Quantum multigraph states and multihypergraph states

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In this paper we propose two classes of multiparticle entangled states: multigraph and multihypergraph states. Two types of operations are designed to correspond to the edges and hyperedges to prepare the states. Furthermore, multigraph and multihypergraph states contain qudit graph and hypergraph states, respectively. More interestingly, we demonstrate the one-to-one correspondence between the proposed multihypergraph states and the generalized real equally weighted states when the dimension of the states is an odd prime.

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I. INTRODUCTION

Multiparticle entangled states are crucial in the fields of quantum computing and quantum information. Hillery *et al.* [1] first proposed a secret sharing scheme using the Greenberger-Horne-Zeilinger state. Briegel and Raussendorf [2] introduced the cluster state, a specific type of multiparticle entangled state. Subsequently, they proposed a new quantum computing approach based on cluster state measurements [3]. Around the same time, cluster states were used to design quantum error-correcting codes [4], which employed graph representations to describe these quantum states for the first time. Raussendorf *et al.* [5] formally employed the term graph state to explore measurement-based quantum computing. Graph states are characterized by connections through controlled-Z (CZ) gates, which can facilitate the simulation of any unitary gate through specific measurement bases and sequences [5]. Rossi *et al.* [6] and Qu *et al.* [7] extended graph states to hypergraph states, and a one-to-one correspondence was discovered between hypergraph states and real equally weighted states (REWSs). Hypergraph states are notable for their Pauli universality in measurement-based computation [8], which enables the simulation of universal unitary gates solely through Pauli measurements of particles. Graph and hypergraph states have been extensively studied [9–22] and are widely used in the fields of quantum computing [23–34] and quantum information [35–38].

The study of two-dimensional quantum systems naturally led to expanding research into d-dimensional quantum systems. Studies have been conducted on qudit graph states [39–41]. The subsequent investigations into qudit hypergraph states [42–44] followed. Xiong *et al.* [43] illuminated the quantitative relationship between qudit hypergraph states and generalized real equally weighted states (GREWSs) and noted that qudit hypergraph states are contained within GREWSs and are significantly fewer in number. The existing research on graph and hypergraph states [6–44] has primarily employed simple and weighted graphs (hypergraphs) [45–47]. However, there are also other classes of graphs in graph theory, such as multigraphs and multihypergraphs, where different edges can connect the same vertices. In this paper we propose quantum states corresponding to this kind of multigraph and multihypergraph and the quantum states are referred to as the multigraph state and the multihypergraph state, respectively. Compared to qudit graph states [39–41] and qudit hypergraph states [42–44], more information can be encoded into multigraph states and multihypergraph states. More interestingly, a one-to-one correspondence exists between the proposed multihypergraph states and the GREWSs when d is an odd prime.

The structure of this paper is outlined as follows. In Sec. II the review of essential graph theory concepts and the definitions of qudit graph and hypergraph states are provided. In Sec. III the definitions of multigraph and multihypergraph states are proposed. In Sec. IV the association between multihypergraph states and GREWSs is detailed. In Sec. V a summary and an outlook for future research are given.

II. PRELIMINARIES

In this section we briefly overview the foundational aspects of graph theory and the definitions of graph and hypergraph states.

A. Fundamentals of graph theory

Graph theory spans a wide range of graph types, such as simple graphs, weighted graphs, multigraphs, hypergraphs, weighted hypergraphs, and multihypergraphs [45–48]. A simple graph is characterized by no more than one edge joining any two vertices. Weighted graphs are an extension of simple graphs, assigning weights to each edge. There are two kinds of multigraphs. One is where multiple edges connecting two vertices are identical, which allows the same edge to appear multiple times. Therefore, the multigraphs can be viewed as weighted graphs [47]. The other type is where edges are

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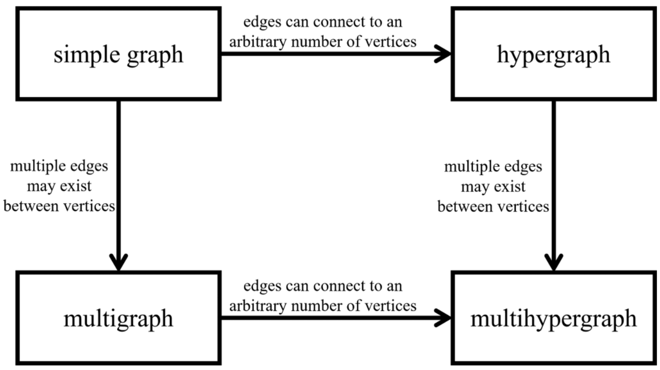


