Universal relations for dilute systems with two-body decays in reduced dimensions

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Physical systems in reduced dimensions exhibit intriguing properties. For instance, the dependences of two-body and many-body physics on scattering lengths are distinct from their counterparts in three dimensions. Whereas many studies of ultracold atoms and molecules in reduced dimensions have been focusing on closed systems, two-body losses may occur in such systems. Here, we show that the two-body inelastic loss rate in reduced dimensions can be expressed in universal relations that are governed by contacts. These universal relations correlate the two-body decay rate with other physical observables at zero and finite temperatures and generic interaction strengths. Our results will provide experimentalists with a new protocol to study inelastic scatterings in both few- and many-body systems in reduced dimensions.

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I. INTRODUCTION

Ultracold atoms have provided physicists with a highly tunable platform to explore quantum few-body and many-body systems in both three dimensions (3D) and reduced dimensions [1–4]. In reduced dimensions, microscopic parameters control few-body and many-body physics in distinct means. For instance, the dependence of phase shifts on energies in one dimension (1D) and two dimensions (2D) is distinct from that in 3D [5,6]. Furthermore, in the celebrated universal relations that underlie ultracold atoms and other dilute quantum systems in arbitrary dimensions, the s-wave scatterings enter these relations in terms of a_0 and $\ln a_0$ in 1D and 2D, respectively, while universal relations in 3D often include $1/a_0$ [7–29]. a_0 is the s-wave scattering length. It is thus an important task to find the counterparts of 3D universal relations in reduced dimensions, which helps us to study how contacts manifest themselves in reduced dimensions. In such studies of universal relations, most works have been focusing on systems with elastic scatterings. However, two-body losses due to inelastic collisions may occur in realistic systems [30]. It is thus desirable to explore how inelastic scatterings may change the universal relations or provide us with conceptually new relations in 3D and reduced dimensions.

In addition to ultracold atoms, ultracold molecules have also been well established as a powerful platform to study a wide range of important topics in condensed matter physics, atomic, molecular and optical physics, and chemical physics [31–34]. One of the key issues emerged in experiments is the inelastic loss of molecules [35–53]. For instance, two reactive molecules can get close and react as $\mathrm{AB} + \mathrm{AB} \to \mathrm{A}_2 + \mathrm{B}_2$, which leads to the loss of AB molecules [43,54]. Even in the absence of reactions, the formation of complexes could also lead to two-body decays [42,44,55–59]. Similar to atoms, ultracold molecules can also be prepared in reduced dimensions [60–66]. A recent pioneering experiment has made an attempt to explore how the two-body decay may change with reducing the dimension by increasing the transverse confinement [60].

In 3D, it has been recognized that universal relations exist in systems with two-body losses [67–69]. Such relations directly correlate two-body decays with other many-body properties such as the momentum distribution and the density-density correlation function. Motivated by the importance of studying ultracold atoms and ultracold molecules in reduced dimensions, in this paper, we explore universal relations in 1D and 2D systems when an arbitrary partial wave scattering is inelastic. We show that the two-body inelastic loss rate can be expressed as contacts multiplied by microscopic parameters determined purely by two-body physics at short range, similar to those obtained in 3D. As such, our results are valid at zero and finite temperatures and generic interaction strengths and will provide experimentalists a useful protocol to explore two-body decays in a many-body environment in reduced dimensions.

The rest of this paper is organized as follows. In Sec. II, we provide a generic method of deriving the two-body inelastic loss rate, the momentum distribution, and the density correlation function in d-dimensional (dD) systems, where $d = 1, 2, 3$. In Sec. III, we consider single-component ultracold reactive molecules in 1D and derive the exact relations between contacts and physical quantities including the two-body inelastic loss rate, the momentum distribution, and the density correlation function. Similar discussions for 2D are given in Sec. IV. Furthermore, we discuss the temperature dependence of the loss rate in both the homogeneous systems and the harmonic traps in Sec. V. We conclude our results in Sec. VI.

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II. TWO-BODY INELASTIC LOSS RATE FOR REACTIVE MOLECULES IN *d* DIMENSIONS

We consider a single-component system of N reactive molecules. The Hamiltonian is written as

$$H=\sum_{i=1}^{N}\left[-\frac{\hbar^2}{2M}\nabla_i^2+V_{\rm ext}(\mathbf{x}_i)\right]+\sum_{i<j}U(\mathbf{x}_{ij}),\tag{1}$$

where M is the mass of each molecule and $\mathbf{x}_i=(x_i^{(1)},x_i^{(2)},\ldots,x_i^{(d)})$ is the coordinate of the ith molecule in dD space. $\mathbf{x}_{ij}=\mathbf{x}_i-\mathbf{x}_j=(x_{ij}^{(1)},x_{ij}^{(2)},\ldots,x_{ij}^{(d)})$. $V_{\rm ext}(\mathbf{x}_i)$ is the external trap. $U(\mathbf{x}_{ij})=U_{\rm R}(\mathbf{x}_{ij})+iU_{\rm I}(\mathbf{x}_{ij})$ is the complex two-body short-range interaction, which captures the two-body inelastic collisions and is nonzero only when $|\mathbf{x}_{ij}|<r_0$. $U_{\rm I}(\mathbf{x}_{ij})$ is nonpositive and nonzero at an even shorter distance characterized by r^*, $|\mathbf{x}_{ij}|<r^*<r_0$, where the chemical reaction happens. The many-body wave function, which is an eigenstate of the system, satisfies the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\Psi(\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_N)=H\Psi(\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_N).\tag{2}$$

We consider a finite system, the net current of which vanishes at large distance. The two-body inelastic loss rate is written as

$$\frac{\partial N}{\partial t}=\frac{4}{\hbar}\sum_{i<j}\int\prod_{i=1}^{N}{\rm d}\mathbf{x}_i U_{\rm I}(\mathbf{x}_i-\mathbf{x}_j)|\Psi(\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_N)|^2,\tag{3}$$

which is consistent with the second quantization form using bosonic (fermionic) operators

$$\frac{\partial N}{\partial t}=\frac{2}{\hbar}\int{\rm d}\mathbf{x}\,{\rm d}\mathbf{x}' U_{\rm I}(\mathbf{x}-\mathbf{x}')\langle\Psi^\dagger(\mathbf{x})\Psi^\dagger(\mathbf{x}')\Psi(\mathbf{x}')\Psi(\mathbf{x})\rangle,\tag{4}$$

which can be derived from the Lindblad master equation [69].

It is clear that a length scale separation exists in ultracold reactive molecules, i.e., the range of interaction r_0 is much shorter than the average interparticle distance characterized by the inverse of the Fermi momentum k_F while the reactive collisions happen in an even shorter distance characterized by r^*, $r^*<r_0\ll k_F^{-1}$. When the distance between any two molecules is much shorter than the average interparticle distance, i.e., $|\mathbf{x}_{ij}|\ll k_F^{-1}$, the possibility of a third molecule to get close to these two molecules and interact together at short distance is negligible. It is thus sufficient to consider only the two-body effect. The many-body wave function has the asymptotical behavior at short distance, which is stated as

$$\Psi(\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_N)\xrightarrow{|\mathbf{x}_{ij}|\ll k_F^{-1}}\sum_{s,\epsilon}\psi_s(\mathbf{x}_{ij};\epsilon)G_s(\mathbf{X}_{ij};E-\epsilon),\tag{5}$$

where ϵ is the two-body collision energy and $\psi_s(\mathbf{x}_{ij};\epsilon)$ is the two-body relative wave function which satisfies

$$\left[-\frac{\hbar^2}{M}\nabla_{\mathbf{x}_{ij}}^2+U(\mathbf{x}_{ij})\right]\psi_s(\mathbf{x}_{ij};\epsilon)=\epsilon\psi_s(\mathbf{x}_{ij};\epsilon).\tag{6}$$

s is the angular momentum quantum number, which denotes (l,m) for 3D and l for 1D and 2D. $\nabla_{\mathbf{x}_{ij}}^2=\sum_{n=1}^{d}[\partial^2/\partial(x_{ij}^{(n)})^2]$. $\mathbf{X}_{ij}=\{(\mathbf{x}_i+\mathbf{x}_j)/2,\mathbf{x}_{k\neq i,j}\}$ denotes the center-of-mass coordinate of the ith and the jth molecules and the coordinates of all the other $N-2$ molecules. $G_s(\mathbf{X}_{ij};E-\epsilon)$ is the many-body wave function, which characterizes the center-of-mass motion of the ith and the jth molecules and the motions of all the other $N-2$ molecules.

