Generalized physically incoherent operations

Ping Li [1] and Yongming Li [2,*]

[1] *College of Computer Science, Shaanxi Normal University, Xi'an 710062, China*

[2] *School of Mathematics and Statistics, Shaanxi Normal University, Xi'an 710062, China*

(Received 9 January 2024; accepted 8 May 2024; published 18 June 2024)

Quantum coherence is an intrinsic property within a single quantum system. Many different coherent resource theories have been proposed to characterize this property within the framework of resource theory. This difference is based on the difference in the identification of "free" operations. In this article, generalized physically incoherent operations, which are strictly contained in the intersection of incoherent operations and dephasing-covariant operations (dephasing-covariant incoherent incoherent operations), are introduced. We characterize the Kraus operators of dephasing-covariant incoherent incoherent operations and generalized physically incoherent operations, and further study the coherent state transformations under generalized physically incoherent operations. Specifically, we investigate the constraints imposed by the rank of the initial state on the extraction of a pure coherent state. Additionally, we present a sufficient condition for the transformation from a mixed state to a pure state ensemble.

DOI: 10.1103/PhysRevA.109.062424

I. INTRODUCTION

The concept of coherence is ubiquitous in quantum systems, and it manifests the superposition principle of quantum mechanics. This is the cornerstone of nonclassicality of the quantum realm. In particular, quantum coherence has also been recognized as a key component of various quantum technology schemes [1,2].

In the past decade, the framework of resource theory has successfully investigated various nonclassical characteristics of quantum systems [3–5], which are regarded as the constituent resources of some operational tasks. This has led many researchers to apply the framework to rigorously characterize quantum coherence [6–12]. In this approach, one defines coherence as the part that cannot be produced by preserving-incoherent operations [6–11,13–18]. These preserving-incoherent operations are proposed based on various physical and mathematical motivations, such as maximal incoherent operations (MIOs) [6], incoherent operations (IOs) [8], dephasing-covariant operations (DIOs) [19,20], strictly incoherent operations (SIOs) [13,14], and physically incoherent operations (PIOs) [17,18]. At the same time, a quantum state is considered incoherent if it presents a diagonal density matrix on the given reference basis. In particular, IOs can be seen as the operation set of generalized measurements that are preserving-incoherent for each measurement outcome and DIOs constitute the largest class of operations that do not detect the coherence of any input state. We focus on two classes of IOs that overlap with DIOs: dephasing-covariant incoherent incoherent operations (DIIOs) [21] and generalized physically incoherent operations (gPIOs), beyond PIOs and SIOs, respectively (see Fig. 1).

References [18,22–24] show that the so-called majorization condition decides pure state transformation feasibility for SIOs, IOs, and DIOs. This implies that the maximally coherent state $|\Psi_m\rangle = \frac{1}{\sqrt{m}}\sum_{i=1}^{m}|i\rangle$ $(m \geqslant 2)$ can be transformed into any coherent state (including any pure coherent state) in a m-dimensional quantum system. In particular, when the target state is determined to be the maximally coherent state Ψ_m, the transformation is called coherent distillation. The protocols of coherent distillation can be divided into two classes: the asymptotic coherence distillation [14,25–28] and the one-shot coherence distillation [21,24,29–41]. It is worth noting that the foundations of an operational theory of coherence were laid by Winter and Yang [14], and that the IO distillation protocol coincides with the DIO one [28], that is, the operation used in both protocols belongs to the intersection of IOs and DIOs (DIIOs). In Refs. [27,38], the authors show that under both the asymptotic and the (one-shot) deterministic framework, SIOs have great limitations in performing coherent distillation. There are a large number of bound coherent states in coherent resource theory with SIO participation. Therefore, it remains an important open question to explore what the smallest physically motivated set of free operations for manipulating coherence without such hindering operational limitations could be. In Ref. [21], the authors present a specific one-shot distillation protocol, which generalizes the protocol in Theorem 6 of Ref. [14], with operations that belong to DIIOs. This suggests that DIIOs are a candidate for the above problem.

The goal of this article is to investigate a special type of DIIOs: gPIOs. On one hand, gPIOs have significant advantages over SIOs and PIOs under the asymptotic coherence distillation. On the other hand, the operation set of gPIOs is smaller compared to DIIOs. The main contributions of this article are as follows:

*liyongm@snnu.edu.cn

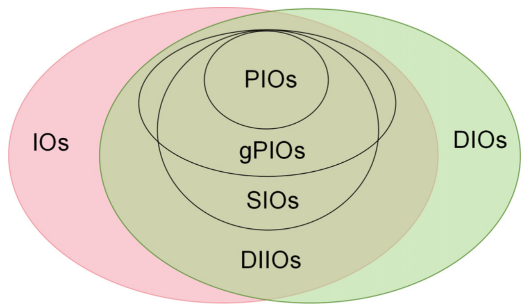

FIG. 1. A heuristic comparison between the six incoherent operations IO/DIO/DIIO/SIO/gPIO/PIO.

(1) For an IO, if a certain incoherent operator has t nonzero elements in a row, then this IO must contain at least t incoherent operators.

(2) For DIIOs, the complete characterization of the IOs that constitute this class of operations is given. Based on this, a class of DIIOs with a simple construction of incoherent operator representation is constructed, and then gPIOs are defined.

(3) It is shown that if the rank of the initial state is greater than the ratio between the rank of the initial state and the target pure state after undergoing complete dephasing, it will prevent the transformation into this desired coherent state.

(4) A sufficient condition for gPIOs to transform a general coherent state into an ensemble of pure coherent states and a new sufficient condition for IOs to transform a general coherent state into a pure coherent state are proposed.

This paper is organized as follows. In Sec. II, we recall some notions of the quantum resource theory of coherence, including IOs and DIOs. In Sec. III, in order to identify suitable physically motivated free operations, we propose gPIOs with Kraus operator representations and corresponding physical implementation scenarios. In Sec. IV, we investigate gPIOs' ability to distill pure coherent states. Finally, in Sec. V we sum up and discuss our results.

II. PRELIMINARIES

Let H be the Hilbert space of a d-dimensional quantum system. A particular basis of H is denoted as $\{|i\rangle\}_{i=1}^{d}$. Specifically, a state σ is said to be incoherent if it is diagonal in the basis $\{|i\rangle\}_{i=1}^{d}$, i.e., $\sigma = \sum_{i=1}^{d} \sigma_i |i\rangle\langle i|$, where the coefficients $\sigma_i \geqslant 0$ form a probability distribution. I is used to represent the set of incoherent states. Any state that is not a diagonal density matrix is defined as a coherent state. For a pure coherent state $|\varphi_{\mathbf{m}}\rangle = \sum_{i=1}^{d} \varphi_i |i\rangle$, we will denote $|\varphi_{\mathbf{m}}\rangle\langle\varphi_{\mathbf{m}}|$ as $\varphi_{\mathbf{m}}$ ($d \geqslant m \geqslant 2$), where the subscript in bold represents the number of nonzero diagonal terms ($\varphi_i \neq 0$). In particular, the maximally coherent state of d-dimension is denoted by $|\Psi_d\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i\rangle$, while we will denote

$$|\Psi_m\rangle = \frac{1}{\sqrt{m}} \sum_{i=1}^{m} |i\rangle, \tag{1}$$

for $d \geqslant m \geqslant 2$, as an m-level maximally coherent state. For arbitrary coherent state ρ, we write $\sqrt{\rho}$ as the column-vector form $(|\rho_1\rangle, |\rho_2\rangle, \ldots, |\rho_d\rangle)$.

Moreover, let Δ denote the completely dephasing map in the basis $\{|i\rangle\}_{i=1}^{d}$, i.e., $\Delta(\cdot) := \sum_{i=1}^{d} |i\rangle\langle i|(\cdot)|i\rangle\langle i|$, and let Π_S denote the incoherent projector with the form $\Pi_S := \sum_{i\in S} |i\rangle\langle i|$ for some subset of indices $S \subseteq [d]$.

An incoherent operation is a completely positive trace-preserving (CPTP) map, expressed as

$$\Lambda(\cdot) = \sum_{\alpha=1}^{N} K_\alpha(\cdot)K_\alpha^\dagger, \tag{2}$$

where the Kraus operators K_α satisfy not only $\sum_{\alpha=1}^{N} K_\alpha^\dagger K_\alpha = \mathbb{I}$ but also $K_\alpha I K_\alpha^\dagger \subseteq I$ for all K_α, i.e., each K_α transforms an incoherent state into an incoherent state, and such K_α is called an incoherent Kraus operator (incoherent operator). Next, we remind the reader of the notion $|V_{j|i}\rangle$ denoted by [18]

$$|V_{j|i}\rangle := \begin{pmatrix} \langle j|K_1|i\rangle \\ \langle j|K_2|i\rangle \\ \vdots \\ \langle j|K_N|i\rangle \end{pmatrix}. \tag{3}$$

The following lemma characterizes Kraus operators belonging to an incoherent operation.

Lemma 1. [18,42–45]

(a) For an incoherent operation $\Lambda = \sum_{\alpha=1}^{N} K_\alpha(\cdot)K_\alpha^\dagger$, the incoherent Kraus operators K_α with the form

$$K_\alpha = \sum_i c_{\alpha i}|f_\alpha(i)\rangle\langle i|$$

satisfy the following equation:

$$\sum_j \langle V_{j|i}|V_{j|i'}\rangle = \delta_{ii'}, \tag{4}$$

where $f_\alpha : \{1, \ldots, d\} \to \{1, \ldots, d\}$.

(b) For an incoherent Kraus operator K_α, there is at most one nonzero element in each column of K_α. In other words, each incoherent Kraus operator can be represented by the following form:

$$K_\alpha = \left(\sum_i c_{\alpha i}|f_\alpha^\star(i)\rangle\langle i|\right)P_\alpha, \tag{5}$$

where $f_\alpha^\star : \{1, \ldots, d\} \to \{1, \ldots, d\}$ is a nondecreasing function, and P_α is a permutation operator.

We note that the coefficients of the incoherent operator $K^\star = \sum_i c_{\alpha i}|f_\alpha^\star(i)\rangle\langle i|$, where $f_\alpha^\star : \{1, \ldots, d\} \to \{1, \ldots, d\}$ is a nondecreasing function, mentioned in Lemma 1(b), are arranged in a ladder type, as shown in Fig. 2.

A dephasing-covariant operation is a CPTP map that possesses covariance with the completely dephasing map, i.e., $\Lambda \circ \Delta = \Delta \circ \Lambda$. The operation class of dephasing-covariant operations can be regarded as inherently classical, since any classical (incoherent) observer is unable to distinguish $\Lambda(\rho)$ from $\Lambda \circ \Delta(\rho)$. For the form of Kraus operators belonging to a dephasing-covariant operation, there is the following lemma.

Lemma 2. [18] For a dephasing-covariant operation $\Lambda = \sum_{\alpha=1}^{N} K_\alpha(\cdot)K_\alpha^\dagger$, there exists a conditional probability

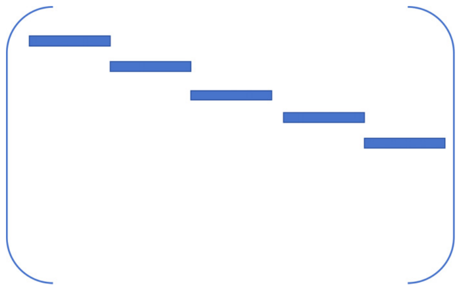

FIG. 2. $K^\star$ is a ladder-form operator.

distribution $\{r_{j|i}\}$ such that

$$\langle V_{j|i}|V_{j'|i}\rangle = r_{j|i}\delta_{jj'}, \tag{6}$$

$$\langle V_{j|i}|V_{j|i'}\rangle = r_{j|i}\delta_{ii'}. \tag{7}$$

In addition, using the linear properties among the column vectors in the square root of coherent state $\sqrt{\rho}$, the problem of outputting any pure coherent state for an individual incoherent operator has the following sufficient and necessary conditions.

Lemma 3. [46] Given a d-dimensional coherent state ρ, write $\sqrt{\rho} = (|\rho_1\rangle, |\rho_2\rangle, \ldots, |\rho_d\rangle)$. Then the following statements are equivalent:

(a) There exists an incoherent operator K_α such that the corresponding output state $\frac{K_\alpha \rho K_\alpha^\dagger}{\mathrm{Tr}(K_\alpha \rho K_\alpha^\dagger)}$ is a pure coherent state $\varphi_{\mathbf{m}}$ $(d \geqslant \mathbf{m} \geqslant 2)$.

(b) There is a subset $S \subseteq \{1, \ldots, d\}$ $(S \neq \emptyset)$ which can be partitioned into m disjoint subsets S_s $(s = 1, \ldots, m)$ such that for all $s = 1, \ldots, m$, the column vectors $|\rho_i\rangle$ $(i \in S_s)$ are nonzero vectors, and the intersection of the subspaces $\mathrm{span}(\{|\rho_i\rangle\}_{i\in S_s})$ includes nonzero vectors, i.e., $\dim[\bigcap_{s=1}^m \mathrm{span}(\{|\rho_i\rangle\}_{i\in S_s})] > 0$.