FIG. 1. Relationships among simple graphs, multigraphs, hypergraphs, and multihypergraphs.

considered primitive entities, which are different when multiple edges connect two vertices. An example is the electrical network multigraph in Ref. [48]. In hypergraphs, edges can connect any number of vertices, with those connected to a single vertex termed rings [45]. In the field of hypergraphs, weighted hypergraphs and multihypergraphs also exist. The definitions of weighted hypergraphs and multihypergraphs are similar to those of weighted graphs and multigraphs except that edges in weighted hypergraphs and multihypergraphs can connect any number of vertices. The relationships among simple graphs, multigraphs, hypergraphs, and multihypergraphs are depicted in Fig. 1.

A graph of N vertices can be described as $G = (V, E)$, where $V = \mathbb{Z}_N$ denotes the set of vertices and E denotes the set of edges. The $e = \{j, k\} \in E$ represents the edge connecting vertices j and k. Similarly, a hypergraph of N vertices is represented as $\tilde{G} = \{V, \tilde{E}\}$, with $\tilde{E} \subseteq \wp(V)$ the set of hyperedges. Here $\wp(V)$ denotes the power set of V and $e = \{v_0, v_1, \dots, v_{t-1}\} \in \tilde{E}$ represents the hyperedge connecting $t \in \mathbb{Z}_{N+1}$ vertices, where $v_0 < v_1 < \cdots < v_{t-1} \in \mathbb{Z}_N$ indicates the connected t vertices. When $t = 0$ the empty set $\varnothing$ represents an empty edge. Simple graphs, hypergraphs, multigraphs, and multihypergraphs can be transformed into weighted graphs, hypergraphs, multigraphs, and multihypergraphs by adding weights to each edge. Within such weighted graphs, the weight assigned to an edge e is denoted by m_e.

B. Review of graph and hypergraph states

1. Graph states

The graph state was initially proposed as a cluster state [2]. Then, once the graph was utilized to characterize this quantum state, it was renamed a graph state [4,5]. Figure 2 illustrates the construction of a graph state for rectangular lattices from Ref. [25].

Following varying graph structures, we can formulate distinct graph states. Before studying the description of such quantum states, it is significant to define the vertices and edges within the graph from a quantum perspective. The formal representation of vertices and edges in graph states was previously delineated in Refs. [5,9,39]. For a graph $G = (V, E)$ comprising N vertices, the corresponding graph state is characterized by N vertices denoted by $|+_d\rangle^{\otimes N} = \underbrace{|+_d\rangle \otimes |+_d\rangle \otimes \cdots \otimes |+_d\rangle}_{N}$. The edge connecting vertices j and k is represented by

$$\mathrm{CZ}_{\{j,k\}} = \sum_{i_0,\dots,i_{N-1}=0}^{d-1} \omega_d^{i_j \times i_k} |i_0, \dots, i_{N-1}\rangle\langle i_0, i_1, \dots, i_{N-1}|, \quad (1)$$

where $\omega_d = e^{2\pi \mathbf{i}/d}$ ($\mathbf{i} = \sqrt{-1}$). The qudit graph state corresponding to graph G is defined as

$$|G\rangle = \prod_{e\in E} \mathrm{CZ}_e^{m_e} |+_d\rangle^{\otimes N}, \quad (2)$$

where $m_e \in \mathbb{Z}_d$ represents both the weight of an edge and the number of operations executed. According to Refs. [6,40], the number of qudit graph states is $d^{\binom{N}{2}}$, where $\binom{N}{2}$ is the binomial coefficient "N choose 2." The stabilizer operator on a vertex k for a qudit graph state is characterized as

$$g_k = \left(\prod_{e\in E} \mathrm{CZ}_e^{m_e}\right) X_k \left(\prod_{e'\in E} \mathrm{CZ}_{e'}^{d-m_{e'}}\right) = X_k \prod_{e\in E, k\in e} \mathrm{CZ}_{e\backslash\{k\}}^{m_e}, \quad (3)$$

where the X_k represents that the qudit Pauli gate $X = \sum_{i=0}^{d-1} |i+1\rangle\langle i|$ is executed on the kth particle, and the $e = \{j, k\}$ and e' are any elements in E. The set g_k generates an Abelian group known as the stabilizer [42].

