# Interacting Bose gas across a narrow Feshbach resonance

Fan Yang<sup>®</sup> and Ran Qi<sup>®</sup>

Department of Physics and Key Laboratory of Quantum State Construction and Manipulation (Ministry of Education), Renmin University of China, Beijing 100872, People's Republic of China

(Received 15 January 2023; accepted 8 May 2024; published 3 June 2024)

We use a two-channel model to investigate an interacting Bose gas across a narrow Feshbach resonance within a field path integral approach. The ground-state properties show strong deviation from that of a broad Feshbach resonance or a single-channel interaction. The deviation can be interpreted by the strong energy dependence of two-body scattering length near a narrow Feshbach resonance. As the density increases, the chemical potential and energy per particle are found to saturate while the inverse compressibility and the phonon velocity undergo a significant reduction. We also take Gaussian fluctuations into account and calculate the ground-state energy correction as well as the quantum depletion.

DOI: 10.1103/PhysRevA.109.063303

## I. INTRODUCTION

The studies of weakly interacting dilute Bose gas have a very long history ever since the pioneer works in the 1950s. A cornerstone progress has been made by Lee, Huang, and Yang in their famous paper [1]. In their work, a low-density expansion was obtained for the equation of state at zero temperature:

$$\frac{E}{N} = \frac{2\pi\hbar^2 a_s}{m} n \left( 1 + \frac{128}{15\sqrt{\pi}} \sqrt{na_s^3} \right),$$
 (1)

where *m* is the mass of one atom, E/N is the energy per particle of the ground state, *n* is the total density of the Bose gas, and  $a_s$  is the *s*-wave scattering length [1]. The first term in Eq. (1) can be obtained by a simple mean-field calculation while the second term includes the contribution from the zero-point energy of quasiparticle excitations above the mean-field ground state, which is often called Lee-Huang-Yang correction nowadays [2]. This low-density expansion was later extended to even higher orders in Refs. [3–8]. Based on Eq. (1), all the thermal dynamic quantities can be derived at zero temperature. For example, the chemical potential and inverse compressibility are given as

$$\mu = \left(\frac{\partial E}{\partial N}\right)_{V} = \frac{4\pi\hbar^{2}a_{s}}{m}n\left(1 + \frac{32}{3\sqrt{\pi}}\sqrt{na_{s}^{3}}\right), \quad (2)$$

$$\kappa^{-1} = \left(\frac{\partial\mu}{\partial n}\right)_N = \frac{4\pi\hbar^2 a_s}{m} \left(1 + \frac{16}{\sqrt{\pi}}\sqrt{na_s^3}\right).$$
 (3)

According to Eqs. (1)–(3), as the density increases with fixed  $a_s$ , both the energy per particle and the chemical potential increase monotonically and faster than linear as a function of n, while the inverse compressibility remains a constant at the mean-field level and shows a weak dependence on density when including the Lee-Huang-Yang correction. Although these results were published more than 60 years ago,

their experimental verification is only possible very recently, thanks to the realization and high-precision measurement of Bose-Einstein condensate in ultracold quantum gases [9–11].

However, one should note that the validity of expansion (1) not only requires the low-density condition  $n^{1/3}a_s \ll 1$  but also relies on the fact that the two-body scattering process is fully determined by a single parameter  $a_s$ . For quantum gases close to a Feshbach resonance, while the second condition is usually fulfilled for a broad Feshbach resonance, it may break down for a very narrow resonance [12-18]. As shown in several previous works, near a Feshbach resonance, the two-body scattering amplitude is generally determined by the energy-dependent scattering length  $a_s(E_{coll}) = a_{bg}[1 +$  $\delta \mu \Delta / (E_{\text{coll}} - \nu_r)$ ], where  $E_{\text{coll}}$  is the total energy of the two particles under collision in the center-of-mass frame,  $a_{bg}$  is the background scattering length,  $v_r$  is the energy detuning of the closed-channel bound state,  $\Delta$  is the resonance width, and  $\delta\mu$  is the difference of magnetic momentum between open and closed channels of the correponding resonance [19–23]. For the magnetic Feshbach resonance,  $v_r =$  $\delta \mu (B - B_{\rm res})$  is magnetic field tunable where  $B_{\rm res}$  labels the resonance position [19,20]. The low-momentum expansion  $1/a_s(E_{\text{coll}}) = 1/a_s(0) - r_{\text{eff}} m E_{\text{coll}}/(2\hbar^2) + \mathcal{O}(E_{\text{coll}}^2)$  gives the effective range  $r_{\rm eff} = -2\hbar^2 \delta \mu \Delta / [ma_{bg}(\delta \mu \Delta - \nu_r)^2]$ , which characterizes the energy dependence of the scattering length. Across a broad resonance,  $n^{1/3}r_{\rm eff}\ll 1$ , the energy dependence in  $a_s(E_{coll})$  can be safely neglected and one has  $a_s(E_{\text{coll}}) \simeq a_s(0)$ . In contrast, for a narrow resonance,  $n^{1/3}r_{\rm eff} \gg 1$  such that  $a_s(E_{\rm coll})$  has a very sensitive dependence on  $E_{\text{coll}}$ , and thus the full functional form must be taken into account [21–25].

In particular, for a Bose gas across a narrow Feshbach resonance, one can qualitatively estimate the effect of the energy-dependent scattering length as follows. In an interacting many-body environment, the scattering energy between any two atoms is uncertain; however, it can be estimated as an average collision energy  $\bar{E}_{coll}$ , which should be on the same order of twice the chemical potential or of twice the energy per particle. As a result, the effective interacting strength

<sup>\*</sup>qiran@ruc.edu.cn

<sup>2469-9926/2024/109(6)/063303(9)</sup> 



FIG. 1. Schematic plot of the energy-dependent scattering length near a narrow Feshbach resonance.

should be determined by an effective scattering length  $\bar{a}_s \simeq a_s(\bar{E}_{coll})$  instead of the zero-energy scattering length  $a_s(0)$ . Obviously, as the density *n* increases,  $\bar{E}_{coll}$  should also increase.

In this paper, we consider a particular case in which  $v_r > 0$ ,  $a_{bg} > 0$ , and  $a_s(0) > 0$ , and a schematic plot of  $a_s(E)$  is shown in Fig. 1. In this case, the function  $a_s(E_{coll})$  decreases from  $a_s(0)$  to 0 as the energy  $E_{coll}$  increases from 0 to the critical value  $E_c = v_r - \delta \mu \Delta$ . At a fixed magnetic field, if one increases the density, the average collision energy will increase such that  $a_s(\bar{E}_{coll})$  decreases towards 0, which results in a rapid reduction of the interaction strength, thereby hindering the further increasing of  $\bar{E}_{coll}$ . As a result, the effect of the energy-dependent  $a_s(\bar{E}_{coll})$  is significantly enhanced in this particular parameter regime, and we expect that chemical potential  $\mu$  and the energy per particle E/N may saturate to  $\mu_c =$  $E_c/2$  as the density increases, consequently yielding a nearly 0 inverse compressibility. In the following, we show that the qualitative analysis above is correct based on a mean-field plus Gaussian fluctuation calculation. We provide a modified low-density expansion for the equation of state, which can be seen as a generalization of Eq. (1), that applies to Bose gases across either broad or narrow Feshbach resonances. The effect of the energy-dependent scattering length on various thermodynamic properties is investigated systematically.