Whereas $G_s(\mathbf{X}_{ij};E-\epsilon)$ is usually very complex and hard to know, $\psi_s(\mathbf{x}_{ij};\epsilon)$ has a universal asymptotic form when $r_0\leqslant|\mathbf{x}_{ij}|\ll k_F^{-1}$. $\psi_s(\mathbf{x}_{ij};\epsilon)=\varphi_s(|\mathbf{x}_{ij}|;\epsilon)Y_s(\hat{\mathbf{x}}_{ij})$, where $\hat{\mathbf{x}}_{ij}=\mathbf{x}_{ij}/|\mathbf{x}_{ij}|$ and $Y_s(\hat{\mathbf{x}}_{ij})$ is the generalized spherical harmonics in dD. Furthermore, $\varphi_s(|\mathbf{x}_{ij}|;\epsilon)$ can be expanded as $\varphi_s(|\mathbf{x}_{ij}|;\epsilon)=\varphi_s^{(0)}(|\mathbf{x}_{ij}|)+\varphi_s^{(1)}(|\mathbf{x}_{ij}|)q_\epsilon^2+{\rm O}(q_\epsilon^4)$, where $q_\epsilon=(M\epsilon/\hbar^2)^{1/2}$. Equation (5) can then be written as

$$\Psi(\mathbf{X}_{ij},\mathbf{x}_{ij})\xrightarrow{|\mathbf{x}_{ij}|\ll k_F^{-1}}\sum_s\left[\varphi_s^{(0)}(|\mathbf{x}_{ij}|)g_s^{(0)}(\mathbf{X}_{ij})+\varphi_s^{(1)}(|\mathbf{x}_{ij}|)g_s^{(1)}(\mathbf{X}_{ij})\right]Y_s(\hat{\mathbf{x}}_{ij}),\tag{7}$$

where $g_s^{(m)}(\mathbf{X}_{ij})=\sum_\epsilon q_\epsilon^{2m}G_s(\mathbf{X}_{ij};E-\epsilon)$. Starting from Eq. (7) and the exact universal asymptotic form of $\varphi_s^{(0)}(|\mathbf{x}_{ij}|)$ and $\varphi_s^{(1)}(|\mathbf{x}_{ij}|)$ at $r_0\leqslant|\mathbf{x}_{ij}|\ll k_F^{-1}$, a number of universal relations determined by contact C, a fundamental quantity in dilute quantum systems, can be derived. $C\sim\int{\rm d}\mathbf{X}_{ij}g_s^{(\nu)*}g_{s'}^{(\nu')}$. In this paper, we will focus on the universal relations of the two-body inelastic decay rate, the momentum distribution, and the density correlation function.

Two-body inelastic loss rate. The two-body inelastic loss rate can be obtained by solving Eq. (3). Starting from Eqs. (5)–(7), the right-hand side of Eq. (3) can be obtained by solving the following equation, which is

$$\begin{aligned}&\mathcal{J}[U_I(\mathbf{x}_{ij})|\Psi(\mathbf{X}_{ij},\mathbf{x}_{ij})|^2]\\&=\frac{1}{2i}\mathcal{J}\left[\Psi^*(\mathbf{X}_{ij},\mathbf{x}_{ij})\sum_{s,\epsilon}\epsilon G_s(\mathbf{X}_{ij};E-\epsilon)\psi_s(\mathbf{x}_{ij};\epsilon)\right]\\&\quad-\frac{1}{2i}\mathcal{J}\left[\Psi(\mathbf{X}_{ij},\mathbf{x}_{ij})\sum_{s,\epsilon}\epsilon^* G_s^*(\mathbf{X}_{ij};E-\epsilon)\psi_s^*(\mathbf{x}_{ij};\epsilon)\right]\\&\quad+\frac{1}{2i}\frac{\hbar^2}{M}\mathcal{J}\big[\Psi^*(\mathbf{X}_{ij},\mathbf{x}_{ij})\nabla_{\mathbf{x}_{ij}}^2\Psi(\mathbf{X}_{ij},\mathbf{x}_{ij})\\&\quad-\Psi(\mathbf{X}_{ij},\mathbf{x}_{ij})\nabla_{\mathbf{x}_{ij}}^2\Psi^*(\mathbf{X}_{ij},\mathbf{x}_{ij})\big],\end{aligned}\tag{8}$$

where $\mathcal{J}$ is the shorthand notation of the summation and integral $\sum_{i<j}\int{\rm d}\mathbf{X}_{ij}\int_0^{r_0}{\rm d}\mathbf{x}_{ij}$. The last term on the right-hand side of Eq. (8) can be further rewritten as the surface integral:

$$\begin{aligned}&\sum_{i<j}\int{\rm d}\mathbf{X}_{ij}\int_0^{r_0}{\rm d}\mathbf{x}_{ij}\big[\Psi^*(\mathbf{X}_{ij},\mathbf{x}_{ij})\nabla_{\mathbf{x}_{ij}}^2\Psi(\mathbf{X}_{ij},\mathbf{x}_{ij})\\&\quad-\Psi(\mathbf{X}_{ij},\mathbf{x}_{ij})\nabla_{\mathbf{x}_{ij}}^2\Psi^*(\mathbf{X}_{ij},\mathbf{x}_{ij})\big]\\&=\sum_{i<j}\int{\rm d}\mathbf{X}_{ij}\oint_{|\mathbf{x}_{ij}|=r_0}[\Psi^*(\mathbf{X}_{ij},\mathbf{x}_{ij})\nabla_{\mathbf{x}_{ij}}\Psi(\mathbf{X}_{ij},\mathbf{x}_{ij})\\&\quad-\Psi(\mathbf{X}_{ij},\mathbf{x}_{ij})\nabla_{\mathbf{x}_{ij}}\Psi^*(\mathbf{X}_{ij},\mathbf{x}_{ij})]\cdot{\rm d}\mathbf{S}.\end{aligned}\tag{9}$$

One can use the mathematics given in the Appendix during the calculation. By taking the explicit expressions of $\varphi_s^{(0)}(|\mathbf{x}_{ij}|)$ and $\varphi_s^{(1)}(|\mathbf{x}_{ij}|)$ at $r_0\leqslant|\mathbf{x}_{ij}|\ll k_F^{-1}$ into Eq. (7) first, and bringing Eq. (7) back to Eqs. (8) and (9) then, Eq. (8) can be calculated explicitly. Finally, by taking Eq. (8) back to Eq. (3), the explicit expression of Eq. (3) can be obtained.

TABLE I. The low-energy expansion of phase shift η_l in different dimensions [70]. $\gamma \approx 0.577$ is the Euler's constant.

l	One dimension	Two dimensions	Three dimensions
$l=0$	$q_\epsilon \tan\eta_0 = \frac{1}{a_0}$	$\frac{\pi}{2}\cot\eta_0 = \ln\left(\frac{q_\epsilon a_0}{2}e^{\gamma}\right)$	$q_\epsilon \cot\eta_0 = -\frac{1}{a_0} + r_0^e q_\epsilon^2$
$l=1$	$q_\epsilon \cot\eta_1 = -\frac{1}{a_1} + r_1^e q_\epsilon^2$	$\frac{\pi}{2}q_\epsilon^2\cot\eta_1 - q_\epsilon^2\ln\left(\frac{q_\epsilon r_0}{2}e^{\gamma-1/2}\right) = -\frac{1}{a_1} + r_1^e q_\epsilon^2$	$q_\epsilon^3\cot\eta_1 = -\frac{1}{a_1} + r_1^e q_\epsilon^2$
$l>1$		$\frac{\pi}{2}q_\epsilon^{2l}\cot\eta_l = -\frac{1}{a_l} + r_l^e q_\epsilon^2$	$q_\epsilon^{2l+1}\cot\eta_l = -\frac{1}{a_l} + r_l^e q_\epsilon^2$

Momentum distribution. The momentum distribution can be obtained by using the first quantization form

$$n(\mathbf{k}) = \sum_{i=1}^{N}\int\prod_{j\neq i}d\mathbf{x}_j\left|\int d\mathbf{x}_i\Psi(\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_N)e^{-i\mathbf{k}\cdot\mathbf{x}_i}\right|^2. \tag{10}$$

Density correlation function. The density correlation function can be obtained by using the definition

$$\begin{aligned} S(\mathbf{x}_{ij}) &= \int d\frac{\mathbf{x}_i+\mathbf{x}_j}{2}\langle n(\mathbf{x}_i)n(\mathbf{x}_j)\rangle \\ &= N(N-1)\int d\mathbf{X}_{ij}|\Psi(\mathbf{X}_{ij},\mathbf{x}_{ij})|^2. \end{aligned} \tag{11}$$

Near the Feshbach resonance of a certain partial-wave scattering, contacts of this partial wave will be dominant. For simplicity, we consider single-component molecules in a single partial-wave scattering channel s in the following discussions. The generalization of these discussions to systems with mixed partial-wave scatterings will be straightforward.

III. UNIVERSAL RELATIONS FOR REACTIVE MOLECULES IN ONE DIMENSION

In this section, we consider single-component reactive molecules in 1D. We need to clarify that by universal relations, we mean the relations between the two-body loss and other quantities that can be expressed in terms of contacts and microscopic parameters independent of the temperature and the total particle number. As such, the meaning of universal is different from the other context, where effects dependent only on the scattering length are called universal and the effective range and other beyond-scattering-length effects are dubbed nonuniversal.