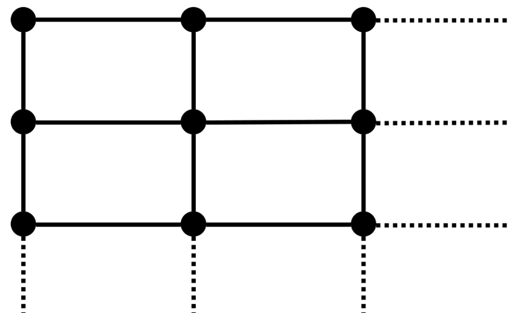


FIG. 2. Graph state. The circles denote quantum states $|+_d\rangle = d^{-1/2} \sum_{i=0}^{d-1} |i\rangle$, solid lines correspond to controlled-Z gates, and dotted lines indicate additional $|+_d\rangle$ states and controlled-Z gates.

2. Hypergraph states

In Ref. [6] the hypergraph state was proposed, which extended the graph states. Within a qudit hypergraph state, vertices are defined as the qudit single-particle states $|+_d\rangle = d^{-1/2} \sum_{i=0}^{d-1} |i\rangle$. The hyperedge connects vertices $v_0, \dots, v_{t-1}$ are characterized by

$$\widetilde{\mathrm{CZ}}_{\{v_0,v_1,\dots,v_{t-1}\}} = \sum_{i_0,i_1,\dots,i_{N-1}=0}^{d-1} \left(\omega_d^{\prod_{j=0}^{t-1} i_{v_j}} |i_0, \dots, i_{N-1}\rangle \langle i_0, \dots, i_{N-1}|\right). \quad (4)$$

The qudit hypergraph state corresponding to graph $\tilde{G} = (V, \tilde{E})$ is defined as

$$|\tilde{G}\rangle = \left(\prod_{e\in\tilde{E}} \widetilde{\mathrm{CZ}}_e^{m_e}\right) |+_d\rangle^{\otimes N}, \quad (5)$$

where $|+_d\rangle^{\otimes N}$ refers to the N vertices comprising the hypergraph state. According to Refs. [42,43], the number of nontrivial qudit hypergraph states is d^{2^N-1} and the stabilizer operator on a vertex k for a qudit hypergraph state is characterized as

$$\tilde{g}_k = \left(\prod_{e\in\tilde{E}} \tilde{CZ}_e^{m_e}\right) X_k \left(\prod_{e'\in\tilde{E}} \tilde{CZ}_{e'}^{d-m_{e'}}\right) = X_k \prod_{e\in\tilde{E},k\in e} \tilde{CZ}_{e\backslash\{k\}}^{m_e}, \quad (6)$$

where the $e = \{v_0, v_1, \dots, v_{t-1}\}$ and e' are any elements in $\tilde{E}$, and $e\backslash\{k\} = \{i \mid i \in e, i \notin \{k\}\}$. The operator $\tilde{g}_k$ stabilizes the hypergraph state $|\tilde{G}\rangle$ [42]. The proof of Eq. (6) is delineated in Appendix A.

