Multichromatic Floquet engineering of quantum dissipation

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The monochromatic driving of a quantum system is a successful technique in quantum simulations, well captured by an effective Hamiltonian approach, and with applications in artificial gauge fields and topological engineering. Here, we investigate multichromatic Floquet driving for quantum simulation. Within a well-defined range of parameters, we show that the time coarse-grained dynamics of such a driven closed quantum system is encapsulated in an effective master equation for the time-averaged density matrix, that evolves under the action of an effective Hamiltonian and tunable Lindblad-type dissipation or quantum gain terms. As an application, we emulate the dissipation induced by phase noise and incoherent emission or absorption processes in the bichromatic driving of a two-level system, and reproduce the phase decoherence in a harmonic oscillator model.

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I. INTRODUCTION

There is currently an intense research effort devoted to the realization of quantum simulators able to reproduce complex quantum dynamics in simpler and controllable setups [1]. In many cases, the quantum systems to be emulated are coupled to an environment, and thus behave as open quantum systems. Such an interaction is usually considered as detrimental. However, a controlled dissipation can be a unique asset for quantum state targeting [2] such as ground states [3], pointer states [4,5], or even excited states [6], for the engineering of correlations [7], and opens many perspectives for many-body quantum simulation [8].

The emulation of quantum dissipation is therefore an important step in the roadmap to accurate quantum simulators [21]. Several mechanisms have been used to produce dissipation in a quantum setup. It includes the driving of two interacting quantum subsystems—one of them acting as a bath on the other [9,10], the use of atom losses [11,12], the Zeno effect [13–17], the bistability of atom transport [18], and time periodic driving [19] to name a few. In this paper, we detail an alternative strategy relying on multichromatic Floquet driving to emulate quantum dissipation while keeping the system conservative.

Periodic Floquet-driven quantum systems have become instrumental to emulate novel interactions, quantum states of matter, or artificial gauge fields [20–32]. Multichromatic Floquet driving has also been applied recently to manipulate topological quantum states [33,34]. In the following, we discuss how an effective quantum dissipation can emerge in a time coarse-grained (TCG) dynamics. In contrast to other studies using a combination of a classical noise with a unitary evolution [35–37] as well as a stochastic averaging, our approach relies exclusively on the unitary quantum dynamics of a Floquet-driven system with no need of extra artificial noise. The emergence of effective Hamiltonians and quantum dissipation in the TCG dynamics was initially discussed in Refs. [38,39]. Nevertheless, the considered approach based on a Dyson series had several flaws and ill-defined approximations which limit its practical applicability. First, it produces an effective equation whose validity is in principle limited to a very short time interval (typically less than a Rabi oscillation for a two-level system) and to moderate dissipation strengths. These assumptions are overly restrictive, as a long time is needed for moderate dissipation to accumulate and alter significantly the dynamics of a given quantum system. Most importantly, in the presence of noncommuting constant and driving Hamiltonians, this previous approach overlooked leading-order nonunitary terms which can significantly influence the effective dissipative dynamics. To address these issues, we exploit a timescale separation formalism [20,22] for multichromatic driving, and derive an effective master equation for the TCG density matrix with well-controlled approximations, and valid over a long time interval. Our treatment provides a robust theoretical framework to tailor effective quantum dissipation in Floquet systems.

This paper is organized as follows. In Sec. II, we first consider a classical harmonic oscillator subject to a bichromatic modulation of the pulsation. This simple example gives physical insights on the emergence of an effective dissipation in the TCG dynamics of a driven system. In Sec. III, we apply the Floquet formalism in order to obtain an effective master equation for the TCG dynamics of a quantum system in the presence of a multichromatic driving. In Sec. IV, we provide examples of emulation of quantum dissipation in the TCG dynamics of simple quantum systems.

II. BICHROMATIC DRIVE OF A CLASSICAL OSCILLATOR

Here, we first take the simple example of a classical oscillator subject to a dual-tone high-frequency drive. A separate treatment of the slow or fast timescales reveals that an effective time-dependent pulsation arises in the low-frequency dynamics. This time dependence corresponds to alternate phases of loss and gain associated respectively to a decrease or increase of the effective pulsation.

Let us first consider the monochromatic drive of a one-dimensional classical harmonic oscillator with a fast and periodic modulation [22], and recall briefly the main results. This system is described by the classical Hamiltonian $H(t) = p_x^2/(2m) + m\omega_0^2 (\cos \omega t)\, x^2/2$, and the modulation frequency is assumed much larger than the harmonic trap frequency $\omega \gg \omega_0$. One thus has a clear timescale separation, and we define a time averaging such that $\overline{\cos \omega t} = 0$ and $\overline{\cos \omega_0 t} = \cos \omega_0 t$. Precisely, we take for the time averaging $\overline{f(t)}$ a low-pass filter that preserves only the Fourier components $\tilde{f}(\omega')$ of frequency lower than a cutoff frequency ω_c, chosen as $\omega_0 < \omega_c < \omega$. The dynamical equation $m\ddot{x} = -m\omega_0^2(\cos \omega t)x$ can then be rewritten by decomposing the global motion $x(t) = \overline{x}(t) + \xi(t)$ as the sum of a slow motion, $\overline{x}(t)$, and a fast micromotion, $\xi(t)$:

$$\ddot{\overline{x}} = -\omega_0^2 \overline{\cos(\omega t)\xi},$$
$$\ddot{\xi} = -\omega_0^2 \cos(\omega t)\overline{x} - \omega_0^2[\cos(\omega t)\xi - \overline{\cos(\omega t)\xi}]. \quad (1)$$

We have used that $\overline{\cos(\omega t)\overline{x}} = \overline{\cos(\omega t)}\,\overline{x} = 0$. By a perturbative treatment of the fast micromotion, one obtains to the leading order $\ddot{\xi} = -\omega_0^2 \cos(\omega t)\overline{x} + O(\omega^2 \epsilon)$. We now use that the time dependence of $\overline{x}$ is very slow, so that $\overline{x}$ can be treated as a constant in the integration with a good approximation. The fast motion can then be integrated as $\xi(t) \simeq \frac{\omega_0^2}{\omega^2} \cos(\omega t)\overline{x}$. One verifies *a posteriori* that the fast motion $\xi(t)$ is of small amplitude as $\omega_0 \ll \omega$. Inserting this expression of $\xi(t)$ in Eq. (1), one obtains a closed equation for the slow motion:

$$\ddot{\overline{x}} = -\frac{\omega_0^4}{\omega^2} \overline{\cos^2 \omega t}\, \overline{x}.$$

As $\overline{\cos^2 \omega t} = 1/2$, this corresponds to a constant effective pulsation $\Omega_{\rm eff} = \omega_0^2/(\sqrt{2}\omega)$. Furthermore, the effective potential energy of the slow motion is stored in the kinetic energy of the micromotion $\frac{1}{2} m \Omega_{\rm eff}^2 \overline{x}^2 = \frac{1}{2} m \overline{\dot{\xi}^2}$.

We now generalize this approach to the bichromatic case, with a quadratic potential $V_F(t) = \frac{1}{2} m\omega_0^2 (\cos(\omega_1 t) + \cos(\omega_2 t))\, x^2$ with fast frequencies ω_1, ω_2 close enough so that $|\omega_2 - \omega_1| \ll \omega_{1,2}$ and $\overline{\cos(\omega_2 - \omega_1)t} = \cos(\omega_2 - \omega_1)t$. The equations of motion for the slow motion take the form

$$\ddot{\overline{x}} = -\omega_0^2 \overline{[\cos(\omega_1 t) + \cos(\omega_2 t)]\xi},$$
$$\ddot{\xi} \simeq -\omega_0^2 [\cos(\omega_1 t) + \cos(\omega_2 t)]\overline{x}. \quad (2)$$

The fast micromotion can be integrated as

$$\xi \simeq \omega_0^2 \left(\frac{\cos(\omega_1 t)}{\omega_1^2} + \frac{\cos(\omega_2 t)}{\omega_2^2} \right) \overline{x}. \quad (3)$$

By inserting this expression in Eq.(2), and taking the time averaging, one finds

$$\ddot{\overline{x}} = -\Omega_{\rm eff}(t)^2 \overline{x}, \quad (4)$$

with

$$\Omega_{\rm eff}(t)^2 = \omega_0^2 \left(\frac{1}{\omega_1^2} + \frac{1}{\omega_2^2} \right)^{1/2} \cos\left(\frac{1}{2}(\omega_1 - \omega_2)t \right). \quad (5)$$

As a result of the bichromatic driving, the effective pulsation $\Omega_{\rm eff}(t)^2$ of the slow motion acquires a time dependence. Differently from the monochromatic case, the potential energy of the low-frequency motion $E = \frac{1}{2} m \dot{\overline{x}}^2 + \frac{1}{2} m \Omega_{\rm eff}^2(t) \overline{x}^2$ is no longer a constant: it oscillates on a timescale corresponding to the beat note $|\omega_1 - \omega_2|$ of the two driving frequencies.

III. OBTENTION OF AN EFFECTIVE MASTER EQUATION FOR THE TCG DYNAMICS WITH THE FLOQUET FORMALISM

We consider from now on a quantum system driven by the sum $\hat{H}(t) = \hat{H}_0 + \hat{H}_F(t)$ of a time-independent Hamiltonian $\hat{H}_0$ and a fast driving Hamiltonian $\hat{H}_F(t) = \sum_m \hat{V}_m e^{i\omega_m t} +$ H.c. The quantum system is closed, and thus the instantaneous quantum state $|\psi(t)\rangle$ undergoes a unitary evolution with a fast time dependence. However, the evolution of the TCG density matrix $\overline{\rho}(t) = \overline{|\psi(t)\rangle\langle\psi(t)|}$ is in general nonunitary, as the TCG procedure wipes out part of the quantum dynamics.

We first recall well-known results on the unitary evolution under Floquet Hamiltonians. The evolution operator corresponding to the Hamiltonian $\hat{H}(t)$ is the product of three unitary transforms [20,22]:

$$\hat{U}(t, t_0) = e^{-i\hat{K}(t)} \hat{U}^{\rm eff}(t) e^{i\hat{K}(t_0)}, \quad (6)$$

where $\hat{U}^{\rm eff}(t) = \mathcal{T}[e^{-i\int_{t_0}^t dt' \hat{H}^{\rm eff}(t')}]$ accounts for the slow dynamics under the effective Hamiltonian $\hat{H}^{\rm eff}(t)$ ($\mathcal{T}$ is the time ordering operator), while the terms involving the kick operator, $\hat{K}(t)$, contain the fast time dependence. The Floquet frequencies ω_m are assumed to be much larger than the eigenfrequencies of $\hat{H}_0$ and $\hat{V}_m$: $\varepsilon = \Omega/\omega \ll 1$ with $\Omega = \max_m\{||\hat{H}_0||, ||\hat{V}_m||\}$ and $\omega = \min_m\{\omega_m\}$. This frequency hierarchy is used to expand $\hat{H}^{\rm eff}(t) = \sum_{n=0}^{+\infty} \hat{H}_n^{\rm eff}(t)$ and $\hat{K}(t) = \sum_{n=1}^{+\infty} \hat{K}_n(t)$.

