the steady state of the system, the onset of the mean-field value (13) of $\rho = |\bar{\psi}_c|^2$ signals a macroscopic occupation of the zero-momentum mode.

So, the pumping and/or two-particle driving induce a transition to a Bose condensate phase at $t_1 = 2|G|$. In fact, this point also marks the boundary between the stable and unstable regions of the noninteracting part of the model ($u = 0$); see Appendix C of the Supplemental Material [70] and Ref. [39].

The two-particle loss ($u > 0$) in the interacting model will stabilize the condensate.

Tree-level scaling analysis. The mean-field theory only describes the system correctly if the quartic terms in the action are irrelevant, i.e., above the upper critical dimension d_c. Let us perform the tree-level RG analysis to determine the engineering dimensions (a.k.a. canonical scaling dimensions) of the fields and coupling parameters such that we can assess the relevance of the (quartic) interaction terms and deduce d_c. To this purpose, one considers the quadratic part of the action and examines how quantities scale under a lowering of the ultraviolet cutoff $\Lambda \mapsto \Lambda/b$, a corresponding rescaling of space/momenta $k \mapsto bk$, and renormalization $\psi_{c/q} \mapsto b^{[\psi_{c/q}]}\psi_{c/q}$, $g_i \mapsto b^{[g_i]}g_i$ of the field variables and coupling parameters. The engineering dimensions $[\psi_c]$, $[\psi_q]$, and $\{[g_i]\}$ are determined such that the action and low-momentum features of all Green's functions are invariant under this tree-level RG transformation.

With the Fourier transformation

$$\phi_{c/q}(\boldsymbol{k},\omega) := \int_{\boldsymbol{x},t} e^{\mathrm{i}(\omega t-\boldsymbol{k}\cdot\boldsymbol{x})}\psi_{c/q}(\boldsymbol{x},t), \tag{15}$$

the quadratic part of the action can be easily diagonalized. The resulting free Green's function is calculated in Appendix B of the Supplemental Material [70], and the dispersion relation for single-particle excitations from the steady state is found to be

$$\omega(\boldsymbol{k}) = -\mathrm{i}(t_1 + Kk^2) \pm \mathrm{i}\sqrt{4G^2 - J^2k^4}. \tag{16}$$

The many-body spectrum of quasi-free systems is fully determined by the single-particle dispersion [36,39] and the *dissipative gap* is

$$\Delta := -\sup_{\lambda\neq 0\in\mathrm{spectrum}(\mathcal{L})} \mathrm{Re}\,\lambda = -\sup_{\boldsymbol{k}} \mathrm{Im}\,\omega(\boldsymbol{k}). \tag{17}$$

So, the noninteracting system becomes gapless at the transition point $t_1 = 2|G|$, where the dissipative gap closes at $k = 0$ with dispersion $\omega \sim -\mathrm{i}Kk^2$. Working with scaling dimension 1 for momenta, we hence have $[k] = 1$ and $[\omega] = 2$.

Let us now perform the tree-level scaling analysis for the critical point $t_1 = 2|G|$. Due to the invariance of the partition function (3), we have $[S] = 0$. Using $[\mathrm{d}^d x] = -d$ and $[\mathrm{d}t] = -2$, it follows that

$$0 = [\textstyle\int_{\boldsymbol{x},t} \psi_c^*\partial_t\psi_q] = -d - 2 + [\psi_c] + 2 + [\psi_q]. \tag{18}$$

From this and the requirement that the terms with coefficients t_1, K_1, K_2, J, and G in the action (7) are also scale invariant, it follows immediately that

$$[t_1] = [G] = 2 \quad \text{and} \quad [K_1] = [J] = 0. \tag{19}$$

Now, on physical grounds it can be argued that t_2 should not scale, i.e., that we have a constant noise vertex in the action [47] with

$$[t_2] = 0 \;\Rightarrow\; 0 = \left[\int_{\boldsymbol{x},t} t_2\psi_q^*\psi_q\right] = -d - 2 + 2[\psi_q]. \tag{20}$$

From this, Eq. (18), and $0 = [\int_{\boldsymbol{x},t} K_2\psi_q^*\nabla^2\psi_q]$ we, finally conclude that

$$[\psi_c] = \frac{d-2}{2}, \quad [\psi_q] = \frac{d+2}{2}, \quad \text{and} \quad [K_2] = -2. \tag{21}$$

This is consistent with the long-range and slow-frequency asymptotic behavior

$$\langle\phi_c\phi_c^*\rangle \sim \frac{t_2}{K_1^2}k^{-d-6} \quad \text{and} \quad \langle\phi_c\phi_q^*\rangle \sim \frac{1}{K_1}k^{-d-4} \tag{22}$$

of the free Green's function at the mean-field transition point as determined in Appendix B of the Supplemental Material [70].