III. PROPOSED MULTIGRAPH AND MULTIHYPERGRAPH STATES

In this section we propose the multigraph and multihypergraph states. Traditional graph and hypergraph states are typically constructed by a series of Z and multiparticle controlled-Z gates. However, our approach involves a series of quantum gates specific to building multigraph and multihypergraph states. First, we define the quantum operations that are generalized from the Z and the controlled-Z gates. Next we employ these gates to construct multigraph and multihypergraph states. In this paper the multigraph and multihypergraph utilized differ in that various edges or hyperedges connecting the same vertices represent distinct entities. We employ N integral variables $s_0, s_1, \dots, s_{N-1} \in \mathbb{Z}_d^*$ ($\mathbb{Z}_d^* = \mathbb{Z}_d\backslash\{0\} = \{a \mid a \in \mathbb{Z}_d, a \notin \{0\}\}$) related to N vertices $v_0, v_1, \dots, v_{N-1}$. The edges (hyperedges) in our multigraph (multihypergraph) are represented as $\dot{e} = (V_{\dot{e}}|S_{\dot{e}}) = (v_0, \dots, v_{t-1}|s_{v_0}, s_{v_1}, \dots, s_{v_{t-1}})$, where $t = 2$ for edges and $t > 2$ for hyperedges. The set $V_{\dot{e}} = \{v_0, \dots, v_{t-1}\}$ represents all the vertices connected by $\dot{e}$, and the different values of t variables $s_{v_0}, s_{v_1}, \dots, s_{v_{t-1}}$ indicate different $\dot{e}$ that contain the same vertices $v_0, v_1, \dots, v_{t-1}$. With this encoding method, we can encode $\sum_{t=1}^{N} \binom{N}{t}(d-1)^t = d^N - 1$ different edges (hyperedges) based on both the different connecting vertices and the different values of $s_{v_0}, \dots, s_{v_{t-1}}$. Here the edge $\dot{e}$ is no longer just a set of vertices which contains two sets $V_{\dot{e}} = \{v_0, \dots, v_{t-1}\}$ and $S_{\dot{e}} = \{s_{v_0}, s_{v_1}, \dots, s_{v_{t-1}}\}$ and is different from the edge in Sec. II A. Therefore, the edges e and $\dot{e}$ are used to distinguish the edges in qudit graph (hypergraph) states and the proposed multigraph (multihypergraph) states, respectively. We introduce $k \in V_{\dot{e}}$ to represent that the edge $\dot{e}$ connects vertex k. However, in regular graphs and hypergraphs, the symbol V_e is not needed because an edge in these kinds of graphs itself is a set containing some vertices, where $e = \{v_0, \dots, v_{t-1}\}$, as described in Sec. II A. Consequently, the multigraph or multihypergraph we use can be represented as a set pair comprising vertices and edges, $\hat{G} = (V, \hat{E})$ or $\hat{\hat{G}} = (V, \hat{\hat{E}})$. Here $t = 2$ characterizes a multigraph, whereas $t \in \mathbb{Z}_{N+1}$ typifies a multihypergraph.

A. Quantum operations generalized from the Z and controlled-Z gates

Howard and Vala [49] explored qudit versions of the qubit $\pi/8$ gate within the Clifford hierarchy [50]. They proposed a formulation for the qudit gate

$$U_h = U_{(h_0,h_1,\dots,h_{d-1})} = \sum_{k=0}^{d-1} \omega_d^{h_k} |k\rangle\langle k|, \quad (7)$$

when $d > 3$. Here U_h based on $h = (h_0, h_1, \dots, h_{d-1})$ forms a series of quantum gates with notable applications in magic-state distillation [51]. We define $h_k = \sum_{j=0}^{\lambda} a_j k^j$, where $\lambda \in \mathbb{Z}_d^*$ and $\sum_{j=1}^{\lambda} a_j k^j$ represents a polynomial of degree λ in the integer residual ring $\mathbb{Z}_d$. For $h_k = k^2/2 + k/2$ and $h_k = k^3/6$, U_h corresponds to the d-dimensional S gate and $\pi/8$ gate in Refs. [52,53], respectively. (We utilize the definition of d-dimensional S in Ref. [52], which is slightly different from that in Ref. [54].) In this paper U_h is employed with $h_k = k^\lambda$. Then we define the generalized Z operation

$$^{\lambda}Z := \sum_{k=0}^{d-1} \omega_d^{k^\lambda} |k\rangle\langle k|. \quad (8)$$

Specifically, for $\lambda = 1$, $^1Z = Z = \sum_{k=0}^{d-1} \omega_d^k |k\rangle\langle k|$. For $\lambda = 2, 3$, the generalized Z operation can be constructed with some other gates, such as the d-dimensional S [52] and T [53] gates. For an odd prime p,

$$S = \sum_{k=0}^{p-1} \omega_d^{k(k+1)/2} |k\rangle\langle k| \quad (9)$$

and

$$T = \begin{cases} \sum_{k=0}^{2} \omega_3^{(1-k)3^{-1}} |k\rangle\langle k|, & p = 3 \\ \sum_{k=0}^{p-1} \omega_p^{k^3 6^{-1}} |k\rangle\langle k|, & p \geqslant 5. \end{cases} \quad (10)$$