Our paper is organized as follows. In Sec. II, we introduce our model Hamiltonian for a spinless Bose gas across a narrow Feshbach resonance and adopt the path integral approach to obtain the thermodynamic partition function [26]. In Sec. III, we present the mean-field results on thermodynamic properties in detail and show the effect of energy-dependent scattering length in uniform systems. In Sec. IV, we calculate the Gaussian fluctuation correction to the mean-field results, which does not show qualitative changes. In Sec. V, we investigate density distributions for a trapped Bose gas. In Sec. VI, we summarize our main findings and conclude the paper.

## **II. MODEL AND PATH INTEGRAL APPROACH**

For spinless bosons interacting across a magnetic Feshbach resonance, we adopt the widely used two-channel model [27-30]. In the open channel, atoms persist in scattering states, while in the closed channel, two atoms form a deeply bound molecular state. The Hamiltonian is

given by

$$\begin{aligned} \hat{H} &= \hat{H}_a + \hat{H}_b + \hat{H}_c, \end{aligned} \tag{4a} \\ \hat{H}_a &= \int d^3 \mathbf{x} \Biggl[ \hat{a}^{\dagger}(\mathbf{x}) \Biggl( -\frac{\nabla^2}{2m} + V_a(\mathbf{x}) \Biggr) \hat{a}(\mathbf{x}) + \frac{g}{2} \hat{a}^{\dagger}(\mathbf{x}) \hat{a}^{\dagger}(\mathbf{x}) \hat{a}(\mathbf{x}) \Biggr], \end{aligned} \tag{4b}$$

$$\hat{H}_b = \int d^3 \mathbf{x} \, \hat{b}^{\dagger}(\mathbf{x}) \left( -\frac{\nabla^2}{2M} + \nu_b + V_b(\mathbf{x}) \right) \hat{b}(\mathbf{x}), \quad (4c)$$

$$\hat{H}_c = \int d^3 \mathbf{x} \left[ \frac{\alpha}{\sqrt{2}} \hat{a}^{\dagger}(\mathbf{x}) \hat{a}^{\dagger}(\mathbf{x}) \hat{b}(\mathbf{x}) + \text{H.c.} \right], \quad (4d)$$

where  $\hat{a}^{\dagger}$  is the creation operator for atoms in the open channel and  $\hat{b}^{\dagger}$  is the creation operator of closed-channel molecules. M = 2m is the mass of one molecule, and we set  $\hbar = 1$ throughout this paper.  $V_a$  and  $V_b$  are external potentials of atoms and molecules. The molecule detuning  $v_b$ , the interchannel coupling  $\alpha$ , and the interaction parameter in open channel g are bare quantities which need to be renormalized as follows:

$$v_b = v_r - [1 - \mathcal{Z}(\Lambda)]\alpha_r^2/g_r, \qquad (5a)$$

$$\alpha = \mathcal{Z}(\Lambda)\alpha_r,\tag{5b}$$

$$g = \mathcal{Z}(\Lambda)g_r,\tag{5c}$$

where  $\mathcal{Z}(\Lambda) = (1 - g_r \Lambda)^{-1}$  and  $\Lambda = \frac{1}{V} \sum_{\mathbf{k}}' (k^2/m)^{-1}$ . The renormalized quantities  $g_r$ ,  $\alpha_r$ , and  $\nu_r$  determine the energy-dependent scattering length as

$$\frac{4\pi a_s(E_{\text{coll}})}{m} = g_r + \frac{|\alpha_r|^2}{E_{\text{coll}} - \nu_r},\tag{6}$$

where  $g_r$  and  $\alpha_r$  are related to the physical parameters  $a_{bg}$ ,  $\Delta$ , and  $\delta\mu$  through the relations  $g_r = 4\pi a_{bg}/m$  and  $|\alpha_r|^2 = g_r \delta \mu \Delta$ . The effective range is expressed as  $r_{\text{eff}} = -2\tilde{\Delta}/mE_c a_s(0)$ , where  $\tilde{\Delta} = \delta\mu\Delta/v_r$ . The partition function of the Hamiltonian Eq. (4a) at arbitrary temperature *T* can be written into the following imaginary-time path integral form

$$Z = \int \mathcal{D}[\phi^*, \phi] \mathcal{D}[\psi^*, \psi] \exp(-S), \qquad (7)$$

where the action S is given as

$$S = \int dx \left[ \phi^*(x) \frac{\partial}{\partial \tau} \phi(x) + \psi^*(x) \frac{\partial}{\partial \tau} \psi(x) + h_a(x) + h_b(x) + h_c(x) \right],$$
(8a)

$$h_a(x) = \phi^*(x) \left[ -\frac{\nabla^2}{2m} + V_a(\mathbf{x}) - \mu \right] \phi(x),$$
  
+  $\frac{g}{2} \phi^*(x) \phi^*(x) \phi(x) \phi(x),$  (8b)

$$h_b(x) = \psi^*(x) \left[ -\frac{\nabla^2}{4m} + \nu_b + V_b(\mathbf{x}) - 2\mu \right] \psi(x),$$
 (8c)

$$h_c(x) = \frac{\alpha}{\sqrt{2}}\phi^*(x)\phi^*(x)\psi(x) + \text{c.c.},$$
(8d)

where  $\phi(x)$  and  $\psi(x)$  are the fields of atoms and molecules and we have defined  $x \equiv (\mathbf{x}, \tau)$  and  $\int dx \equiv \int d^3 \mathbf{x} \int_0^\beta d\tau$ , with  $\beta = (k_B T)^{-1}$ .

The path integral (7) cannot be performed exactly due to the interaction terms. However, for a weakly interacting Bose gas with very low density, it is a good approximation to expand the action S in Eq. (8a) around its saddle point solution [26]:

$$\phi(x) = \phi_0(\mathbf{x}) + \phi'(x), \tag{9}$$

$$\psi(x) = \psi_0(\mathbf{x}) + \psi'(x), \tag{10}$$

where  $\phi_0(\mathbf{x})$  and  $\psi_0(\mathbf{x})$  are the saddle point solutions which minimize the action *S*, while  $\psi'(x)$  and  $\psi'(x)$  are the fluctuation fields of atoms and molecules.

Later in Sec. III, we take the mean-field approximation by neglecting all the fluctuation terms. The approximate action is then given by

$$S^{(0)} = \beta \int d^{3}\mathbf{x} \bigg\{ \phi_{0}^{*}(\mathbf{x}) \bigg[ -\frac{\nabla^{2}}{2m} + V_{a}(\mathbf{x}) - \mu \bigg] \phi_{0}(\mathbf{x}) + \frac{g}{2} |\phi_{0}(\mathbf{x})|^{4} + \psi_{0}^{*}(\mathbf{x}) \bigg[ -\frac{\nabla^{2}}{4m} + V_{b}(\mathbf{x}) - 2\mu + \nu_{b} \bigg] \psi_{0}(\mathbf{x}) + \frac{\alpha}{\sqrt{2}} \phi_{0}^{*}(\mathbf{x})^{2} \psi_{0}(\mathbf{x}) + \frac{\alpha^{*}}{\sqrt{2}} \psi_{0}^{*}(\mathbf{x}) (\phi_{0}(\mathbf{x}))^{2} \bigg\}.$$
(11)