The considered TCG procedure works as a low-pass filter in frequency space involving a cutoff frequency ω_c: $\overline{\hat{O}}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\omega_c}^{\omega_c} \hat{O}(\omega) e^{-i\omega t} d\omega$, where $\hat{O}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \hat{O}(t) e^{i\omega t} dt$ is the Fourier transform of the considered operator $\overline{\hat{O}}(t)$. As in the previous example, the cutoff frequency ω_c is chosen as to leave invariant the slow Hamiltonian dynamics, i.e., $\overline{e^{\pm i\hat{H}_0 t}} = e^{\pm i\hat{H}_0 t}$, while filtering out the fast Floquet frequencies $\forall m\ \overline{e^{\pm i\omega_m t}} = 0$ $[\overline{\hat{H}_F(t)} = 0]$. We assume that for the slow operators considered throughout this paper [such that $\overline{\hat{O}_{\rm slow}(t)} = \hat{O}_{\rm slow}(t)$] one always has $\overline{\hat{O}_{\rm slow}(t)\hat{O}(t)} = \hat{O}_{\rm slow}(t)\overline{\hat{O}(t)}$ and $\overline{\hat{O}(t)\hat{O}_{\rm slow}(t)} = \overline{\hat{O}(t)}\hat{O}_{\rm slow}(t)$. This property is fulfilled if the $\hat{O}_{\rm slow}$ operator oscillates at frequencies $\omega_{\rm slow} \ll \omega_c$ and if the $\hat{O}(t)$ operator does not have frequencies nearby the cutoff ω_c. These assumptions are realistic for a sufficient large separation between the slow and fast timescales.

Let us now proceed to derive an effective master equation for the TCG density matrix. From Eq. (6), we obtain $\overline{\rho}(t) = \overline{e^{-i\hat{K}(t)} \rho_e(t) e^{i\hat{K}(t)}}$ with $\rho_e(t) = \hat{U}^{\rm eff}(t) e^{i\hat{K}(t_0)} |\psi(t_0)\rangle\langle\psi(t_0)| e^{-i\hat{K}(t_0)} \hat{U}^{\rm eff}(t)^\dagger$ evolving under an effective Hamiltonian $\hat{H}^{\rm eff}(t)$ (see Appendix A). By construction of the effective Hamiltonian, the density matrix $\rho_e(t)$ follows slow dynamics and fulfills $\overline{\rho_e(t)} = \rho_e(t)$. We subsequently expand the fast unitary transforms $e^{\pm i\hat{K}(t)}$ in terms of the small parameter $\varepsilon = \Omega/\omega$. The TCG density matrix then reads

$$\overline{\rho}(t) = \rho_e(t) + \sum_{n=1}^{N} \overline{\delta\rho^{(m)}}(t) + O(\varepsilon^{N+1}). \tag{7}$$

Each term $\overline{\delta\rho^{(m)}(t)}$ represents a correction of order $O(\varepsilon^m)$ and depends linearly on the density matrix $\rho_e(t)$. In order to derive these corrections, one needs explicit expressions for the fast kick operators $\hat{K}_m(t)$. These are used to cancel the fast time dependence in the effective Hamiltonian, and can be obtained at each order through a systematic procedure (Ref. [20] and Appendix A). For instance, $\hat{K}_1(t)$ fulfills $\dot{\hat{K}}_1(t) = \hat{H}_F(t)$ and reads $\hat{K}_1(t) = \sum_m \frac{1}{i\omega_m}(\hat{V}_m e^{i\omega_m t} - h.c)$. The lowest-order correction is of second order as $\delta\overline{\rho^{(1)}}(t) = -i[\overline{\hat{K}_1(t)}, \rho_e(t)] = 0$ and is given by $\delta\overline{\rho^{(2)}}(t) = -\frac{1}{2}\{\overline{\hat{K}_1(t)^2}, \rho_e(t)\} + \overline{\hat{K}_1(t)\rho_e(t)\hat{K}_1(t)}$. An effective equation for the time-averaged density matrix is obtained by taking the time derivative of Eq. (7). Special care is, however, needed in order to gather consistently corrections to the same order. For instance, the contribution $\overline{\delta\rho^{(2)}(t)}$ involves a product of fast-evolving $[\hat{K}_1(t)]$ and slow-evolving [the density matrix $\rho_e(t)$] functions. When applied to the latter, the time derivative yields terms which are smaller by one order in the small parameter ε. This leads us to distinguish the slow and fast time dependence by setting τ and t for the corresponding time variables, with $\partial_\tau = O(\Omega)$ and $\partial_t = O(\omega)$ [40,41]. We note $\overline{\delta\rho^{(m)}}(t, \tau)$ the corresponding corrections to the density matrix, so that the second-order correction reads $\overline{\delta\rho^{(2)}}(t, \tau) = -\frac{1}{2}\{\overline{\hat{K}_1(t)^2}, \rho_e(\tau)\} + \overline{\hat{K}_1(t)\rho_e(\tau)\hat{K}_1(t)}$. The complete effective master equation can be written to second order as $\frac{\partial}{\partial t}\overline{\rho} = -i[\hat{H}^{\text{eff}}, \rho_e] + \partial_t\overline{\delta\rho^{(2)}(t, \tau)} + \partial_\tau\overline{\delta\rho^{(2)}(t, \tau)} + \partial_t\overline{\delta\rho^{(3)}(t, \tau)} + O(\Omega\varepsilon^3)$. The contribution $\partial_t\overline{\delta\rho^{(2)}(t, \tau)}$ yields the operator $\mathcal{L}_2^{FF}[\overline{\rho}]$ (9), while the terms $\partial_\tau\overline{\delta\rho^{(2)}(t, \tau)}$ and $\partial_t\overline{\delta\rho^{(3)}(t, \tau)}$ yield altogether the Lindblad-like term $\mathcal{L}_2^{FSF}[\overline{\rho}]$ (10) accounting for the interplay between the slow and fast degrees of freedom.

We eventually obtain the effective master equation for the TCG density matrix, which constitutes the central result of this paper (see Appendix B):

$$\frac{\partial\overline{\rho}}{\partial t} = -i[\hat{H}^{\text{eff}}, \overline{\rho}] + \mathcal{L}_2^{FF}[\overline{\rho}] + \mathcal{L}_2^{FSF}[\overline{\rho}] + O(\Omega\varepsilon^3) \tag{8}$$

with

$$\mathcal{L}_2^{FF}[\overline{\rho}] = \sum_{m,n} \frac{2\,\overline{e^{i\Delta\omega_{mn}t}}}{i\omega_{mn-}} \mathcal{D}[\hat{V}_m^\dagger, \hat{V}_n][\overline{\rho}], \tag{9}$$

$$\mathcal{L}_2^{FSF}[\overline{\rho}] = \sum_{m,n} \frac{\overline{e^{i\Delta\omega_{mn}t}}}{i}\Bigg[\frac{1}{\omega_n^2}\mathcal{D}[\hat{V}_m, [\hat{V}_n^\dagger, \hat{H}_0]][\overline{\rho}] + \frac{1}{\omega_m^2}\mathcal{D}[\hat{V}_n^\dagger, [\hat{V}_m, \hat{H}_0]][\overline{\rho}]\Bigg], \tag{10}$$

with $\Delta\omega_{mn} = \omega_m - \omega_n$, $1/\omega_{mn-} = \frac{1}{2}(1/\omega_m - 1/\omega_n)$, and $\mathcal{D}[\hat{V}, \hat{V}'][\overline{\rho}] = \hat{V}\overline{\rho}\hat{V}' + \hat{V}'\overline{\rho}\hat{V} - \frac{1}{2}\{\{\hat{V}, \hat{V}'\}, \overline{\rho}\}$.

This effective master equation contains two nonunitary contributions encapsulated in the Lindblad-like terms $\mathcal{L}_2^{FF}[\overline{\rho}]$ and $\mathcal{L}_2^{FSF}[\overline{\rho}]$ provided that the fast driving Hamiltonian contains at least two different frequencies $\{\omega_m, \omega_n\}$ close enough to ensure $\overline{e^{\pm i(\omega_m-\omega_n)t}} = e^{\pm i(\omega_m-\omega_n)t}$. We assume from now on that this property is fulfilled, i.e., that the Floquet frequencies ω_m belong to a narrow bandwidth: $\forall(m, n)\ |\omega_m - \omega_n| < \omega_c \ll \omega$. Under this assumption, the beat notes between these Floquet modes generate tunable non-Hermitian contributions to the time-averaged dynamics. The nonunitary operator $\mathcal{L}_2^{FF}[\overline{\rho}]$ scales as $1/\omega_{mn-} \simeq |\Delta\omega_{mn}|/\omega^2$. For usual situations where $|\Delta\omega_{mn}| \leqslant \Omega$, which corresponds to dissipation terms oscillating at a comparable pace (or slower) as the effective Hamiltonian dynamics, the nonunitary operators of Eq. (8) are of second order. The extra contribution $\mathcal{L}_2^{FSF}[\overline{\rho}]$ (10) arises when $[\hat{V}_m, \hat{H}_0] \neq 0$, and takes into account the interaction between slow and fast quantum dynamics in the time-averaged evolution. This term can be of the same magnitude as $\mathcal{L}_2^{FF}[\overline{\rho}]$ and significantly affect the effective dissipative dynamics. In order to have a clear physical interpretation, Eq. (9) must yield a complete positive trace-preserving (CPTP) evolution and thus involve positive jump rates in the Lindblad operators (9) and (10). Here, these rates oscillate as a function of time between negative and positive values. As discussed below, a suitable phase choice can nevertheless ensure their positivity over a long time window for close Floquet frequencies ω_m and ω_n.

The effective master equation derived in the present framework is valid over an arbitrary long time interval. This is an essential benefit from our approach based on the exact expression (6) followed by an expansion in terms of the Floquet frequencies. We obtain (see Refs. [20,22] and Appendixes A and B) $\hat{H}_0^{\text{eff}} = \hat{H}_0$, $\hat{H}_1^{\text{eff}} = \frac{1}{2}\sum_{m,n}(\frac{1}{\omega_m} + \frac{1}{\omega_n})[\hat{V}_m, \hat{V}_n^\dagger]e^{i(\omega_m-\omega_n)t}$ and $\hat{H}_2^{\text{eff}} = \frac{1}{2}\sum_{m,n}(\frac{1}{\omega_m^2}[[\hat{V}_m, \hat{H}_0], \hat{V}_n^\dagger] + \frac{1}{\omega_n^2}[[\hat{V}_n^\dagger, \hat{H}_0], \hat{V}_m])e^{i(\omega_m-\omega_n)t}$. As in the classical oscillator of Sec. II, the time dependence of the effective Hamiltonian results from the multichromatic driving. At the considered second order and for Floquet frequencies taken in a narrow bandwidth, kick operators must be grouped pairwise in order to generate low-frequency harmonics that survive the time averaging. This is why the bichromatic case considered below contains the phenomenology of the nonunitary effects that arise in any multichromatic Floquet driving.

IV. EMULATION OF DISSIPATION WITH TCG DYNAMICS IN SIMPLE QUANTUM SYSTEMS

We provide in this section an illustration of the emulation of quantum dissipation in the TCG dynamics of simple quantum systems. We investigate the TCG dynamics in two-level systems, and in a quantum harmonic oscillator. Finally, we discuss the class of quantum dissipative processes that can be simulated by this approach.