We label $s=l$, $\mathbf{x}_i = z_i$, $\mathbf{x}_{ij} = z_{ij}$, $\mathbf{k} = k_z$, and $\mathbf{X}_{ij} = \mathbf{R}_{ij}^z$ in the following discussions. The generalized spherical harmonics in 1D is $Y_l(\hat{z}_{ij}) = (z_{ij}/|z_{ij}|)^l/\sqrt{2}$. The two-body wave function $\psi_l(z_{ij};\epsilon) = \varphi_l(|z_{ij}|;\epsilon)Y_l(\hat{z}_{ij})$ has the universal asymptotic form when $r_0 \leqslant |z_{ij}| \ll k_F^{-1}$, which is

$$\begin{aligned} \varphi_l(|z_{ij}|;\epsilon) \xrightarrow{r_0\leqslant|z_{ij}|\ll k_F^{-1}} \frac{q_\epsilon^{l-1}}{\tan\eta_l}\Bigg[&\cos\left(q_\epsilon|z_{ij}| - \frac{l\pi}{2}\right) \\ &- \tan\eta_l\sin\left(q_\epsilon|z_{ij}| - \frac{l\pi}{2}\right)\Bigg], \end{aligned} \tag{12}$$

where η_l is the 1D lth partial-wave phase shift and can be expanded under the low-energy limit $q_\epsilon r_0 \ll 1$, as shown in Table I.

A. Even-wave scatterings with $l = 0$

We first consider the even-wave scatterings with $l=0$. For even-wave scatterings, it is sufficient to take the zero-energy limit, i.e., we consider only the even-wave scattering length a_0 and $\varphi_0^{(0)}(|z_{ij}|)$. Based on Eq. (12), we obtain

$$\varphi_0^{(0)}(|z_{ij}|) \xrightarrow{r_0\leqslant|z_{ij}|\ll k_F^{-1}} a_0 - |z_{ij}|. \tag{13}$$

By taking Eq. (13) into Eq. (7) first, and then bringing Eq. (7) back to Eqs. (3), (10), and (11), respectively, we obtain the following universal relations.

Two-body inelastic loss rate. From the calculation of Eq. (3), we obtain that

$$\frac{\partial N}{\partial t} = -\frac{\hbar}{2M}\mathrm{Im}(-a_0)C_1^{(0)}, \tag{14}$$

where $C_1^{(0)}$ is the 1D even-wave contact defined in Table II.

Momentum distribution: From Eq. (10), we obtain that

$$n(k_z) \xrightarrow{k_F\leqslant|k_z|\ll r_0^{-1}} \frac{C_1^{(0)}}{|k_z|^4}|Y_0(\hat{k}_z)|^2. \tag{15}$$

Density correlation function. From Eq. (11), we obtain

$$S(z_{ij}) \xrightarrow{r_0\leqslant|z_{ij}|\ll k_F^{-1}} \tfrac{1}{4}|a_0|^2C_1^{(0)}|Y_0(\hat{z}_{ij})|^2. \tag{16}$$

B. Odd-wave scatterings with $l = 1$

Next, we consider the odd-wave scatterings with $l=1$. Based on Eq. (12), we obtain

$$\varphi_1^{(0)}(|z_{ij}|) \xrightarrow{r_0\leqslant|z_{ij}|\ll k_F^{-1}} 1 - \frac{|z_{ij}|}{a_1}, \tag{17}$$

$$\varphi_1^{(1)}(|z_{ij}|) \xrightarrow{r_0\leqslant|z_{ij}|\ll k_F^{-1}} r_1^e|z_{ij}| - \frac{|z_{ij}|^2}{2} + \frac{1}{a_1}\frac{|z_{ij}|^3}{6}, \tag{18}$$

where a_1 and r_1^e are the 1D odd-wave scattering length and effective range, respectively. By taking Eqs. (17) and (18) into Eq. (7) first, and then bringing Eq. (7) back to Eqs. (3), (10), and (11), respectively, we obtain the following universal relations.

TABLE II. The two-body inelastic loss rate in different dimensions. $\tilde{\varphi}_s^{(0)}(r)$ is a wave function obtained from extending the actual wave function $\varphi_s^{(0)}(r)$ outside the potential ($r > r_0$) into the regime $r < r_0$. $\mathbf{R}_{ij} = (\mathbf{R}_{ij}^{\rho}, \mathbf{R}_{ij}^{z})$.

	One dimension ($s = l$)	Two dimensions ($s = l$)	Three dimensions [69] ($s = lm$)
$\partial_t N$	$-\frac{2\hbar}{2^2 M}\sum_{\nu=1}^{3}\kappa_\nu C_\nu^{(s)}$	$-\frac{2\hbar}{(2\pi)^2 M}\sum_{\nu=1}^{3}\kappa_\nu C_\nu^{(s)}$	$-\frac{2\hbar}{(4\pi)^2 M}\sum_{\nu=1}^{3}\kappa_\nu C_\nu^{(s)}$
$C_1^{(s)}$	$2^2 N(N-1)\int \mathrm{d}\mathbf{R}_{ij}^{z}\left\|g_s^{(0)}\right\|^2$	$(2\pi)^2 N(N-1)\int \mathrm{d}\mathbf{R}_{ij}^{\rho}\left\|g_s^{(0)}\right\|^2$	$(4\pi)^2 N(N-1)\int \mathrm{d}\mathbf{R}_{ij}\left\|g_s^{(0)}\right\|^2$
$C_2^{(s)}$	$2(2^2)N(N-1)\int \mathrm{d}\mathbf{R}_{ij}^{z}\mathrm{Re}\left(g_s^{(0)*}g_s^{(1)}\right)$	$2(2\pi)^2 N(N-1)\int \mathrm{d}\mathbf{R}_{ij}^{\rho}\mathrm{Re}\left(g_s^{(0)*}g_s^{(1)}\right)$	$2(4\pi)^2 N(N-1)\int \mathrm{d}\mathbf{R}_{ij}\mathrm{Re}\left(g_s^{(0)*}g_s^{(1)}\right)$
$C_3^{(s)}$	$2(2^2)N(N-1)\int \mathrm{d}\mathbf{R}_{ij}^{z}\mathrm{Im}\left(g_s^{(0)*}g_s^{(1)}\right)$	$2(2\pi)^2 N(N-1)\int \mathrm{d}\mathbf{R}_{ij}^{\rho}\mathrm{Im}\left(g_s^{(0)*}g_s^{(1)}\right)$	$2(4\pi)^2 N(N-1)\int \mathrm{d}\mathbf{R}_{ij}\mathrm{Im}\left(g_s^{(0)*}g_s^{(1)}\right)$
κ_1	$-\frac{M}{\hbar^2}\int_0^\infty \left\|\varphi_s^{(0)}(r)\right\|^2 U_I(r)\mathrm{d}r$	$-\frac{M}{\hbar^2}\int_0^\infty \left\|\varphi_s^{(0)}(r)\right\|^2 U_I(r) r\,\mathrm{d}r$	$-\frac{M}{\hbar^2}\int_0^\infty \left\|\varphi_s^{(0)}(r)\right\|^2 U_I(r) r^2\mathrm{d}r$
κ_2	$-\frac{M}{\hbar^2}\mathrm{Re}\left[\int_0^\infty \varphi_s^{(0)*}(r)\varphi_s^{(1)}(r)U_I(r)\mathrm{d}r\right]$	$-\frac{M}{\hbar^2}\mathrm{Re}\left[\int_0^\infty \varphi_s^{(0)*}(r)\varphi_s^{(1)}(r)U_I(r) r\,\mathrm{d}r\right]$	$-\frac{M}{\hbar^2}\mathrm{Re}\left[\int_0^\infty \varphi_s^{(0)*}(r)\varphi_s^{(1)}(r)U_I(r) r^2\mathrm{d}r\right]$
κ_3	$\frac{M}{\hbar^2}\mathrm{Im}\left[\int_0^\infty \varphi_s^{(0)*}(r)\varphi_s^{(1)}(r)U_I(r)\mathrm{d}r\right]$	$\frac{M}{\hbar^2}\mathrm{Im}\left[\int_0^\infty \varphi_s^{(0)*}(r)\varphi_s^{(1)}(r)U_I(r) r\,\mathrm{d}r\right]$	$\frac{M}{\hbar^2}\mathrm{Im}\left[\int_0^\infty \varphi_s^{(0)*}(r)\varphi_s^{(1)}(r)U_I(r) r^2\mathrm{d}r\right]$
κ_1	$\mathrm{Im}(-a_0)$	$\mathrm{Im}[\ln(1/a_0)]$	$\mathrm{Im}(1/a_0)$
κ_1	$\mathrm{Im}(1/a_{l>0})$	$\mathrm{Im}(1/a_{\|l\|>0})$	$\mathrm{Im}(1/a_{l\geqslant 0})$
κ_2	$\mathrm{Im}(-r_{l>0}^e/2)$	$\mathrm{Im}\left(-r_{\|l\|>0}^e/2\right)$	$\mathrm{Im}\left(-r_{l\geqslant 0}^e/2\right)$
κ_3	$\int_0^{r_0}\left\{\left[\mathrm{Im}\tilde{\varphi}_{l>0}^{(0)}(r)\right]^2 - \left[\mathrm{Im}\varphi_{l>0}^{(0)}(r)\right]^2\right\}\mathrm{d}r$	$\int_0^{r_0}\left\{\left[\mathrm{Im}\tilde{\varphi}_{\|l\|>0}^{(0)}(r)\right]^2 - \left[\mathrm{Im}\varphi_{\|l\|>0}^{(0)}(r)\right]^2\right\} r\,\mathrm{d}r$	$\int_0^{r_0}\left\{\left[\mathrm{Im}\tilde{\varphi}_s^{(0)}(r)\right]^2 - \left[\mathrm{Im}\varphi_s^{(0)}(r)\right]^2\right\} r^2\mathrm{d}r$