Then in Sec. IV we include the contribution from Gaussian fluctuation around the saddle point. For a uniform system, the saddle point solution is uniform, i.e.,  $\phi_0(\mathbf{x}) \equiv \phi_0$  and  $\psi_0(\mathbf{x}) \equiv \psi_0$ , and we obtain the quadratic action as

$$S \approx S^{(0)} + S^{(2)},$$
 (12)

$$S^{(2)} = \int dx \phi'^{*}(x) \left[ \frac{\partial}{\partial \tau} - \frac{\nabla^{2}}{2m} - \mu + 2g |\phi_{0}|^{2} \right] \phi'(x) + \frac{1}{2} \int dx \{ \phi'^{*}(x)^{2} [g \phi_{0}^{2} + \sqrt{2} \alpha \psi_{0}] + \text{c.c.} \} + \int dx \psi'^{*}(x) \left[ \frac{\partial}{\partial \tau} - \frac{\nabla^{2}}{4m} - 2\mu + \nu_{b} \right] \psi'(x) + \int dx \left[ 2 \frac{\alpha}{\sqrt{2}} \phi'^{*}(x) \phi_{0}^{*} \psi'(x) + \text{c.c.} \right].$$
(13)

To perform the path integral for this quadratic action, it is more convenient to first transform the action into momentumfrequency space,

$$\phi'(\mathbf{x}) = \frac{1}{\sqrt{\beta V}} \sum_{\mathbf{k},n} \phi'_{\mathbf{k},n} \exp[i(\mathbf{k} \cdot \mathbf{x} - \omega_n \tau)], \qquad (14)$$

$$\psi'(x) = \frac{1}{\sqrt{\beta V}} \sum_{\mathbf{k},n} \psi'_{\mathbf{k},n} \exp[i(\mathbf{k} \cdot \mathbf{x} - \omega_n \tau)], \quad (15)$$

where  $\omega_n$  is the bosonic Matsubara frequency, and **k** is the free wave vector. Then the action can be written more compactly as a matrix multiplication in Nambu space as

$$S^{(2)} = -\frac{\beta}{2} \sum_{\mathbf{k}}^{\prime} \left( \epsilon_{\mathbf{k}}^{b} + \epsilon_{\mathbf{k}}^{a} + 2g|\phi_{0}|^{2} \right) -\frac{1}{2} \sum_{\mathbf{k}\neq0,n} \Phi_{k}^{\dagger} \mathbf{G}^{-1}(\mathbf{k}, i\omega_{n}) \Phi_{k}, \qquad (16)$$

where  $k \equiv (\mathbf{k}, i\omega_n)$ ,  $\sum'_{\mathbf{k}} \equiv \sum_{\mathbf{k}\neq 0}, \ \epsilon^b_{\mathbf{k}} = \frac{\mathbf{k}^2}{4m} + \nu_b - 2\mu$ , and  $\epsilon^a_{\mathbf{k}} = \frac{\mathbf{k}^2}{2m} - \mu$  and we define the following vector:

$$\Phi_k^{\dagger} = [\psi_k^{\prime*}, \phi_k^{\prime*}, \phi_{-k}^{\prime}, \psi_{-k}^{\prime}].$$
(17)

The first summation in Eq. (16) comes from the order exchange between the creation and annihilation fields during the transformation into Nambu space [26]. And the  $4 \times 4$  matrix  $\mathbf{G}^{-1}$  gives the inverse Green's function

$$\mathbf{G}^{-1} = \mathbf{G}_0^{-1} - \mathbf{\Sigma},\tag{18}$$

where

$$\mathbf{G}_{0}^{-1} = \begin{bmatrix} G_{b0}^{-1}(k) & & & \\ & G_{a0}^{-1}(k) & & \\ & & & G_{a0}^{-1}(-k) & \\ & & & & & G_{b0}^{-1}(-k) \end{bmatrix}$$
(19)

and

$$\boldsymbol{\Sigma} = \begin{bmatrix} 0 & \tilde{\alpha}^* & & \\ \tilde{\alpha} & 2g|\phi_0|^2 & \tilde{g} & \\ & \tilde{g}* & 2g|\phi_0|^2 & \tilde{\alpha}^* \\ & & \tilde{\alpha} & 0 \end{bmatrix}.$$
(20)

Here we have defined  $G_{a0}(k) = (i\omega_n - \epsilon_{\mathbf{k}}^a)^{-1}$ ,  $G_{b0}(k) = (i\omega_n - \epsilon_{\mathbf{k}}^b)^{-1}$ ,  $\tilde{\alpha} = \sqrt{2\alpha}\phi_0^*$ , and  $\tilde{g} = g\phi_0^2 + \sqrt{2\alpha}\psi_0$ .

In the limit  $\alpha \to 0$  with fixed  $\mu < \mu_c$ , the last two terms in Eq. (11) and the elements  $\tilde{\alpha}$  in Eq. (20) become negligible, leading to decoupling between atomic and molecular states, and the effect of the closed-channel molecule can be neglected. In this limit, the system can be approximately described by a single-channel model corresponding to  $\hat{H}_a$  in Eq. (4a). We call this limit the single-channel limit. On the other hand in the limit  $\mu \rightarrow 0$  corresponding to a vanishingly small density, we have  $\overline{a}_s \simeq a_s(0)$  and thus the system is approximately described by a constant scattering length,  $a_{\rm s}(0)$ . We call this limit the extremely low-density limit. Consequently in these two limits, the energy dependence of  $a_s(\bar{E}_{coll})$  can be safely neglected and the equation of states reduced to Eqs. (1)–(3) with  $a_s = a_{bg}$  and  $a_s = a_s(0)$  for the single-channel limit and the extremely low-density limit, respectively. We show in the following sections that this is indeed true based on our general results which provide a nontrivial check for our calculation.

It is known that the quasiparticle excitation corresponds to the poles of the Green's function. By diagonalizing Eq. (18) we obtain two branches of excitations given as

$$\omega_{\mathbf{k}}^{\pm} = \sqrt{(B \pm \sqrt{B^2 - 4C})/2},$$
 (21)

where

$$B = \left(\epsilon_{\mathbf{k}}^{a} + 2g|\phi_{0}|^{2}\right)^{2} + \left(\epsilon_{\mathbf{k}}^{b}\right)^{2} + 2|\tilde{\alpha}|^{2} - \mu^{2}, \qquad (22)$$

$$C = \left[ \left( \epsilon_{\mathbf{k}}^{a} + 2g|\phi_{0}|^{2} \right) \epsilon_{\mathbf{k}}^{b} - |\tilde{\alpha}|^{2} \right]^{2} - \mu^{2} \left( \epsilon_{\mathbf{k}}^{b} \right)^{2}.$$
(23)

It is straightforward to check that  $\omega_{\mathbf{k}}^{-}$  has a linear dependence on  $|\mathbf{k}|$  as  $\mathbf{k} \to 0$  and thus represents the phonon mode of this Bose superfluid. On the other hand,  $\omega_{\mathbf{k}}^+$  is gapped at  $\mathbf{k} = 0$ , corresponding to the density fluctuation of the closed-channel molecule.