As a first example, we consider a two-level system with $\hat{H}_0 = \omega_0\sigma_z$, and the Floquet operators $\hat{V}_m = \Omega_m\sigma_x$ for $m = 1, 2$ ($\sigma_{x,y,z}$ are the Pauli matrices, and we set here $\Omega_{1,2} = \Omega > 0$). This choice yields $\mathcal{L}_2^{FF}[\overline{\rho}] = 8\Omega^2(\sin(\Delta\omega_{21}t)/\omega_{12-})(\overline{\rho} - \sigma_x\overline{\rho}\sigma_x)$ and $\mathcal{L}^{FSF}[\overline{\rho}] = -8\omega_0\Omega^2 (\frac{1}{\omega_1^2} + \frac{1}{\omega_2^2})\cos^2(\frac{1}{2}\Delta\omega_{21}t)(\sigma_x\overline{\rho}\sigma_y + \sigma_y\overline{\rho}\sigma_x)$. The effective Hamiltonian contributions are $\hat{H}_1^{\text{eff}} = 0$ and $\hat{H}_2^{\text{eff}} = -8\omega_0 \Omega^2(\frac{1}{\omega_1^2} + \frac{1}{\omega_2^2})\cos^2(\frac{1}{2}\Delta\omega_{21}t)\sigma_z$. To emphasize the role of the

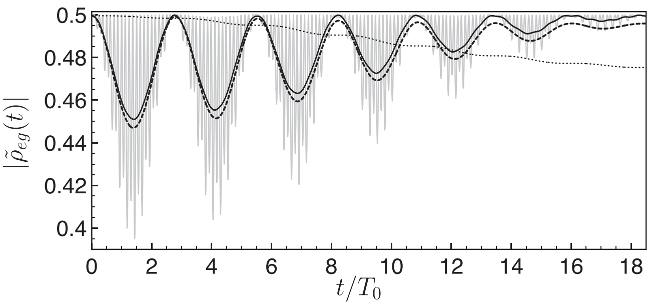

FIG. 1. Quantum dynamics in the interaction picture with $\hat{H}_0 = \omega_0\sigma_z$ and $\hat{V}_{1,2} = \Omega_{1,2}\sigma_x$. Shown are the instantaneous density-matrix profile $|\tilde{\rho}_{eg}(t)|$ as a function of time (solid gray line) with $\tilde{\rho}(t) = U^{\text{eff}}(t)^\dagger \rho(t) U^{\text{eff}}(t)$ and coarse-grained density-matrix coherence $|\overline{\rho}_{eg}(t)|$ as a function of time obtained from a time averaging of the instantaneous solution $\tilde{\rho}(t)$ (solid black line), or from a resolution of the full effective master equation (dashed black line), of an effective master equation without the contribution $\mathcal{L}^{FSF}[\overline{\rho}]$ (see text) to the quantum dissipation term (dotted line). Parameters: $\rho(0) = |\psi_+\rangle\langle\psi_+|$ where $|\psi_+\rangle = (|e\rangle + |g\rangle)/\sqrt{2}$, $\omega_0 = 0.1 \times (2\pi)/T_0$, $\Omega_1 = \Omega_2 = 2/T_0$, $\omega_1 = 4 \times (2\pi)/T_0$, $\Delta\omega_{21} = 0.025 \times (2\pi)/T_0$, $\epsilon = 0.1$.

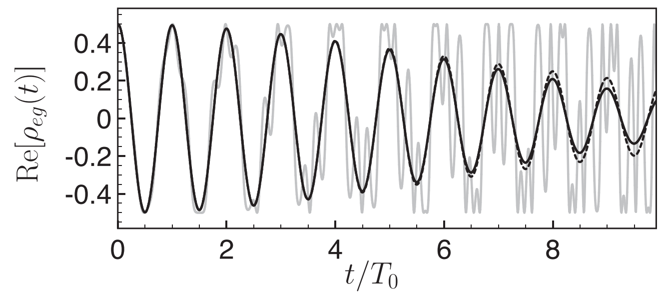

FIG. 2. Emulation of phase noise: results of the effective master equation vs full quantum evolution. Shown are the instantaneous density-matrix coherence $\text{Re}[\rho_{eg}(t)]$ as a function of time (solid gray line), time coarse-grained coherence $\text{Re}[\overline{\rho}_{eg}(t)]$ (solid black line), and the density-matrix coherence $\text{Re}[\overline{\rho}_{eg}(t)]$ (dashed black line) obtained from the effective master equation [Eq. (8)]. Parameters: $\hat{H}_0 = \omega_0\sigma_z$, $\hat{V}_{1,2} = \Omega_{1,2}\sigma_z$, $\omega_0 = 0.5 \times (2\pi)/T_0$, $\Omega_1 = -\Omega_2 = 7/T_0$, $\omega_1 = \sqrt{10} \times (2\pi)/T_0$, and $\varepsilon = 0.35$. Other parameters are identical to Fig. 1.

dissipative dynamics, we provide hereafter the quantum evolution within the interaction picture with respect to the second-order effective Hamiltonian $\hat{H}^{\text{eff}} = \hat{H}_0 + \hat{H}_2^{\text{eff}}$. Figure 1 pictures the time evolution of the instantaneous density-matrix coherence $\tilde{\rho}_{eg}(t)$ with $\tilde{\rho}(t) = \hat{U}^{\text{eff}}(t)^\dagger \rho(t) \hat{U}^{\text{eff}}(t)$. We express all time-related quantities using an arbitrary time unit T_0. We also provide the TCG evolution using a convolution with the cardinal sine function $f(t) = \sin(\omega_c t)/(\pi t)$ [with $\omega_c = 2 \times (2\pi)/T_0$ in all numerical examples]. We subsequently compare this time-averaged density matrix $\overline{\rho}(t) = \int dt' f(t'-t)\tilde{\rho}(t)$ with the predictions of the effective master equation with the initial condition $\overline{\rho}_0 = e^{i\hat{K}_1(t_0)}[\int_{-\infty}^{+\infty} dt f(-t)\tilde{\rho}(t)]e^{-i\hat{K}_1(t_0)}$. We have also added the prediction from the master equation in the absence of the $\mathcal{L}^{FSF}[\overline{\rho}]$ term, i.e., as derived in Ref. [39]; this latter approach is only valid over a short time interval and for moderate dissipation forces, two too restrictive assumptions.

Our second example illustrates the emulation of phase noise in the time-averaged quantum dynamics. In NMR, such a dissipative physical mechanism results from fluctuations of the magnetic field. The phenomenological equation for the average spin dynamics, $\dot{\mathbf{M}} = \gamma \mathbf{M} \times \mathbf{B} + (M_0 - M_z)/T_1 \hat{\mathbf{z}} - \mathbf{M}_\perp/T_2$, accounts for the dissipative effects through two times T_1 and T_2, associated respectively to the longitudinal (M_z) and transverse ($\mathbf{M}_\perp$) relaxations. In terms of the density matrix, the former corresponds to the population difference $\rho_{ee} - \rho_{gg}$ while the latter involves the density-matrix coherences ρ_{eg}, ρ_{ge}. The phase noise is accounted for with decay times $T_1 = +\infty$ and $T_2 = 1/\gamma$ [42]. The master equation that models the phase noise reads $\partial_t \rho = -i[\hat{H}_0, \rho] + \frac{\gamma}{2}\mathcal{L}_{\text{phase}}[\rho]$ with the Liouvillian $\mathcal{L}_{\text{phase}}[\rho] = \sigma_z \rho \sigma_z - \frac{1}{2}\{\sigma_z\sigma_z, \rho\}$.

To emulate such a dissipative dynamics, we consider a bichromatic driving with $\hat{H}_0 = \omega_0\sigma_z$ and $\hat{V}_m = \Omega_m\sigma_z$ for $m = 1, 2$. In this particular case, the contribution of the $\mathcal{L}^{FSF}[\overline{\rho}]$ term vanishes and the resulting master equation coincides with the desired form with $\gamma(t) = -(16/\omega_{12-})\text{Im}[\Omega_1^*\Omega_2 e^{i(\omega_2-\omega_1)t}]$. Using close and noncommensurate frequencies ω_1 and ω_2 enables us to accumulate decoherence (or gain) over a significant time interval depending on the sign of γ.

As previously, we validate numerically our findings by resolving the full unitary quantum dynamics driven by the Hamiltonian $\hat{H}(t) = \hat{H}_0 + \hat{H}_F(t)$ (see Fig. 2). The seemingly erratic oscillations of the instantaneous density-matrix coherence generate a TCG dynamics that follows very accurately the effective master equation, i.e., the one of a damped Rabi oscillation. This averaging effect on the Floquet-induced peaks is reminiscent of the averaging on individual stochastic trajectories involving quantum jumps in the Monte Carlo wave-function formalism [43]. Floquet-induced peaks accumulate periodically at a pace determined by the beat frequency $\Delta\omega_{21}$ between the two involved Floquet modes. This periodic increase or decrease of sharp peaks provokes an oscillation of the effective damping rate $\gamma(t)$ at the same frequency $\Delta\omega_{21}$. An initial loss (gain) phase can be obtained by setting a specific phase difference ϕ between the two Floquet modes. Using $\Omega_1 \in \mathbf{R}^+$ and $\Omega_2 = |\Omega_2|e^{i\phi}$, the choice $\Omega_2 = -\Omega_1$ yields $\gamma(t) > 0$ over the time interval $0 \leqslant t \leqslant \pi/|\omega_2 - \omega_1|$, and thus a physical CPTP evolution for the TCG density matrix (see Fig. 2).

This observed excellent agreement is not obvious, as Eq. (8) is a mere second-order approximation, and discards several contributions associated to the higher-order kick operators $\hat{K}_m(t)$. Actually, the operators $\hat{K}_m(t)$ vanish here for $m \geqslant 2$ as a result of the commutation between the fast and time-independent Hamiltonians. Thus, the expansion of the unitary operators $e^{\pm i\hat{K}(t)}$ boils down to a simple power expansion in the operator $\hat{K}_1(t)$. Furthermore, odd powers of $\hat{K}_1(t)$ do not generate any low-frequency harmonics, and the effective master equation only receives contributions from even-order terms. Incidentally, the fourth-order contribution cancels (Appendix C). Here, Eq. (8) is thus accurate to the fifth order, which explains the remarkable agreement between the TCG (8) and exact quantum dynamics, which still holds for moderate values of the parameter ε ($\varepsilon = 0.35$ in Fig. 2).

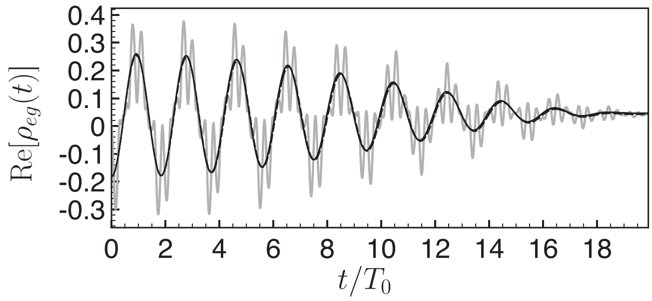

FIG. 3. Incoherent absorption or emission in the time-averaged dynamics. Shown are the instantaneous density-matrix coherence $\mathrm{Re}[\rho_{eg}(t)]$ as a function of time (solid gray line) and time-averaged coherence $\mathrm{Re}[\overline{\rho}_{eg}(t)]$ (solid black line) vs density-matrix coherence $\mathrm{Re}[\overline{\rho}_{eg}(t)]$ (dashed black line) obtained from the effective master equation. Parameters: $\hat{H}_0 = \omega_0\sigma_x$, $\hat{V}_{1,2} = \Omega_{1,2}\sigma_+$, $\rho(0) = |e\rangle\langle e|$, $\omega_0 = 0.25 \times (2\pi)/T_0$, $\Omega_{1,2} = 2/T_0$, $\epsilon = 0.1$. Other parameters are identical to Fig. 2.