(c) There is a subset $S \subseteq \{1, \ldots, d\}$ $(S \neq \emptyset)$ which can be partitioned into m disjoint subsets S_s $(s = 1, \ldots, m)$ such that for all $s = 1, \ldots, m$, the column vectors $|\rho_i\rangle$ $(i \in S_s)$ are not only nonzero vectors but also linearly independent, and the intersection of the subspaces $\mathrm{span}(\{|\rho_i\rangle\}_{i\in S_s})$ includes nonzero vectors, i.e., $\dim[\bigcap_{s=1}^m \mathrm{span}(\{|\rho_i\rangle\}_{i\in S_s})] > 0$.

III. GENERALIZED PHYSICALLY INCOHERENT OPERATIONS

According to the definition of DIOs, their intrinsic property is the commutativity with the completely dephasing map, while the intrinsic property of IOs is that there is a Kraus operator representation preserving-incoherent for every operator. Combining these two concepts, a genuine dephasing-covariant incoherent operation is both an incoherent operation and a dephasing-covariant operation. That is DIIOs, which are first proposed in Ref. [21]. Formally, DIIOs have the following definition.

Definition 1. [21] A dephasing-covariant incoherent IO is a CPTP map that admits a Kraus operator representation as $\Lambda(\cdot) = \sum_{\alpha=1}^N K_\alpha(\cdot)K_\alpha^\dagger$, where $\Lambda \circ \Delta = \Delta \circ \Lambda$ and $K_\alpha I K_\alpha^\dagger \subseteq I$ for each α.

Combining Lemma 1 and Lemma 2, we can get the following lemma.

Lemma 4. Let Λ be an incoherent operation with Kraus incoherent operators $\{K_\alpha\}_{\alpha=1}^N$. Λ is a dephasing-covariant incoherent IO if and only if, for all $i \neq i'$, the following equation holds:

$$\langle V_{j|i}|V_{j|i'}\rangle = 0. \tag{8}$$

Proof. According to Lemma 1(b), we obtain that the nonzero entries of vectors $V_{j|i}$ and $V_{j'|i}$ $(j \neq j')$ are in different rows. Thus, $\langle V_{j|i}|V_{j'|i}\rangle = 0$ holds, for all $j \neq j'$. And, Eq. (4) in Lemma 1(a) shows that there is a conditional probability distribution $\{r_{j|i}\}$ that satisfies

$$\langle V_{j|i}|V_{j'|i}\rangle = r_{j|i}\delta_{jj'}. \tag{9}$$

Combining Lemma 2, we get that $\Lambda(\cdot)$ is also a dephasing-covariant operation if and only if the above conditional probability distribution $\{r_{j|i}\}$ satisfies

$$\langle V_{j|i}|V_{j|i'}\rangle = r_{j|i}\delta_{ii'}. \tag{10}$$

Combining Eqs. (9) and (10), we obtain that if an IO Λ satisfies

$$\langle V_{j|i}|V_{j|i'}\rangle = 0 \quad (i \neq i'),$$

then Λ is also a DIO. ■

It should be noted that all CPTP maps satisfying $K_\alpha \circ \Delta = \Delta \circ K_\alpha$ for all α are referred to as SIOs. Therefore, any SIO is a dephasing-covariant incoherent IO, i.e., SIOs $\subset$ DIIOs.

As a supplement, we remind the reader that, for any two IOs Λ_1 and Λ_2, their concatenation $\Lambda_1 \circ \Lambda_2$ and convex combination $p\Lambda_1 + (1-p)\Lambda_2$ $(0 \leqslant p \leqslant 1)$ remain IOs. These properties occur simultaneously in the operation set of DIOs. With this knowledge, it is easy to show that, for any two DIIOs Λ_1 and Λ_2, their concatenation $\Lambda_1 \circ \Lambda_2$ and convex combination $p\Lambda_1 + (1-p)\Lambda_2$ $(0 \leqslant p \leqslant 1)$ remain DIIOs.

We now introduce a class of DIIOs with a simple Kraus operator structure. And, the set of operations induced by this class of operations can be regarded as a generalization of PIOs.

Lemma 5. Let $\Lambda^\star$ be an incoherent operation with Kraus incoherent operators $\{K_\alpha^\star\}_{\alpha=1}^N$. $\Lambda^\star$ is a dephasing-covariant incoherent IO, if there is a permutation operator P such that for all α, the incoherent operator $K_\alpha^\star$ has the following form:

$$K_\alpha^\star = \left(\sum_i c_{\alpha i}\big|f^\star(i)\big\rangle\langle i|\right)P, \tag{11}$$

where $f^\star : \{1, \ldots, d\} \to \{1, \ldots, d\}$ is a nondecreasing function and coefficients $c_{\alpha i}$ satisfy Eq. (4).

Proof. With the definition of Eq. (11), note that all incoherent operators $K_\alpha^\star$ share the same coefficient arrangement, as shown in Fig. 2, with respect to the permutation operator P. In other words, if $f^\star(i) \neq j$, the vectors $V_{j|i}$ are zero vectors. According to Lemma 1(a), we have the following derivation:

$$\sum_{j=1}^d \langle V_{j|i}|V_{j|i'}\rangle = \langle V_{j^\star|i}|V_{j^\star|i'}\rangle = \delta_{ii'}, \tag{12}$$

where $j^\star = f^\star(i) = f^\star(i')$. Therefore, there is a conditional probability distribution $\{r_{j|i}\}$, such that

$$\langle V_{j|i}|V_{j|i'}\rangle = r_{j|i}\delta_{ii'},$$

where if $j = f^\star(i)$, then $r_{j|i} = 1$; otherwise, $r_{j|i} = 0$. According to Lemma 4, we obtain that $\Lambda^\star$ is a DIIO. ■

It should be noted that when any incoherent projection operator Π_S ($S \subseteq \{1, \ldots, d\}$) acts on a DIIO $\Lambda^\star$ formed by the Kraus operators in Lemma 5, on the corresponding subspace $\Pi_S(H)$, the resulting mapping given by

$$K^\star_\alpha = \left(\sum_i c_{\alpha i}\big|f^\star(i)\big\rangle\langle i|\right)P\Pi_S \tag{13}$$

is still a DIIO with the same form. The reason is that there exists a conditional probability distribution $\{r_{j|i}\}_{i\in S}$, such that

$$\langle V_{j|i}|V_{j|i'}\rangle = r_{j|i}\delta_{ii'},$$
$$\langle V_{j'|i}|V_{j|i}\rangle = r_{j|i}\delta_{jj'},$$

where $i, i' \in S$ and if $j = f^\star(i)$, then $r_{j|i} = 1$; otherwise, $r_{j|i} = 0$.

Practically, PIOs are a class of operations obtained by manipulating both the primary and ancillary systems, and these operations do not generate coherence on either system. The Kraus operators belonging to PIOs can be characterized as follows.

Lemma 6. [26] A physically incoherent operation can be thought of as a convex combination of "elementary PIOs" each acting as

$$\Lambda(\cdot) = \sum_{\alpha=1}^{N} U_\alpha \Pi_{S_\alpha}(\cdot)\Pi_{S_\alpha}U_\alpha^\dagger, \tag{14}$$

where $U_\alpha = \sum_i e^{i\theta_i}|P_\alpha(i)\rangle\langle i|$ are incoherent unitary operators and $\{\Pi_{S_\alpha}\}_\alpha$ form a complete set of incoherent projectors $\sum_{\alpha=1}^N \Pi_{S_\alpha} = \mathbb{I}$.

Below we present the definition of gPIOs by using a similar construction in Lemma 6.

Definition 2. A generalized physically incoherent operation can be thought of as a convex combination of "elementary gPIOs" each acting as

$$\Lambda(\cdot) = \sum_{\beta\alpha} K^\star_{\beta\alpha}\Pi_{S_\beta}(\cdot)\Pi_{S_\beta}K^{\star\dagger}_{\beta\alpha}, \tag{15}$$

where $\Lambda^\star_\beta(\cdot) = \sum_\alpha K^\star_{\beta\alpha}(\cdot)K^{\star\dagger}_{\beta\alpha}$ are DIIOs, as shown in Lemma 5, and $\{\Pi_{S_\beta}\}_\beta$ form a complete set of incoherent projectors $\sum_\beta \Pi_{S_\beta} = \mathbb{I}$.

Note that for any two gPIOs Λ_1 and Λ_2, their concatenation $\Lambda_1 \circ \Lambda_2$ and convex combination $p\Lambda_1 + (1-p)\Lambda_2$ ($0 \leqslant p \leqslant 1$) remain gPIOs. Furthermore, it should be noted that the convex combination of elementary gPIOs in Definition 2 includes all PIOs, because incoherent unitary operator $U_\alpha = \sum_i e^{i\theta_i}|P_\alpha(i)\rangle\langle i|$, satisfying the form of Eq. (11).

However, gPIOs do not include all SIOs. For example, the SIOs that implement the conversions between pure coherent states in Ref. [47] do not belong to gPIOs—specifically, when we want to bring a pure coherent state $|\psi\rangle = \sum_{i=1}^d \psi_i|i\rangle$ to another pure coherent state $|\varphi\rangle = \sum_{i=1}^d \varphi_i|i\rangle$ with $\Delta(\psi)^\downarrow \prec \Delta(\varphi)^\downarrow$ using a SIO. Here, the superscript $\downarrow$ represents that all elements are arranged in a nonincreasing order. For two probability vectors $\mathrm{p} = \{\mathrm{p}_i\}$ and $\mathrm{q} = \{\mathrm{q}_i\}(i = 1, \ldots, d)$ arranged in nonincreasing order, p is said to be majorized by q, i.e., $\mathrm{p} \prec \mathrm{q}$ if $\sum_{i=1}^l \mathrm{q}_i \leqslant \sum_{i=1}^l \mathrm{p}_i$ for $l = 1, \ldots, d-1$ and $\sum_{i=1}^d \mathrm{p}_i = 1 = \sum_{i=1}^d \mathrm{q}_i$. Since pure states are extreme points of the set of states, in the task of pure-state transformation, SIO has the following Kraus incoherent operator form:

$$K_\alpha = P_\alpha\left(\sqrt{p_\alpha}\sum_{i=1}^d \frac{a_{\alpha i}}{\psi_i}|i\rangle\langle i|\right),$$

where $P_\alpha(a_{\alpha 1}, a_{\alpha 2}, \ldots, a_{\alpha d})^T = (\varphi_1, \varphi_2, \ldots, \varphi_d)^T$ and $p_\alpha = \mathrm{Tr}(K_\alpha\rho K_\alpha^\dagger)$. Observe that the single operator $P_\alpha(\sum_{i=1}^d \frac{a_{\alpha i}}{\psi_i}|i\rangle\langle i|)$ cannot form a trace-preserving operation, thus, this SIO is not a gPIO. In a sense, we will learn that gPIOs cannot transform one pure coherent state into another pure coherent state, but directly excavate the pure coherence properties inherent in the quantum state itself.

The block coherence theory was introduced in Ref. [6]. We adopt the framework proposed in Refs. [48,49] to physically implement arbitrary gPIO $\Lambda^\star$ in Lemma 5, which is the core operation used to construct gPIOs.

Consider a quantum system associated with a d-dimensional Hilbert space H. One has partition $H = \oplus_{s=1}^m H_s$ into orthogonal subspaces H_s of dimension $\dim(H_s) = d_s$, for which $\sum_{s=1}^m d_s = d$. Correspondingly, one gets a projective measurement $\Pi = \{\Pi_s\}_{s=1}^m$, with each projector satisfying $\Pi_s(H) = H_s$. A state σ^B on H is called block incoherent (BI) with respect to Π if

$$\Pi_s\sigma^B\Pi_{s'} = 0, \quad \forall \quad s \neq s'.$$

A CPTP map Φ is called block incoherent if it admits an expression of Kraus operators $\Phi = \{K_\alpha\}_\alpha$ such that the following equation holds:

$$\Pi_s K_\alpha^B\sigma^B K_\alpha^{B\dagger}\Pi_{s'} = 0, \quad \forall \quad s \neq s'$$

for any block incoherent state σ^B.