The Green's function can be diagonalized with a transformation matrix U [31]:

$$-\mathbf{U}^{\mathrm{T}}\mathbf{G}^{-1}(\mathbf{k}, i\omega_{n})\mathbf{U} = \begin{bmatrix} -i\omega_{n} + \omega_{\mathbf{k}}^{+} & & \\ & -i\omega_{n} + \omega_{\mathbf{k}}^{-} & & \\ & & i\omega_{n} + \omega_{\mathbf{k}}^{-} & \\ & & & i\omega_{n} + \omega_{\mathbf{k}}^{+} \end{bmatrix}.$$
(24)

Finally, we obtain the following Gaussian action:

$$S = S^{(0)} + \frac{\beta}{2} \sum_{\mathbf{k}}^{\prime} (\omega_{\mathbf{k}}^{+} + \omega_{\mathbf{k}}^{-}) - \frac{\beta}{2} \sum_{\mathbf{k}}^{\prime} \left( \epsilon_{\mathbf{k}}^{a} + 2g|\phi_{0}|^{2} + \epsilon_{\mathbf{k}}^{b} \right) + \sum_{\mathbf{k}}^{\prime} (-i\omega_{n} + \omega_{\mathbf{k}}^{+}) \psi_{\mathbf{k},n}^{*} \psi_{\mathbf{k},n} + \sum_{\mathbf{k}}^{\prime} (-i\omega_{n} + \omega_{\mathbf{k}}^{-}) \phi_{\mathbf{k},n}^{*} \phi_{\mathbf{k},n},$$
(25)

where  $[\psi_k^*, \phi_k^*, \phi_{-k}, \psi_{-k}]$  is related to  $[\psi_k^{\prime*}, \phi_k^{\prime*}, \phi_{-k}^{\prime}, \psi_{-k}^{\prime}]$  through

$$[\psi_k^*, \phi_k^*, \phi_{-k}, \psi_{-k}] = [\psi_k'^*, \phi_k'^*, \phi_{-k}', \psi_{-k}']\mathbf{U}^{-1}.$$
 (26)

Again the summation of  $\omega_{\mathbf{k}}^+$  and  $\omega_{\mathbf{k}}^-$  in Eq. (25) comes from the exchange of the field operators [26].

In the following sections we calculate the zero-temperature thermodynamic potential  $\Omega$  and the total density *n* with the action given by either Eq. (11) (in Sec. III) or Eq. (25) (in Sec. IV) through the following thermodynamic relations:

$$\frac{\Omega}{V} = -\frac{1}{\beta V} \ln(Z), \qquad (27)$$

$$n = -\frac{\partial(\Omega/V)}{\partial\mu},\tag{28}$$

$$\frac{E}{V} = \frac{\Omega}{V} + n\mu.$$
(29)

# **III. MEAN-FIELD CALCULATION**

In this section, we neglect the contribution of Gaussian fluctuation, and the action is governed by  $S^{(0)}$  (11). After minimizing the mean-field action  $S^{(0)}$  with  $\phi_0(\mathbf{x})$  and  $\psi_0(\mathbf{x})$ ,  $\delta S^{(0)}/\delta \phi_0^*(\mathbf{x}) = \delta S^{(0)}/\delta \psi_0^*(\mathbf{x}) = 0$ , we obtain the following two-channel Gross-Pitaevskii equations:

$$\left[-\frac{\nabla^2}{2m} + V_a + g_r |\phi_0(\mathbf{x})|^2 - \mu\right] \phi_0(\mathbf{x}) + \sqrt{2}\alpha_r \phi_0^*(\mathbf{x})\psi_0(\mathbf{x}) = 0,$$
(30a)

$$\left[-\frac{\nabla^2}{4m} + \nu_r + V_b - 2\mu\right]\psi_0(\mathbf{x}) + \frac{\alpha_r^*}{\sqrt{2}}\phi_0(\mathbf{x})^2 = 0.$$
(30b)

The renormalization should be taken to the same order, so that at mean-field level we should use the renormalized parameters directly in Eq. (30) [26].



FIG. 2. (a) The chemical potential, (b) the molecular fraction, (c) the energy per particle, and (d) the inverse compressibility  $\kappa^{-1}$  as a function of the total density *n* for narrow resonance (red lines) with  $\tilde{\Delta} = 0.01$  and  $v_r a_s^2(0) = 0.01$  and for broad resonance (blue deshed lines) with a constant scattering length  $a_s = a_s(0)$ . The gray dotdashed lines donates the position of  $\mu_c$ . The inset in panel (c) shows the ratio between energy per particle and chemical potential as a function of the density.

We now consider a uniform system with  $V_a = V_b = 0$ . In this case, the solutions of Eq. (30) for the ground state are also uniform, and the atomic density  $n_a$  and the molecular density  $n_b$  are given by

$$n_a = |\phi_0|^2 = \frac{\mu}{g(2\mu)},\tag{31}$$

$$u_{b} = |\psi_{0}|^{2} = \frac{1}{2} \left[ \frac{1}{g(2\mu)} \frac{\mu |\alpha_{r}|}{2\mu - \nu_{r}} \right]^{2},$$
(32)

where we define the energy-dependent interacting strength  $g(\mathcal{E}) = 4\pi \hbar^2 a_s(\mathcal{E})/m$ . It is easy to see that both  $n_a$  and  $n_b$ , and thus the total atomic density  $n = n_a + 2n_b$ , diverge as  $g(2\mu) \rightarrow 0$  corresponding to  $\mu \rightarrow \mu_c = E_c/2 = (v_r - |\alpha_r|^2/g_r)/2$ . As a result, the chemical potential  $\mu$  saturates to  $\mu_c$  as *n* increases. This behavior is shown exactly in Fig. 2(a).

Substituting Eqs. (31) and (32) into the mean-field action  $S^{(0)}$  (11), we can obtain the partition function *Z* through Eq. (7) by replacing *S* by  $S^{(0)}$  and, subsequently, the total energy per particle through Eqs. (27)–(29) as

$$\frac{E}{N} = \frac{g(2\mu)n}{2} \left(1 - \gamma_b^2\right) = \frac{1 + \gamma_b}{2} \mu,$$
(33)

where  $\gamma_b = 2n_b/n$  is the molecular fraction representing the fraction of atoms occupying the closed channel. The behavior of  $\gamma_b$  as a function of *n* is shown in Fig. 2(b). As density grows,  $\gamma_b$  increases from 0 to 1, and Eq. (33) indicates that E/N also saturates to  $\mu_c$  together with the chemical potential as shown in Fig. 2(c). The inset of Fig. 2(c) shows the ratio between



FIG. 3. The phonon velocity as a function of the total density *n* for a broad resonance (black dashed line) and for narrow resonances (solid lines) with  $v_r a_s^2(0) = 0.01$  and  $\tilde{\Delta} = \{0.01, 0.1, 0.5\}$ .

energy per particle and chemical potential  $(E/N)/\mu$ , which increases from 1/2 to 1 as the density grows. These behaviors of the energy per particle and the chemical potential are in qualitative difference with those given by Eqs. (1) and (2) as shown with dashed lines in Figs. 2(a) and 2(c).

This difference can be attributed to the energy dependence of the scattering length. As discussed in Sec. I,  $\mu$  and E/Nshould roughly be determined by the effective interacting strength given by  $\overline{a}_s \simeq a_s(E_{\text{coll}})$ . For a broad resonance,  $a_s$ is a constant and the energies increase monotonically as *n* increases according to Eqs. (1) and (2). When close to a narrow resonance, if density *n* increases, then  $\mu$  also increases such that  $\overline{a}_s$  decreases as shown in Fig. 1. This decreasing of  $\overline{a}_s$  suppresses the further increasing of  $\mu$  and E/N. Since  $\overline{a}_s$  approaches 0 as the average collision energy  $E_{\text{coll}} \rightarrow E_c$ , one may expect that  $\mu$  and E/N saturate to  $\mu_c$  as *n* increases, which is indeed the case as shown in Fig. 2.