With the canonical dimensions (21) of the fields and the condition $[S] = 0$, we can determine the RG relevance of all terms in the action (7) of the interacting model. According to Eqs. (19)–(21), the t_1 and G terms are RG relevant, the t_2, K_1, and J are marginal, and the K_2 term is irrelevant. For the quartic (interaction) terms, Eq. (21) implies

$$\left[\int_{\boldsymbol{x},t} \psi_c^2\psi_c^*\psi_q^*\right] = d-4, \quad \left[\int_{\boldsymbol{x},t} \psi_q^2\psi_c^*\psi_q^*\right] = d,$$
$$\text{and} \quad \left[\int_{\boldsymbol{x},t} \psi_c\psi_c^*\psi_q\psi_q^*\right] = d-2. \tag{23}$$

Although these terms have the same coefficient u in the bare action, they scale differently under RG due to the different scaling dimensions of ψ_c and ψ_q. The upper critical dimension d_c is defined such that all quartic terms are irrelevant for $d > d_c$. So, in this model, we have $d_c = 4$, and the Gaussian fixed point

$$\tilde{t}_1 = 2|G|, \quad \tilde{u} = 0 \tag{24}$$

with dynamical critical exponent $z = 2$ [Eq. (16)] is stable for $d > d_c$.

Gaussian approximation above the upper critical dimension. For $d > d_c$, let us now consider the Gaussian fluctuations around the mean-field solution (13). Specifically, we substitute

$$\psi_c = \bar{\psi}_c + \delta\psi_c \quad \text{and} \quad \psi_q = \delta\psi_q, \tag{25}$$

and expand the action (7) to second order in the fluctuations $\delta\psi_c$, $\delta\psi_q$. The resulting Green's function is computed in Appendix D, and we find the single-particle dispersion relation

$$\omega_\pm = -\mathrm{i}(t_1 + Kk^2 + u\rho) \pm \mathrm{i}\sqrt{\left|2\mathrm{i}G - u\bar{\psi}_c^2/2\right|^2 - J^2k^4}. \tag{26}$$

For $k = 0$, one has $\omega_\pm = -\mathrm{i}(t_1 + u\rho \mp |2\mathrm{i}G - u\bar{\psi}_c^2/2|)$. So, the symmetric phase with $t_1 > 2|G|$ and $\rho = |\bar{\psi}_c|^2 = 0$ is gapped. For the symmetry-broken Bose condensate phase with $t_1 < 2|G|$ and $\rho > 0$, let us first consider the case $G = 0$, i.e., systems without two-particle driving. Then, the condensate has a gapless excitation with $\omega_+ = 0$ at $k = 0$, corresponding to the Goldstone mode arising due to the spontaneous breaking of the continuous U(1) symmetry (5). In contrast, with two-particle driving $G \neq 0$, both ω_+ and ω_- are nonzero at $k = 0$, meaning that this $\mathcal{PT}$-symmetry-broken Bose condensate is gapped. In summary, the symmetry broken phase is gapped for $G \neq 0$ but gapless for $G = 0$, and the symmetric phase is always gapped.

There is a connection of these properties to general limitations on driven-dissipative phase transitions [74] in quasi-free and quadratic open systems [75]: In their stable region, the steady states of such systems cannot undergo transitions between gapped phases unless one imposes symmetry constraints beyond invariance under single-particle basis transformations [66]. In fact, one can connect any two gapped quadratic Liouvillians through a continuous path of gapped Liouvillians by tuning single-particle loss terms.

The model that describes the order-parameter fluctuations around the mean-field solution for $d > d_c$ is Gaussian, i.e., quasi-free. As the fluctuations $\delta\psi_{c/q}$ are just linear in the (original) microscopic field variables $\psi_\pm$, we can also establish a direct connection between the gap-opening terms in the Gaussian model and corresponding terms in the quasi-free part of the original Liouvillian $\mathcal{L}$. So, a phase transition between gapped phases could only occur inside or at the boundary to the unstable region of the noninteracting theory ($u = 0$) and, indeed, the Bose condensation transition was found to occur at that boundary.