The operational protocol concerning a gPIO $\Lambda^\star$, as shown in Lemma 5, consists of three main steps. The first step finds the partition $H = \oplus_{s=1}^m H_{S_s}$ associated with the incoherent projective measurement $\Pi = \{\Pi_{S_s}\}$, where all subsets $S_s \subset \{1, \ldots, d\}$ ($s = 1, \ldots, m$) satisfy $P(i) = s$ ($i \in S_s$) for the permutation operator P in Eq. (11), the second step performs block incoherent operation $\Phi_\alpha = \{K_\alpha^B\}_{\alpha=1}^{\lceil \frac{N}{\min\{d_1,\ldots,d_m\}}\rceil}$, and the third step destructively measures $\Pi = \{|i_1\rangle\langle i_1|_{i_1\in H_{S_1}} + |i_2\rangle\langle i_2|_{i_2\in H_{S_2}} + \cdots + |i_m\rangle\langle i_m|_{i_m\in H_{S_m}}\}$ in every subspaces H_{S_s}. Here, K_1^B has the following form:

$$K_1^B = \begin{pmatrix} |K_1^\star\rangle^{(1)} \\ |K_2^\star\rangle^{(1)} \\ \vdots \\ |K^\star_{\min\{d_1,\ldots,d_m\}}\rangle^{(1)} \\ \Theta \\ \vdots \\ |K_1^\star\rangle^{(m)} \\ |K_2^\star\rangle^{(m)} \\ \vdots \\ |K^\star_{\min\{d_1,\ldots,d_m\}}\rangle^{(m)} \\ \Theta \end{pmatrix} P,$$

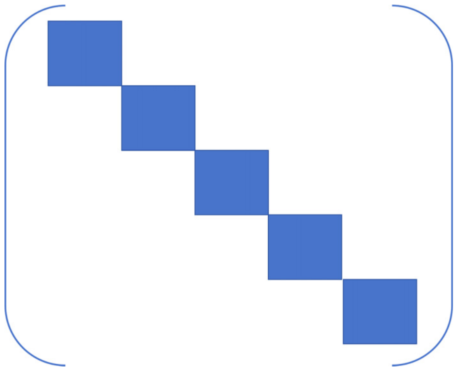

FIG. 3. $K^{\star\dagger}K^{\star}$ is a block matrix.

where $|K^{\star}_{\alpha})^{(s)}$ is the sth row vector of Kraus operator $K^{\star}_{\alpha}P^{\dagger} = \begin{pmatrix} |K^{\star}_{\alpha})^{(1)} \\ |K^{\star}_{\alpha})^{(2)} \\ \vdots \\ |K^{\star}_{\alpha})^{(m)} \end{pmatrix}$ and Θ represents the all-zeros matrix of appropriate size. Other operators K^{B}_{α} $(\alpha \geqslant 2)$ are constructed similarly.

IV. COHERENT STATES' TRANSFORMATION VIA GENERALIZED PHYSICALLY INCOHERENT OPERATIONS

We first provide an observation that shows the relationship between the nonzero coefficients of a single incoherent operator and the number of incoherent operators involved in the entire IO. In other words, for the given incoherent operator, the complementary incoherent operators need to be investigated to meet the requirement of completion identity. For the class of ladder incoherent operator $K^{\star}$ in Eq. (5), the matrix of $K^{\star\dagger}K^{\star}$ is a block matrix, as shown in Fig. 3. Therefore, it is enough to consider the properties on one of the submatrix blocks. We get the following lemma concerning the number of incoherent operators.

Lemma 7. Let incoherent Kraus operators $\{K_{\alpha}\}_{\alpha=1}^{N}$ compose an incoherent operation Λ. Suppose K_1 has the following form of

$$\begin{pmatrix} \underbrace{\star \ \cdots \ \star}_{t} & 0 \cdots 0 \\ \vdots & \vdots \\ 0 \ \cdots \ 0 & \cdots \end{pmatrix} P_1,$$

where $\star$ represents a nonzero number in some row, then the number of incoherent operators $\{K_{\alpha}\}_{\alpha=1}^{N}$ is at least t, i.e., $N \geqslant t$.

The proof is given in Appendix B. Lemma 7 shows that, for a certain incoherent operator K_{α}, which has at most t nonzero elements in the same row, the number of its complementary incoherent operators, which are needed to construct an IO, exceed $t-1$.

To investigate the operational capability of gPIOs in Definition 2, we arrive at the following proposition.

Proposition 1. Given a d-dimensional coherent state ρ, write $\sqrt{\rho} = (|\rho_1\rangle, |\rho_2\rangle, \ldots, |\rho_d\rangle)$. Then, the implication $(a) \Rightarrow (b)$ holds:

(a) There is a subset $S \subseteq \{1, \ldots, d\}$ with $\mathrm{Tr}(\rho[S]) = 1$ which contains a subset S_1 with $|S_1| > \lfloor \frac{|S|}{m} \rfloor$ such that

$$\Delta(\rho)[S] = \sum_{i} \langle \rho_i|\rho_i\rangle |l(i)\rangle\langle l(i)| \tag{16}$$

is positive definite and the following equations hold:

$$\det(\rho[S]) = 0, \tag{17}$$

$$\det(\rho[S_1]) > 0, \tag{18}$$

where $l: \ S \to \{1, \ldots, |S|\}$ is a strictly increasing bijection.

(b) There exists no generalized physically incoherent operation $\Lambda(\cdot)$ such that the corresponding output state $\Lambda(\rho)$ is a pure coherent state $\varphi_{\mathbf{m}}$ $(d \geqslant m \geqslant 2)$.

For the proof, see Appendix C. Proposition 1 shows that if the rank of the initial state ρ is greater than $\frac{\mathrm{rank}[\Delta(\rho)]}{\mathrm{rank}[\Delta(\varphi_{\mathbf{m}})]}$, it will prevent the transformation into this desired coherent state $\varphi_{\mathbf{m}}$, even a pure-coherent state ensemble $\{p_{\alpha}, \varphi^{\alpha}_{\mathbf{m}}\}$. Specifically, this conclusion provides a no-go theory for deterministic coherence distillation [38] under gPIOs, namely, when $\mathrm{rank}(\rho) > \frac{\mathrm{rank}[\Delta(\rho)]}{m}$, one cannot certainly distill the maximally coherent state Ψ_m through gPIOs.

Based on the Lemma 9 (see Appendix D), we have the following theorem, which describes a sufficient condition for obtaining a pure-state ensemble $\{p_{\alpha}, \varphi^{\alpha}_{\mathbf{m}}\}$ $(d \geqslant m \geqslant 2)$ using gPIOs, as shown in Lemma 5.

Theorem 1. Given a d-dimensional coherent state ρ, write $\sqrt{\rho} = (|\rho_1\rangle, |\rho_2\rangle, \ldots, |\rho_d\rangle)$. Then the implication $(a) \Rightarrow (b)$ holds,

(a) There is a subset $S \subseteq \{1, \ldots, d\}$ with $\mathrm{Tr}(\rho[S]) = 1$ which can be partitioned into m $(m \geqslant 2)$ disjoint subsets S_s $(s = 1, \ldots, m)$ with $|S_1| = |S_2| = \cdots = |S_m|$, and $\mathrm{rank}[(|\rho_i\rangle)_{i\in S_1}] = \mathrm{rank}[(|\rho_i\rangle)_{i\in S_2}] = \cdots = \mathrm{rank}[(|\rho_i\rangle)_{i\in S_m}]$ such that

$$\Delta(\rho)[S] = \sum_{i\in S} \langle \rho_i|\rho_i\rangle |l(i)\rangle\langle l(i)| \tag{19}$$

is positive definite and, for $s = 1, \ldots, m-1$, the following equations hold:

$$\rho[S_s]\sqrt{\rho[S_s, S_{s+1}]\rho[S_{s+1}, S_s]} = \sqrt{\rho[S_s, S_{s+1}]\rho[S_{s+1}, S_s]}\rho[S_s], \tag{20}$$

$$\lambda(\rho[S_s])\lambda(\rho[S_{s+1}]) = \lambda(\sqrt{\rho[S_s, S_{s+1}]\rho[S_{s+1}, S_s]})\lambda(\sqrt{\rho[S_{s+1}, S_s]\rho[S_s, S_{s+1}]}), \tag{21}$$

where $l: \ S \to \{1, \ldots, |S|\}$ is a strictly increasing bijection and $\lambda(M)$ is the eigenvalues diagonal matrix of positive-semidefinite matrix M in the corresponding eigenvector basis.

(b) There exists a generalized physically incoherent operation $\Lambda^{\star}(\cdot)$, as shown in Lemma 5, such that the corresponding output state $\Lambda^{\star}(\rho)$ is a pure-state ensemble $\{p_{\alpha}, \ \varphi^{\alpha}_{\mathbf{m}}\}$ $(d \geqslant m \geqslant 2)$.

Proof. Without loss of generality, we assume directly that $\Delta(\rho) = \sum_{i=1}^{d} \langle \rho_i|\rho_i\rangle |i\rangle\langle i| > 0$. According to Lemma 9, there are m $t \times t-$ unitary matrices V_s and m $t \times t-$ diagonal matrices D_s $(t = \frac{d}{m}; \ s = 1, \ldots, m)$ such that $(|\rho_i\rangle)_{i\in S_1}V_1D_1 = (|\rho_i\rangle)_{i\in S_2}V_2D_2 = \cdots = (|\rho_i\rangle)_{i\in S_m}V_mD_m$, where the disjoint subsets satisfy $\bigcup_{s=1}^{m} S_s = \{1, \ldots, d\}$. Therefore, there is a disjoint-subset partition of $S = \{1, \ldots, d\}$ such that ρ has a permutation-equivalence density matrix with the following form:

$$P\rho P^\dagger = \begin{pmatrix} V_1A_{11}V_1^\dagger & V_1A_{12}V_2^\dagger & \cdots & V_1A_{1m}V_m^\dagger \\ V_2A_{21}V_1^\dagger & V_2A_{22}V_2^\dagger & \cdots & V_2A_{2m}V_m^\dagger \\ \vdots & \vdots & & \vdots \\ V_mA_{m1}V_1^\dagger & V_mA_{m2}V_2^\dagger & \cdots & V_mA_{mm}V_m^\dagger \end{pmatrix}, \quad (22)$$

where P is a permutation operator, A_{ij} are diagonal matrices, $A_{ii}A_{jj} = A_{ij}A_{ji}$, and $i, j = 1, \ldots, m$.

Without loss of generality, suppose that

$$\rho = \begin{pmatrix} V_1A_{11}V_1^\dagger & V_1A_{12}V_2^\dagger & \cdots & V_1A_{1m}V_m^\dagger \\ V_2A_{21}V_1^\dagger & V_2A_{22}V_2^\dagger & \cdots & V_2A_{2m}V_m^\dagger \\ \vdots & \vdots & & \vdots \\ V_mA_{m1}V_1^\dagger & V_mA_{m2}V_2^\dagger & \cdots & V_mA_{mm}V_m^\dagger \end{pmatrix}, \quad (23)$$

where A_{ij} are diagonal matrices, $A_{ii}A_{jj} = A_{ij}A_{ji}$, and $i, j = 1, \ldots, m$. We construct a gPIO $\Lambda^\star$, whose incoherent operators $\{K_\alpha^\star\}$ have the following form:

$$K_\alpha^\star = \begin{pmatrix} \langle v_{1\alpha}| & \Theta & \cdots & \Theta \\ \Theta & \langle v_{2\alpha}| & \cdots & \Theta \\ \vdots & \vdots & \ddots & \vdots \\ \Theta & \Theta & \cdots & \langle v_{m\alpha}| \end{pmatrix}, \quad (24)$$

where $|v_{s\alpha}\rangle$ is the αth column vector of V_s, Θ represents the all-zeros matrix of appropriate size, $s = 1, \ldots, m$, and $\alpha = 1, \ldots, t$. That is to say, we express V_s as $(|v_{s1}\rangle, |v_{s2}\rangle, \ldots, |v_{st}\rangle)$.

According to the above definition of incoherent operators $\{K_\alpha^\star\}_{\alpha=1}^m$, we get

$$K_\alpha^\star \rho K_\alpha^{\star\dagger} = \begin{pmatrix} \langle\alpha|A_{11}|\alpha\rangle & \langle\alpha|A_{12}|\alpha\rangle & \cdots & \langle\alpha|A_{1m}|\alpha\rangle \\ \langle\alpha|A_{21}|\alpha\rangle & \langle\alpha|A_{22}|\alpha\rangle & \cdots & \langle\alpha|A_{2m}|\alpha\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle\alpha|A_{m1}|\alpha\rangle & \langle\alpha|A_{m2}|\alpha\rangle & \cdots & \langle\alpha|A_{mm}|\alpha\rangle \end{pmatrix} = \begin{pmatrix} a_\alpha^{11} & a_\alpha^{12} & \cdots & a_\alpha^{1m} \\ a_\alpha^{21} & a_\alpha^{22} & \cdots & a_\alpha^{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_\alpha^{m1} & a_\alpha^{m2} & \cdots & a_\alpha^{mm} \end{pmatrix},$$

where $a_\alpha^{ij} = \langle\alpha|A_{ij}|\alpha\rangle$ and $i, j = 1, \ldots, m$. For all α,

$$\frac{1}{\sum_{s=1}^m a_\alpha^{ss}} \begin{pmatrix} a_\alpha^{11} & a_\alpha^{12} & \cdots & a_\alpha^{1m} \\ a_\alpha^{21} & a_\alpha^{22} & \cdots & a_\alpha^{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_\alpha^{m1} & a_\alpha^{m2} & \cdots & a_\alpha^{mm} \end{pmatrix}$$

is a pure coherent state, due to $A_{ii}A_{jj} = A_{ij}A_{ji}$, i.e., $a_\alpha^{ii}a_\alpha^{jj} = a_\alpha^{ij}a_\alpha^{ji}$ for all $i, j = 1, \ldots, m$.