Now we analyze the behavior of  $\mu$  and E/N at different limits in detail. In the extremely low-density limit where we have  $\mu \ll \mu_c$ , the open channel dominates the scattering since the closed channel is nearly unoccupied. As a result,  $\gamma_b$ tends to 0 and Eq. (33) reduces to the broad resonance result  $E/N = \mu/2$ , which can be derived from a weakly interacting single-channel model. This is shown in Fig. 2 where the curves for narrow resonance and broad resonance approach each other in the extremely low-density regime. As the density increases, the system reaches the opposite limit  $\mu \rightarrow \mu_c$  in which the effective scattering length  $\bar{a}_s$  vanishes. This leads to a saturated energy per particle as well as a vanishingly small inverse compressibility  $\kappa^{-1}$  as shown in Fig. 2(d) where the compressibility  $\kappa$  is given as

$$\kappa = \left(\frac{\partial n}{\partial \mu}\right)_N = \frac{n}{\mu} \left(1 + \gamma_b \frac{3\mu_c - \mu}{\mu_c - \mu}\right). \tag{34}$$

In the single-channel limit  $\Delta \ll 1$ , the crossover between the above two limits roughly takes place around  $n = v_r/g_r$ .

Furthermore, the sound velocity  $v_p$  of this system can be obtained through the thermodynamic relation  $v_p^2 = n\kappa^{-1}$ . The results are shown in Fig. 3. We can see that  $v_p$  develops a

PHYSICAL REVIEW A 109, 063303 (2024)

pronounced peak as the density increases. This nonmonotonic behavior can be explained by the strong density dependence of the inverse compressibility  $\kappa^{-1}$ , as shown in Fig. 2(d). In the zero-density limit  $(n \to 0)$ , we have  $\kappa^{-1} \to \text{const.}$  and thus  $v_p^2 = n\kappa^{-1}$  increases linearly as density *n*. In the highdensity regime, we find  $\kappa^{-1} \to 0$  faster than 1/n and thus we have  $v_p^2 \to 0$ . In the intermediate-density region around  $n \sim v_r/g_r$ , the significant drop in  $\kappa^{-1}$  combined with the linear increase of *n* contributes the pronounced peak in  $v_p$ . As analyzed earlier, the abnormal density dependence in  $\kappa^{-1}$ is a direct consequence of the strong energy dependence in  $a_s(E_{\text{coll}})$ , as a result, the peak behavior in  $v_p$  is only pronounced for very narrow resonance. We have also verified that this value of  $v_p$  is fully consistent with the value obtained directly from the quasiparticle spectrum given in Eq. (21) through

$$v_p = \lim_{k \to 0} \frac{\omega_{\mathbf{k}}}{k}.$$
 (35)

#### **IV. GAUSSIAN FLUCTUATION CALCULATION**

In this section, we take the correction of Gaussian fluctuation into account, and the action is governed by Eq. (25) which gives the thermodynamic potential

$$\frac{\Omega}{V} = \frac{\Omega_0}{V} + \frac{1}{V\beta} \sum_{\mathbf{k}}^{\prime} \ln(1 - e^{-\beta\omega_{\mathbf{k}}^-}) + \frac{1}{V\beta} \sum_{\mathbf{k}}^{\prime} \ln(1 - e^{-\beta\omega_{\mathbf{k}}^+}).$$
(36)

At zero temperature, the ground-state thermodynamic potential  $\Omega_0$  is given by

$$\frac{\Omega_0}{V} = \frac{F_L}{V} + \frac{F_{\rm fluc}}{V}.$$
(37)

 $F_L$  is the mean-field thermodynamic potential given as

$$\frac{F_L}{V} = -\frac{\mu^2}{2g(2\mu)}.$$
(38)

 $F_{\text{fluc}}$  is the zero-point energy correction induced by Gaussian fluctuation. After the renormalization procedure, we obtain

$$\frac{F_{\text{fluc}}}{V} = \frac{1}{2V} \sum_{\mathbf{k}}^{\prime} \left[ (\omega_{\mathbf{k}}^{-} + \omega_{\mathbf{k}}^{+}) - (\epsilon_{\mathbf{k}}^{a} + 2g_{r}|\phi_{0}|^{2} + \epsilon_{\mathbf{k}}^{b}) + \frac{m\mu^{2}}{k^{2}} \right] \\
= \frac{1}{4\pi^{2}} (2m)^{3/2} \mu^{5/2} F(\tilde{\mu}, \tilde{\Delta}).$$
(39)

where the term  $m\mu^2/k^2$  in the square brackets comes from the renormalization of the bare parameters and cancels the divergence in the momentum summation  $\sum'_{\mathbf{k}}$ . Here  $F(\tilde{\mu}, \tilde{\Delta})$ is a dimensionless function defined as

$$F(\tilde{\mu}, \tilde{\Delta}) = \int_0^\infty \left[\frac{1}{2} + x^2 g(x, \tilde{\mu}, \tilde{\Delta})\right] dx, \qquad (40)$$



FIG. 4. Behavior of  $F(\tilde{\mu}, \tilde{\Delta}) - F_{\text{LHY}}$  when close to three different limits. (a) The extremely low-density limit with  $\tilde{\Delta} = 0.4$ . (b) The single-channel limit with  $\tilde{\mu} = 0.2$ . (c) The saturation limit with  $\tilde{\Delta} = 0.01$ . The red lines in panels (a)–(c) show the results given by Eq. (40), while the blue dashed lines are the asymptotic results in the corresponding limits given by Eqs. (43), (44), and (48).

where  $\tilde{\mu} = \mu/\nu_r$ , and  $g(x, \tilde{\mu}, \tilde{\Delta})$  is another dimensionless function defined as

$$g(x, \tilde{\mu}, \tilde{\Delta}) = \sqrt{\omega_a^2 + \omega_b^2 + 4\xi - 1 + 2\sqrt{(\omega_a \omega_b - 2\xi)^2 - \omega_b^2}} - \omega_a - \omega_b,$$
(41)

with

$$\omega_a = x^2 + 1 - 2\frac{\tilde{\Delta}}{2\tilde{\mu} - 1 + \tilde{\Delta}},\tag{42a}$$

$$\omega_b = \frac{x^2}{2} + \frac{1}{\tilde{\mu}} - 2,$$
 (42b)

$$\xi = \frac{\tilde{\Delta}}{\tilde{\mu}} \frac{2\tilde{\mu} - 1}{2\tilde{\mu} - 1 + \tilde{\Delta}}.$$
(42c)

Based on the asymptotic behavior of  $F(\tilde{\mu}, \tilde{\Delta})$  as is shown below, in the extremely low-density limit  $(n \to 0 \text{ such that } \tilde{\mu} \to 0 \text{ with finite } \tilde{\Delta})$  or the single-channel limit  $(\alpha_r \to 0 \text{ such that } a_s(E) = mg_r/(4\pi) \equiv a_{bg}$  and we have  $\tilde{\Delta} \to 0$  with  $\tilde{\mu}$  remaining finite), *F* approaches a constant given by  $F_{\text{LHY}} = 8\sqrt{2}/15$ . We have verified that in these two limits the energy per particle obtained from our Eqs. (49) and (50) recovers the LHY result in Eq. (1) with  $a_s$  replaced by  $a_s(0)$  for the extremely low-density limit and by  $a_{bg}$  for the single-channel limit (see also the comparison in Fig. 4). This should be the case since in both limits the energy dependence of  $a_s(E)$  can be safely neglected.