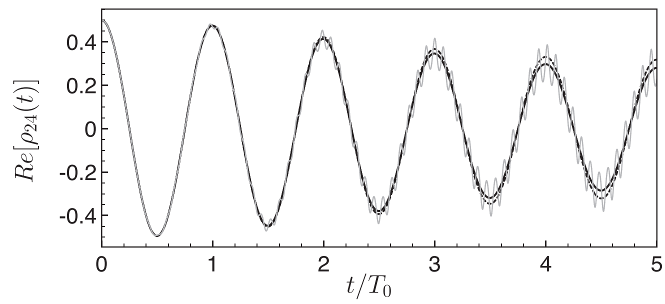

FIG. 4. Emulated decoherence in a harmonic oscillator with $\hat{H}_0 = \omega_0\hat{a}^\dagger\hat{a}$ and $\hat{V}_{1,2} = \pm\Omega\hat{a}^\dagger$. Shown are the instantaneous density matrix $\mathrm{Re}[\rho_{24}(t)]$ (solid gray line) and exact TCG density matrix $\mathrm{Re}[\overline{\rho}_{24}(t)]$ (solid black line) elements in the Fock state basis as a function of time compared to the prediction of the effective master equation (dashed black line). Parameters: $\omega_1 = 40/T_0$, $\Delta\omega_{21} = 0.1 \times (2\pi)/T_0$, $\epsilon = \Omega/\omega_1 = 0.15$.

In our third example, we emulate a quantum dynamics reminiscent of incoherent emission and absorption processes in the TCG evolution of a two-level system. These processes are described respectively by the Liouvillians $\mathcal{L}_{\mathrm{em}}[\rho] = \sigma_-\rho\sigma_+ - \frac{1}{2}\{\sigma_-\sigma_+, \rho\}$ and $\mathcal{L}_{\mathrm{ab}}[\rho] = \sigma_+\rho\sigma_- - \frac{1}{2}\{\sigma_+\sigma_-, \rho\}$, where $\sigma_+ = |e\rangle\langle g|$ and $\sigma_- = \sigma_+^\dagger$. By symmetry of the dissipative term $\mathcal{D}[\hat{V}, \hat{V}'][\rho]$ in the effective master equation, if the TCG dynamics contains the Liouvillian $\mathcal{L}_{\mathrm{em}}[\rho]$, it also contains the Liouvillian $\mathcal{L}_{\mathrm{ab}}[\rho]$ associated to the reverse process. It occurs, for example, in the dynamics of a two-level atom illuminated by an intense light field where stimulated emission predominates over spontaneous emission [44]. The emission and absorption rates then approximately equal $\gamma_{\mathrm{em}} \simeq \gamma_{\mathrm{ab}} \simeq \gamma$.

To emulate such a dissipation, we take the time-independent $\hat{H}_0 = \omega_0\sigma_x$ and fast Hamiltonians with $\hat{V}_m = \Omega_m\sigma_+$ for $m = 1, 2$ ($\Omega_{1,2} = \Omega > 0$). With this choice, the bilinear term $\mathcal{L}_2^{FF}[\overline{\rho}]$ accounts for these two incoherent processes as $\mathcal{L}_2^{FF}[\overline{\rho}] = \gamma(t)(\mathcal{L}_{\mathrm{em}}[\overline{\rho}] + \mathcal{L}_{\mathrm{ab}}[\overline{\rho}])$ with the time-dependent effective emission or absorption rate $\gamma(t) = -4\Omega^2 \sin(\Delta\omega_{21}t)/\omega_{12-}$. The remaining contribution reads $\mathcal{L}_2^{FSF}[\overline{\rho}] = -\omega_0\Omega^2(\frac{1}{\omega_1^2} + \frac{1}{\omega_2^2})\cos^2(\frac{1}{2}\Delta\omega_{21}t)(\sigma_y\rho\sigma_z + \sigma_z\rho\sigma_y) + \Omega O(\varepsilon^3)$. One finds the effective Hamiltonian corrections $\hat{H}_1^{\mathrm{eff}} = 2\Omega^2(\frac{1}{\omega_1} + \frac{1}{\omega_2})\cos^2(\frac{1}{2}\Delta\omega_{21}t)\sigma_z$ and $\hat{H}_2^{\mathrm{eff}} = -2\omega_0\Omega^2(\frac{1}{\omega_1^2} + \frac{1}{\omega_2^2})\cos^2(\frac{1}{2}\Delta\omega_{21}t)\sigma_x + \Omega O(\varepsilon^3)$. In Fig. 3, we observe an excellent agreement between the exact instantaneous density-matrix coherence and its corresponding TCG evolution. As a final example of a higher-dimensional system, we discuss the TCG evolution of a quantum harmonic oscillator ($\hat{H}_0 = \omega_0\hat{a}^\dagger\hat{a}$) subjected to a bichromatic driving with $\hat{V}_{1,2} = \pm\Omega\hat{a}^\dagger$. Dissipative TCG dynamics are given by $\mathcal{L}_2^{FF}[\overline{\rho}] = -(4/\omega_{12-})\mathcal{L}_2^{\mathrm{har}}[\overline{\rho}]$ with $\mathcal{L}_2^{\mathrm{har}}[\rho] = \mathrm{Im}[\Omega_1^*\Omega_2 e^{i(\omega_2-\omega_1)t}]\mathcal{D}[\hat{a}, \hat{a}^\dagger][\rho]$ and $|\mathcal{L}_2^{FSF}| \ll |\mathcal{L}_2^{FF}|$. The effective Hamiltonian reads $\hat{H}^{\mathrm{eff}} = \hat{H}_0$. Starting from the initial Fock state superposition $\rho_0 = |\psi_+\rangle\langle\psi_+|$ with $|\psi_+\rangle = (|2\rangle + |4\rangle)/\sqrt{2}$, the expected effective decoherence is well reproduced (see Fig. 4).

More generally, our approach enables one to emulate a Lindblad master equation of the form $\dot{\rho} = -i[\hat{H}, \hat{\rho}] + \sum_{m=1}^{N} \gamma_m[\hat{L}_m\rho\hat{L}_m^\dagger + \hat{L}_m^\dagger\rho\hat{L}_m - \frac{1}{2}\{\{\hat{L}_m, \hat{L}_m^\dagger\}, \rho\}]$, i.e., involving, for each quantum jump operator $\hat{L}_m$, the reverse jump $\hat{L}_m^\dagger$ at the same rate γ_m [45]. It requires a driving with well-separated pairs of close frequencies $\{\omega_m, \omega_m + \Delta\omega_m\}$, such that $\Delta\omega_m \ll \omega_c$ and $|\omega_m - \omega_n| > \omega_c$ for $m \neq n$. Interestingly, the effective time-dependent rates $\gamma_m(t) \simeq -4(|\Omega_m|^2\Delta\omega_m/\omega_m^2)\sin(\Delta\omega_m t + \varphi_m)$ can be shaped independently by a suitable choice of the Rabi pulsations (Ω_m), frequency ($\Delta\omega_m$), and phase (φ_m) differences. Regarding the $\mathcal{L}_2^{FSF}[\overline{\rho}]$ term, its contribution can be attenuated by an appropriate choice of $\hat{H}_0$, e.g., the third example detailed above.

V. CONCLUSION

In summary, we have used the formalism of kick operators and effective Hamiltonians to derive an effective master equation for the TCG dynamics in a multichromatic Floquet system, that exploits the beat modes between pairs of Floquet frequencies. In contrast to previous studies, our treatment holds in the long-time limit. Different driving Hamiltonians and time-averaging procedures can be considered to emulate a wide range of dynamics. Perspectives for this paper include the application of this method to diverse quantum systems such as fermionic chains [46], or the emulation of effective Lindbladians presenting different symmetries [47].

ACKNOWLEDGMENTS

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APPENDIX A: PERTURBATIVE EXPANSION OF THE KICK OPERATORS AND EFFECTIVE HAMILTONIANS

The exact Hamiltonian under consideration reads $\hat{H}(t) = \hat{H}_0 + \hat{H}_F(t)$ with $\hat{H}_F(t) = \sum_m \hat{V}_m e^{i\omega_m t} + \mathrm{H.c.}$ We assume that all Floquet frequencies ω_m belong to a narrow bandwidth, i.e., fulfill $\forall m, \forall n$, $|\omega_m - \omega_n| \ll \omega_c \ll \omega$. Following the procedure of Refs. [20,22], we search for a unitary operator $e^{i\hat{K}(t)}$ such that the state expressed in the new gauge $|\phi(t)\rangle = e^{iK(t)}|\psi(t)\rangle$ follows a slow dynamics. The Hamiltonian in the new gauge frame is given by

$$\hat{H}^{\rm eff}(t) = e^{i\hat{K}(t)}\hat{H}(t)e^{-i\hat{K}(t)} + i\frac{\partial e^{i\hat{K}(t)}}{\partial t}e^{-i\hat{K}(t)}, \quad \text{(A1)}$$

and must be such that $\overline{H^{\rm eff}(t)} = H^{\rm eff}(t)$ at any time t. For the considered Floquet frequencies, one has $\overline{e^{i\pm(\omega_m-\omega_n)t}} = e^{i\pm(\omega_m-\omega_n)t}$ and $\overline{e^{\pm i\omega_m t}} = \overline{e^{\pm i(\omega_m+\omega_n)t}} = 0$. Thus, only terms rotating at a difference between two Floquet frequencies (or constant terms) will contribute to the effective Hamiltonian $H^{\rm eff}(t)$. In this section, we determine iteratively the first contributions to the expansion $\hat{H}^{\rm eff}(t) = \sum_{n=0}^{+\infty}\hat{H}_n^{\rm eff}(t)$ and $\hat{K}(t) = \sum_{n=1}^{+\infty}\hat{K}_n(t)$ using the identities provided by the Baker-Campbell-Hausdorff formula:

$$\begin{aligned} e^{i\hat{K}(t)}\hat{H}(t)e^{-i\hat{K}(t)} &= \hat{H}(t) + i[\hat{K}(t),\hat{H}(t)] \\ &- \frac{1}{2}[\hat{K}(t),[\hat{K}(t),\hat{H}(t)]] \\ &- \frac{i}{6}[\hat{K}(t),[\hat{K}(t),[\hat{K}(t),\hat{H}(t)]]] + \dots, \end{aligned} \quad \text{(A2)}$$

$$\begin{aligned} \left(\frac{\partial e^{i\hat{K}(t)}}{\partial t}\right)e^{-i\hat{K}(t)} &= i\frac{\partial \hat{K}}{\partial t} - \frac{1}{2}\left[\hat{K}(t), \frac{\partial K}{\partial t}\right] \\ &- \frac{i}{6}\left[\hat{K}(t),\left[\hat{K}(t),\frac{\partial K}{\partial t}\right]\right] + \dots. \end{aligned} \quad \text{(A3)}$$