Finally, we can verify the completion identity of gPIO $\Lambda^\star$ with $\{K_\alpha^\star\}_\alpha$ in H through the following steps. Because

$$K_\alpha^{\star\dagger} K_\alpha^\star = \begin{pmatrix} |v_{1\alpha}\rangle\langle v_{1\alpha}| & \Theta & \cdots & \Theta \\ \Theta & |v_{2\alpha}\rangle\langle v_{2\alpha}| & \cdots & \Theta \\ \vdots & \vdots & \ddots & \vdots \\ \Theta & \Theta & \cdots & |v_{m\alpha}\rangle\langle v_{m\alpha}| \end{pmatrix},$$

we obtain

$$\sum_\alpha K_\alpha^{\star\dagger} K_\alpha^\star = \begin{pmatrix} \sum_\alpha |v_{1\alpha}\rangle\langle v_{1\alpha}| & \Theta & \cdots & \Theta \\ \Theta & \sum_\alpha |v_{2\alpha}\rangle\langle v_{2\alpha}| & \cdots & \Theta \\ \vdots & \vdots & \ddots & \vdots \\ \Theta & \Theta & \cdots & \sum_\alpha |v_{m\alpha}\rangle\langle v_{m\alpha}| \end{pmatrix} = \mathbb{I}.$$

■

The transformation protocol presented in Theorem 1 is consistent with the distillation protocol in Refs. [14,21].

Consider partial coherence theory, which is an extension of coherence theory [50,51]. For a bipartite quantum system AB with Hilbert space $H = H_A \otimes H_B$, partial coherence theory based on Lüders measurement $\Pi^L = \{|i_A\rangle\langle i_A| \otimes \mathbb{I}_B\}$, where $\{i_A\}$ is the fixed incoherent basis of party A. The free states σ^L in partial coherence theory are defined by

$$\sigma^L = \Pi^L(\sigma^L),$$

where $\Pi^L(\cdot) = \sum_{i_A} (|i_A\rangle\langle i_A| \otimes \mathbb{I}_B)(\cdot)(|i_A\rangle\langle i_A| \otimes \mathbb{I}_B)$. And, a CPTP map Φ^L with Kraus operators $\{K_\alpha^L\}$ is called partially incoherent if

$$K_\alpha^L \sigma^L K_\alpha^{L\dagger} = \Pi^L(K_\alpha^L \sigma^L K_\alpha^{L\dagger}),$$

for any partial coherent state σ^L. Meanwhile, it is obvious that Lüders measurement is a projective measurement. Therefore, when the partition of the Hilbert space H is an "isometric" partition, the framework of block coherence is equivalent to the framework of partial coherence. Specifically, for $m|d$ ($d > m \geqslant 2$), we have $G(|i\rangle) = |i_A\rangle|i_B\rangle$, where $i_A = i \pmod m$, $i_B = \lceil \frac{i}{m} \rceil$, and $i = 1, \ldots, d$.

The operation in Theorem 1 can be performed in three main steps. The first step finds the isometric partition $H = \oplus_{s=1}^m H_{S_s}$ associated with disjoint-subset partition $\{S_s\}_{s=1}^m$, the second step performs the unitary operator $U = V_1^\dagger \oplus V_2^\dagger \oplus \cdots \oplus V_m^\dagger$, and the third step destructively measures $\Pi = \{|i_B\rangle\langle i_B|\}$ in system B, for which $\{|i_B\rangle\}$ is the fixed incoherent basis of party B. The above process is consistent with the asymptotic distillation protocol in Theorem 6 of Ref. [14] and the one-shot distillation protocol in Theorem 5 of Ref. [21].

Therefore, Theorem 1 can be seen as the extension of the distillation protocol to precisely distill any pure coherent state. The one-shot distillation protocol in Ref. [21], which can be used to recover the asymptotic limit in Ref. [14], belongs to the operation set of gPIOs, which is strictly contained in the operation set of DIIOs. This means that gPIOs overcome the limitations of SIOs, including PIOs, under the asymptotic coherent distillation task. In this sense, gPIOs can be considered a smaller physically motivated set of coherent resource theory, compared to DIIOs.

In addition, gPIOs can improve SIO-based bound coherence, i.e., gPIOs can convert some SIO-based bound coherent states to pure coherent states. In particular, the operation of Theorem 4 in Ref. [43] is also a gPIO. We illustrate this fact by the following example.

Example 1. Let $\rho = \frac{1}{2}|\varphi_1\rangle\langle\varphi_1| + \frac{1}{2}|\varphi_2\rangle\langle\varphi_2|$, where

$$|\varphi_1\rangle = \frac{1}{\sqrt{5}}\left(1, 0, \frac{4\sqrt{5}}{5}, \frac{2\sqrt{5}}{5}\right)^T,$$

$$|\varphi_2\rangle = \frac{1}{\sqrt{2}}\left(0, 1, \frac{\sqrt{5}}{5}, \frac{2\sqrt{5}}{5}\right)^T.$$

We can obtain

$$\sqrt{\rho} = \frac{1}{\sqrt{2}}\begin{pmatrix} \frac{1}{5} & 0 & \frac{4\sqrt{5}}{25} & \frac{2\sqrt{5}}{25} \\ 0 & \frac{1}{2} & -\frac{\sqrt{5}}{10} & \frac{\sqrt{5}}{5} \\ \frac{4\sqrt{5}}{25} & -\frac{\sqrt{5}}{10} & \frac{37}{50} & \frac{3}{25} \\ \frac{2\sqrt{5}}{25} & \frac{\sqrt{5}}{5} & \frac{3}{25} & \frac{14}{25} \end{pmatrix},$$

and $\text{rank}(\sqrt{\rho}) = 2$. We get the following vectors with $\sqrt{\rho} = (|\rho_1\rangle\ |\rho_2\rangle\ |\rho_3\rangle\ |\rho_4\rangle)$:

$$|\rho_1\rangle = \frac{1}{\sqrt{2}}\left(\frac{1}{5}, 0, \frac{4\sqrt{5}}{25}, \frac{2\sqrt{5}}{25}\right)^T,$$

$$|\rho_2\rangle = \frac{1}{\sqrt{2}}\left(0, \frac{1}{2}, -\frac{\sqrt{5}}{10}, \frac{\sqrt{5}}{5}\right)^T,$$

$$|\rho_3\rangle = \frac{1}{\sqrt{2}}\left(\frac{4\sqrt{5}}{25}, -\frac{\sqrt{5}}{10}, \frac{37}{50}, \frac{3}{25}\right)^T,$$

$$|\rho_4\rangle = \frac{1}{\sqrt{2}}\left(\frac{2\sqrt{5}}{25}, \frac{\sqrt{5}}{5}, \frac{3}{25}, \frac{14}{25}\right)^T.$$

The disjoint subset partition of $\{1, 2, 3, 4\}$ is provided,

$$S_1 = \{1, 2\},\ S_2 = \{3, 4\}.$$

There are two unitary matrices $\mathbb{I}$, $\begin{pmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}$, and two diagonal matrices $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$, $\mathbb{I}$, such that the following equation holds:

$$(|\rho_1\rangle\ |\rho_2\rangle)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = (|\rho_3\rangle\ |\rho_4\rangle)\begin{pmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Here are the corresponding gPIO with Kraus operators $K_0^\star$ and $K_1^\star$, where

$$K_1^\star = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix},$$

$$K_2^\star = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}.$$

We have

$$K_1^\star \rho K_1^{\star\dagger} = \frac{1}{10}\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix},$$

$$K_2^\star \rho K_2^{\star\dagger} = \frac{1}{2}\Psi_2,$$

and $K_1^{\star\dagger}K_1^\star + K_2^{\star\dagger}K_2^\star = \mathbb{I}$.

As a supplement, we show the following equation:

$$\rho = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ 0 & 0 & -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}\begin{pmatrix} \frac{1}{10} & 0 & \frac{2}{10} & 0 \\ 0 & \frac{1}{4} & 0 & \frac{1}{4} \\ \frac{2}{10} & 0 & \frac{4}{10} & 0 \\ 0 & \frac{1}{4} & 0 & \frac{1}{4} \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ 0 & 0 & \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}.$$

In short, the gPIO $\Lambda^\star$ can be regarded as applying a von Neumann measurement on each subspace $H_{S_1} \oplus H_{S_2} \oplus \cdots \oplus H_{S_m}$ of the Hilbert space H, under given disjoint-subset partition $\{S_s\}_{s=1}^m$ of $S = \{1, \ldots, d\}$. When the target states are respectively determined as an arbitrary pure coherent state and the maximally coherent state, Theorem 1 can derive the following two corollaries.

Corollary 1. Given a d-dimensional coherent state ρ, write $\sqrt{\rho} = (|\rho_1\rangle, |\rho_2\rangle, \ldots, |\rho_d\rangle)$. Then the implication $(a) \Rightarrow (b)$ holds:

(a) There is a subset $S \subseteq \{1, \ldots, d\}$ satisfying $\text{Tr}(\rho[S]) = 1$, which can be partitioned into m $(m \geqslant 2)$ disjoint subsets S_s with $|S_1| = |S_2| = \cdots = |S_m| = t$, and m $t \times t$ − unitary matrices V_s $(s = 1, \ldots, m)$ such that

$$\Delta(\rho)[S] = \sum_{i \in S}\langle\rho_i|\rho_i\rangle|l(i)\rangle\langle l(i)| \tag{25}$$

is positive definite and the following equations hold:

$$\Pi_S\rho\Pi_S = P\begin{pmatrix} V_1A_{11}V_1^\dagger & V_1A_{12}V_2^\dagger & \cdots & V_1A_{1m}V_m^\dagger \\ V_2A_{21}V_1^\dagger & V_2A_{22}V_2^\dagger & \cdots & V_mA_{2m}V_m^\dagger \\ \vdots & \vdots & \ddots & \vdots \\ V_mA_{m1}V_1^\dagger & V_mA_{m2}V_2^\dagger & \cdots & V_mA_{mm}V_m^\dagger \end{pmatrix}P^\dagger, \tag{26}$$

where $l:\ S \to \{1, \ldots, |S|\}$ is a strictly increasing bijection, P is a permutation operator, A_{ij} are diagonal matrices, $A_{ii}A_{jj} = A_{ij}A_{ji}$, $\varphi_j\varphi_j^*A_{ii} = \varphi_i\varphi_i^*A_{jj}$, $\sum_{i=1}^m|\varphi_i|^2 = 1$, and $i, j = 1, 2, \ldots, m$.

(b) There exists a generalized physically incoherent operation $\Lambda^\star(\cdot)$, as shown in Lemma 5, such that the corresponding output state $\Lambda^\star(\rho)$ is a pure coherent state $\varphi_{\mathbf{m}}$, which is $|\varphi_{\mathbf{m}}\rangle = \sum_{i=1}^m\varphi_i|i\rangle$ $(d \geqslant m \geqslant 2)$.

Corollary 2. Given a d-dimensional coherent state ρ, write $\sqrt{\rho} = (|\rho_1\rangle, |\rho_2\rangle, \ldots, |\rho_d\rangle)$. Then the implication $(a) \Rightarrow (b)$ holds:

(a) There is a subset $S \subseteq \{1, \ldots, d\}$ satisfying $\text{Tr}(\rho[S]) = 1$, which can be partitioned into m disjoint subsets S_s with

$|S_1| = |S_2| = \cdots = |S_m|$, such that

$$\Delta(\rho)[S] = \sum_{i\in S} \langle \rho_i|\rho_i\rangle |l(i)\rangle\langle l(i)| \quad (27)$$

is positive definite and, for $s = 1, \ldots, m-1$, the following equations hold:

$$\rho[S_s] = \sqrt{\rho[S_s, S_{s+1}]\rho[S_{s+1}S_s]}, \quad (28)$$

where $l:\ S \to \{1, \ldots, |S|\}$ is a strictly increasing bijection.

(b) There exists a generalized physically incoherent operation $\Lambda^\star(\cdot)$, as shown in Lemma 5, such that the corresponding output state $\Lambda^\star(\rho)$ is the maximally coherent state $\Psi_{\mathbf{m}}$ ($d \geqslant m \geqslant 2$).

Compared to the operational protocol of Theorem 1, the third step involved in Corollaries 1 and 2 is to take the partial trace over system B.