Below, we analyze the asymptotic behavior of  $F(\tilde{\mu}, \tilde{\Delta})$  close to the above two limits as well as to the saturation limit where  $\mu \to \mu_c$  as discussed in Sec. III.

(i) In the limit  $\tilde{\mu} \ll 1$  while  $\tilde{\Delta}$  remains finite, i.e., the extremely low-density limit, we have

$$F(\tilde{\mu}, \tilde{\Delta}) = \frac{8\sqrt{2}}{15} + \frac{\sqrt{6}\pi}{4} \frac{\tilde{\Delta}}{1 - \tilde{\Delta}} \sqrt{\tilde{\mu}} + O(\tilde{\mu}).$$
(43)

(ii) In the limit  $\tilde{\Delta} \ll 1$ , i.e., the single-channel limit, we have

$$F(\tilde{\mu}, \tilde{\Delta}) = \frac{8\sqrt{2}}{15} + \frac{G(\tilde{\mu})}{1 - 2\tilde{\mu}}\tilde{\Delta} + O(\tilde{\Delta}^2), \qquad (44)$$

$$G(\tilde{\mu}) = \int_0^\infty \eta(x, \tilde{\mu}) dx, \qquad (45)$$

where

$$\eta(x,\tilde{\mu}) = \frac{1}{\sqrt{(x^2+1)^2 - 1} + (x^2+1)} \\ \times \frac{2x^2}{\sqrt{(x^2+1)^2 - 1} + \frac{x^2}{2} - 2 + \frac{1}{\tilde{\mu}}} \\ \times \left(1 + \frac{x^2}{2\sqrt{(x^2+1)^2 - 1}}\right).$$
(46)

If  $\tilde{\mu}$  also approaches 0 in this case, then we have

$$G(\tilde{\mu}) = \frac{\sqrt{6\pi}}{4} \sqrt{\tilde{\mu}} + O(\tilde{\mu}). \tag{47}$$

(iii) In the limit  $\tilde{\mu} \to (1 - \tilde{\Delta})/2 - 0^+$  such that  $g(2\mu) \to 0^+$ , which is called the saturation limit (since this is the limit where  $\mu$  and E/N saturate to  $\mu_c$ ), we have

$$F(\tilde{\mu}, \tilde{\Delta}) = \frac{\pi\sqrt{\tilde{\Delta}}}{4} \frac{1}{\sqrt{z}} + \frac{1 - (21 - 8\sqrt{6})\tilde{\Delta}}{8(1 - \tilde{\Delta})\sqrt{\tilde{\Delta}}} \pi\sqrt{z} + O(z^{3/2}),$$
(48)

where  $z = (1 - \tilde{\Delta})/2 - \tilde{\mu}$ . The comparison between the above three asymptotic behaviors and the full results of *F* is shown in Fig. 4.

The correction of energy per particle due to the Gaussian fluctuation is given by

$$\frac{E}{N} = \frac{1}{n} \left[ -\frac{\mu^2}{2g(2\mu)} + \frac{(2m)^{\frac{3}{2}}}{4\pi^2} \mu^{\frac{5}{2}} F(\tilde{\mu}, \tilde{\Delta}) \right] + \mu, \quad (49)$$

$$n = n_0 - \frac{(2\mu)^{\frac{3}{2}}}{4\pi^2} \bigg[ \tilde{\mu} F'(\tilde{\mu}, \tilde{\Delta}) + \frac{5}{2} F(\tilde{\mu}, \tilde{\Delta}) \bigg], \quad (50)$$

where  $n_0 = n_a + 2n_b$  is the mean-field density given by Eqs. (31) and (32). As shown in Fig. 5, the Gaussian fluctuation correction does not change the behavior of the mean-field results qualitatively. At low density, again the results approximately coincide with that of the broad resonance case with a constant  $a_s = a_s(0)$ . As density increases towards the saturation limit, the contribution of Gaussian fluctuation is small compared with the mean-field part, and the energy per particle still saturates to  $\mu_c$  and the energy per particle remains on the same order of the chemical potential.



FIG. 5. Energy per particle as a function of density *n* at  $v_r a_s^2(0) = 0.1$  and  $\tilde{\Delta} = 0.1$ . Close to a narrow resonance, E/N with a Gaussian fluctuation correction (red line) coincides with that of the broad resonance case with a constant  $a_s = a_s(0)$  (blue dot-dashed line) at very small density and saturates to  $\mu_c$  (indicated by the dotted line) at the saturation limit. The gray long-dashed line donates the results of the chemical potential  $\mu$ , and the black short-dashed line donates the mean-field results of the energy per particle.

In Fig. 6(a), we show  $\delta \bar{E} = E/N - E_{\text{LHY}}/N$  at fixed  $na_s^3(0)$  with different detuning  $v_r$  and resonance width  $\Delta$ , which is the difference between the energy per particle given by Eq. (49) and that given by the LHY formula in Eq. (1) with  $a_s = a_s(0)$ . This discrepancy results from the energy dependence of the scattering length and, therefore, is anticipated to be characterized by the effective range  $r_{\text{eff}}$  depicted in Fig. 6(b). It is evident from Fig. 6 that, in general, an increase in the effective range  $|r_{\text{eff}}|$  corresponds to a more pronounced energy difference,  $\delta \bar{E}$ . As an example, consider the narrow resonance of <sup>87</sup>Rb at 406.2 G with  $\Delta = 0.0004$  G [19,32]. Tuning the magnetic field to  $\tilde{\Delta} = 0.5$  gives an effective range



FIG. 6. (a)  $\delta \bar{E} / \bar{E}_{LHY}$  as a function of  $v_r$  and  $\Delta$  at fixed  $na_s^3(0) = 0.002$ .  $\bar{E}_{LHY} = E_{LHY}/N$  is the energy per particle for the singlechannel model given by Eq. (1). (b)  $r_{\text{eff}}$  as a function of  $v_r$  and  $\Delta$  at fixed  $na_s^3(0) = 0.002$ .



FIG. 7. The ratio between the density of quantum depletion and the total density  $n_{dp}/n$  as a function of the total density *n*. The solid lines from top to bottom stand for  $v_r a_s^2(0) = 0.01$  with  $\tilde{\Delta} =$ {0.01, 0.1, 0.5}, respectively. The black dashed line shows  $n_{dp}/n$  for the single-channel model with  $v_r a_s^2 = 0.01$ .

of  $r_{\rm eff}/a_s(0) = -15748$  and leads to  $\delta \bar{E}/\bar{E}_{\rm LHY} = -30.2\%$  at density  $n = 10^{14} \text{ cm}^{-3}$ .

We have also investigated the quantum depletion  $n_{dp} = \sum_{\mathbf{k}}' \langle G | \hat{a}_{\mathbf{k}}^{\dagger} \hat{a}_{\mathbf{k}} + 2 \hat{b}_{\mathbf{k}}^{\dagger} \hat{b}_{\mathbf{k}} | G \rangle$  as shown in Fig. 7. In particular, we find

(

$$G|\hat{a}_{\mathbf{k}}^{\dagger}\hat{a}_{\mathbf{k}}|G\rangle = \xi_1^2 + v_1^2, \qquad (51a)$$

$$\langle G|\hat{b}_{\mathbf{k}}^{\dagger}\hat{b}_{\mathbf{k}}|G\rangle = \xi_2^2 + v_2^2, \tag{51b}$$

where

$$\hat{a}_{\mathbf{k}} = \frac{1}{\sqrt{V}} \int \hat{a}(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}} d\mathbf{x},$$
$$\hat{b}_{\mathbf{k}} = \frac{1}{\sqrt{V}} \int \hat{b}(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}} d\mathbf{x},$$
(52)

and  $|G\rangle$  is the ground state of quasiparticles at zero temperature, and  $\xi_1$ ,  $\xi_2$ ,  $v_1$ , and  $v_2$  are the matrix elements of the quasiparticle transformation [31].