Two-body inelastic loss rate. As shown in Sec. II, to calculate Eq. (3), we could calculate Eqs. (8) and (9) first. From the calculation of Eq. (9), we have (see Appendix)

$$\begin{aligned}
&\varphi_1^*(|z_{ij}|;\epsilon)\frac{\partial}{\partial|z_{ij}|}\varphi_1(|z_{ij}|;\epsilon)\Big|_{|z_{ij}|=r_0} \\
&\quad - \varphi_1(|z_{ij}|;\epsilon)\frac{\partial}{\partial|z_{ij}|}\varphi_1^*(|z_{ij}|;\epsilon)\Big|_{|z_{ij}|=r_0} \\
&= -\frac{1}{a_1} + \frac{1}{a_1^*} + r_1^e q_\epsilon^2 - r_1^{e*}\left(q_\epsilon^2\right)^* \\
&\quad + \left[-r_0 + \frac{r_0}{2}\left(\frac{1}{a_1} + \frac{1}{a_1^*}\right) - \frac{r_0^3}{3}\frac{1}{a_1 a_1^*}\right]\left[q_\epsilon^2 - \left(q_\epsilon^2\right)^*\right] \\
&\quad + \mathrm{O}\left(q_\epsilon^4\right).
\end{aligned}$$

By using a trick that the first term on the right-hand side of the 1D odd-wave effective range in Table III can be rewritten as

$$\begin{aligned}
r_0 = &\frac{r_1^e + r_1^{e*}}{2} + \frac{r_0^2}{2}\left(\frac{1}{a_1} + \frac{1}{a_1^*}\right) - \frac{r_0^3}{6}\left(\frac{1}{a_1^2} + \frac{1}{(a_1^*)^2}\right) \\
&+ \frac{1}{2}\int_0^{r_0}\left\{\left[\varphi_1^{(0)}(r)\right]^2 + \left[\varphi_1^{(0)*}(r)\right]^2\right\}\mathrm{d}r, \qquad (19)
\end{aligned}$$

one has

$$\begin{aligned}
&\varphi_1^*(|z_{ij}|;\epsilon)\frac{\partial}{\partial|z_{ij}|}\varphi_1(|z_{ij}|;\epsilon)\Big|_{|z_{ij}|=r_0} \\
&\quad - \varphi_1(|z_{ij}|;\epsilon)\frac{\partial}{\partial|z_{ij}|}\varphi_1^*(|z_{ij}|;\epsilon)\Big|_{|z_{ij}|=r_0} \\
&= -\frac{1}{a_1} + \frac{1}{a_1^*} + \frac{1}{2}\left(r_1^e - r_1^{e*}\right)\left[q_\epsilon^2 + \left(q_\epsilon^2\right)^*\right] \\
&\quad - 2\int_0^{r_0}\left\{\left[\mathrm{Im}\tilde{\varphi}_1^{(0)}(r)\right]^2 + \frac{\left[\varphi_1^{(0)}(r)\right]^2 + \left[\varphi_1^{(0)*}(r)\right]^2}{4}\right\}\mathrm{d}r \\
&\quad \times \left[q_\epsilon^2 - \left(q_\epsilon^2\right)^*\right] + \mathrm{O}(q_\epsilon^4),
\end{aligned}$$

where $\tilde{\varphi}_1^{(0)}(r)$ is obtained by extending the universal asymptotic form of $\varphi_1^{(0)}(r)$ in the range $r_0 \leqslant r \ll k_F^{-1}$ into the range $r < r_0$, i.e., $\tilde{\varphi}_1^{(0)}(r) = 1 - r/a_1$. Following the procedure given in Sec. II, Eq. (3) can be written as

$$\frac{\partial N}{\partial t} = -\frac{\hbar}{2M}\sum_{\nu=1}^{3}\kappa_\nu C_\nu^{(1)}, \qquad (20)$$

where $C_\nu^{(1)}$ are the 1D odd-wave contacts that fully capture the many-body physics and κ_ν are the microscopic parameters determined purely by the two-body short-range physics. Both $C_\nu^{(1)}$ and κ_ν are defined in Table II. κ_1 and

TABLE III. The effective range r_l^e in different dimensions [70].

Dimension	$l = 1$	$l > 1$
1D	$r_1^e = r_0 - r_0^2\frac{1}{a_1} + \frac{r_0^3}{3}\frac{1}{a_1^2} - \int_0^{r_0}\left[\varphi_1^{(0)}(r)\right]^2\mathrm{d}r$	
2D	$r_1^e = \frac{1}{2} - \frac{r_0^2}{2}\frac{1}{a_1} + \frac{r_0^4}{16}\frac{1}{a_1^2} - \int_0^{r_0}\left[\varphi_1^{(0)}(r)\right]^2 r\mathrm{d}r$	$r_{l>1}^e = -\frac{(2l-2)!!(2l-4)!!}{r_0^{2l-2}} - \frac{r_0^2}{2l}\frac{1}{a_l} + \frac{r_0^{2l+2}}{(2l)!!(2l+2)!!}\frac{1}{a_l^2} - \int_0^{r_0}\left[\varphi_l^{(0)}(r)\right]^2 r\,\mathrm{d}r$
3D	$r_{l\geqslant 0}^e = -\frac{(2l-1)!!(2l-3)!!}{r_0^{2l-1}} - \frac{r_0^2}{2l+1}\frac{1}{a_l} + \frac{r_0^{2l+3}}{(2l+1)!!(2l+3)!!}\frac{1}{a_l^2} - \int_0^{r_0}\left[\varphi_l^{(0)}(r)\right]^2 r^2\mathrm{d}r$	

κ_2 can simply be expressed as $\mathrm{Im}(1/a_1)$ and $\mathrm{Im}(-r_1^e/2)$, respectively. κ_3, however, is a new microscopic parameter emerged in the system with inelastic losses. As shown in Table II, the physical meaning of κ_3 is the integration of $[\mathrm{Im}\tilde{\varphi}_1^{(0)}(r)]^2$ subtracted by $[\mathrm{Im}\varphi_1^{(0)}(r)]^2$ with respect to r from 0 to r_0. We note that κ_2 is the real part of the integral regarding $\varphi^{(0)*}\varphi^{(1)}U_I$, while κ_3 is the imaginary part of this integral. In general, the real and imaginary parts of $\varphi^{(0)*}\varphi^{(1)}U_I$ are independent functions. κ_2 and κ_3 are thus independent parameters. Whereas it is possible to express κ_3 in terms of other microscopic parameters characterizing the two-body interactions, here, we keep κ_3 in the expression of the universal relation since all these parameters, $\kappa_{1,2,3}$, are independent on the particle number and temperature. As such, $\kappa_{1,2,3}$ are measurable quantities.

Momentum distribution. From Eq. (10), we obtain that

$$n(k_z) \xrightarrow{k_F \leqslant |k_z| \ll r_0^{-1}} \frac{C_1^{(1)}}{|k_z|^2} |Y_1(\hat{k}_z)|^2. \tag{21}$$

Density correlation function. From Eq. (11), we obtain

$$\begin{aligned} S(z_{ij}) \xrightarrow{r_0 \leqslant |z_{ij}| \ll k_F^{-1}} \frac{1}{4}\Bigg\{ & C_1^{(1)} + \left[\mathrm{Re}\left(r_1^e\right)C_2^{(1)} - \mathrm{Re}\left(\frac{2}{a_1}\right)C_1^{(1)} - \mathrm{Im}\left(r_1^e\right)C_3^{(1)}\right]|z_{ij}| \\ & + \left[\frac{1}{|a_1|^2}C_1^{(1)} - \mathrm{Re}\left(\frac{1}{2} + \frac{r_1^e}{a_1^*}\right)C_2^{(1)} + \mathrm{Im}\left(\frac{r_1^e}{a_1^*}\right)C_3^{(1)}\right]|z_{ij}|^2 \\ & + \left[\mathrm{Re}\left(\frac{2}{a_1}\right)C_2^{(1)} + \mathrm{Im}\left(\frac{1}{a_1}\right)C_3^{(1)}\right]\frac{|z_{ij}|^3}{3} - \frac{1}{|a_1|^2}C_2^{(1)}\frac{|z_{ij}|^4}{6}\Bigg\}|Y_1(\hat{z}_{ij})|^2. \end{aligned} \tag{22}$$

By fitting the data of quantities such as the momentum distribution and the density correlation function obtained in experiment, all quantities in Eqs. (14) and (20) can be measured.

For s-wave inelastic scatterings, only the leading term determined by scattering length a_0 is important in the low-energy expansion of phase shift (See Table I). $C_1^{(1)}$ alone is enough to describe physics in such systems. Generally speaking, for high-partial-wave scatterings with a generic short-range interaction, other microscopic parameters like the effective range are required in the low-energy expansion of phase shift. The low-energy expansion of the wave function needs to be kept up to the q_ϵ^2 term in Eq. (7). As such, all three contacts $C_\nu^{(1)}$ are required in the complete expressions of the universal relations that apply to all parameter regimes. Nevertheless, in certain parameter regimes, the terms including $C_2^{(1)}$ and $C_2^{(1)}$ in Eq. (20) may be less important. For instance, in weakly interacting systems when $a_1 \to 0$, the contribution to the phase shift is dominated by the scattering length and other microscopic parameters can be neglected. As such, the universal relations are mainly governed by $C_1^{(1)}$, similar to the original universal relations for s-wave scatterings. This could simplify data analysis in experiments as fewer parameters are required to fit the experimental results. This argument works for 2D and 3D systems as well.