Formally, a SIO $\Lambda^\alpha_{\rm SIO}$ can be written as

$$\Lambda^{\rm SIO}_\alpha(\cdot) = \sum_\beta P_{\beta\alpha} D_{\beta\alpha}(\cdot) D^\dagger_{\beta\alpha} P^\dagger_{\beta\alpha}, \quad (29)$$

where $P_{\beta\alpha}$ are permutation operators and $D_{\beta\alpha}$ are diagonal matrices. Then, we can give an IO Λ, which is neither a gPIO or a DIIO, with the following form:

$$\Lambda(\cdot) = \sum_\alpha \Lambda^{\rm SIO}_\alpha\big[K^\star_\alpha(\cdot)K^{\star\dagger}_\alpha\big], \quad (30)$$

where $\Lambda^{\rm SIO}_\alpha$ are SIOs and $\Lambda^\star(\cdot) = \sum_\alpha K^\star_\alpha(\cdot)K^{\star\dagger}_\alpha$ is a gPIO in Lemma 5.

The following theorem characterizes a sufficient condition of a mixed state to a pure state transformation by an IO $\Lambda(\cdot)$, as shown in Eq. (30).

Theorem 2. Given a d-dimensional coherent state ρ, write $\sqrt{\rho} = (|\rho_1\rangle, |\rho_2\rangle, \ldots, |\rho_d\rangle)$. Then the implication $(a) \Rightarrow (b)$ holds:

(a) There is a subset $S \subseteq \{1, \ldots, d\}$ with $\mathrm{Tr}(\rho[S]) = 1$, which can be partitioned into m ($d \geqslant m \geqslant 2$) disjoint subsets S_s with $|S_1| = |S_2| = \cdots = |S_m|$, and $\mathrm{rank}[(|\rho_i\rangle)_{i\in S_1}] = \mathrm{rank}[(|\rho_i\rangle)_{i\in S_2}] = \cdots = \mathrm{rank}[(|\rho_i\rangle)_{i\in S_m}]$, such that

$$\Delta(\rho)[S] = \sum_{i\in S} \langle \rho_i|\rho_i\rangle |l(i)\rangle\langle l(i)| \quad (31)$$

is positive definite and, for $s = 1, \ldots, m-1$, the following equations hold:

$$\rho[S_s]\sqrt{\rho[S_s, S_{s+1}]\rho[S_{s+1}, S_s]} = \sqrt{\rho[S_s, S_{s+1}]\rho[S_{s+1}, S_s]} \times \rho[S_s], \quad (32)$$

$$\lambda(\rho[S_s])\lambda(\rho[S_{s+1}]) = \lambda(\sqrt{\rho[S_s, S_{s+1}]\rho[S_{s+1}, S_s]})\lambda(\sqrt{\rho[S_{s+1}, S_s]\rho[S_s, S_{s+1}]}), \quad (33)$$

where $l:\ S \to \{1, \ldots, |S|\}$ is a strictly increasing bijection and $\lambda(M)$ is the eigenvalues diagonal matrix of positive-semidefinite matrix M in the corresponding eigenvector basis.

(b) There exists an incoherent operation $\Lambda(\cdot)$ such that the corresponding output state $\Lambda(\rho)$ is a pure coherent state $\varphi_{\mathbf{n}}$ ($m \geqslant n \geqslant 2$).

Proof. Without loss of generality, suppose that

$$\rho = \begin{pmatrix} V_1A_{11}V_1^\dagger & V_1A_{12}V_2^\dagger & \cdots & V_1A_{1m}V_m^\dagger \\ V_2A_{21}V_1^\dagger & V_2A_{22}V_2^\dagger & \cdots & V_2A_{2m}V_m^\dagger \\ \vdots & \vdots & & \vdots \\ V_mA_{m1}V_1^\dagger & V_mA_{m2}V_2^\dagger & \cdots & V_mA_{mm}V_m^\dagger \end{pmatrix}, \quad (34)$$

where A_{ij} are diagonal matrices and $A_{ii}A_{jj} = A_{ij}A_{ji},\ i, j = 1, \ldots, m$. According to Theorem 1, we construct a gPIO $\Lambda^\star$, whose incoherent operators $\{K^\star_\alpha\}$ have the following form:

$$K^\star_\alpha = \begin{pmatrix} \langle v_{1\alpha}| & \Theta & \cdots & \Theta \\ \Theta & \langle v_{2\alpha}| & \cdots & \Theta \\ \vdots & \vdots & \ddots & \vdots \\ \Theta & \Theta & \cdots & \langle v_{m\alpha}| \end{pmatrix}, \quad (35)$$

where $|v_{s\alpha}\rangle$ is the αth column vector of V_s, Θ represents the all-zeros matrix of appropriate size, $s = 1, \ldots, m$, and $\alpha = 1, \ldots, t$. That is to say, we express V_s as $(|v_{s1}\rangle, |v_{s2}\rangle, \ldots, |v_{st}\rangle)$.

According to the above definition of incoherent operators $\{K^\star_\alpha\}^t_{\alpha=1}$, we get

$$K^\star_\alpha\rho K^{\star\dagger}_\alpha = \begin{pmatrix} a^{11}_\alpha & a^{12}_\alpha & \cdots & a^{1m}_\alpha \\ a^{21}_\alpha & a^{22}_\alpha & \cdots & a^{2m}_\alpha \\ \vdots & \vdots & \ddots & \vdots \\ a^{m1}_\alpha & a^{m2}_\alpha & \cdots & a^{mm}_\alpha \end{pmatrix},$$

where $a^{ij}_\alpha = \langle\alpha|A_{ij}|\alpha\rangle$ and $i, j = 1, \ldots, m$.

For all α,

$$\frac{1}{\sum_{s=1}^m a^{ss}_\alpha}\begin{pmatrix} a^{11}_\alpha & a^{12}_\alpha & \cdots & a^{1m}_\alpha \\ a^{21}_\alpha & a^{22}_\alpha & \cdots & a^{2m}_\alpha \\ \vdots & \vdots & \ddots & \vdots \\ a^{m1}_\alpha & a^{m2}_\alpha & \cdots & a^{mm}_\alpha \end{pmatrix}$$

is a pure coherent state, due to $A_{ii}A_{jj} = A_{ij}A_{ji}$, i.e., $a^{ii}_\alpha a^{jj}_\alpha = a^{ij}_\alpha a^{ji}_\alpha$ for $i, j = 1, \ldots, m$. Because all probability distributions of m entries in decreasing order form a complete lattice under majorization [52,53], there is a coherent pure state φ_n satisfying the condition of

$$(\langle\alpha|A_{11}|\alpha\rangle, \langle\alpha|A_{22}|\alpha\rangle, \ldots, \langle\alpha|A_{mm}|\alpha\rangle)^\downarrow \prec \Delta(\varphi_{\mathbf{n}})^\downarrow,$$

such that there is a SIO $\Lambda^{\rm SIO}_\alpha$ with $\{K_{\beta\alpha}\}_\beta$ in H to realize $K_\alpha\rho K^\dagger_\alpha \to \varphi_{\mathbf{n}}$ definitively.

Finally, we can verify the completion identity of IO Λ with $\{K_{\beta\alpha}K_\alpha^\star\}_{\alpha\beta}$ in H through the following steps. Because

$$K_\alpha^{\star\dagger}\left(\sum_\beta K_{\beta\alpha}^\dagger K_{\beta\alpha}\right)K_\alpha^\star = K_\alpha^{\star\dagger}\mathbb{I}K_\alpha^\star = \begin{pmatrix} |v_{1\alpha}\rangle\langle v_{1\alpha}| & \Theta & \cdots & \Theta \\ \Theta & |v_{2\alpha}\rangle\langle v_{2\alpha}| & \cdots & \Theta \\ \vdots & \vdots & \ddots & \vdots \\ \Theta & \Theta & \cdots & |v_{m\alpha}\rangle\langle v_{m\alpha}| \end{pmatrix},$$

we obtain

$$\sum_\alpha K_\alpha^{\star\dagger}\left(\sum_\beta K_{\beta\alpha}^\dagger K_{\beta\alpha}\right)K_\alpha^\star = \sum_\alpha K_\alpha^{\star\dagger}K_\alpha^\star = \begin{pmatrix} \sum_\alpha |v_{1\alpha}\rangle\langle v_{1\alpha}| & \Theta & \cdots & \Theta \\ \Theta & \sum_\alpha |v_{2\alpha}\rangle\langle v_{2\alpha}| & \cdots & \Theta \\ \vdots & \vdots & \ddots & \vdots \\ \Theta & \Theta & \cdots & \sum_\alpha |v_{m\alpha}\rangle\langle v_{m\alpha}| \end{pmatrix} = \mathbb{I}.$$

■

Theorem 2 provides a sufficient condition for transforming a general quantum state into a pure coherent state through IOs. It covers Theorem 2 in Ref. [46] that presents sufficient conditions for the conversion. We illustrate the strategy in Theorem 2 by the following example.

Example 2. Let $\rho = \frac{1}{2}|\varphi_1\rangle\langle\varphi_1| + \frac{1}{2}|\varphi_2\rangle\langle\varphi_2|$, where

$$|\varphi_1\rangle = \frac{1}{\sqrt{5}}\left(1, 0, \frac{4}{5}, \frac{2}{5}\right)^T,$$

$$|\varphi_2\rangle = \frac{1}{\sqrt{2}}\left(0, 1, \frac{\sqrt{5}}{5}, \frac{2\sqrt{5}}{5}\right)^T.$$

Here are the IO with Kraus operators $K_0,\ K_1$, and K_2, where

$$K_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix},$$

$$K_2 = \begin{pmatrix} 0 & \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix},$$

$$K_3 = \begin{pmatrix} \frac{1}{\sqrt{5}} & 0 \\ 0 & \frac{2}{\sqrt{5}} \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}.$$

Thus, we have

$$K_1\rho K_1^\dagger = \frac{1}{10}\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix},$$

$$K_2\rho K_2^\dagger = \frac{1}{20}\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix},$$

$$K_3\rho K_3^\dagger = \frac{1}{20}\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix},$$

and $K_1^\dagger K_1 + K_2^\dagger K_2 + K_3^\dagger K_3 = \mathbb{I}$.

V. CONCLUSIONS

In summary, we have defined a new preserving-incoherent operations, i.e., gPIO, and investigated the mixed state to pure state transformation by using gPIOs. In the physical implementation of gPIOs, it is considered free to partition the subsystems of a quantum system. We have shown that there is an obstacle for PIOs to output a pure coherent state caused by the rank of the initial state. We have proved a sufficient condition for the conversion, where a pure state ensemble $\{p_\alpha,\ \varphi_{\mathbf{m}}^\alpha\}$ is selected as the output state, via the gPIOs. Consequently, the corresponding sufficient conditions for extracting both a general pure coherent state and the maximally coherent state Ψ_m ($m \geqslant 2$) under the gPIOs are presented. Particularly, we have shown the formal unity of these operations for extracting pure states with the well-known two coherent distillation protocols [14,21]. Furthermore, our work advances the research of the mixed state to pure state transformation under IOs. We have reached a sufficient condition for the mixed state to pure state transformation using IOs. In addition to further exploration of the operational properties of gPIOs, for DIIOs, we have another question of interest, namely, whether DIIOs as a whole can be regarded as the generalization of SIOs.

ACKNOWLEDGMENTS

We sincerely thank the associate editor and anonymous reviewers for their valuable time and insights, which have made important contributions to the improvement of this article. This paper was supported by the National Natural Science Foundation of China (Grants No. 12071271, No. 11671244, and No. 62171266.

APPENDIX A: LEMMA 8

Here, we need to extend detailed discussions about the characters of $\sqrt{\rho} = (|\rho_1\rangle, |\rho_2\rangle, \ldots, |\rho_d\rangle)$. Besides $\rho = \sqrt{\rho}^\dagger\sqrt{\rho} = (\langle\rho_i|\rho_j\rangle)_{i,j\in\{1,\ldots,d\}}$, the following lemma gives general properties of these column vectors $|\rho_1\rangle, |\rho_2\rangle, \ldots, |\rho_d\rangle$.

Lemma 8. [54] Let $|v_1\rangle, \ldots, |v_d\rangle$ be vectors in an inner product space V with inner product $\langle\cdot|\cdot\rangle$, and let $G := (\langle v_i|v_j\rangle)_{i,j=1,\ldots,d}$, named Gram matrix. Then,

(a) G is Hermitian and positive semidefinite;

(b) G is positive definite if and only if the vectors $|v_1\rangle, \ldots, |v_d\rangle$ are linearly independent.

APPENDIX B: THE PROOF OF LEMMA 7

Here, we present a detailed proof of Lemma 7.