Then, based on the d-dimensional S and T gates, we can obtain

$$^{\lambda}Z = \begin{cases} Z^\dagger SS, & \lambda = 2,\ p \geqslant 3 \\ T^6, & \lambda = 3,\ p \geqslant 5 \end{cases} \quad (11)$$

for the p-dimensional quantum system. Based on the generalized Z operation, we define the generalized t-particle qudit controlled-Z operation as

$$^{\Lambda}C^tZ_{\{0,\dots,t-1\}} := \sum_{j_0,\dots,j_{t-1}=0}^{d-1} \left(\omega_d^{\prod_{k=0}^{t-1}(j_k)^{\lambda_k}} |j_0, \dots, j_{t-1}\rangle \langle j_0, \dots, j_{t-1}|\right), \quad (12)$$

where $\Lambda = \{\lambda_0, \dots, \lambda_{t-1}\}$, the superscript t represents a t-particle controlled operation, and the subscript $\{0, \dots, t-1\}$ indicates the position of all the particles. Next we construct the multigraph and multihypergraph states by the generalized Z and controlled-Z operations.

B. Multigraph state

In the multigraph state, the vertex is represented by a qudit single-particle state $|+_d\rangle = d^{-1/2} \sum_{i=0}^{d-1} |i\rangle$. The edges within this framework are delineated through the operation

$$\widehat{CZ}_{(j,k|s_j,s_k)} = {}^{\Lambda}C^2Z_{\{j,k\}} \otimes I_{N-2} = \sum_{i_0,\ldots,i_{N-1}=0}^{d-1} \left(\omega_d^{(i_j)^{s_j}\times(i_k)^{s_k}} |i_0,\ldots,i_{N-1}\rangle\langle i_0,\ldots,i_{N-1}|\right), \tag{13}$$

where $\Lambda = \{s_j, s_k\}$ and I_{N-2} represents the $(N-2)$-particle identity operation. The multigraph state corresponding to multigraph $\hat{G} = (V, \hat{E})$ can be represented by

$$|\hat{G}\rangle = \prod_{\dot{e}\in\hat{E}} \widehat{CZ}_{\dot{e}}^{m_{\dot{e}}} |+_d\rangle^{\otimes N}, \tag{14}$$

where $m_{\dot{e}}$ represents both the weight of an edge and the number of operations performed. The multigraph comprises $\binom{N}{2} \times (d-1)^2$ varieties of edges, and each edge represents a quantum operation that can be executed d times. Therefore, the number of multigraph states is $d^{\binom{N}{2}\times(d-1)^2}$. The stabilizer operator on vertex k for a multigraph state is characterized as

$$\hat{g}_k = \left(\prod_{\dot{e}\in\hat{E}} \widehat{CZ}_{\dot{e}}^{m_{\dot{e}}}\right) X_k \left(\prod_{\dot{e}'\in\hat{E}} \widehat{CZ}_{\dot{e}'}^{d-m_{\dot{e}'}}\right) = X_k \prod_{\dot{e}\in\hat{E},k\in V_{\dot{e}}} \left(\sum_{i_k=0}^{d-1} |i_k\rangle\langle i_k| \otimes ({}^{s_j}Z_{\{j\}})^{m_{\dot{e}}[\sum_{l=0}^{s_k-1}\binom{s_k}{l}(i_k)^l]}\right), \tag{15}$$

where $\dot{e} = (j,k|s_j,s_k)$ and $\dot{e}'$ are any two edges in $\hat{E}$, and ${}^{s_j}Z_{\{j\}}$ represents that the operation ${}^{\lambda}Z$ for $\lambda = s_j$ is executed on the jth particle. The operator $\hat{g}_k$ stabilizes the multigraph state $|\hat{G}\rangle$ as $|\hat{G}\rangle = \hat{g}_k|\hat{G}\rangle$. The proof of Eq. (15) is contained in the proof of Eq. (20) in Appendix B. We provide an illustrative example of a multigraph state with $N = 3$, $d = 3$, and the edge set $\hat{E} = \{\dot{e}_1, \dot{e}_2, \dot{e}_3, \dot{e}_4\}$. Here $\dot{e}_1 = (0,1|1,1)$, $\dot{e}_2 = (0,1|1,2)$, $\dot{e}_3 = (0,1|2,2)$, and $\dot{e}_4 = (0,2|1,1)$ with the weights of edges $m_{\dot{e}_1} = 1$, $m_{\dot{e}_2} = 2$, $m_{\dot{e}_3} = 2$, and $m_{\dot{e}_4} = 1$, and the corresponding quantum operations are