IV. UNIVERSAL RELATIONS FOR REACTIVE MOLECULES IN TWO DIMENSIONS

We now consider single-component reactive molecules in 2D. We label $s = l$, $\mathbf{x}_i = \boldsymbol{\rho}_i$, $\mathbf{x}_{ij} = \boldsymbol{\rho}_{ij} = (x_{ij}, y_{ij})$, and $\mathbf{X}_{ij} = \mathbf{R}_{ij}^{\rho}$ in the following discussions. The generalized spherical harmonics in 2D is $Y_l(\hat{\boldsymbol{\rho}}_{ij}) = [(x_{ij} + iy_{ij})/\rho_{ij}]^l/\sqrt{2\pi}$, where $\rho_{ij} = |\boldsymbol{\rho}_{ij}|$. The two-body wave function $\psi_l(\boldsymbol{\rho}_{ij};\epsilon) = \varphi_l(\rho_{ij};\epsilon)Y_l(\hat{\boldsymbol{\rho}}_{ij})$ has the universal asymptotic form when $r_0 \leqslant \rho_{ij} \ll k_F^{-1}$, which is

$$\begin{aligned} \varphi_l(\rho_{ij};\epsilon) \xrightarrow{r_0 \leqslant \rho_{ij} \ll k_F^{-1}} & \frac{\pi}{2}\frac{q_\epsilon^l}{\tan\eta_l}[J_l(q_\epsilon\rho_{ij}) \\ & - \tan\eta_l N_l(q_\epsilon\rho_{ij})], \end{aligned} \tag{23}$$

where J_l (N_l) is the Bessel function of the first (second) kind and η_l is the 2D lth partial-wave phase shift and can be expanded under the low-energy limit $q_\epsilon r_0 \ll 1$, as shown in Table I.

A. s-wave scatterings with $l = 0$

We first consider the s-wave scatterings with $l = 0$. For s-wave scatterings, it is sufficient to take the zero-energy limit, i.e., we consider only the s-wave scattering lengths a_0 and $\varphi_0^{(0)}(\rho_{ij})$. We obtain

$$\varphi_0^{(0)}(\rho_{ij}) \xrightarrow{r_0 \leqslant \rho_{ij} \ll k_F^{-1}} \ln a_0 - \ln \rho_{ij}. \tag{24}$$

By taking Eq. (24) into Eq. (7) first, and then bringing Eq. (7) back to Eqs. (3), (10), and (11), respectively, we obtain the following universal relations.

Two-body inelastic loss rate. From the calculation of Eq. (3), we obtain that

$$\frac{\partial N}{\partial t} = -\frac{\hbar}{2\pi^2 M}\mathrm{Im}\left(\ln\frac{1}{a_0}\right)C_1^{(0)}, \tag{25}$$

where $C_1^{(0)}$ is the 2D s-wave contact defined in Table II.

Momentum distribution. From Eq. (10), we obtain that

$$n(\mathbf{k}_\rho) \xrightarrow{k_F \leqslant |\mathbf{k}_\rho| \ll r_0^{-1}} \frac{C_1^{(0)}}{|\mathbf{k}_\rho|^4}|Y_0(\hat{\mathbf{k}}_\rho)|^2. \tag{26}$$

Density correlation function. From Eq. (11), we obtain

$$S(\boldsymbol{\rho}_{ij}) \xrightarrow{r_0\leqslant\rho_{ij}\ll k_F^{-1}} \frac{1}{(2\pi)^2}|\ln\rho_{ij}|^2 C_1^{(0)}|Y_0(\hat{\boldsymbol{\rho}}_{ij})|^2. \tag{27}$$

B. High-partial-wave scatterings with $l > 0$

Next, we consider the high-partial-wave scatterings with $l > 0$. From Eq. (23), we obtain

$$\varphi_{l>0}^{(0)}(\rho_{ij}) \xrightarrow{r_0\leqslant\rho_{ij}\ll k_F^{-1}} -\frac{1}{a_l}\frac{\rho_{ij}^l}{(2l)!!} + \frac{(2l-2)!!}{\rho_{ij}^l}, \tag{28}$$

$$\varphi_{l=1}^{(1)}(\rho_{ij}) \xrightarrow{r_0\leqslant\rho_{ij}\ll k_F^{-1}} r_1^e\frac{\rho_{ij}}{2} - \ln\left(\frac{\rho_{ij}}{r_0}\right)\frac{\rho_{ij}}{2} + \frac{1}{a_1}\frac{\rho_{ij}^3}{16}, \tag{29}$$

$$\varphi_{l>1}^{(1)}(\rho_{ij}) \xrightarrow{r_0\leqslant\rho_{ij}\ll k_F^{-1}} r_l^e\frac{\rho_{ij}^l}{(2l)!!} + \frac{1}{a_l}\frac{\rho_{ij}^{l+2}}{2(2l+2)!!} + \frac{(2l-4)!!}{2}\frac{1}{\rho_{ij}^{l-2}}, \tag{30}$$

where a_l and r_l^e are the 2D lth partial-wave scattering length and effective range, respectively. By taking Eqs. (28)–(30) into Eq. (7) first, and then bringing Eq. (7) back to Eqs. (3), (10), and (11), respectively, we obtain the following universal relations.

Two-body inelastic loss rate. As shown in Sec. II, to calculate Eq. (3), we could calculate Eqs. (8) and (9) first. From the calculation of Eq. (9), we have (see Appendix)

$$\begin{aligned}&\rho_{ij}\varphi_1^*(\rho_{ij};\epsilon)\frac{\partial}{\partial\rho_{ij}}\varphi_1(\rho_{ij};\epsilon)\Big|_{\rho_{ij}=r_0} - \rho_{ij}\varphi_1(\rho_{ij};\epsilon)\frac{\partial}{\partial\rho_{ij}}\varphi_1^*(\rho_{ij};\epsilon)\Big|_{\rho_{ij}=r_0}\\ &= -\frac{1}{a_1} + \frac{1}{a_1^*} + r_1^e q_\epsilon^2 - r_1^{e*}\left(q_\epsilon^2\right)^* + \left[-\frac{1}{2} + \frac{r_0^2}{4}\left(\frac{1}{a_1}+\frac{1}{a_1^*}\right) - \frac{r_0^4}{16}\frac{1}{a_1 a_1^*}\right]\left[q_\epsilon^2 - \left(q_\epsilon^2\right)^*\right] + \mathrm{O}\left(q_\epsilon^4\right)\end{aligned}$$

for the p-wave scattering and

$$\begin{aligned}&\rho_{ij}\varphi_l^*(\rho_{ij};\epsilon)\frac{\partial}{\partial\rho_{ij}}\varphi_l(\rho_{ij};\epsilon)\Big|_{\rho_{ij}=r_0} - \rho_{ij}\varphi_l(\rho_{ij};\epsilon)\frac{\partial}{\partial\rho_{ij}}\varphi_l^*(\rho_{ij};\epsilon)\Big|_{\rho_{ij}=r_0}\\ &= -\frac{1}{a_l} + \frac{1}{a_l^*} + r_l^e q_\epsilon^2 - r_l^{e*}\left(q_\epsilon^2\right)^* + \left[\frac{(2l-2)!!(2l-4)!!}{r_0^{2l-2}} + \frac{r_0^2}{4l}\left(\frac{1}{a_l}+\frac{1}{a_l^*}\right) - \frac{r_0^{2l+2}}{(2l)!!(2l+2)!!}\frac{1}{a_l a_l^*}\right]\left[q_\epsilon^2 - \left(q_\epsilon^2\right)^*\right] + \mathrm{O}\left(q_\epsilon^4\right)\end{aligned}$$

for the higher-partial-wave scatterings with $l > 1$. By using the same trick in 1D case that the first term on the right-hand side of the 2D effective range in Table III can be rewritten as

$$\frac{1}{2} = \frac{r_1^e + r_1^{e*}}{2} + \frac{r_0^2}{4}\left(\frac{1}{a_1}+\frac{1}{a_1^*}\right) - \frac{r_0^4}{32}\left(\frac{1}{a_1^2}+\frac{1}{(a_1^*)^2}\right) + \frac{1}{2}\int_0^{r_0}\left\{\left[\varphi_1^{(0)}(r)\right]^2 + \left[\varphi_1^{(0)*}(r)\right]^2\right\} r\,\mathrm{d}r \tag{31}$$