Since $\sum_{\alpha=1}^{N} K_\alpha^\dagger K_\alpha = \sum_{\alpha=1}^{N} P_1 K_\alpha^\dagger K_\alpha P_1^\dagger = \mathbb{I}$, without loss of generality, we assume that the incoherent operator K_1 have the following form:

$$\begin{pmatrix} \underbrace{\star \ \cdots \ \star}_{t} & 0 \cdots 0 \\ \vdots & \vdots \\ 0 \ \cdots \ 0 & \cdots \end{pmatrix}.$$

Then, according to Lemma 1(b), we obtain that the nonzero entries of vectors $|V_{j|i}\rangle$ and $|V_{j'|i}\rangle$, $(j \neq j')$, are in different rows. Thus, $\langle V_{j|i}|V_{j'|i}\rangle = 0$ holds for all $j \neq j'$. Let $|V_i\rangle = \sum_{j=1}^{d} |V_{j|i}\rangle$, then we will prove that vectors $\{|V_i\rangle\}_{i=1}^{t}$ are linearly independent. Equipped with this property, we can get that each vector $|V_i\rangle$ has at least t elements. By the definition of Eq. (3), we prove that the number of the incoherent operators belonging to the IO Λ, which includes the incoherent operator K_1, exceed t.

Because of Lemma 1(a), a CPTP map Λ is an IO, and we have

$$\sum_{j=1}^{d} \langle V_{j|i}|V_{j|i'}\rangle = \delta_{ii'}. \quad \text{(B1)}$$

On the one hand, this means the vectors $\{|V_i\rangle\}_{i=1}^{d}$ are unit vectors, i.e., $\langle V_i|V_i\rangle = \sum_{j=1}^{d}\langle V_{j|i}|V_{j|i}\rangle = 1$. On the other hand, for $i \neq i'$, $|V_i\rangle$ and $|V_{i'}\rangle$ are linearly independent, since the product of both vectors' entry in the first row, i.e., the nonzero entry represented by $\star$, must be removed by the sum of the product of some other rows' entries, due to $\sum_{j=1}^{d}\langle V_{j|i}|V_{j|i'}\rangle = 0$.

Then, we calculate the Gram determinant of the vectors $\{|V_i\rangle\}_{i=1}^{t}$,

$$\det(V^\dagger V) = \det\begin{pmatrix} 1 & \langle V_1|V_2\rangle & \cdots & \langle V_1|V_t\rangle \\ \langle V_2|V_1\rangle & 1 & \cdots & \langle V_2|V_t\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle V_t|V_1\rangle & \langle V_t|V_2\rangle & \cdots & 1 \end{pmatrix}_{t\times t} = \det\begin{pmatrix} \langle V_2|(\mathbb{I}-|V_1\rangle\langle V_1|)|V_2\rangle & \langle V_2|(\mathbb{I}-|V_1\rangle\langle V_1|)|V_3\rangle & \cdots \\ \langle V_3|(\mathbb{I}-|V_1\rangle\langle V_1|)|V_2\rangle & \langle V_3|(\mathbb{I}-|V_1\rangle\langle V_1|)|V_3\rangle & \cdots \\ \vdots & \vdots & \\ \langle V_t|(\mathbb{I}-|V_1\rangle\langle V_1|)|V_2\rangle & \langle V_t|(\mathbb{I}-|V_1\rangle\langle V_1|)|V_3\rangle & \cdots \end{pmatrix}_{(t-1)\times(t-1)},$$

where the second equation comes from diagonalizing the first row and the first column of matrix $V^\dagger V$ and $\langle V_2|V_i\rangle - \langle V_2|V_1\rangle\langle V_1|V_i\rangle = \langle V_2|(\mathbb{I}-|V_1\rangle\langle V_1|)|V_i\rangle$, for all $i = 1, \ldots, d$.

Let $|V_i^1\rangle = \frac{(\mathbb{I}-|V_1\rangle\langle V_1|)|V_i\rangle}{\sqrt{\langle V_i|(\mathbb{I}-|V_1\rangle\langle V_1|)|V_i\rangle}}$, $i = 2, \ldots, t$. These vectors $\{|V_i^1\rangle\}_{i=2}^{t}$ are unit vectors and have nonzero numbers in the first row. Note that for $i \neq i'$, $|V_i^1\rangle$ and $|V_{i'}^1\rangle$ are also linearly independent, since

$$\langle V_i|(\mathbb{I}-|V_1\rangle\langle V_1|)^2|V_{i'}\rangle = \langle V_i|(\mathbb{I}-|V_1\rangle\langle V_1|)|V_{i'}\rangle = \langle V_i|V_{i'}\rangle - \langle V_i|V_1\rangle\langle V_1|V_{i'}\rangle,$$

where the product of both vectors' first entry will be removed by the sum of the product of some other rows' entries.

Thus, we establish the following equation by deduction:

$$\det(V^\dagger V) = \prod_{i=2}^{t}\langle V_i|(\mathbb{I}-|V_1\rangle\langle V_1|)|V_i\rangle \det\begin{pmatrix} 1 & \langle V_2^1|V_3^1\rangle & \cdots & \langle V_2^1|V_t^1\rangle \\ \langle V_3^1|V_2^1\rangle & 1 & \cdots & \langle V_3^1|V_t^1\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle V_t^1|V_2^1\rangle & \langle V_t^1|V_3^1\rangle & \cdots & 1 \end{pmatrix}$$

$$= \prod_{i=2}^{t}\langle V_i|(\mathbb{I}-|V_1\rangle\langle V_1|)|V_i\rangle \prod_{i=3}^{t}\langle V_i^1|(\mathbb{I}-|V_2^1\rangle\langle V_2^1|)|V_i^1\rangle \cdots \prod_{i=t}^{t}\langle V_i^{t-2}|(\mathbb{I}-|V_{t-1}^{t-2}\rangle\langle V_{t-1}^{t-2}|)|V_i^{t-2}\rangle > 0,$$

where the last inequality comes from the Cauchy-Schwarz inequality. The important detail in the above process is that the first vector in each of the sets $\{|V_i\rangle\}_{i=1}^{t}$, $\{|V_i^1\rangle\}_{i=2}^{t}$, ..., $\{|V_i^{t-1}\rangle\}_{i=t}^{t}$ needs to have the minimum absolute value among the first-row elements of the vectors in each corresponding set. One can achieve this through a permutation operator. Simultaneously, this will not change the value of the determinant of $V^\dagger V$, because the elementary transformations applied to the rows of the determinant need to be applied again to the corresponding columns, i.e., $\det(V^\dagger V) = \det(P^T V^\dagger V P)$, where P is a permutation operator. Therefore, it is ensured that the first-row element of each vector in each set is nonzero. For example, if the first-row element of vector $|V_1\rangle$ has the minimum absolute value among the first-row elements of all vectors $\{|V_i\rangle\}_{i=1}^{t}$, then, according to $(\mathbb{I}-|V_1\rangle\langle V_1|)|V_i\rangle = |V_i\rangle - t|V_1\rangle$, where $t = \langle V_1|V_i\rangle$, for which $1 > |t| > 0$, and $i = 2, \ldots, t$, the first-row elements of all vectors $\{|V_i^1\rangle\}_{i=2}^{t}$ are nonzero.

According to Lemma 8(b), we have proved that the vectors $|V_i\rangle_{i=1}^{t}$ are linearly independent.

APPENDIX C: THE PROOF OF PROPOSITION 1

We proceed to prove Proposition 1.

Without loss of generality, we assume directly that $\Delta(\rho) = \sum_{i=1}^{d}\langle\rho_i|\rho_i\rangle|i\rangle\langle i| > 0$. Here, $S = \{1, \ldots, d\}$ and $|S| = d$.

First, we consider the presentation of gPIOs in Lemma 5. Since the incoherent operators $K_\alpha^\star$ of the gPIO $\Lambda^\star(\cdot)$ in Lemma 5 have the same ladder form under the permutation operator P, the construction of these incoherent operators $K_\alpha^\star$ requires sharing the same disjoint-subset partition. According to Lemma 3, we discuss the following two cases separately.

According to Lemma 3(c), consider that there is a disjoint-subset partition of $S = \{1, \dots, d\}$, i.e., S can be partitioned into m disjoint subsets S_s $(s = 1, \dots, m)$, such that for all $s = 1, \dots, m$, the column vectors of $|\rho_i\rangle$, $(i \in S_s)$ are not only nonzero vectors but also linearly independent, and the intersection of the spaces span$[\{|\rho_i\rangle\}\ (i \in S_s)]$ includes nonzero vectors, i.e.,

$$\dim\left\{\bigcap_{s=1}^{m} \text{span}[\{|\rho_i\rangle\}\ (i \in S_s)]\right\} > 0.$$

To output the pure coherent state $\varphi_{\mathbf{m}}$, there are m nonzero vectors $|x_s\rangle = (x_{\mathbf{s}1}, x_{\mathbf{s}2}, \dots, x_{\mathbf{s}t_s})^T$ such that the continued equality $(|\rho_i\rangle)_{i\in S_1}|x_1\rangle = (|\rho_i\rangle)_{i\in S_2}|x_2\rangle = \cdots = (|\rho_i\rangle)_{i\in S_m}|x_m\rangle$ holds. The corresponding incoherent operator, which outputs the pure coherent state $\varphi_{\mathbf{m}}$, has the following form (see the proof of Definition 2 in Ref. [46]):

$$K_\alpha = \begin{pmatrix} y_{\mathbf{1}1}^* & \cdots & y_{\mathbf{1}t_1}^* & \Theta & & \cdots & & \Theta & \\ & \Theta & & y_{\mathbf{2}1}^* & \cdots & y_{\mathbf{2}t_2}^* & \cdots & \Theta & \\ & \vdots & & & \vdots & & & \vdots & \\ & \Theta & & & \Theta & & \cdots & y_{\mathbf{m}1}^* & \cdots & y_{\mathbf{m}t_m}^* \end{pmatrix} P,$$

where P is a permutation operator, $\text{rank}(K_\alpha) = m$, Θ represents the all-zeros matrix of appropriate size, and $(y_{\mathbf{s}1}, y_{\mathbf{s}2}, \dots, y_{\mathbf{s}t_s})^T = c_s(x_{\mathbf{s}1}, x_{\mathbf{s}2}, \dots, x_{\mathbf{s}t_s})^T$. Here, the vector $|c\rangle = (c_1, \dots, c_m)^T$ is linearly dependent on the vector $|\varphi_{\mathbf{m}}\rangle$ and there is a bijection

$$f : \{1, \dots, d\} \to \{\mathbf{1}1, \dots, \mathbf{1}t_1, \mathbf{2}1, \dots, \mathbf{2}t_2, \dots, \mathbf{r}1, \dots, \mathbf{m}t_m\}$$

such that $f(i) = \mathbf{s}j$, where $i = 1, \dots, d,\ j > 0$, and the boldface type of $s = 1, \dots, m$ is used to mark the partition. Based on Lemma 7, in order to construct a complete gPIO $\Lambda^\star(\cdot)$, for each linear system

$$(|\rho_i\rangle)_{i\in S_k}|\mathbf{x}_k\rangle = (|\rho_i\rangle)_{i\in S_{k'}}|\mathbf{x}_{k'}\rangle,$$

for which $k, k' = 1, \dots, m$ and $k \neq k'$, t_k linearly independent solutions are required. Without loss of generality, we assume

$$\text{rank}[(|\rho_i\rangle)_{i\in S_k}] \geqslant \text{rank}[(|\rho_i\rangle)_{i\in S_{k'}}]. \tag{C1}$$

Here, rank$[(|\rho_i\rangle)_{i\in S_s}]$ represents the number of vectors corresponding to S_s, i.e., $t_s = \text{rank}[(|\rho_i\rangle)_{i\in S_s}]$, for all $s = 1, \dots, m$. To eliminate the $t_k = \text{rank}[(|\rho_i\rangle)_{i\in S_k}]$ nonzero entries in the kth row of K_α, we need at least t_k Kraus operators. Therefore, the dimension of solution space span$[(|\rho_i\rangle)_{i\in S_{k'}}]$ needs to be greater than the rank of coefficient matrix $(|\rho_i\rangle)_{i\in S_k}$, i.e.,

$$\begin{aligned}\dim[\text{span}(\{|\rho_i\rangle\}_{i\in S_{k'}})] &= \text{rank}[(|\rho_i\rangle)_{i\in S_{k'}}]\\ &\geqslant \text{rank}[(|\rho_i\rangle)_{i\in S_k}].\end{aligned} \tag{C2}$$

Combining Eq. (C1), we can conclude that for all $k, k' \in \{1, \dots, m\}$, $\text{rank}[(|\rho_i\rangle)_{i\in S_k}] = \text{rank}[(|\rho_i\rangle)_{i\in S_{k'}}]$ holds, which means that the disjoint-subset partition is an isometric partition, i.e., $t_k = t_{k'}$.