RG analysis below the upper critical dimension. Just below the upper critical dimension, the quartic term

$$\int_{\boldsymbol{x},t} \frac{u}{2}\left(\psi_c^2\psi_c^*\psi_q^* - \text{c.c.}\right) \tag{27}$$

becomes relevant and can alter the phase diagram. We analyze this using an ϵ expansion [67–69,76,77] to perform the one-loop RG in $d = 4 - \epsilon$ dimensions. For simplicity, we set the two-particle driving $G = 0$, which restores the U(1) symmetry (5). We can drop all terms that have been identified as RG irrelevant in the tree-level scaling analysis, i.e., we only retain the interaction term (27) and also discard the K_2 term. For brevity, we use $K \equiv K_1$ in the following. Splitting the resulting action S_R of all nonirrelevant terms into its Gaussian and quartic parts, we have

$$\begin{aligned} S_R &= S_G + S_u \quad \text{with} \\ S_G &= \int_{\boldsymbol{x},t} [\psi_c^*\partial_t\psi_q + \psi_q^*\partial_t\psi_c + (t_1\psi_q^*\psi_c - t_1^*\psi_c^*\psi_q) \\ &\quad + (K - \mathrm{i}J)\psi_c^*\nabla^2\psi_q - (K + \mathrm{i}J)\psi_q^*\nabla^2\psi_c + t_2\psi_q^*\psi_q], \\ S_u &= \frac{1}{2}\int_{\boldsymbol{x},t} \left[u\psi_c^2\psi_c^*\psi_q^* - u^*\psi_c^{*2}\psi_c\,\psi_q\right]. \end{aligned} \tag{28}$$

In the RG process, two additional terms are generated. The first corresponds to a detuning $\sim\hat{a}^\dagger\hat{a}$ and the second to a Bose-Hubbard interaction $\sim\hat{a}^{\dagger 2}\hat{a}^2$ in the Hamiltonian. We have included them in the action S_G and S_u, with coupling coefficients $i\,\mathrm{Im}\,t_1$ and $i\,\mathrm{Im}\,u$, respectively. So, while t_1 and u are real in the initial model (7), they generally flow to complex values during the RG.

As deduced in Appendix E of the Supplemental Material [70], the one-loop RG flow equations for S_R read

$$\frac{\mathrm{d}t_1}{\mathrm{d}\ell} = 2t_1 + \frac{\mathcal{S}_d t_2}{2K + t_1 + t_1^*}u + O(u^2), \tag{29a}$$

$$\frac{\mathrm{d}u}{\mathrm{d}\ell} = \epsilon u - \frac{\mathcal{S}_d t_2}{2K^2}\left(u^2\frac{3K + 2\mathrm{i}J}{2(K + \mathrm{i}J)} + uu^*\right) + O(u^3), \tag{29b}$$

where we consider an infinitesimal momentum rescaling $k \to bk$ with $b = 1 + \mathrm{d}\ell$, $\mathcal{S}_d := 2/[(4\pi)^{d/2}\Gamma(d/2)]$ is a phase-space factor, and we have set the ultraviolet cutoff to $\Lambda = 1/a = 1$. Here, we see that the edge-correlated pumping $\sim\gamma_p$ (K) is needed to make the theory renormalizable. The field renormalization has been chosen such that rates t_2 and K as well as the inverse mass J are RG invariant.

The Gaussian fixed point at $t_1 = u = 0$ is stable for $d > d_c$ and the critical physics is described by the Gaussian field theory with dynamical critical exponent $z = 2$ and the correlation-length exponent $\nu = 1/2$ assuming their mean-field values. For $d < d_c$, the Gaussian fixed point is unstable and the system now features an additional Wilson-Fisher-like fixed point at

$$\tilde{t}_1 = -\epsilon\,\frac{K + \mathrm{i}J}{5} + O(\epsilon^2), \tag{30a}$$

$$\tilde{u} = \epsilon\,\frac{4K(K + \mathrm{i}J)}{5\mathcal{S}_d t_2} + O(\epsilon^2). \tag{30b}$$

To analyze the flow in its vicinity, we express $t_1 = \tilde{t}_1 + \delta t_1$ and $u = \tilde{u} + \delta u$ and expand the flow equations (29) to linear order in the deviations from the fixed point (30), finding

$$\frac{\mathrm{d}}{\mathrm{d}\ell}\begin{pmatrix}\mathrm{Re}\,\delta t_1\\ \mathrm{Im}\,\delta t_1\\ \mathrm{Re}\,\delta u\\ \mathrm{Im}\,\delta u\end{pmatrix} = \begin{bmatrix} 2 - \frac{2\epsilon}{5} & 0 & * & * \\ -\frac{2\epsilon J}{5K} & 2 & * & * \\ 0 & 0 & -\epsilon & 0 \\ 0 & 0 & -\frac{4J\epsilon}{5K} & -\frac{\epsilon}{5} \end{bmatrix}\begin{pmatrix}\mathrm{Re}\,\delta t_1\\ \mathrm{Im}\,\delta t_1\\ \mathrm{Re}\,\delta u\\ \mathrm{Im}\,\delta u\end{pmatrix}.$$