for the p-wave scattering and

$$\begin{aligned}&-\frac{(2l-2)!!(2l-4)!!}{r_0^{2l-2}} = \frac{r_l^e + r_l^{e*}}{2} + \frac{r_0^2}{4l}\left(\frac{1}{a_l}+\frac{1}{a_l^*}\right) - \frac{r_0^{2l+2}}{2(2l)!!(2l+2)!!}\left(\frac{1}{a_l^2}+\frac{1}{(a_l^*)^2}\right)\\ &\quad + \frac{1}{2}\int_0^{r_0}\left\{\left[\varphi_l^{(0)}(r)\right]^2 + \left[\varphi_l^{(0)*}(r)\right]^2\right\} r\,\mathrm{d}r\end{aligned} \tag{32}$$

for the higher-partial-wave scatterings with $l > 1$, one has

$$\begin{aligned}&\rho_{ij}\varphi_l^*(\rho_{ij};\epsilon)\frac{\partial}{\partial\rho_{ij}}\varphi_l(\rho_{ij};\epsilon)\Big|_{\rho_{ij}=r_0} - \rho_{ij}\varphi_l(\rho_{ij};\epsilon)\frac{\partial}{\partial\rho_{ij}}\varphi_l^*(\rho_{ij};\epsilon)\Big|_{\rho_{ij}=r_0}\\ &= -\frac{1}{a_l} + \frac{1}{a_l^*} + \frac{1}{2}\left(r_l^e - r_l^{e*}\right)\left[q_\epsilon^2 + \left(q_\epsilon^2\right)^*\right] - 2\int_0^{r_0}\left\{\left[\mathrm{Im}\tilde{\varphi}_l^{(0)}(r)\right]^2 + \frac{\left[\varphi_l^{(0)}(r)\right]^2 + \left[\varphi_l^{(0)*}(r)\right]^2}{4}\right\} r\,\mathrm{d}r\\ &\quad\times\left[q_\epsilon^2 - \left(q_\epsilon^2\right)^*\right] + \mathrm{O}\left(q_\epsilon^4\right)\end{aligned}$$

for $l > 0$, where $\tilde{\varphi}_l^{(0)}(r)$ is obtained from extending the universal asymptotic form of $\varphi_l^{(0)}(r)$ in the range $r_0 \leqslant r \ll k_F^{-1}$ into the range $r < r_0$, i.e., $\tilde{\varphi}_{l>0}^{(0)}(\rho_{ij}) = -(1/a_l)[\rho_{ij}^l/(2l)!!] + (2l-2)!!/\rho_{ij}^l$. It is interesting to notice that, for high-partial-wave scatterings, the above formula suits for 3D systems as well, which gives

$$\begin{aligned}&|\mathbf{x}_{ij}|^{d-1}\varphi_l^*(|\mathbf{x}_{ij}|;\epsilon)\frac{\partial}{\partial|\mathbf{x}_{ij}|}\varphi_l(|\mathbf{x}_{ij}|;\epsilon)\Big|_{|\mathbf{x}_{ij}|=r_0} - |\mathbf{x}_{ij}|^{d-1}\varphi_l(|\mathbf{x}_{ij}|;\epsilon)\frac{\partial}{\partial|\mathbf{x}_{ij}|}\varphi_l^*(|\mathbf{x}_{ij}|;\epsilon)\Big|_{|\mathbf{x}_{ij}|=r_0}\\ &= -\frac{1}{a_l} + \frac{1}{a_l^*} + \frac{1}{2}\left(r_l^e - r_l^{e*}\right)\left[q_\epsilon^2 + \left(q_\epsilon^2\right)^*\right] - 2\int_0^{r_0}\left\{\left[\mathrm{Im}\tilde{\varphi}_l^{(0)}(r)\right]^2 + \frac{\left[\varphi_l^{(0)}(r)\right]^2 + \left[\varphi_l^{(0)*}(r)\right]^2}{4}\right\}\\ &\quad\times r^{d-1}\mathrm{d}r\left[q_\epsilon^2 - \left(q_\epsilon^2\right)^*\right] + \mathrm{O}\left(q_\epsilon^4\right).\end{aligned}$$

Following the procedure given in Sec. II, Eq. (3) can be written as

$$\frac{\partial N}{\partial t} = -\frac{\hbar}{2\pi^2 M}\sum_{\nu=1}^{3}\kappa_\nu C_\nu^{(l)}, \tag{33}$$

where $C_\nu^{(l)}$ are the 2D lth partial-wave contacts that fully capture the many-body physics and κ_ν are the microscopic parameters determined purely by the two-body short-range physics. Both $C_\nu^{(l)}$ and κ_ν are defined in Table II. κ_1 and κ_2 can simply be expressed as $\mathrm{Im}(1/a_l)$ and $\mathrm{Im}(-r_l^e/2)$, respectively. Similar to that for 1D odd-wave scatterings κ_3, however, is a new microscopic parameter emerged in the system with inelastic losses. As shown in Table II, again, the physical meaning of κ_3 is the integration of $[\mathrm{Im}\tilde{\varphi}_l^{(0)}(r)]^2$ subtracted by $[\mathrm{Im}\varphi_l^{(0)}(r)]^2$ with respect to r from 0 to r_0.

Momentum distribution. From Eq. (10), we obtain that

$$n(\mathbf{k}_{\boldsymbol{\rho}}) \xrightarrow{k_F \leqslant |\mathbf{k}_{\boldsymbol{\rho}}| \ll r_0^{-1}} C_1^{(l)} |\mathbf{k}_{\boldsymbol{\rho}}|^{2l-4} |Y_l(\hat{\mathbf{k}}_{\boldsymbol{\rho}})|^2. \tag{34}$$

Density correlation function. From Eq. (11), we obtain

$$\begin{aligned} S(\boldsymbol{\rho}_{ij}) \xrightarrow{r_0 \leqslant \rho_{ij} \ll k_F^{-1}} & \frac{1}{(2\pi)^2}\Big\{C_1^{(1)}\frac{1}{\rho_{ij}^2} - C_2^{(1)}\frac{\ln(\rho_{ij}/r_0)}{2} \\ & + \Big[\mathrm{Re}\big(r_1^e\big)C_2^{(1)} - \mathrm{Re}\Big(\frac{2}{a_1}\Big)C_1^{(1)} - \mathrm{Im}\big(r_1^e\big)C_3^{(1)}\Big]\frac{1}{2} \\ & + \Big[\mathrm{Re}\Big(\frac{1}{a_1}\Big)C_2^{(1)} + \mathrm{Im}\Big(\frac{1}{a_1}\Big)C_3^{(1)}\Big]\frac{\ln(\rho_{ij}/r_0)\rho_{ij}^2}{4} \\ & + \Big[\frac{1}{|a_1|^2}C_1^{(1)} - \mathrm{Re}\Big(\frac{r_1^e - 1/4}{a_1^*}\Big)C_2^{(1)} \\ & + \mathrm{Im}\Big(\frac{r_1^e + 1/4}{a_1^*}\Big)C_3^{(1)}\Big]\frac{\rho_{ij}^2}{4} \\ & - \frac{1}{|a_1|^2}C_2^{(1)}\frac{\rho_{ij}^4}{32}\Big\}\big|Y_1(\hat{\boldsymbol{\rho}}_{ij})\big|^2 \end{aligned} \tag{35}$$

for the p-wave scattering and

$$\begin{aligned} S(\boldsymbol{\rho}_{ij}) \xrightarrow{r_0 \leqslant \rho_{ij} \ll k_F^{-1}} & \frac{1}{(2\pi)^2}\Big\{C_1^{(l)}\frac{[(2l-2)!!]^2}{\rho_{ij}^{2l}} \\ & + C_2^{(l)}\frac{(2l-4)!!(2l-2)!!}{2\rho_{ij}^{2l-2}} \\ & + \Big[\mathrm{Re}\big(r_l^e\big)C_2^{(l)} - \mathrm{Re}\Big(\frac{2}{a_l}\Big)C_1^{(l)} - \mathrm{Im}\big(r_l^e\big)C_3^{(l)}\Big]\frac{1}{2l} \\ & - \Big[\frac{1}{l}\mathrm{Re}\Big(\frac{1}{a_l}\Big)C_2^{(l)} + \mathrm{Im}\Big(\frac{1}{a_l}\Big)C_3^{(l)}\Big]\frac{\rho_{ij}^2}{(2l-2)(2l+2)} \\ & + \Big[\frac{1}{|a_l|^2}C_1^{(l)} - \mathrm{Re}\Big(\frac{r_l^e}{a_l^*}\Big)C_2^{(l)} + \mathrm{Im}\Big(\frac{r_l^e}{a_l^*}\Big)C_3^{(l)}\Big]\frac{\rho_{ij}^{2l}}{[(2l)!!]^2} \\ & - \frac{1}{|a_l|^2}C_2^{(l)}\frac{\rho_{ij}^{2l+2}}{2(2l)!!(2l+2)!!}\Big\}\big|Y_l(\hat{\boldsymbol{\rho}}_{ij})\big|^2 \end{aligned} \tag{36}$$

for the higher-partial-wave scatterings with $l > 1$.