According to Lemma 3(b), consider that there is a disjoint-subset partition of $S = \{1, \dots, d\}$, i.e., S can be partitioned into m disjoint subsets S_s such that for all $s = 1, \dots, m$, the column vectors of $|\rho_i\rangle$ $(i \in S_s)$ are nonzero vectors, and the intersection of the spaces span$[\{|\rho_i\rangle\}\ (i \in S_s)]$ includes nonzero vectors, i.e.,

$$\dim\left\{\bigcap_{s=1}^{m} \text{span}[\{|\rho_i\rangle\}\ (i \in S_s)]\right\} > 0.$$

Based on Lemma 7, in order to construct a complete gPIO $\Lambda^\star(\cdot)$, for each linear system

$$(|\rho_i\rangle)_{i\in S_k}|\mathbf{x}_k\rangle = (|\rho_i\rangle)_{i\in S_{k'}}|\mathbf{x}_{k'}\rangle,$$

for which $k, k' = 1, \dots, m$ and $k \neq k'$, t_k linearly independent solutions are required. Without loss of generality, we assume

$$\text{rank}[(|\rho_i\rangle)_{i\in S_k}] \geqslant \text{rank}[(|\rho_i\rangle)_{i\in S_{k'}}]. \tag{C3}$$

Since the solution space of the nonhomogeneous linear equations $(|\rho_i\rangle)_{i\in S_k}|\mathbf{x}_k\rangle = |\gamma\rangle$ $(|\gamma\rangle \neq 0)$ is an affine subspace of the solution space of the homogeneous linear equations $(|\rho_i\rangle)_{i\in S_k}|x_k\rangle = 0$, there are only t_k linearly independent solutions $|x_k\rangle$. Specifically, the homogeneous linear equations $(|\rho_i\rangle)_{i\in S_k}|\mathbf{x}_k\rangle = 0$ have

$$t_k - \text{rank}[(|\rho_i\rangle)_{i\in S_k}] \tag{C4}$$

linearly independent solutions. Moreover, for a nonzero vector $|\gamma\rangle \neq 0$, the nonhomogeneous linear equations $(|\rho_i\rangle)_{i\in S_k}|\mathbf{x}_k\rangle = |\gamma\rangle$ only need to provide one solution, as the other solutions can be expressed as linear combinations of it and the solutions of the homogeneous linear equations $(|\rho_i\rangle)_{i\in S_k}|\mathbf{x}_k\rangle = 0$. In fact, for the nonhomogeneous linear equations $(|\rho_i\rangle)_{i\in S_k}|\mathbf{x}_k\rangle = |\gamma\rangle$, the nonzero vector on the right side can be expressed as a linear combination of the vectors $|\rho_i\rangle$ $(i \in S_{k'})$. Therefore, at most there are

$$\text{rank}[(|\rho_i\rangle)_{i\in S_{k'}}] = \dim[\text{span}(\{|\rho_i\rangle\}_{i\in S_{k'}})] \tag{C5}$$

linearly independent nonzero vectors. Combining Eqs. (C4) and (C5), we can construct more than t_k linearly independent solutions only when

$$\text{rank}[(|\rho_i\rangle)_{i\in S_{k'}}] \geqslant \text{rank}[(|\rho_i\rangle)_{i\in S_k}] \tag{C6}$$

holds. In summary, we obtain that, for all $k, k' \in \{1, \dots, m\}$, $\text{rank}[(|\rho_i\rangle)_{i\in S_k}] = \text{rank}[(|\rho_i\rangle)_{i\in S_{k'}}]$ holds. At the same time, for $k, k' = 1, \dots, m$, the linear system

$$(|\rho_i\rangle)_{i\in S_k}|\mathbf{x}_k\rangle = (|\rho_i\rangle)_{i\in S_{k'}}|\mathbf{x}_{k'}\rangle,\ (k \neq k')$$

only has t_k linearly independent solutions $|\mathbf{x}_k\rangle$.

Therefore, when $\text{rank}(\rho) \geqslant |S_1| > \lfloor \frac{d}{m} \rfloor$, we get that the maximal linearly independent group of $\sqrt{\rho}$ contains rank(ρ) vectors. Because of $m \cdot \text{rank}(\rho) > d$, some vectors of the maximal linearly independent set cannot be included in the disjoint-subset partition of $\{1, \dots, d\}$ satisfying $\text{rank}[(|\rho_k\rangle)_{i\in S_k}] = \text{rank}[(|\rho_i\rangle)_{i\in S_{k'}}]$ for all $k, k' \in \{1, \dots, m\}$. It means that $\text{Tr}(\sum_\alpha K_\alpha \rho K_\alpha) = \text{Tr}(\sum_\alpha K_\alpha \Pi_{S'} \rho \Pi_{S'} K_\alpha) < 1$ holds, where $S' \subset \{1, \dots, d\}$. Here, vectors $|\rho_i\rangle$ $(i \in S')$ do not contain some of the column vectors in the maximal linearly independent set of $\sqrt{\rho}$. Therefore, we cannot find a gPIO $\Lambda^\star(\cdot)$, with incoherent operators $K_\alpha^\star$ sharing the form as Eq. (11), such that the resultant state $\Lambda^\star(\rho)$ is the pure coherent state $\varphi_{\mathbf{m}}$ $(d \geqslant m \geqslant 2)$.

According to Eq. (13), for all β, the operation $\Lambda_\beta^\star[\Pi_{S_\beta}(\cdot)\Pi_{S_\beta}]$, for which $\Lambda_\beta^\star(\cdot) = \sum_\alpha K_{\beta\alpha}^\star(\cdot)K_{\beta\alpha}^{\star\dagger}$ is a gPIO, as shown in Lemma 5, and Π_{S_β} is an incoherent projector, maintains the form manifested in Lemma 5 on the corresponding subspace $\Pi_{S_\beta}(H)$. Thus, it satisfies the aforementioned conclusion. That is, in order to output the pure coherent state $\varphi_{\mathbf{m}}$, the inequation $\text{rank}(\Pi_{S_\beta}\rho\Pi_{S_\beta}) \leqslant \lfloor \frac{|S_\beta|}{m} \rfloor$ needs to be satisfied, for all β. Thus, for the elementary gPIO,

$$\Lambda(\rho) = \sum_\beta \Lambda_\beta^\star(\Pi_{S_\beta}\rho\Pi_{S_\beta}), \tag{C7}$$

where $\Lambda_\beta^\star(\cdot) = \sum_\alpha K_{\beta\alpha}(\cdot)K_{\beta\alpha}^\dagger$ are gPIOs, as shown in Lemma 5, and $\sum_\beta \Pi_{S_\beta} = \mathbb{I}$, we can use the inequation $\text{rank}(\rho) \leqslant \sum_\beta \text{rank}(\Pi_{S_\beta}\rho\Pi_{S_\beta}) \leqslant \sum_\beta \lfloor \frac{|S_\beta|}{m} \rfloor \leqslant \lfloor \frac{d}{m} \rfloor$ to prove the proposition holds. That is, the inequation $\text{rank}(\rho) > \lfloor \frac{d}{m} \rfloor$ implies the existence of an incoherent projector Π_{S_β} that makes the inequation $\text{rank}(\Pi_{S_\beta}\rho\Pi_{S_\beta}) > \lfloor \frac{|S_\beta|}{m} \rfloor$ hold. Finally, we can directly derive that the proposition holds for every gPIO, which is a convex combination of elementary PIOs.

APPENDIX D: THE PROOF OF LEMMA 9

Here, we provide the equivalent conditions for the existence of isometric disjoint-subset partitions, which can be viewed as a generalization of Lemma 6 in Ref. [46]. For two index sets $S_1, S_2 \subseteq \{1, \dots, d\}$, we denote by $M[S_1, S_2]$ the submatrix of entries that lie in the rows of M indexed by M_1 and the columns of M indexed by M_2.

Lemma 9. Let $|v_1\rangle, \dots, |v_d\rangle$ be vectors in an inner product space V with inner product $\langle\cdot|\cdot\rangle$, and let $G = (\langle v_i|v_j\rangle)_{i,j=1,\dots,d}$. For two disjoint subsets $S_1, S_2 \subseteq \{1, \dots, d\}$ with $|S_1| = |S_2|$ and $\text{rank}[(|v_i\rangle)_{i\in S_1}] = \text{rank}[(|v_i\rangle)_{i\in S_2}]$, the following statements are equivalent,

(a) There are two unitary matrices U_1, U_2 and three diagonal matrices D_1, D_2, D which are $|S_1| \times |S_1|$ matrices, such that

$$(|v_i\rangle)_{i\in S_1} U_1 D_1 = (|v_j\rangle)_{j\in S_2} U_2 D_2 \tag{D1}$$

and

$$\begin{aligned} D &= D_1 U_1^\dagger (\langle v_i|v_j\rangle)_{i,j\in S_1} U_1 D_1 \\ &= D_2 U_2^\dagger (\langle v_i|v_j\rangle)_{i,j\in S_2} U_2 D_2. \end{aligned} \tag{D2}$$

(b) The principal submatrices $G[S_1]$ and $G[S_2]$ satisfy the following two equations:

$$G[S_1]\sqrt{G[S_1, S_2]G[S_2, S_1]} = \sqrt{G[S_1, S_2]G[S_2, S_1]}G[S_1], \tag{D3}$$

$$\begin{aligned} &\lambda(G[S_1])\lambda(G[S_2]) \\ &\quad = \lambda(\sqrt{G[S_1, S_2]G[S_2, S_1]})\lambda(\sqrt{G[S_2, S_1]G[S_1, S_2]}), \end{aligned} \tag{D4}$$

where $\lambda(M)$ is the eigenvalues diagonal matrix of positive-semidefinite matrix M in the corresponding eigenvector basis. Here, the left singular vectors of $G[S_1, S_2]$ (the eigenvectors of $G[S_1]$) uniquely determine the right singular vectors of $G[S_1, S_2]$ (the eigenvectors of $G[S_2]$).

Proof. First, let us show our proof with the following example, where vector sets $\{|v_a\rangle, |v_b\rangle, |v_c\rangle\}$ and $\{|v_{a'}\rangle, |v_{b'}\rangle, |v_{c'}\rangle\}$ are considered for which $\{a, b, c\}$, $\{a', b', c'\} \subseteq \{1, \dots, d\}$, and $\{a, b, c\} \bigcap \{a', b', c'\} = \emptyset$.

According to the singular value theorem, there are unitary matrices U_L and V_R such that

$$\begin{aligned} G[\{a, b, c\}, \{a', b', c'\}] &= \begin{pmatrix} \langle v_a| \\ \langle v_b| \\ \langle v_c| \end{pmatrix} \begin{pmatrix} |v_{a'}\rangle & |v_{b'}\rangle & |v_{c'}\rangle \end{pmatrix} \\ &= U_L D V_R^\dagger \\ &= U_L D_{11}^{\frac{1}{2}} \mathbb{I} D_{22}^{\frac{1}{2}} V_R^\dagger \end{aligned} \tag{D5}$$

and

$$\begin{aligned} G[\{a', b', c'\}, \{a, b, c\}] &= \begin{pmatrix} \langle v_{a'}| \\ \langle v_{b'}| \\ \langle v_{c'}| \end{pmatrix} \begin{pmatrix} |v_a\rangle & |v_b\rangle & |v_c\rangle \end{pmatrix} \\ &= V_R D U_L^\dagger \\ &= V_R D_{22}^{\frac{1}{2}} \mathbb{I} D_{11}^{\frac{1}{2}} U_L^\dagger \end{aligned} \tag{D6}$$

due to $G[\{a, b, c\}, \{a', b', c'\}] = G[\{a', b', c'\}, \{a, b, c\}]^\dagger$ for the Hermitian matrix G, following Lemma 8(a), where the matrices D, D_{11}, and D_{22} are diagonal matrices. Here, $(\cdot)^a$ $(a \in \mathbb{R})$ is an operator function and the diagonal matrices D_{11} and D_{22} satisfy

$$D_{11} = \lambda(G[S_1]), \quad D_{22} = \lambda(G[S_2]).$$

According to Eqs. (D3) and (D4), let us show if

$$\begin{aligned} G[\{a, b, c\}] &= \begin{pmatrix} \langle v_a| \\ \langle v_b| \\ \langle v_c| \end{pmatrix} \begin{pmatrix} |v_a\rangle & |v_b\rangle & |v_c\rangle \end{pmatrix} \\ &= U_L D_{11} U_L^\dagger \\ &= U_L D_{11}^{\frac{1}{2}} \mathbb{I} D_{11}^{\frac{1}{2}} U_L^\dagger \end{aligned} \tag{D7}$$

and

$$D_{11} D_{22} = D^2 \tag{D8}$$

hold, then vector sets $\{|v_a\rangle, |v_b\rangle, |v_c\rangle\}$ and $\{|v_{a'}\rangle, |v_{b'}\rangle, |v_{c'}\rangle\}$ are quasi-unitary equivalent.