The upper-right block is $\mathcal{S}_d t_2 \mathbb{1}_{2\times 2}/2K + O(\epsilon)$ and does not affect the eigenvalues of the matrix. The flow of δu is independent of δt_1 and is characterized by the eigenvalues

$$\lambda_3 = -\epsilon + O(\epsilon^2) \quad \text{and} \quad \lambda_4 = -\epsilon/5 + O(\epsilon^2) \tag{31}$$

of the lower-right 2×2 submatrix. So, u will always flow towards the fixed-point value (30b). Since the generating matrix of the linearized RG flow is already in block-triangular form, the remaining two eigenvalues can be read off as

$$\lambda_1 = 2 - 2\epsilon/5 + O(\epsilon^2) \quad \text{and} \quad \lambda_2 = 2 + O(\epsilon^2). \tag{32}$$

These correspond to two relevant directions concerning the real and imaginary parts of t_1. However, as already pointed out in Ref. [47], the U(1) symmetry (5) of the model for $G = 0$ can be used to impose a gauge where t_1 is real by going to a suitable rotating frame. In particular, the transformation $\phi_{c/q}(\boldsymbol{x}, t) \mapsto \phi_{c/q}(\boldsymbol{x}, t)e^{-\mathrm{i}\omega_0 t}$ generates the term $-\mathrm{i}(\omega_0\psi_c^*\psi_q + \psi_q^*\psi_c)$ in S_G, which cancels the term $\propto \mathrm{Im}\,t_1$ for $\omega_0 = \mathrm{Im}\,t_1$. With real t_1, we are left with only one physically relevant direction, determining the boundary between the normal state and the condensate in the remaining three-dimensional parameter space. The corresponding eigenvalue λ_1 in Eq. (32) yields the correlation-length exponent [59]

$$\nu = \frac{1}{\lambda_1} = \frac{1}{2} + \frac{\epsilon}{10} + O(\epsilon^2) \text{ such that } \xi \sim |\delta t_1|^{-\nu} \tag{33}$$

for the correlation length near the critical point. To see this, note that, for $u = \tilde{u}$ and a rescaling factor $b = e^\ell$, the RG flow

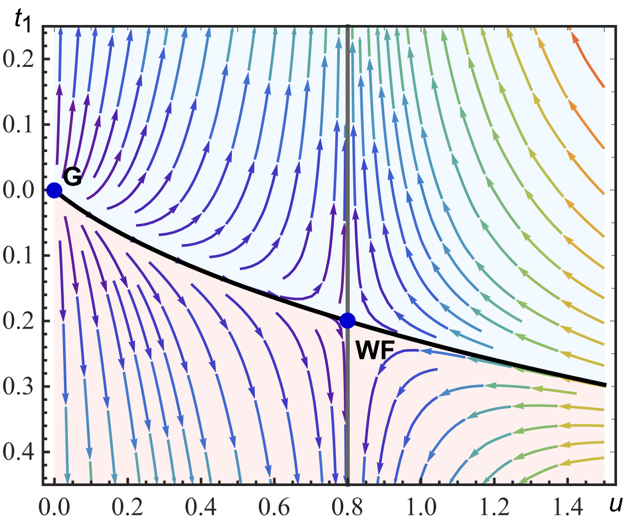

FIG. 1. RG flow diagram for $d < d_c$ and $J = G = 0$. The figure shows the RG flow (29) for $d = 3$ spatial dimensions which, for $J = 0$, remains in the two-dimensional plane spanned by real u and t_1. In addition to the unstable Gaussian fixed point (G), there is now a Wilson-Fisher-like fixed point (WF) which determines the critical behavior. The two-particle loss rate always flows to $\tilde{u}$ [Eq. (30)], while the difference of the single-particle loss and pumping $t_1 = d(\gamma_l - \gamma_p)$ flows to plus or minus infinity. The critical manifold separates the symmetric phase (upper blue region) from the lower region with a finite steady-state condensate density. For the figure, we have chosen $K = \mathcal{S}_d t_2 = 1$.

equations imply that two-point correlation functions obey homogeneity relations of the form [3]

$$C(e^{-\ell}\Delta\boldsymbol{x}, e^{\lambda_1\ell}\delta t_1) = e^{(d-2+\eta)\ell}C(\Delta\boldsymbol{x}, \delta t_1). \tag{34}$$