At very small density, the behavior of quantum depletion also approaches that of broad resonance with a constant  $a_s = a_s(0)$ . In the saturation limit ( $\mu \rightarrow \mu_c$ ), the depletion vanishes due to the vanishing of effective interaction strength represented by  $\bar{a}_s$ . The phonon velocity measures the collective excitation due to these fluctuations, and the **k** components of the quantum depletion (51) donate the momentum distribution of the density fluctuations. In the broad resonance limit, one has  $n_{dp} \propto v_p^3$ . As a result, the overall behavior of quantum depletion is similar to that of the phonon velocity in Fig. 3.

## V. DENSITY PROFILE IN HARMONIC TRAP

In this section, we investigate the density distribution of a narrow Feshbach resonance Bose gas trapped in a harmonic potential with  $V_b(r)/2 = V_a(r) = \frac{1}{2}m\omega^2 r^2$ . For trapped gas with a large particle number, if one assumes the density fluctuation is negligible, the kinetic terms in Eq. (30) can



FIG. 8. Density distribution  $na_s^3(0)$  of a Bose gas in a harmonic trap with fixed particle number  $N = 2 \times 10^5$ . (a) The comparison between the narrow resonance (red line) with  $\tilde{\Delta} = 0.001$  and  $a_o = \sqrt{m\omega/\hbar} = 60a_s(0)$  and the broad resonance (blue dashed line) with  $a_s \equiv a_s(0)$  at the mean-field level. (b) and (c) The comparison of the density distribution with (red lines) and without (blue dashed lines) Gaussian correction for broad resonance (b) and narrow resonance (c).

be neglected by local-density approximation and Eq. (30) becomes

$$[V_a(\mathbf{x}) + g_r n_a(\mathbf{x}) - \mu_0] + \alpha_r \sqrt{2n_b(\mathbf{x})} = 0, \quad (53a)$$

$$[\nu_r + V_b(\mathbf{x}) - 2\mu_0]\sqrt{2n_b(\mathbf{x})} + \alpha_r^* n_a(\mathbf{x}) = 0.$$
 (53b)

where  $\mu_0$  is the chemical potential determined by the total particle number. At the edge **R** of the gas cloud, the density vanishes and  $n_a(\mathbf{R}) = n_b(\mathbf{R}) = 0$ . One obtains  $\mu_0 = V_a(\mathbf{R})$  according to Eq. (53a). Then the density distribution just becomes Eqs. (31) and (32) by replacing  $\mu$  with the local chemical potential  $\mu_0 - V_a(\mathbf{x})$ .

For an interacting Bose gas across a broad Feshbach resonance, the distribution in a harmonic trap is approximately parabolic as a consequence of the competition between the interaction energy and the trap potential [33,34]. Here, as shown in Fig. 8(a), the distribution is similar to that of a broad resonance Bose gas at the edge of the trap while more particles can be accommodated in the center where the gas has higher density and thus a smaller effective scattering length  $\bar{a}_s$  as discussed in Sec. III. A sharp peak appears if the chemical potential at the trap center  $\mu_0$  approaches  $\mu_c$ . This density bump at the trap center is related to the vanishing of inverse compressibility in the saturation limit and the increase in the amount of the noninteracting molecules in the closed channel.

Finally, we take account of the correction by fluctuations and evaluate the density distribution by substituting  $\mu$  in Eq. (50) with  $\mu_0 - V(r)$ . The distribution compared with mean-field results with a fixed particle number is shown in Fig. 8(c). The density at the trap center is further increased after the Gaussian fluctuation is included.

#### VI. CONCLUSION

We studied the ground-state properties of a weakly interacting Bose gas close to a narrow Feshbach resonance. With the help of a path integral approach, we established a low-density expansion for the equation of state of this system. As a consequence of the energy dependence for narrow Feshbach resonance, the gas behaves very differently, especially for higher density. As the density increases, the energy dependence in scattering length leads to a saturated energy, a vanishingly small inverse compressibility, and a highly suppressed quantum depletion. When the Bose gas is trapped in a harmonic potential, this effect leads to a density bump in the trap center. Such phenomena should be able to be observed in current cold-atom experiments.

#### ACKNOWLEDGMENTS

This work was supported by the National Key Research and Development Program of China (Grants No. 2022YFA1405301 and No. 2018YFA0306502), the National Natural Science Foundation of China (Grants No. 12022405 and No. 11774426), and the Beijing Natural Science Foundation (Grant No. Z180013).

# APPENDIX: DIAGONALIZATION OF THE INVERSE GREEN'S FUNCTION $G^{-1}$

In this Appendix, we provide the details on how to obtain the transformation matrix U that appears in Eq. (24) in the main text. Considering symmetry, the transformation matrix U can be written as

$$\begin{pmatrix} \eta_2 & u_2 & v_2 & \xi_2 \\ \eta_1 & u_1 & v_1 & \xi_1 \\ \xi_1 & v_1 & u_1 & \eta_1 \\ \xi_2 & v_2 & u_2 & \eta_2 \end{pmatrix},$$
(A1)

where for convenience we set matrix elements as real numbers. Because the quasiparticles are bosonic, it is convenient to set

$$\mathbf{U}^{T} \cdot \begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \\ & & & 1 \end{pmatrix} \cdot \mathbf{U} = \begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \\ & & & 1 \end{pmatrix},$$
(A2)

and the diagonalization of the inverse Green's function is accomplished by

$$\mathbf{U}^{T} \cdot \begin{pmatrix} \epsilon_{b} & \alpha & & \\ \alpha & \epsilon_{a} & g & \\ & g & \epsilon_{a} & \alpha \\ & & \alpha & \epsilon_{b} \end{pmatrix} \cdot \mathbf{U} = \begin{pmatrix} \omega^{+} & & & \\ & \omega^{-} & & \\ & & \omega^{-} & & \\ & & & \omega^{+} \end{pmatrix},$$
(A3)

where

$$\epsilon_a = \frac{k^2}{2m} + 2g\phi_0^2 - \mu, \qquad (A4a)$$

$$\epsilon_b = \frac{k^2}{4m} + \nu_b - 2\mu, \qquad (A4b)$$

$$g = g\phi_0^2 + \sqrt{2\alpha\psi_0},\tag{A4c}$$

$$\alpha = \sqrt{2\alpha\phi_0}.\tag{A4d}$$

It is expedient to first take the transformation as follows:

$$u_1 = (x_1 + w_1)/2, \quad v_1 = (x_1 - w_1)/2,$$
 (A5a)

$$\eta_1 = (y_1 + z_1)/2, \quad \xi_1 = (y_1 - z_1)/2,$$
 (A5b)

$$u_2 = (x_2 + w_2)/2, \quad v_2 = (x_2 - w_2)/2,$$
 (A5c)

$$\eta_2 = (y_2 + z_2)/2, \quad \xi_2 = (y_2 - z_2)/2.$$
 (A5d)