Then, combined with Eqs. (D5), (D7), and (D8), the following reasoning process is presented:

$$\begin{pmatrix} \mathbb{I} & \Theta \\ \Theta & \Theta \end{pmatrix} = D_{11}^{-\frac{1}{2}} U_L^{\dagger} \begin{pmatrix} \langle v_a| \\ \langle v_b| \\ \langle v_c| \end{pmatrix} (|v_{a'}\rangle\ |v_{b'}\rangle\ |v_{c'}\rangle) V_R D_{22}^{-\frac{1}{2}}$$

$$= D_{11}^{-\frac{1}{2}} U_L^{\dagger} \begin{pmatrix} \langle v_a| \\ \langle v_b| \\ \langle v_c| \end{pmatrix} (|\rho_a\rangle\ |\rho_b\rangle\ |v_c\rangle) U_L D_{11}^{-\frac{1}{2}},$$

where Θ represents the all-zeros matrix of appropriate size.

We get the desired result,

$$(|v_a\rangle\ |v_b\rangle\ |v_c\rangle) U_L D_{11}^{-\frac{1}{2}} = (|v_{a'}\rangle\ |v_{b'}\rangle\ |v_{c'}\rangle) V_R D_{22}^{-\frac{1}{2}},$$

and

$$D_{22}^{-\frac{1}{2}} V_R^{\dagger} \begin{pmatrix} \langle v_{a'}| \\ \langle v_{b'}| \\ \langle v_{c'}| \end{pmatrix} (|v_{a'}\rangle\ |v_{b'}\rangle\ |v_{c'}\rangle) V_R D_{22}^{-\frac{1}{2}} = \begin{pmatrix} \mathbb{I} & \Theta \\ \Theta & \Theta \end{pmatrix}$$

$$= D_{11}^{-\frac{1}{2}} U_L^{\dagger} \begin{pmatrix} \langle v_a| \\ \langle v_b| \\ \langle v_c| \end{pmatrix} (|\rho_a\rangle\ |\rho_b\rangle\ |v_c\rangle) U_L D_{11}^{-\frac{1}{2}}.$$

Conversely, let us assume there are two unitary matrices U_1, U_2 and three diagonal matrices D_1, D_2, D which are $|S_1| \times |S_1|$ matrices, such that

$$(|v_a\rangle\ |v_b\rangle\ |v_c\rangle) U_1 D_1 = (|v_{a'}\rangle\ |v_{b'}\rangle\ |v_{c'}\rangle) U_2 D_2,$$

and

$$D = D_1 U_1^{\dagger} \begin{pmatrix} \langle v_a| \\ \langle v_b| \\ \langle v_c| \end{pmatrix} (|v_a\rangle\ |v_b\rangle\ |v_c\rangle) U_1 D_1$$

$$= D_2 U_2^{\dagger} \begin{pmatrix} \langle v_{a'}| \\ \langle v_{b'}| \\ \langle v_{c'}| \end{pmatrix} (|v_{a'}\rangle\ |v_{b'}\rangle\ |v_{c'}\rangle) U_2 D_2. \tag{D9}$$

Then, we have the following equation

$$G[\{a,b,c\},\{a',b',c'\}] = \begin{pmatrix} \langle v_a| \\ \langle v_b| \\ \langle v_c| \end{pmatrix} \big(|v_{a'}\rangle\ |v_{b'}\rangle\ |v_{c'}\rangle\big) = U_1 D_1^{-1} D D_2^{-1} U_2^{\dagger} \tag{D10}$$

and

$$G[\{a',b',c'\},\{a,b,c\}] = \begin{pmatrix} \langle v_{a'}| \\ \langle v_{b'}| \\ \langle v_{c'}| \end{pmatrix} \big(|v_a\rangle\ |v_b\rangle\ |v_c\rangle\big) = U_2 D_2^{-1} D D_1^{-1} U_1^{\dagger} \tag{D11}$$

Thus, we can deduce that there are unitary matrices V_i and diagonal matrices A_{ij} $(i, j = 1, 2)$ such that $G[S_1 \cup S_2]$ has the following permutation-equivalence density matrix:

$$\begin{pmatrix} V_1 A_{11} V_1^{\dagger} & V_1 A_{12} V_2^{\dagger} \\ V_2 A_{21} V_1^{\dagger} & V_2 A_{22} V_2^{\dagger} \end{pmatrix}, \tag{D12}$$

where $A_{11}A_{22} = A_{12}A_{21}$. According to Eq. (D12), the corresponding submatrices are easily checked to satisfy Eqs. (D3) and (D4).

And, for more general cases, it can be proved by the same method as mentioned above. ■

[1] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2010).
[2] J. Watrous, *The Theory of Quantum Information* (Cambridge University Press, Cambridge, 2018).
[3] E. Chitambar and G. Gour, Quantum resource theories, Rev. Mod. Phys. **91**, 025001 (2019).
[4] F. G. S. L. Brandão and G. Gour, Reversible framework for quantum resource theories, Phys. Rev. Lett. **115**, 070503 (2015).
[5] M. Horodecki and J. Oppenheim, (Quantumness in the context of) resource theories, Int. J. Mod. Phys. B **27**, 1345019 (2013).
[6] J. Åberg, Quantifying superposition, arXiv:quant-ph/0612146.
[7] F. Levi and F. Mintert, A quantitative theory of coherent delocalization, New J. Phys. **16**, 033007 (2014).
[8] T. Baumgratz, M. Cramer, and M. B. Plenio, Quantifying coherence, Phys. Rev. Lett. **113**, 140401 (2014).
[9] M. L. Hu, X. Hu, J. Wang, Y. Peng, Y. R. Zhang, and H. Fan, Quantum coherence and geometric quantum discord, Phys. Rep. **762**, 1 (2018).
[10] A. Streltsov, G. Adesso, and M. B. Plenio, *Colloquium:* Quantum coherence as a resource, Rev. Mod. Phys. **89**, 041003 (2017).
[11] J. I. de Vicente and A. Streltsov, Genuine quantum coherence, J. Phys. A: Math. Theor. **50**, 045301 (2016).
[12] X. D. Yu, D. J. Zhang, G. F. Xu, and D. M. Tong, Alternative framework for quantifying coherence, Phys. Rev. A **94**, 060302(R) (2016).
[13] B. Yadin, J. Ma, D. Girolami, M. Gu, and V. Vedral, Quantum processes which do not use coherence, Phys. Rev. X **6**, 041028 (2016).
[14] A. Winter and D. Yang, Operational resource theory of coherence, Phys. Rev. Lett. **116**, 120404 (2016).
[15] I. Marvian and R. W. Spekkens, How to quantify coherence: Distinguishing speakable and unspeakable notions, Phys. Rev. A **94**, 052324 (2016).
[16] A. Streltsov, S. Rana, P. Boes, and J. Eisert, Structure of the resource theory of quantum coherence, Phys. Rev. Lett. **119**, 140402 (2017).
[17] E. Chitambar and G. Gour, Critical examination of incoherent operations and a physically consistent resource

theory of quantum coherence, Phys. Rev. Lett. **117**, 030401 (2016).

[18] E. Chitambar and G. Gour, Comparison of incoherent operations and measures of coherence, Phys. Rev. A **94**, 052336 (2016).

[19] I. Marvian and R. W. Spekkens, The theory of manipulations of pure state asymmetry: I. Basic tools, equivalence classes and single copy transformations, New J. Phys. **15**, 033001 (2013).

[20] I. Marvian and R. W. Spekkens, Extending Noether's theorem by quantifying the asymmetry of quantum states, Nat. Commun. **5**, 3821 (2014).

[21] Q. Zhao, Y. Liu, X. Yuan, E. Chitambar, and A. Winter, One-shot coherence distillation: Towards completing the picture, IEEE Trans. Inf. Theory **65**, 6441 (2019).

[22] S. Du, Z. Bai, and Y. Guo, Conditions for coherence transformations under incoherent operations, Phys. Rev. A **91**, 052120 (2015).

[23] H. Zhu, Z. Ma, Z. Cao, S. M. Fei, and V. Vedral, Operational one-to-one mapping between coherence and entanglement measures, Phys. Rev. A **96**, 032316 (2017).

[24] B. Regula, V. Narasimhachar, F. Buscemi, and M. Gu, Coherence manipulation with dephasing-covariant operations, Phys. Rev. Res. **2**, 013109 (2020).

[25] X. Yuan, H. Zhou, Z. Cao, and X. Ma, Intrinsic randomness as a measure of quantum coherence, Phys. Rev. A **92**, 022124 (2015).

[26] L. Lami, Completing the grand tour of asymptotic quantum coherence manipulation, IEEE Trans. Inf. Theory **66**, 2165 (2019).

[27] L. Lami, B. Regula, and G. Adesso, Generic bound coherence under strictly incoherent operations, Phys. Rev. Lett. **122**, 150402 (2019).

[28] E. Chitambar, Dephasing-covariant operations enable asymptotic reversibility of quantum resources, Phys. Rev. A **97**, 050301(R) (2018).

[29] K. Fang, X. Wang, L. Lami, B. Regula, and G. Adesso, Probabilistic distillation of quantum coherence, Phys. Rev. Lett. **121**, 070404 (2018).

[30] K. Bu, U. Singh, S. M. Fei, A. K. Pati, and J. Wu, Maximum relative entropy of coherence: An operational coherence measure, Phys. Rev. Lett. **119**, 150405 (2017).

[31] B. Regula, L. Lami, and A. Streltsov, Nonasymptotic assisted distillation of quantum coherence, Phys. Rev. A **98**, 052329 (2018).

[32] B. Regula, K. Fang, X. Wang, and G. Adesso, One-shot coherence distillation, Phys. Rev. Lett. **121**, 010401 (2018).

[33] Z. W. Liu, K. Bu, and R. Takagi, One-shot operational quantum resource theory, Phys. Rev. Lett. **123**, 020401 (2019).

[34] G. Torun, L. Lami, G. Adesso, and A. Yildiz, Optimal distillation of quantum coherence with reduced waste of resources, Phys. Rev. A **99**, 012321 (2019).

[35] S. Du, Z. Bai, and X. Qi, Coherence manipulation under incoherent operations, Phys. Rev. A **100**, 032313 (2019).

[36] S. Chen, X. Zhang, Y. Zhou, and Q. Zhao, One-shot coherence distillation with catalysts, Phys. Rev. A **100**, 042323 (2019).

[37] S. Zhang, Y. Luo, L. H. Shao, Z. Xi, and H. Fan, One-shot assisted distillation of coherence via one-way local quantum-incoherent operations and classical communication, Phys. Rev. A **102**, 052405 (2020).

[38] C. L. Liu and D. L. Zhou, Deterministic coherence distillation, Phys. Rev. Lett. **123**, 070402 (2019).

[39] C. L. Liu, Y. Q. Guo, and D. M. Tong, Enhancing coherence of a state by stochastic strictly incoherent operations, Phys. Rev. A **96**, 062325 (2017).

[40] C. L. Liu and D. L. Zhou, Catalyst-assisted probabilistic coherence distillation for mixed states, Phys. Rev. A **101**, 012313 (2020).

[41] C. L. Liu and C. P. Sun, Optimal probabilistic distillation of quantum coherence, Phys. Rev. Res. **3**, 043220 (2021).

[42] C. L. Liu, D. L. Zhou, and C. P. Sun, Criterion for a state to be distillable via stochastic incoherent operations, Phys. Rev. A **105**, 032448 (2022).

[43] C. L. Liu and D. L. Zhou, Increasing the dimension of the maximal pure coherent subspace of a state via incoherent operations, Phys. Rev. A **102**, 062427 (2020).

[44] Y. Yao, X. Xiao, L. Ge, and C. P. Sun, Quantum coherence in multipartite systems, Phys. Rev. A **92**, 022112 (2015).

[45] Y. Luo, Y. Li, and M. H. Hsieh, Inequivalent multipartite coherence classes and two operational coherence monotones, Phys. Rev. A **99**, 042306 (2019).

[46] P. Li and Y. Li, Coherence manipulation with stochastic incoherent operation, Phys. Rev. A **108**, 042415 (2023).

[47] G. Torun and A. Yildiz, Deterministic transformations of coherent states under incoherent operations, Phys. Rev. A **97**, 052331 (2018).

[48] F. Bischof, H. Kampermann, and D. Bruß, Quantifying coherence with respect to general quantum measurements, Phys. Rev. A **103**, 032429 (2021).

[49] J. Xu, L. H. Shao, and S. M. Fei, Coherence measures with respect to general quantum measurements, Phys. Rev. A **102**, 012411 (2020).

[50] S. Luo and Y. Sun, Quantum coherence versus quantum uncertainty, Phys. Rev. A **96**, 022130 (2017).

[51] S. Kim, L. Li, A. Kumar, and J. Wu, Interrelation between partial coherence and quantum correlations, Phys. Rev. A **98**, 022306 (2018).

[52] Y. Xiao, Y. Yang, X. Wang, Q. Liu, and M. Gu, Quantum uncertainty principles for measurements with interventions, Phys. Rev. Lett. **130**, 240201 (2023).

[53] Y. Xiao, K. Sengupta, S. Yang, and G. Gour, Uncertainty principle of quantum processes, Phys. Rev. Res. **3**, 023077 (2021).

[54] R. A. Horn and C. R. Johnson, *Matrix Analysis* (Cambridge University Press, Cambridge, 2012).