In the following, we explicitly use the fact that $\hat{H}_n^{\rm eff}(t)$, $\hat{K}_n(t)$, and $\frac{\partial}{\partial t}\hat{K}_{n+1}$ are of the same order $O(\varepsilon^n)$ where $\varepsilon = \Omega/\omega$. The zeroth-order contribution is obtained by taking $e^{i\hat{K}(t)}\hat{H}(t)e^{-i\hat{K}(t)} = H(t) + O(\varepsilon)$. Using Eqs. (A1) and (A3), we find

$$\hat{H}_0^{\rm eff}(t) = \hat{H}_0 + \hat{H}_F - \frac{\partial \hat{K}_1}{\partial t}.$$

From now on, we remove the explicit time dependence of the operators on the right-hand side to lighten notations when needed. As $\overline{\hat{H}_0} = \hat{H}_0$ and $\overline{\hat{H}_F(t)} = 0$, we set $\hat{H}_0^{\rm eff}(t) = \hat{H}_0$ and

$$\frac{\partial \hat{K}_1}{\partial t} = \hat{H}_F. \quad \text{(A4)}$$

The kick operator $\hat{K}_1(t)$ removes all the fast time dependence from the effective Hamiltonian, and can be chosen as

$$\hat{K}_1(t) = \sum_m \frac{1}{i\omega_m}(\hat{V}_m e^{i\omega_m t} - \hat{V}_m^\dagger e^{-i\omega_m t}), \quad \text{(A5)}$$

up to an arbitrary constant operator.

To obtain the result to next order, we introduce the two lowest-order kick operators $\hat{K}_{1,2}(t)$ into Eqs. (A1)–(A3). We find

$$H_1^{\rm eff}(t) = i[\hat{K}_1,\hat{H}] - \frac{i}{2}\left[\hat{K}_1, \frac{\partial \hat{K}_1}{\partial t}\right] - \frac{\partial \hat{K}_2}{\partial t},$$

which can be recast thanks to Eq. (A4) as

$$\hat{H}_1^{\rm eff}(t) = i[\hat{K}_1,\hat{H}_0] + \frac{i}{2}[\hat{K}_1,\hat{H}_F] - \frac{\partial \hat{K}_2}{\partial t}.$$

We infer

$$\begin{aligned} \hat{H}_1^{\rm eff}(t) &= \overline{i[\hat{K}_1,\hat{H}_0] + \frac{i}{2}[\hat{K}_1,\hat{H}_F]}, \\ \frac{\partial \hat{K}_2}{\partial t} &= i[\hat{K}_1,\hat{H}_0] + \frac{i}{2}[\hat{K}_1,\hat{H}_F] - \hat{H}_1^{\rm eff}. \end{aligned} \quad \text{(A6)}$$

As a product of a kick operator with a slow Hamiltonian yields a null average, we have $\overline{[\hat{K}_1(t),\hat{H}_0]} = 0$ and get the first-order expression to the effective Hamiltonian,

$$\begin{aligned} \hat{H}_1^{\rm eff}(t) = \sum_{m,n}\frac{1}{2}\left(\frac{1}{\omega_m} + \frac{1}{\omega_n}\right)[\hat{V}_m,\hat{V}_n^\dagger] \\ \times e^{i(\omega_m-\omega_n)t + i(\varphi_m-\varphi_n)}, \end{aligned} \quad \text{(A7)}$$

and the second-order expression for the kick operator,

$$\begin{aligned} \hat{K}_2(t) &= \sum_m \frac{1}{i\omega_m^2}([\hat{V}_m,\hat{H}_0]e^{i\omega_m t} - \text{H.c.}) \\ &+ \sum_{m,n}\frac{1}{2i\omega_m(\omega_m+\omega_n)} \\ &\times([\hat{V}_m,\hat{V}_n]e^{i(\omega_m+\omega_n)t} - \text{H.c.}), \end{aligned} \quad \text{(A8)}$$

up to an arbitrary constant operator.

To find $\hat{H}_2^{\rm eff}(t)$, we iterate the very same procedure. Using Eqs. (A2) and (A3), we find

$$\begin{aligned} \hat{H}_2^{\rm eff}(t) &= i[\hat{K}_2,\hat{H}] - \frac{1}{2}[\hat{K}_1,[\hat{K}_1,\hat{H}]] - \frac{i}{2}\left[\hat{K}_2,\frac{\partial \hat{K}_1}{\partial t}\right] \\ &- \frac{i}{2}\left[\hat{K}_1,\frac{\partial \hat{K}_2}{\partial t}\right] + \frac{1}{6}\left[\hat{K}_1,\left[\hat{K}_1,\frac{\partial \hat{K}_1}{\partial t}\right]\right] - \frac{\partial \hat{K}_3}{\partial t}. \end{aligned} \quad \text{(A9)}$$

This equation can be recast using Eqs. (A4) and (A6) to express the derivatives of the kick operators $K_{1,2}(t)$ in terms of commutators:

$$\begin{aligned} \hat{H}_2^{\rm eff}(t) &= i[\hat{K}_2,\hat{H}_0] + \frac{i}{2}[\hat{K}_2,\hat{H}_F] + \frac{i}{2}[\hat{K}_1,\hat{H}_1^{\rm eff}] \\ &- \frac{1}{12}[\hat{K}_1,[\hat{K}_1,\hat{H}_F]] - \frac{\partial \hat{K}_3}{\partial t}. \end{aligned} \quad \text{(A10)}$$

We find $\overline{[\hat{K}_2,\hat{H}_0]} = \overline{[\hat{K}_1,\hat{H}_1^{\rm eff}]} = 0$. As the product of three fast operators with similar Floquet frequencies does not generate any slow harmonics, we have $\overline{[\hat{K}_1,[\hat{K}_1,\hat{H}_F]]} = 0$. The second-order effective Hamiltonian contribution is then given by $\hat{H}_2^{\rm eff}(t) = \frac{i}{2}\overline{[\hat{K}_2,\hat{H}_F]}$, or equivalently

$$\hat{H}_2^{\rm eff}(t) = \sum_{m,n}\frac{1}{2\omega_m^2}[[\hat{V}_m,\hat{H}_0],\hat{V}_n^\dagger]e^{i(\omega_m-\omega_n)t+i(\varphi_m-\varphi_n)} + \text{H.c.} \quad \text{(A11)}$$

$\hat{K}_3(t)$ can be obtained by an integration of Eq. (A10) using the expression (A11) for $\hat{H}_2^{\rm eff}(t)$. The third-order contribution to the kick operator is derived in Sec. C in the specific case $\hat{H}_0 = \omega_0\sigma_z$ and $\hat{V}_m = \Omega\sigma_x$.

APPENDIX B: DERIVATION OF THE SECOND-ORDER EFFECTIVE MASTER EQUATION

We provide here additional details on the derivation of the effective master equation, starting from the following expression obtained in Sec. III:

$$\frac{\partial\overline{\rho}}{\partial t} = -i[\hat{H}^{\rm eff}(t), \rho_e] + \partial_t\overline{\delta\rho^{(2)}(t,\tau)} + \partial_\tau\overline{\delta\rho^{(2)}(t,\tau)} + \partial_t\overline{\delta\rho^{(3)}(t,\tau)} + O(\Omega\varepsilon^3). \quad \text{(B1)}$$

Using the evolution of the instantaneous state [Eq. (1) of the main text]; the following expansion of the fast unitary transform in terms of the kick operators,

$$e^{i\hat{K}(t)} = 1 - i\hat{K}_1(t) - \frac{1}{2}\hat{K}_1(t)^2 - i\hat{K}_2(t) - \frac{1}{2}\{\hat{K}_1(t), \hat{K}_2(t)\} + \frac{i}{6}\hat{K}_1(t)^3 - i\hat{K}_3(t) + O\left(\frac{\Omega^4}{\omega^4}\right); \quad \text{(B2)}$$

and the properties $\overline{\hat{K}_m(t)} = 0$ for $m \geqslant 1$ and $\overline{\hat{K}_1(t)^{2n+1}} = 0$ for $n \geqslant 0$ mentioned in the main text, we obtain the following expressions for $\delta\rho^{(2)}(t,\tau), \delta\rho^{(3)}(t,\tau)$:

$$\begin{aligned}\overline{\delta\rho^{(2)}}(t,\tau) &= -\tfrac{1}{2}\{\overline{\hat{K}_1(t)^2}, \rho_e(\tau)\} + \overline{\hat{K}_1(t)\rho_e(\tau)\hat{K}_1(t)},\\ \overline{\delta\rho^{(3)}}(t,\tau) &= \overline{\hat{K}_1(t)\rho_e(\tau)\hat{K}_2(t)} + \overline{\hat{K}_2(t)\rho_e(\tau)\hat{K}_1(t)}\\ &\quad - \tfrac{1}{2}\{\overline{\{\hat{K}_1(t), \hat{K}_2(t)\}}, \rho_e(\tau)\}.\end{aligned} \quad \text{(B3)}$$

Regarding the expression of $\overline{\delta\rho^{(3)}}(t,\tau)$, we have used the fact that the time averaging eliminates the isolated contribution of the fast operator $\hat{K}_3(t)$, and that cubic terms of the kind $\hat{K}_1(t)^3\rho_e(t), \hat{K}_1(t)^2\rho_e(t)\hat{K}_1(t), \ldots$ do not contain low-frequency harmonics and thus also disappear upon time averaging. This property holds for the narrow-bandwidth case $\forall(m,n)\ |\omega_m - \omega_n| < \omega_c \ll \omega$ considered throughout the paper.