By fitting the data of quantities such as the momentum distribution and the density correlation function obtained in experiment, all quantities in Eqs. (25) and (33) can be measured. It is worth pointing out that it might be difficult to distinguish certain terms such as $\ln(\rho_{ij}/r_0)\rho_{ij}^2$ and $\ln(\rho_{ij}/r_0)$ in practice. It is nevertheless useful to keep the full expression of the universal relation as a complete description, which shall be useful even for purely theoretical studies. In experiments, despite that the full expression may lead to difficulties in fitting the experimental data, a unique feature is that the same universal relation applies to both the weakly and strongly interacting regimes and also any particle numbers. Furthermore, in certain parameter regimes, some terms may be more important than others. For instance, when $C_2^{(1)} \ll \mathrm{Re}(1/a_1)C_2^{(1)} + \mathrm{Im}(1/a_1)C_3^{(1)}$, the term dependent on $\ln(\rho_{ij}/r_0)\rho_{ij}^2$ shall be more important than that dependent on $\ln(\rho_{ij}/r_0)$. This may simplify the fitting procedures.

V. DISCUSSION

In Table II, we list the two-body inelastic loss rate in 1D, 2D, and 3D. One can recognize that the two-body inelastic loss rate has exactly the same form in all dD, which is

$$\frac{\partial N}{\partial t} = -\frac{2\hbar}{\Omega_d^2 M}\sum_{\nu=1}^{3}\kappa_\nu C_\nu^{(s)} \tag{37}$$

or, equivalently,

$$\frac{\partial n}{\partial t} = -\frac{2\hbar}{\Omega_d^2 M}\sum_{\nu=1}^{3}\kappa_\nu \mathcal{C}_\nu^{(s)}, \tag{38}$$

where Ω_d is the solid angle in dD, which is $\Omega_1 = 2$, $\Omega_2 = 2\pi$, and $\Omega_3 = 4\pi$, respectively. $n = N/L_d$ and $\mathcal{C}_\nu^{(s)} = C_\nu^{(s)}/L_d$ are the molecular density and the contact density of the system, respectively. L_d is the size of the system in dD. Whereas κ_ν behave very differently in different dimensions for s-wave (even-wave for 1D) scatterings, which is originated from the distinct behavior of the low-energy expansion of the phase shift in different dimensions as shown in Table I, they are exactly the same for high-partial-wave scatterings, regardless of the dimension of the system.

We need to emphasize that we have considered the short-range interactions $U(\mathbf{r})$ with a cutoff length r_0 throughout this work to demonstrate the physics underlying the universal relations in lossy systems at low dimensions. When an electric field is applied, the dipole moment of a polar molecule becomes finite, and the dipole-dipole interaction $\sim A/|\mathbf{r}|^n$ with $n = 3$ would become important. Generally, for dilute systems with the power-law interaction $\sim A/|\mathbf{r}|^n$ where $n > 2$, a characteristic length $\tilde{r} = (M|A|/\hbar^2)^{1/(n-2)}$ can be defined [71]. When $\tilde{r} \ll |\mathbf{r}| \ll k_F^{-1}$, due to such a length scale separation, the many-body wave function has universal asymptotic behavior Eq. (7) as well [72]. Following the method presented in our paper, universal relation (37) can also be obtained. While the low-energy expansion of the phase shift might be very different that the scattering length and effective range may not be well defined [73,74], new microscopic parameters determined by the details of the interactions, such as n and l, need to be used. For instance, without losses, universal relations for systems with dipole-dipole interactions have been studied [72]. In lossy systems like reactive molecules, it will be interesting to study how the power-law interactions influence contacts, universal relations, and the decay rates.

Equations (37) and (38) are exact for any many-body eigenstates. Thus, it is invariant under the thermal average.

A. Temperature dependence of the loss rate in homogeneous systems

We take a two-body system in free space as an example. In this case, ϵ becomes a good quantum number. The two-body wave function can be written as $\Psi(\mathbf{x}_1,\mathbf{x}_2)=\phi_c(\mathbf{X}_{12})\psi_s(\mathbf{x}_{12})$, where $\phi_c(\mathbf{X}_{12})$ is the normalized wave function of the center-of-mass motion of the two molecules. $\psi_s(\mathbf{x}_{12})$ is

$$\psi_l(z_{12})=\left[\sqrt{\frac{2\Omega_1}{L_1}}\frac{q_\epsilon^l}{q_\epsilon^{2l-1}(\cot\eta_l-i)}\right]\frac{q_\epsilon^{l-1}}{\tan\eta_l}\times\left[\cos\left(q_\epsilon|z_{12}|-\frac{l\pi}{2}\right)-\tan\eta_l\sin\left(q_\epsilon|z_{12}|-\frac{l\pi}{2}\right)\right]Y_l(\hat{z}_{12}) \tag{39}$$

for 1D systems and

$$\psi_l(\boldsymbol{\rho}_{12})=\left[\sqrt{\frac{2\Omega_2}{L_2}}\frac{q_\epsilon^l}{(\pi/2)q_\epsilon^{2l}(\cot\eta_l-i)}\right]\frac{\pi}{2}\frac{q_\epsilon^l}{\tan\eta_l}[J_l(q_\epsilon\rho_{12})-\tan\eta_l N_l(q_\epsilon\rho_{12})]Y_l(\hat{\boldsymbol{\rho}}_{12}) \tag{40}$$

for 2D systems, where L_1 is the length of the 1D system and L_2 is the area of the 2D system. Recall that, in 3D,

$$\psi_{lm}(\mathbf{r}_{12})=\left[\sqrt{\frac{2\Omega_3}{L_3}}\frac{q_\epsilon^l}{q_\epsilon^{2l+1}(\cot\eta_l-i)}\right]\frac{q_\epsilon^{l+1}}{\tan\eta_l}[j_l(q_\epsilon|\mathbf{r}_{12}|)-\tan\eta_l n_l(q_\epsilon|\mathbf{r}_{12}|)]Y_{lm}(\hat{\mathbf{r}}_{12}), \tag{41}$$

where $\mathbf{r}_{12}=(\boldsymbol{\rho}_{12},z_{12})$, j_l (n_l) is the spherical Bessel function of the first (second) kind, and L_3 is the volume of the 3D system.

By denoting $C_\nu^{[l]}$ as $C_\nu^{(l)}$ for 1D, $C_\nu^{(l)}+C_\nu^{(-l)}$ for 2D, and $\sum_m C_\nu^{(lm)}$ for 3D, respectively, and based on the definition in Table II, $C_\nu^{[l]}$ in dD is expressed as

$$C_1^{[l]}=\sigma_d\frac{4\Omega_d^3}{L_d}\left|q_\epsilon^l f_{l,d}(q_\epsilon)\right|^2, \tag{42}$$

$$C_2^{[l]}=2\,\mathrm{Re}\left(q_\epsilon^2\right)C_1^{[l]}, \tag{43}$$

$$C_3^{[l]}=2\,\mathrm{Im}\left(q_\epsilon^2\right)C_1^{[l]}, \tag{44}$$

where $f_{l,d}(q_\epsilon)\equiv 1/\{[1+\delta_{d,2}(\pi/2-1)]q_\epsilon^{2l+d-2}(\cot\eta_l-i)\}$ and $\delta_{d,d'}$ is the Kronecker delta. σ_d is the fold of degeneracy for the lth partial-wave scatterings in dD, which is $\sigma_1=1$, $\sigma_2=2$, and $\sigma_3=2l+1$, respectively. Based on the results shown in Table I, $f_{l,d}(q_\epsilon)$ can be expanded in the low-energy limit, $f_{l,d}(q_\epsilon)=f_{l,d}^{(0)}+\mathrm{O}(q_\epsilon)$, where $f_{l,d}^{(0)}$ is q_ϵ independent and relates only to the scattering length a_l. Note that $f_{0,2}^{(0)}=1/[\ln(a_0e^\gamma/2)]$. As an example, we consider the scattering states only and the case that only the term $f_{l,d}^{(0)}$ in $f_{l,d}(q_\epsilon)$ is important, where q_ϵ can treated as a real quantity and $C_3^{[l]}=0$.

By considering the second-order virial expansion only and based on the two-body results as shown in Eqs. (42)–(44), the thermal averaged contacts can be obtained by doing the calculation [69]

$$\left\langle C_\nu^{[l]}\right\rangle_T=Z^{-1}e^{\frac{2\mu}{k_BT}}\sum_{E_c}e^{-\frac{E_c}{k_BT}}\sum_n C_\nu^{[l]}e^{-\frac{\epsilon_n}{k_BT}}, \tag{45}$$

where Z is the partition function, $E_c=\hbar^2q_c^2/(4M)$ is the energy of the center-of-mass motion with momentum q_c, $\epsilon_n=\hbar^2q_{\epsilon_n}^2/M$ is the eigenenergy of the relative motion with momentum q_{ϵ_n}, and k_B is the Boltzmann constant. μ is the chemical potential, which can be extracted from $N=k_BT\partial_\mu\ln Z$. In the high-temperature regime, $N/L_d\approx\exp[\mu/(k_BT)]/\lambda_T^d$, where $\lambda_T=[2\pi\hbar^2/(k_BTM)]^{1/2}$ is the thermal wavelength. We have

$$\left\langle C_\nu^{[l]}\right\rangle_T=2^{\frac{d}{2}-1}N^2\lambda_T^d\frac{\Omega_d}{(2\pi)^d}\int_0^\infty C_\nu^{[l]}\exp\left(-\frac{\lambda_T^2}{2\pi}k^2\right)k^{d-1}\mathrm{d}k. \tag{46}$$