Evaluating this with ℓ chosen such that $e^{\lambda_1\ell}\delta t_1 = \pm 1$ gives the scaling form

$$C(\Delta\boldsymbol{x}, \delta t_1) = \xi^{-(d-2+\eta)}F_{\pm}\left(\frac{\Delta\boldsymbol{x}}{\xi}\right) \text{ with } \xi = |\delta t_1|^{-\frac{1}{\lambda_1}}.$$

The RG flow is illustrated in Fig. 1. Its structure is very similar to that of the celebrated Wilson-Fisher phase diagram [67–69] in the scalar ϕ^4 theory [78]. Depending on its initial value, t_1 will flow to $+\infty$ or $-\infty$, separating the symmetric and Bose condensate phases. The critical manifold lies in the unstable region of the noninteracting theory (cf. Appendix C of the Supplemental Material [70]).

Discussion. We have seen how incoherent pumping and/or coherent two-particle driving in competition with single-particle and two-particle loss can stabilize a Bose-Einstein condensate as a nonequilibrium steady state. Above the upper critical dimension $d_c = 4$ of the associated driven-dissipative phase transition, the fluctuations around the mean-field solution are captured by a Gaussian theory. According to a general result discussed in Ref. [66], transitions between two gapped phases can never occur inside the stable region of a noninteracting Markovian theory. For $d > d_c$, our bosonic system is a specific example. The transition then occurs right at the boundary between the stable and unstable regions of the noninteracting theory. With two-particle driving, the condensate is gapped, i.e., we have a transition between two distinct gapped phases. Without two-particle driving, the U(1) symmetry results in gapless Goldstone-mode excitations from the steady-state condensate.

For systems below the upper critical dimension ($d < d_c$), the Gaussian fixed point becomes unstable, and we have carried out the one-loop RG analysis using ϵ expansion. Interestingly, the transition still occurs in the unstable region of the noninteracting theory, and the physics of the critical point is described by the universal field theory of a Wilson-Fisher-like fixed point [59,67–69]. As shown by Eq. (29), coupling coefficients in the Keldysh action can flow to complex values during the RG. This is due to the non-Hermiticity of the Liouvillian. The one-loop analysis yields a correlation-length exponent of $\nu = 1/2 + (d_c - d)/10 + O((d_c - d)^2)$. The value $\nu = 0.6$ for $d = 3$ dimensions lies between the mean-field value $1/2$ and the value $\nu_{\text{fRG}} \approx 0.716$ found in a functional-RG analysis [50,51].

Let us shortly contrast these results with BEC in closed systems, where we are dealing with a single complex field ψ: A dilute interacting Bose gas can undergo BEC at low temperatures, where a single-particle state gets macroscopically occupied [79,80]. The transition to the normal (symmetric) phase is caused by thermal fluctuations. The long-range physics of nonzero-temperature transitions in d-dimensional quantum systems are described by *classical* field theories in d dimensions [1–3,81]. In the case of BEC with the U(1) symmetry (5), this is the O(2) model (a.k.a. XY model) which has upper critical dimension $d_c = 4$ and correlation-length exponent $\nu \approx 0.67$ for $d = 3$ [82–87]. Condensation can also be driven by quantum fluctuations in closed systems at zero temperature. Such quantum phase transitions can be captured by classical $(d + 1)$-dimensional field theories [1–3]. In the Bose-Hubbard model, the competition between the coherent kinetic and on-site repulsion terms leads to a transition between the Mott insulator and a superfluid (BEC) [3,88]. Coming from a Mott lobe with integer particle density ρ, one has to distinguish two cases. Generic transitions with continuously changing ρ are in the dilute-Bose-gas universality class with dynamical exponent $z = 2$ (quadratic dispersion for the excess particles), $d_c = 2$, and the mean-field value $1/2$ for ν in $d \geqslant 2$ dimensions. Transitions with fixed ρ are described by the classical $(d + 1)$-dimensional O(2) model with $z = 1$ due to space-time isotropy (linear dispersion), $d_c + 1 = 4$, and the mean-field value $\nu = 1/2$ in $d \geqslant 3$ dimensions [3,88].

It would be valuable to probe the field-theoretical predictions for the driven-dissipative BEC in numerical simulations. To this purpose it may be useful to consider the limit $u \to \infty$ of infinitely strong two-particle loss, restricting the maximum number of bosons per site to one. Above the upper critical dimension, the u term is RG irrelevant and should not affect the critical behavior.

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