Let us first derive the term $\partial_t\overline{\delta\rho^{(2)}}(t,\tau)$. Using Eq. (A4) and the substitution $\rho_e(\tau) = \overline{\rho} + O(\varepsilon^2)$, we find

$$\partial_t\overline{\delta\rho^{(2)}}(t,\tau) = -\frac{1}{2}\{\overline{\{\hat{H}_F(t), \hat{K}_1(t)\}}, \overline{\rho}\} + \overline{\hat{H}_F(t)\overline{\rho}\hat{K}_1(t)} + \overline{\hat{K}_1(t)\overline{\rho}\hat{H}_F(t)} + O(\Omega\varepsilon^3). \quad \text{(B4)}$$

Using Eq. (A5), we compute the second contribution of the right-hand side as

$$\begin{aligned}\overline{\hat{H}_F(t)\overline{\rho}\hat{K}_1(t)} &= \sum_{m,n}\frac{-1}{i\omega_n}\hat{V}_m\overline{\rho}\hat{V}_n^\dagger\overline{e^{i(\omega_m-\omega_n)t}} + \sum_{m,n}\frac{1}{i\omega_n}\hat{V}_m^\dagger\overline{\rho}\hat{V}_n\overline{e^{i(\omega_n-\omega_m)t}}\\ &= \sum_{m,n}\frac{1}{i}\left(\frac{\hat{V}_n^\dagger\overline{\rho}\hat{V}_m}{\omega_m} - \frac{\hat{V}_m\overline{\rho}\hat{V}_n^\dagger}{\omega_n}\right)\overline{e^{i(\omega_m-\omega_n)t}},\end{aligned} \quad \text{(B5)}$$

where we have exchanged the indices m and n in the second term. Similarly, we have

$$\overline{\hat{K}_1(t)\overline{\rho}\hat{H}_F(t)} = \sum_{m,n}\frac{1}{i}\left(\frac{\hat{V}_m\overline{\rho}\hat{V}_n^\dagger}{\omega_m} - \frac{\hat{V}_n^\dagger\overline{\rho}\hat{V}_m}{\omega_n}\right)\overline{e^{i(\omega_m-\omega_n)t}}. \quad \text{(B6)}$$

Summing up both contributions, we finally obtain

$$\begin{aligned}&\overline{\hat{H}_F(t)\overline{\rho}\hat{K}_1(t)} + \overline{\hat{K}_1(t)\overline{\rho}\hat{H}_F(t)}\\ &= \sum_{m,n}\frac{1}{i}\left(\frac{1}{\omega_m} - \frac{1}{\omega_n}\right)\\ &\quad\times e^{i(\varphi_m-\varphi_n)}\overline{e^{i(\omega_m-\omega_n)t}}(\hat{V}_m\overline{\rho}\hat{V}_n^\dagger + \hat{V}_n^\dagger\overline{\rho}\hat{V}_m).\end{aligned} \quad \text{(B7)}$$

The term $\{\overline{\{\hat{H}_F(t), \hat{K}_1(t)\}}, \overline{\rho}\}$ can be obtained along similar lines. Finally, from Eqs. (B1) and (B4), the contribution $\mathcal{L}^{FF}[\overline{\rho}]$ can be expressed as $\mathcal{L}^{FF}[\overline{\rho}] = \partial_t\overline{\delta\rho^{(2)}}(t,\tau)$ and

$$\mathcal{L}^{FF}(\overline{\rho}) = \sum_{m,n}\frac{2\,\overline{e^{i\Delta\omega_{mn}t}}}{i\omega_{mn-}}\left(\hat{V}_m\overline{\rho}\hat{V}_n^\dagger + \hat{V}_n^\dagger\overline{\rho}\hat{V}_m - \frac{1}{2}\{\{\hat{V}_m^\dagger, \hat{V}_n\}, \rho\}\right) + O(\Omega\varepsilon^3), \quad \text{(B8)}$$

with $\Delta\omega_{mn} = \omega_m - \omega_n$ and $1/\omega_{mn-} = \frac{1}{2}(1/\omega_m - 1/\omega_n)$, which yields Eq. (9).

In the following, we assume that $\overline{e^{\pm i(\omega_m-\omega_n)t}} = e^{\pm i(\omega_m-\omega_n)t}$, and derive the term $\mathcal{L}_2^{FSF}[\overline{\rho}] \equiv \partial_\tau\overline{\delta\rho^{(2)}}(t,\tau) + \partial_t\overline{\delta\rho^{(3)}}(t,\tau)$—the replacement of $\rho_e(t)$ by $\overline{\rho}(t)$ is valid up to third-order corrections. We begin by deriving explicitly $\partial_\tau\overline{\delta\rho^{(2)}}(t,\tau)$. Special care is needed with respect to the operator ordering. As an example, we find

$$\begin{aligned}\frac{\partial}{\partial\tau}[\{\overline{\hat{K}_1(t)^2}, \overline{\rho(\tau)}\}] &= -i\{\overline{\hat{K}_1(t)^2}, [\hat{H}^{\rm eff}(t), \overline{\rho(\tau)}]\}\\ &= -i[\hat{H}^{\rm eff}(t), \{\overline{\hat{K}_1(t)^2}, \overline{\rho(\tau)}\}]\\ &\quad - i\{[\overline{\hat{K}_1(t)^2}, \hat{H}^{\rm eff}(t)], \overline{\rho(\tau)}\},\end{aligned} \quad \text{(B9)}$$

where we have used the generic identity $\hat{A}[\hat{B},\hat{C}] = [\hat{A},\hat{B}]\hat{C} + [\hat{B},\hat{A}]\hat{C}$. The other terms can be evaluated in a similar manner:

$$\begin{aligned}\frac{\partial\overline{\delta\rho^{(2)}(t,\tau)}}{\partial\tau} &= -i[\hat{H}^{\rm eff}, \overline{\delta\rho^{(2)}(t)}] - i\left\{\left[\frac{1}{2}\overline{\hat{K}_1(t)^2}, \hat{H}_0\right], \overline{\rho(\tau)}\right\}\\ &\quad - i\overline{[\hat{K}_1(t), \hat{H}_0]\overline{\rho}(\tau)\hat{K}_1(t)}\\ &\quad - i\overline{\hat{K}_1(t)\overline{\rho}(\tau)[\hat{K}_1(t), \hat{H}_0]}.\end{aligned} \quad \text{(B10)}$$

We have taken $\hat{H}^{\rm eff} = \hat{H}_0$ in the dissipative terms, which is valid to the considered order.

We now derive the contribution $\partial_t \overline{\delta\rho^{(3)}(t,\tau)}$. Let us compute the term $\partial_t \overline{\hat{K}_1(t)\rho_e(\tau)\hat{K}_2(t)}$:

$$\frac{\partial}{\partial t}\overline{\hat{K}_1(t)\rho_e(\tau)\hat{K}_2(t)} = \overline{\hat{H}_F(t)\rho_e(\tau)\hat{K}_2(t)} + i\overline{\hat{K}_1(t)\rho_e(\tau)[\hat{K}_1(t),\hat{H}_0]} + \frac{i}{2}\overline{\hat{K}_1(t)\rho_e(\tau)[\hat{K}_1(t),\hat{H}_F(t)]} - \overline{\hat{K}_1(t)\rho_e(\tau)\hat{H}_1^{\mathrm{eff}}(t)}, \tag{B11}$$

where we have used Eqs. (A4) and (A6). We find that the two last contributions of the right-hand side vanish upon averaging, i.e., $\overline{\hat{K}_1\overline{\rho}\hat{H}_1^{\mathrm{eff}}} = \overline{\hat{K}_1\overline{\rho}[\hat{K}_1,\hat{H}_F]} = 0$. Note that the second contribution of the right-hand side cancels a term from $\frac{\partial\overline{\delta\rho^{(2)}(t,\tau)}}{\partial\tau}$. Other contributions to $\partial_t\overline{\delta\rho^{(3)}(t,\tau)}$ are obtained along similar lines, and one obtains a one by one cancellation of the dissipative terms in $\partial_\tau\overline{\delta\rho^{(2)}(t,\tau)}$. Using Eqs. (B1) and (B10), we find

$$\frac{\partial\overline{\rho}}{\partial t} = -i[\hat{H}_{\mathrm{eff}}, \rho_e + \overline{\delta\rho^{(2)}}] + \mathcal{L}^{FF}(\overline{\rho}) + \mathcal{L}^{FSF}(\overline{\rho}) + O(\Omega\varepsilon^3). \tag{B12}$$

By writing $\overline{\rho} = \rho_e + \overline{\delta\rho^{(2)}} + O(\Omega\varepsilon^3)$, the equation above becomes a close equation in $\overline{\rho}$ at the considered order. The contribution coupling the fast and slow quantum dynamics is expressed as

$$\mathcal{L}^{FSF}[\overline{\rho}] = \overline{\hat{H}_F(t)\overline{\rho}(\tau)\hat{K}_2(t)} + \overline{\hat{K}_2(t)\overline{\rho}(\tau)\hat{H}_F(t)} - \frac{1}{2}\{\overline{\{\hat{H}_F(t),\hat{K}_2(t)\}},\overline{\rho}(\tau)\}. \tag{B13}$$

Let us evaluate one of these terms, for instance $\overline{\hat{H}_F(t)\overline{\rho}(\tau)\hat{K}_2(t)}$. From Eq. (A8), the kick operator $\hat{K}_2(t)$ contains contributions oscillating approximately at the Floquet frequency and at twice the Floquet frequency respectively. The latter does not contribute as it vanishes upon time averaging. The considered contribution eventually boils down to

$$\overline{\hat{H}_F(t)\overline{\rho}(\tau)\hat{K}_2(t)} = \sum_{m,n}\left(\frac{1}{i\omega_n^2}\hat{V}_m\overline{\rho}[\hat{V}_n^\dagger,\hat{H}_0] + \frac{1}{i\omega_m^2}\hat{V}_n^\dagger\overline{\rho}[\hat{V}_m,\hat{H}_0]\right)\times e^{i(\omega_m-\omega_n)t}. \tag{B14}$$

Other terms are derived in a similar manner. Gathering all the contributions, we have

$$\mathcal{L}^{FSF}[\overline{\rho}] = -\frac{1}{2i}\sum_{m,n}\left(\frac{1}{\omega_n^2}\{\{\hat{V}_m,[\hat{V}_n^\dagger,\hat{H}_0]\},\overline{\rho}\} + \frac{1}{\omega_m^2}\{\{\hat{V}_n^\dagger,[\hat{V}_m,\hat{H}_0]\},\overline{\rho}\}\right)e^{i(\omega_m-\omega_n)t} + \frac{1}{i}\sum_{m,n}\left[\frac{1}{\omega_n^2}(\hat{V}_m\overline{\rho}[\hat{V}_n^\dagger,\hat{H}_0] + [\hat{V}_n^\dagger,\hat{H}_0]\overline{\rho}\hat{V}_m) + \frac{1}{\omega_m^2}(\hat{V}_n^\dagger\overline{\rho}[\hat{V}_m,\hat{H}_0] + [\hat{V}_m,\hat{H}_0]\overline{\rho}\hat{V}_n^\dagger)\right]e^{i(\omega_m-\omega_n)t},$$

which can be written more concisely as Eq. (10).

APPENDIX C: HIGHER-ORDER CONTRIBUTIONS TO THE EFFECTIVE MASTER EQUATION

In this Appendix, we apply our method to get the next-order terms to improve the accuracy of the effective quantum master equation for larger values of the parameters ε (and larger dissipation strengths), and we provide a few applications related to the examples developed in the paper.

At higher orders, one can no longer substitute ρ_e by $\overline{\rho}$ in the second-order quantum dissipative terms—this would be equivalent to ignoring terms of similar magnitude as the corrections that we seek to obtain. Consequently, we rely on the relation $\rho_e = \overline{\rho} - \overline{\delta\rho^{(2)}(t,\tau)} + O(\varepsilon^3)$ within the second-order dissipative contributions.