Thus, we obtain $\langle C_\nu^{[l]}\rangle_T$ as a function of N and T by substituting Eqs. (42) and (43) into Eq. (46). Based on the fact that $\int_0^\infty\exp(-nx^2)x^{\mathrm{w}-1}\mathrm{d}x=2^{-1}\Gamma(\mathrm{w}/2)n^{-\mathrm{w}/2}$, we obtain

$$\left\langle C_1^{[l]}\right\rangle_T=2^l\pi^{l-\frac{d}{2}}\Gamma\left(l+\frac{d}{2}\right)\sigma_d\Omega_d^4\left|f_{l,d}^{(0)}\right|^2\frac{N^2}{L_d}\lambda_T^{-2l}, \tag{47}$$

$$\left\langle C_2^{[l]}\right\rangle_T=2^{l+2}\pi^{l-\frac{d}{2}+1}\Gamma\left(l+\frac{d}{2}+1\right)\sigma_d\Omega_d^4\left|f_{l,d}^{(0)}\right|^2\frac{N^2}{L_d}\lambda_T^{-2l-2}. \tag{48}$$

B. Temperature dependence of the loss rate in harmonic traps

When a harmonic trap $V_{\mathrm{ext}}(\mathbf{x})=(1/2)M\omega^2(\mathbf{x}\cdot\mathbf{x})$ is applied, under the local density approximation, we can replace μ by the local chemical potential $\mu(\mathbf{x})=\mu(0)-V_{\mathrm{ext}}(\mathbf{x})$ and write $n(\mathbf{x})=\exp[\mu(\mathbf{x})/(k_BT)]/\lambda_T^d$ in the high-temperature regime. ω is the harmonic frequency. $\mu(0)$ is the chemical potential at the center of the trap. At any point $\mathbf{x}$ in the trap, Eq. (38) still applies, we have

$$\frac{\partial n(\mathbf{x})}{\partial t}=-\frac{2\hbar}{\Omega_d^2M}\sum_{\nu=1}^3\kappa_\nu\left\langle C_\nu^{[l]}(\mathbf{x})\right\rangle_T. \tag{49}$$

Thus, by taking the integration over $\mathbf{x}$ on both sides of Eq. (49), the two-body inelastic loss rate in dD traps can be written as

$$\frac{\partial N^{\mathrm{trap}}}{\partial t}=-\frac{2\hbar}{\Omega_d^2M}\sum_{\nu=1}^3\kappa_\nu\left\langle C_\nu^{[l]\mathrm{trap}}\right\rangle_T. \tag{50}$$

Based on Eqs. (47) and (48), at the center of the trap, we still have

$$\left\langle C_1^{[l]}(0)\right\rangle_T=2^l\pi^{l-\frac{d}{2}}\Gamma\left(l+\frac{d}{2}\right)\sigma_d\Omega_d^4\left|f_{l,d}^{(0)}\right|^2n^2(0)\lambda_T^{-2l}, \tag{51}$$

$$\left\langle C_2^{[l]}(0)\right\rangle_T=2^{l+2}\pi^{l-\frac{d}{2}+1}\Gamma\left(l+\frac{d}{2}+1\right)\sigma_d\Omega_d^4\left|f_{l,d}^{(0)}\right|^2\times n^2(0)\lambda_T^{-2l-2}, \tag{52}$$

where $n(0)$ can be expressed by N^{trap} and T, which is

$$N^{\mathrm{trap}}=\int n(\mathbf{x})\mathrm{d}\mathbf{x}=\left(\frac{2\pi k_BT}{M\omega^2}\right)^{d/2}n(0). \tag{53}$$

The total contacts $\langle C_\nu^{\rm trap}\rangle_T$ can be determined by integrating the local contacts in the trap,

$$\left\langle C_\nu^{[l]\rm trap}\right\rangle_T = \left\langle C_\nu^{[l]}(0)\right\rangle_T \int e^{-2V_{\rm ext}(\mathbf{x})/(k_BT)}\mathrm{d}\mathbf{x} = \left(\frac{\pi k_BT}{M\omega^2}\right)^{d/2}\left\langle C_\nu^{[l]}(0)\right\rangle_T. \tag{54}$$

Thus, based on Eqs. (50)–(54), one can map the loss rate in a harmonic trap to the one in a homogeneous system by setting the effective size of the homogeneous system to be $\tilde{L}_d = [4\pi k_BT/(M\omega^2)]^{d/2}$. One has

$$\frac{\partial N^{\rm trap}}{\partial t} = -\beta_{l,d}\frac{(N^{\rm trap})^2}{\tilde{L}_d} \tag{55}$$

or, equivalently,

$$\frac{\partial \tilde{n}}{\partial t} = -\beta_{l,d}\tilde{n}^2, \tag{56}$$

where $\tilde{n} = N^{\rm trap}/\tilde{L}_d$ is the average molecular density of the system and $\beta_{l,d}$ is the loss-rate coefficient for dD lth partial-wave scatterings.

VI. CONCLUSION

In conclusion, we have established universal relations for the two-body inelastic loss rate, which are controlled by contacts $C_\nu^{(s)}$ in 1D and 2D, respectively. Whereas κ_ν have different forms in different dimensions for s-wave (even-wave for 1D) scatterings, the loss rate can be written as exactly the same form in arbitrary dimensions for high-partial-wave (odd-wave for 1D) scatterings. Moreover, the two-body inelastic loss rate can be related to other physical quantities such as the momentum distribution and the density correlation function through contacts. While we considered single-component ultracold atoms or reactive molecules, discussions can be generalized to multicomponent systems straightforwardly. It will also be interesting to consider a finite confinement in the transverse direction such that the dimension crossover can be explored in the presence of two-body losses. We hope that our work could inspire more efforts of using contacts and universal relations to study novel phenomena in lossy quantum systems in condensed matter physics, atomic, molecular, and optical physics, and chemical physics.

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APPENDIX: MATHEMATICS USED IN THE CALCULATION OF EQ. (9)

For a specific partial-wave scattering only, to calculate Eq. (9), one can first calculate

$$\begin{aligned}&\int_0^{r_0}\mathrm{d}\mathbf{x}_{ij}\left[\psi_s^*(\mathbf{x}_{ij};\epsilon)\nabla^2_{\mathbf{x}_{ij}}\psi_s(\mathbf{x}_{ij};\epsilon) - \psi_s(\mathbf{x}_{ij};\epsilon)\nabla^2_{\mathbf{x}_{ij}}\psi_s^*(\mathbf{x}_{ij};\epsilon)\right]\\ &= \int_0^{r_0}\mathrm{d}\mathbf{x}_{ij}\left[\varphi_s^*(|\mathbf{x}_{ij}|;\epsilon)Y^*_{\{0\}}(\hat{\mathbf{x}}_{ij})\nabla^2_{\mathbf{x}_{ij}}\varphi_s(|\mathbf{x}_{ij}|;\epsilon)Y_{\{0\}}(\hat{\mathbf{x}}_{ij})\right.\\ &\quad\left.-\varphi_s(|\mathbf{x}_{ij}|;\epsilon)Y_{\{0\}}(\hat{\mathbf{x}}_{ij})\nabla^2_{\mathbf{x}_{ij}}\varphi_s^*(|\mathbf{x}_{ij}|;\epsilon)Y^*_{\{0\}}(\hat{\mathbf{x}}_{ij})\right]\\ &= \frac{1}{\Omega_d}\oint_{|\mathbf{x}_{ij}|=r_0}\left[\varphi_s^*(|\mathbf{x}_{ij}|;\epsilon)\frac{\partial}{\partial|\mathbf{x}_{ij}|}\varphi_s(|\mathbf{x}_{ij}|;\epsilon)\right.\\ &\quad\left.-\varphi_s(|\mathbf{x}_{ij}|;\epsilon)\frac{\partial}{\partial|\mathbf{x}_{ij}|}\varphi_s^*(|\mathbf{x}_{ij}|;\epsilon)\right]\hat{\mathbf{e}}_{\mathbf{x}}\cdot\mathrm{d}\mathbf{S},\end{aligned} \tag{A1}$$

where $Y_{\{0\}} = Y_{s=\{0\}}$ means that all the quantum numbers in s are zero. $\hat{\mathbf{e}}_{\mathbf{x}}$ is the outgoing unit vector perpendicular to $\mathbf{S}$. $|Y_{\{0\}}|^2 = 1/\Omega_d$ is also used. Thus, to calculate Eq. (9), it is helpful to first calculate

$$\begin{aligned}&|\mathbf{x}_{ij}|^{d-1}\varphi_s^*(|\mathbf{x}_{ij}|;\epsilon)\frac{\partial}{\partial|\mathbf{x}_{ij}|}\varphi_s(|\mathbf{x}_{ij}|;\epsilon)\Big|_{|\mathbf{x}_{ij}|=r_0}\\ &-|\mathbf{x}_{ij}|^{d-1}\varphi_s(|\mathbf{x}_{ij}|;\epsilon)\frac{\partial}{\partial|\mathbf{x}_{ij}|}\varphi_s^*(|\mathbf{x}_{ij}|;\epsilon)\Big|_{|\mathbf{x}_{ij}|=r_0}.\end{aligned} \tag{A2}$$

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