Taking  $x_2 = s_1x_1$ ,  $w_2 = t_1w_1$  and  $y_1 = s_2y_2$ ,  $z_1 = t_2z_2$ , we can obtain the following from Eqs. (A2) and (A3):

$$x_1 = \left(\frac{1}{1+s_1t_1}\sqrt{\frac{\epsilon_a - g + \epsilon_b t_1^2 + 2\alpha t_1}{\epsilon_a + g + \epsilon_b s_1^2 + 2\alpha s_1}}\right)^{1/2}, \quad (A6a)$$

$$w_1 = \left(\frac{1}{1+s_1t_1}\sqrt{\frac{\epsilon_a + g + \epsilon_b s_1^2 + 2\alpha s_1}{\epsilon_a - g + \epsilon_b t_1^2 + 2\alpha t_1}}\right)^{1/2}, \quad (A6b)$$

$$y_{2} = \left(\frac{1}{1+s_{2}t_{2}}\sqrt{\frac{(\epsilon-g)t_{2}^{2}+\epsilon_{b}+2\alpha t_{2}}{(\epsilon+g)s_{2}^{2}+\epsilon_{b}+2\alpha s_{2}}}\right)^{1/2}, \quad (A7a)$$
$$z_{2} = \left(\frac{1}{1+s_{2}t_{2}}\sqrt{\frac{(\epsilon+g)s_{2}^{2}+\epsilon_{b}+2\alpha s_{2}}{(\epsilon-g)t_{2}^{2}+\epsilon_{b}+2\alpha t_{2}}}\right)^{1/2}, \quad (A7b)$$

There are two sets of solutions of  $s_1$ ,  $t_1$ ,  $s_2$ , and  $t_2$ ; however, using the conditions that  $u_2$ ,  $v_2$ ,  $\xi_2$ ,  $\eta_1$ , and  $\xi_1$  should tend to 0 in the limit  $\alpha \rightarrow 0$ , we can determine the correct solutions as follows,

$$s_{1} = -t_{2} = \frac{g^{2} - \epsilon_{a}^{2} + \epsilon_{b}^{2} - \lambda s_{0}}{2\alpha(\epsilon_{a} + \epsilon_{b} - g)},$$
  

$$t_{1} = -s_{2} = \frac{g^{2} - \epsilon_{a}^{2} + \epsilon_{b}^{2} - \lambda s_{0}}{2\alpha(g + \epsilon_{a} + \epsilon_{b})},$$
(A8)

where  $\lambda$  is the sign of  $g^2 - \epsilon_a^2 + \epsilon_b^2$  and

$$s_0 = \sqrt{\left(g^2 - \epsilon_a^2 + \epsilon_b^2\right)^2 - 4\alpha^2(g - \epsilon_a - \epsilon_b)(g + \epsilon_a + \epsilon_b)}.$$
(A9)

Substituting the solutions of x, w, y, and z [Eqs. (A6) and (A7)] into Eq. (A5), we can finally obtain the transformation matrix and quasiparticle excitations.

- [1] T. D. Lee, K. Huang, and C. N. Yang, Phys. Rev. 106, 1135 (1957).
- [2] C. J. Pethick and H. Smith, *Bose-Einstein Condensation in Dilute Gases*, 2nd ed. (Cambridge University, Cambridge, England, 2008).
- [3] S. Tan, Phys. Rev. A 78, 013636 (2008).
- [4] T. T. Wu, Phys. Rev. **115**, 1390 (1959).
- [5] K. Sawada, Phys. Rev. 116, 1344 (1959).
- [6] N. Hugenholtz and D. Pines, Phys. Rev. 116, 489 (1959).
- [7] E. Braaten and A. Nieto, Eur. Phys. J. B 11, 143 (1999).
- [8] E. Braaten, H.-W. Hammer, and T. Mehen, Phys. Rev. Lett. 88, 040401 (2002).
- [9] M. H. Anderson, J. R. Ensher, M. R. Matthews, C. E. Wieman and E. A. Cornell, Science 269, 198 (1995).
- [10] K. B. Davis, M.-O. Mewes, M. R. Andrews, N. J. van Druten, D. S. Durfee, D. M. Kurn, and W. Ketterle, Phys. Rev. Lett. 75, 3969 (1995).
- [11] N. Navon, S. Piatecki, K. Günter, B. Rem, T. C. Nguyen, F. Chevy, W. Krauth, and C. Salomon, Phys. Rev. Lett. 107, 135301 (2011).
- [12] A. Parola, L. Salasnich, and L. Reatto, Phys. Rev. A 57, R3180 (1998).
- [13] E. Braaten, H.-W. Hammer, and S. Hermans, Phys. Rev. A 63, 063609 (2001).
- [14] H. Fu, Y. Wang, and B. Gao, Phys. Rev. A **67**, 053612 (2003).
- [15] A. Collin, P. Massignan, and C. J. Pethick, Phys. Rev. A 75, 013615 (2007).
- [16] H. Veksler, S. Fishman, and W. Ketterle, Phys. Rev. A 90, 023620 (2014).

- [17] L. Salasnich, Phys. Rev. Lett. 118, 130402 (2017).
- [18] A. Cappellaro and L. Salasnich, Phys. Rev. A 95, 033627 (2017).
- [19] C. Chin, R. Grimm, P. Julienne, and E. Tiesinga, Rev. Mod. Phys. 82, 1225 (2010).
- [20] H. Zhai, Ultracold Atomic Physics (Cambridge University, Cambridge, England, 2021).
- [21] T.-L. Ho, X. Cui, and W. Li, Phys. Rev. Lett. 108, 250401 (2012).
- [22] R. Qi and H. Zhai, Phys. Rev. A 85, 041603(R) (2012).
- [23] C. Trefzger and Y. Castin, Phys. Rev. A 85, 053612 (2012).
- [24] L. Pricoupenko, Phys. Rev. Lett. 110, 180402 (2013).
- [25] Y. Nishida, Phys. Rev. Lett. 114, 115302 (2015).
- [26] H. T. C. Stoof, D. B. M. Dickerscheid, and K. Gubbels, *Ultra-cold Quantum Fields* (Springer, Dordrecht, The Netherlands, 2009).
- [27] M. Holland, S. J. J. M. F. Kokkelmans, M. L. Chiofalo, and R. Walser, Phys. Rev. Lett. 87, 120406 (2001).
- [28] S. J. J. M. F. Kokkelmans, J. N. Milstein, M. L. Chiofalo, R. Walser, and M. J. Holland, Phys. Rev. A 65, 053617 (2002).
- [29] S. J. J. M. F. Kokkelmans and M. J. Holland, Phys. Rev. Lett. 89, 180401 (2002).
- [30] D. J. M. Ahmed-Braun, S. Musolino, V. E. Colussi, and S. J. J. M. F. Kokkelmans, Phys. Rev. A 106, 013315 (2022).
- [31] Details on how to diagonalize the **G** matrix can be found in the Appendix.
- [32] A. Marte, T. Volz, J. Schuster, S. Dürr, G. Rempe, E. G. M. van Kempen, and B. J. Verhaar, Phys. Rev. Lett. 89, 283202 (2002).
- [33] M. Edwards and K. Burnett, Phys. Rev. A 51, 1382 (1995).
- [34] G. Baym and C. J. Pethick, Phys. Rev. Lett. 76, 6 (1996).