Then, the effective master equation can be written as

$$\frac{\partial\overline{\rho}}{\partial t} = -i[\hat{H}_{\mathrm{eff}},\rho_e] + \partial_t\overline{\delta\rho^{(2)}(t,\tau)} + \partial_\tau\overline{\delta\rho^{(2)}(t,\tau)} + \partial_t\overline{\delta\rho^{(3)}(t,\tau)} + \partial_\tau\overline{\delta\rho^{(3)}(t,\tau)} + \partial_t\overline{\delta\rho^{(4)}(t,\tau)} + O(\Omega\varepsilon^4) \tag{C1}$$

where

$$\begin{aligned}\partial_t\overline{\delta\rho^{(2)}(t,\tau)} &= \dot{\mathcal{E}}_2[\overline{\rho} - \mathcal{E}_2[\overline{\rho}]] + O(\Omega\varepsilon^5),\\ \partial_\tau\overline{\delta\rho^{(2)}(t,\tau)} &= -i\mathcal{E}_2[[\hat{H}^{\mathrm{eff}},\overline{\rho} - \mathcal{E}_2[\overline{\rho}]]] + O(\Omega\varepsilon^5),\\ \partial_t\overline{\delta\rho^{(3)}(t,\tau)} &= \dot{\mathcal{E}}_3[\overline{\rho} - \mathcal{E}_2[\overline{\rho}]] + O(\Omega\varepsilon^5).\end{aligned} \tag{C2}$$

The linear maps $\mathcal{E}_m[\rho]$ are defined in a similar way as in Ref. [39]: $\mathcal{E}_m[\rho]$ is associated to the mth-order correction, i.e., these maps are defined by the relation $\mathcal{E}_m[\rho_e] = \overline{\delta\rho^{(m)}}$. From Eqs. (B3) and (B3), we have

$$\mathcal{E}_2[\rho] = -\tfrac{1}{2}\{\overline{\hat{K}_1(t)^2},\rho\} + \overline{\hat{K}_1(t)\rho\hat{K}_1(t)}, \tag{C3}$$

$$\mathcal{E}_3[\rho] = \overline{\hat{K}_1(t)\rho\hat{K}_2(t)} + \overline{\hat{K}_2(t)\rho\hat{K}_1(t)} - \tfrac{1}{2}\{\overline{\{\hat{K}_1(t),\hat{K}_2(t)\}},\rho\}. \tag{C4}$$

The time dependence of the linear maps $\mathcal{E}_m$ comes from the operators $\hat{K}_m(t)$—ρ is used as a simple variable on which the map is applied. We have also used the fact that the term $\dot{\mathcal{E}}_2[\overline{\rho}] \equiv \mathcal{L}^{FF}[\overline{\rho}]$ is already of second order in ε.

1. Third- and fourth-order contributions in the phase noise configuration ($\hat{H}_0 \propto \sigma_z$ and $\hat{H}_F \propto \sigma_z$)

In the specific case of phase noise, one can easily obtain the effective equation up to the fifth order. Indeed, one has $\hat{K}_m = 0$ for $m \geqslant 2$, so the total kick operator is simply $\hat{K}(t) = \hat{K}_1(t)$. Then, the different nonunitary terms arise from an expansion of the unitary operators $e^{\pm i\hat{K}_1(t)}$. Odd powers of the kick operator $\hat{K}_1$ disappear upon time averaging, so that $\mathcal{E}_1 = \mathcal{E}_3 = \mathcal{E}_5 = 0$, and the fourth-order equation is given by

$$\dot{\overline{\rho}} = -i[\hat{H}_{\rm eff}, \overline{\rho}] + \dot{\mathcal{E}}_2[\overline{\rho}] + \dot{\mathcal{E}}_4[\overline{\rho}] - \dot{\mathcal{E}}_2[\mathcal{E}_2[\overline{\rho}]] + O(\Omega\varepsilon^5). \tag{C5}$$

As $\mathcal{E}_5 = 0$, the next-order terms correspond to $\dot{\mathcal{E}}_6[\rho]$ and are thus of fifth order. We have used the fact that for the present case for which the operators $\hat{K}_1$ and $\hat{H}^{\rm eff}$ are both proportional to σ_z, one has $[\hat{H}^{\rm eff}, \mathcal{E}_2[\overline{\rho}]] = 0$. By expanding $e^{i\hat{K}_1(t)}$ in the expression of the evolved quantum state [Eq. (1) of the main text], one finds the fourth-order expansion [we note $\mathcal{E}_4[\rho] \equiv \mathcal{E}_4^{FF}[\rho]$ and this contribution is associated to the kick operators $\hat{K}_1(t)$ alone]:

$$\mathcal{E}_4^{FF}[\rho] = \tfrac{1}{24}\{\overline{\hat{K}_1^4}, \rho\} - \tfrac{1}{6}\overline{\hat{K}_1^3\rho\hat{K}_1} - \tfrac{1}{6}\overline{\hat{K}_1\rho\hat{K}_1^3} + \tfrac{1}{4}\overline{\hat{K}_1^2\rho\hat{K}_1^2}. \tag{C6}$$

We then express the corresponding time derivatives:

$$\begin{aligned}\dot{\mathcal{E}}_4^{FF}[\rho] = {}& \tfrac{1}{6}\big\{\overline{\hat{H}_F\hat{K}_1^3}, \rho\big\} - \tfrac{1}{2}\overline{\hat{H}_F\hat{K}_1^2\rho\hat{K}_1} \\ & - \tfrac{1}{6}\overline{\hat{K}_1^3\rho\hat{H}_F} - \tfrac{1}{2}\overline{\hat{K}_1\rho\hat{H}_F\hat{K}_1^2} - \tfrac{1}{6}\overline{\hat{H}_F\rho\hat{K}_1^3} \\ & + \tfrac{1}{2}\overline{\hat{K}_1^2\rho\hat{K}_1\hat{H}_F} + \tfrac{1}{2}\overline{\hat{H}_F\hat{K}_1\rho\hat{K}_1^2},\end{aligned} \tag{C7}$$

$$\begin{aligned}\dot{\mathcal{E}}_2[\mathcal{E}_2[\rho]] = {}& -\{\overline{\hat{H}_F\hat{K}_1}, -\tfrac{1}{2}\{\overline{\hat{K}_1^2}, \rho\} + \overline{\hat{K}_1\rho\hat{K}_1}\,\} \\ & + \overline{\hat{H}_F\big(-\tfrac{1}{2}\big\{\overline{\hat{K}_1^2}, \rho\big\} + \overline{\hat{K}_1\rho\hat{K}_1}\big)\hat{K}_1} \\ & + \overline{\hat{K}_1\big(-\tfrac{1}{2}\big\{\overline{\hat{K}_1^2}, \rho\big\} + \overline{\hat{K}_1\rho\hat{K}_1}\big)\hat{H}_F},\end{aligned} \tag{C8}$$

where we have used the commutation relation $[\hat{K}_1, \hat{H}_F] = 0$. One can drastically simplify these expressions by writing $\hat{K}_1(t) = F(t)\sigma_z$, $\hat{H}_F(t) = f(t)\sigma_z$, and using the identity $\sigma_z^2 = 1_{2\times 2}$:

$$\dot{\mathcal{E}}_4^{FF}[\rho] = \tfrac{4}{3}\,\overline{f(t)F(t)^3}(\rho - \sigma_z\rho\sigma_z), \tag{C9}$$

$$\dot{\mathcal{E}}_2[\mathcal{E}_2[\rho]] = 2\,\overline{f(t)F(t)}\ \overline{F(t)^2}(\rho - \sigma_z\rho\sigma_z). \tag{C10}$$

We take as in the main text $\hat{V}_m = \Omega\sigma_z$. The functions respectively associated to the fast Hamiltonian and kick operator (A5) are given by $f(t) = \Omega\sum_{m,\varepsilon_m} e^{i\varepsilon_m\omega_m t}$ and $F(t) = \Omega\sum_{m,\varepsilon_m}\frac{\varepsilon_m}{i\omega_m}e^{i\varepsilon_m\omega_m t}$, where for each label m the sum is extended over all the Floquet frequencies, and the label ε_m takes the two values $\{-1, 1\}$.

From previous results, the second-order time-averaged functions read

$$\overline{f(t)F(t)} = \Omega^2\sum_{m,n}\left(\frac{e^{i(\omega_n-\omega_m)t} - e^{i(\omega_m-\omega_n)t}}{i\omega_n}\right), \tag{C11}$$

$$\overline{F(t)^2} = 2\Omega^2\sum_{m,n}\frac{1}{\omega_m\omega_n}e^{i(\omega_m-\omega_n)t}. \tag{C12}$$

Let us evaluate the fourth-order time-averaged function

$$\overline{f(t)F(t)^3} = \Omega^4\sum_{m,\varepsilon_m}\sum_{n,\varepsilon_m}\sum_{p,\varepsilon_p}\sum_{q,\varepsilon_q} \times e^{i\varepsilon_m\omega_m t}\frac{\varepsilon_n e^{i\varepsilon_n\omega_n t}}{i\omega_n}\frac{\varepsilon_p e^{i\varepsilon_p\omega_p t}}{i\omega_p}\frac{\varepsilon_q e^{i\varepsilon_q\omega_q t}}{i\omega_q}\delta_{\varepsilon_m+\varepsilon_n+\varepsilon_p+\varepsilon_q,0}. \tag{C13}$$

The Kronecker symbol $\delta_{\varepsilon_m+\varepsilon_n+\varepsilon_p+\varepsilon_q,0}$ accounts for the time averaging and retains only the slow-rotating contributions such that $\varepsilon_m + \varepsilon_n + \varepsilon_p + \varepsilon_q = 0$. For a given $\varepsilon_m = \pm 1$, there are only three sets $\{\varepsilon_m, \varepsilon_p, \varepsilon_q\}$ in $\{-1, 1\}^3$ that yield $\varepsilon_m + \varepsilon_n + \varepsilon_p + \varepsilon_q = 0$, so that

$$\begin{aligned}\overline{f(t)F(t)^3} = 3\,\Omega^4\sum_{m,n,p,q}\Bigg(& e^{i\omega_m t}\frac{e^{i\omega_n t}}{i\omega_n}\frac{e^{-i\omega_p t}}{i\omega_p}\frac{e^{-i\omega_q t}}{i\omega_q} \\ & - e^{-i\omega_m t}\frac{e^{i\omega_n t}}{i\omega_n}\frac{e^{-i\omega_p t}}{i\omega_p}\frac{e^{i\omega_q t}}{i\omega_q}\Bigg),\end{aligned} \tag{C14}$$

which can be rewritten as

$$\begin{aligned}\overline{f(t)F(t)^3} &= 3\,\Omega^4\sum_{n,p}\left(\frac{e^{i\omega_n t}}{i\omega_n}\frac{e^{-i\omega_p t}}{i\omega_p}\right) \\ &\quad\times\sum_{m,q}\left(e^{i\omega_m t}\frac{e^{-i\omega_q t}}{i\omega_q} - e^{-i\omega_m t}\frac{e^{i\omega_q t}}{i\omega_q}\right) \\ &= 3\,\frac{\overline{F(t)^2}}{2}\ \overline{f(t)F(t)}.\end{aligned} \tag{C15}$$

From Eqs. (C9) and (C10), one finds $\dot{\mathcal{E}}_4^{FF}[\rho] - \dot{\mathcal{E}}_2[\mathcal{E}_2[\rho]] = 0$, i.e., the two third-order terms of the equation cancel each other. As there are no fourth-order terms, the equation presented in the main text is accurate to the fifth order.

2. Third-order contribution to the effective equation in the configuration $\hat{H}_0 \propto \sigma_z$ and $\hat{H}_F \propto \sigma_x$

From Eq. (1) of the main text and expansion of the exponential of kick operators, one obtains the fourth-order contribution to the density matrix $\overline{\delta\rho^{(4)}} = \mathcal{E}_4^{FF}[\overline{\rho}] + \mathcal{E}_4^{FSF}[\overline{\rho}]$ where

$$\mathcal{E}_4^{FSF}[\overline{\rho}] = -\tfrac{1}{2}\{\overline{\{\hat{K}_1, \hat{K}_3\}}, \overline{\rho}\} + \overline{\hat{K}_1\overline{\rho}\hat{K}_3} + \overline{\hat{K}_3\overline{\rho}\hat{K}_1} + \overline{\hat{K}_2\overline{\rho}\hat{K}_2}. \tag{C16}$$

This contribution arises in the presence of noncommuting fast and constant Hamiltonians $[\hat{H}_F(t), \hat{H}_0] \neq 0$. As seen previously, the contribution $\dot{\mathcal{E}}_4^{FF}[\overline{\rho}]$ to the equation of motion is canceled by the term $-\dot{\mathcal{E}}_2[\mathcal{E}_2[\overline{\rho}]]$ (the calculation performed above relies on $[\hat{H}_F, \hat{K}_1] = 0$ and thus still holds here).

Using Eq. (B1), the additional third-order terms to the effective master equation correspond to

$$\mathcal{L}^{(3)}[\overline{\rho}] = i[\hat{H}^{\rm eff}, \overline{\delta\rho^{(3)}}] + \partial_\tau\overline{\delta\rho_3}(t, \tau) + \dot{\mathcal{E}}_4^{FSF}[\overline{\rho}]. \tag{C17}$$

The first term arises from the identity $\rho_e = \overline{\rho} - \overline{\delta\rho^{(2)}} - \overline{\delta\rho^{(3)}} + O(\varepsilon^4)$ in the commutator $[\hat{H}^{\rm eff}, \rho_e]$ of Eq. (C5).

Before evaluating the third-order contributions, we give the expression for the kick operators in this case. With $\hat{H}_0 = \omega_0\sigma_z$

and $\hat{V}_m = \Omega\sigma_x$, the two leading kick operators read $\hat{K}_1(t) = \Omega\sum_m \frac{e^{i\omega_m t}}{i\omega_m}\sigma_x + \text{H.c.}$ and $\hat{K}_2(t) = -2\omega_0\Omega\sum_m \frac{e^{i\omega_m t}}{\omega_m^2}\sigma_y + \text{H.c.}$ From Eqs. (A10) and (A11), one obtains the third-order kick operator:

$$\hat{K}_3(t) = \sum_m \frac{4\omega_0^2\Omega}{i\omega_m^3}(e^{i\omega_m t} - \text{H.c.})\sigma_x - 2\sum_{mn} \frac{\omega_0\Omega^2}{i\omega_m^2(\omega_m+\omega_n)}(e^{i(\omega_m+\omega_n)t} - \text{H.c.})\sigma_z. \quad \text{(C18)}$$

The third-order correction (B3) reads here

$$\overline{\delta\rho_3}(t,\tau) = \overline{\hat{K}_1(t)\rho_e(\tau)\hat{K}_2(t)} + \overline{\hat{K}_2(t)\rho_e(\tau)\hat{K}_1(t)} \quad \text{(C19)}$$

as $\{\hat{K}_1(t), \hat{K}_2(t)\} = 0$ in this specific case.

Let us first evaluate

$$\partial_\tau \overline{\delta\rho^{(3)}}(t,\tau) = -i[\hat{H}_{\text{eff}}, \overline{\delta\rho^{(3)}}] - i\overline{[\hat{K}_1, \hat{H}_0]\overline{\rho}\hat{K}_2} - i\overline{\hat{K}_2\overline{\rho}[\hat{K}_1, \hat{H}_0]} - i\overline{\hat{K}_1\overline{\rho}[\hat{K}_2, \hat{H}_0]} - i\overline{[\hat{K}_2, \hat{H}_0]\overline{\rho}\hat{K}_1} \quad \text{(C20)}$$

where at this order it is valid to use the equality $\rho_e = \overline{\rho}$ and $\hat{H}_{\text{eff}} \equiv \hat{H}_0$ in the dissipative terms. The first term of the right-hand side yields the correct unitary dynamics. We now focus on nonunitary terms:

$$\dot{\mathcal{E}}_4^{FSF}[\overline{\rho}] = -\frac{1}{2}\left\{\overline{\left\{\hat{K}_1, \frac{d\hat{K}_3}{dt}\right\}}, \rho\right\} - \frac{1}{2}\left\{\overline{\left\{\frac{d\hat{K}_1}{dt}, \hat{K}_3\right\}}, \rho\right\} + \overline{\frac{d\hat{K}_1}{dt}\rho\hat{K}_3} + \overline{\hat{K}_1\rho\frac{d\hat{K}_3}{dt}} + \overline{\frac{d\hat{K}_3}{dt}\rho\hat{K}_1} + \overline{\hat{K}_3\rho\frac{d\hat{K}_1}{dt}} + \overline{\frac{d\hat{K}_2}{dt}\rho\hat{K}_2} + \overline{\hat{K}_2\rho\frac{d\hat{K}_2}{dt}}. \quad \text{(C21)}$$

From Eq. (A6), we infer $\frac{d\hat{K}_2}{dt} = i[\hat{K}_1, \hat{H}_0]$ as $[\hat{K}_1, \hat{H}_F] = 0$ for the considered fast driving Hamiltonian. Hence, the terms $\overline{\hat{K}_2\rho\frac{d\hat{K}_2}{dt}}$ and $\overline{\frac{d\hat{K}_2}{dt}\rho\hat{K}_2}$ simply cancel part of the nonunitary contributions of Eq. (C20). Thanks to the relation

$$\frac{d\hat{K}_3}{dt} = i[\hat{K}_2, \hat{H}_0] + \frac{i}{2}[\hat{K}_2, \hat{H}_F] - \hat{H}_{\text{eff}}^{(2)}, \quad \text{(C22)}$$

one can express the contributions of Eq. (C21) as

$$\overline{\frac{d\hat{K}_3}{dt}\rho\hat{K}_1} = i\overline{[\hat{K}_2, \hat{H}_0]\rho\hat{K}_1} + \frac{i}{2}\overline{[\hat{K}_2, \hat{H}_F]\rho\hat{K}_1} - \overline{\hat{H}_{\text{eff}}^{(2)}\rho\hat{K}_1}. \quad \text{(C23)}$$

The first member of the right-hand side cancels nonunitary contributions of Eq. (C20). The second and third terms of the right-hand side yield a null time average. We eventually get the simple relation

$$\mathcal{L}^{(3)}[\overline{\rho}] = -\tfrac{1}{2}\{\overline{\{\hat{H}_F, \hat{K}_3\}}, \rho\} + \overline{\hat{H}_F\rho\hat{K}_3} + \overline{\hat{K}_3\rho\hat{H}_F}, \quad \text{(C24)}$$

where we have used Eq. (A4). These contributions can be evaluated thanks to Eq. (C18). Only the terms rotating at a single Floquet frequency contribute to the time averaging. One obtains $\mathcal{L}^{(3)}[\overline{\rho}] = h(t)(\overline{\rho} - \sigma_x\overline{\rho}\sigma_x)$ with

$$h(t) = 8\omega_0^2\Omega^2 \sum_{m,\varepsilon_m,n,\varepsilon_n} \frac{\varepsilon_m e^{i(\varepsilon_m\omega_m+\varepsilon_n\omega_n)t}}{i\omega_m^3}\delta_{\varepsilon_m+\varepsilon_n,0}. \quad \text{(C25)}$$

In the bichromatic case, we find the additional third-order nonunitary contribution

$$\mathcal{L}^{(3)}[\overline{\rho}] = 16\omega_0^2\Omega^2\left(\frac{1}{\omega_1^3} - \frac{1}{\omega_2^3}\right) \times \sin[(\omega_2-\omega_1)t](\overline{\rho} - \sigma_x\overline{\rho}\sigma_x). \quad \text{(C26)}$$

This "third-order" correction is actually of fourth order in ε. A complete treatment of the fourth-order corrections should also include the following contributions in the right-hand side of the effective equation: $i\mathcal{E}_2[\,[\hat{H}_0, \mathcal{E}_2[\overline{\rho}]]\,] - \dot{\mathcal{E}}_3[\mathcal{E}_2[\overline{\rho}]] + \partial_\tau\overline{\delta\rho^{(4)}(t,\tau)} + \partial_t\overline{\delta\rho^{(5)}(t,\tau)}$. The derivation of these fourth-order terms is a long but straightforward calculation, beyond the scope of this paper.

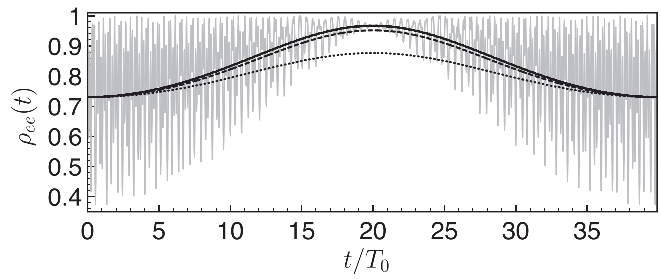

FIG. 5. Higher-order effective master equation vs full quantum evolution. Shown are the instantaneous density-matrix population $\rho_{ee}(t)$ as a function of time (solid gray line, in arbitrary unit T_0), time-convoluted population $\overline{\rho}_{\text{ex}}(t)$ (solid black line) obtained from the exact unitary evolution, and time coarse-grained density-matrix population $\overline{\rho}_{ee}(t)$ obtained from the second-order (dotted black line) and third-order (dashed black line) effective master equation. Parameters: initial density matrix $\rho_0 = |e\rangle\langle e|$, constant and fast Hamiltonians $\hat{H}_0 = \omega_0\sigma_z$ and $\hat{H}_F(t) = \Omega[\cos(\omega_1 t) + \cos(\omega_2 t)]\sigma_x$. Results obtained for the frequencies: $\omega_0 = 0.5 \times (2\pi)/T_0$, $\Omega = 3.5/T_0$, and $\varepsilon \simeq 0.18$. The frequencies ω_1, ω_2, and ω_c are the same as in Figs. 2 and 3 of the main text.

Figure 5 represents the instantaneous (solid gray line) and time-convoluted density-matrix population (solid black line) as a function of time, confronted to the predictions of the second- (black dotted line) and third-order (dashed black line) effective master equations. The latter corresponds to the addition of the contribution $\mathcal{L}_3[\rho]$ (C26) to the right-hand side of the second-order effective master equation (8).

We have used an initial density matrix $\rho_0 = |e\rangle\langle e|$ corresponding to a pure eigenstate of the constant Hamiltonian $\hat{H}_0$. This initial state is also an eigenstate of the effective Hamiltonian $\hat{H}^{\text{eff}}(t)$ in the presence of the Floquet driving. Hence, the unitary part of the quantum dynamics leaves the initial density matrix invariant, and the observed time dependence in the population comes exclusively from the nonunitary contributions. As in Figs. 1–3 of the main text, the initial condition for the considered effective equations is obtained from a convolution with the instantaneous solution $\rho(t)$ as $\overline{\rho}_0 = \int_{-\infty}^{+\infty} dt\, f(-t)\rho(t)$. Figure 5 reveals that the higher-order correction $\mathcal{L}^{(3)}[\overline{\rho}]$ to the effective master equation considerably enhances its accuracy in the prediction of the time coarse-grained dynamics.

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