$$\begin{aligned}
\widehat{CZ}_{\dot{e}_1} &= \sum_{i_0,i_1,i_2=0}^{2} \omega_3^{i_0\times i_1} |i_0,i_1,i_2\rangle\langle i_0,i_1,i_2|,\\
\widehat{CZ}_{\dot{e}_2}^2 &= \sum_{i_0,i_1,i_2=0}^{2} \omega_3^{i_0\times i_1^2\times 2} |i_0,i_1,i_2\rangle\langle i_0,i_1,i_2|,\\
\widehat{CZ}_{\dot{e}_3}^2 &= \sum_{i_0,i_1,i_2=0}^{2} \omega_3^{i_0^2\times i_1^2\times 2} |i_0,i_1,i_2\rangle\langle i_0,i_1,i_2|,\\
\widehat{CZ}_{\dot{e}_4} &= \sum_{i_0,i_1,i_2=0}^{2} \omega_3^{i_0\times i_2} |i_0,i_1,i_2\rangle\langle i_0,i_1,i_2|.
\end{aligned} \tag{16}$$

Then we can obtain the multigraph state

$$\begin{aligned}
|\hat{G}\rangle =& \widehat{CZ}_{\dot{e}_4}\widehat{CZ}_{\dot{e}_3}^2\widehat{CZ}_{\dot{e}_2}^2\widehat{CZ}_{\dot{e}_1}|+_3\rangle^{\otimes 3}\\
=& \frac{1}{3^{3/2}}\big(|000\rangle + |001\rangle + |002\rangle + |010\rangle + |011\rangle + |012\rangle + |020\rangle + |021\rangle + |022\rangle\\
&+ |100\rangle + \omega_3^{1\times 1}|101\rangle + \omega_3^{1\times 2}|102\rangle + \omega_3^{1\times 1+1\times 1^2\times 2+1^2\times 1^2\times 2}|110\rangle\\
&+ \omega_3^{1\times 1+1\times 1^2\times 2+1^2\times 1^2\times 2+1\times 1}|111\rangle + \omega_3^{1\times 1+1\times 1^2\times 2+1^2\times 1^2\times 2+1\times 2}|112\rangle\\
&+ \omega_3^{1\times 2+1\times 2^2\times 2+1^2\times 2^2\times 2}|120\rangle + \omega_3^{1\times 2+1\times 2^2\times 2+1^2\times 2^2\times 2+1\times 1}|121\rangle\\
&+ \omega_3^{1\times 2+1\times 2^2\times 2+1^2\times 2^2\times 2+1\times 2}|122\rangle + |200\rangle + \omega_3^{2\times 1}|201\rangle + \omega_3^{2\times 2}|202\rangle\\
&+ \omega_3^{2\times 1+2\times 1^2\times 2+2^2\times 1^2\times 2}|210\rangle + \omega_3^{2\times 1+2\times 1^2\times 2+2^2\times 1^2\times 2+2\times 1}|211\rangle\\
&+ \omega_3^{2\times 1+2\times 1^2\times 2+2^2\times 1^2\times 2+2\times 2}|212\rangle + \omega_3^{2\times 2+2\times 2^2\times 2+2^2\times 2^2\times 2}|220\rangle\\
&+ \omega_3^{2\times 2+2\times 2^2\times 2+2^2\times 2^2\times 2+2\times 1}|221\rangle + \omega_3^{2\times 2+2\times 2^2\times 2+2^2\times 2^2\times 2+2\times 2}|222\rangle\big).
\end{aligned} \tag{17}$$

To explain this equation more clearly, we take the phase $|122\rangle$ as an example. For the phase $|122\rangle$, $\widehat{CZ}_{\dot{e}_1}|122\rangle = \omega_3^{1\times 2}|122\rangle$, $\widehat{CZ}_{\dot{e}_2}^2|122\rangle = \omega_3^{1\times 2^2\times 2}|122\rangle$, $\widehat{CZ}_{\dot{e}_3}^2|122\rangle = \omega_3^{1^2\times 2^2\times 2}|122\rangle$, and $\widehat{CZ}_{\dot{e}_4}\ |122\rangle = \omega_3^{1\times 2}|122\rangle$. Consequently, $\widehat{CZ}_{\dot{e}_4}\widehat{CZ}_{\dot{e}_3}^2\widehat{CZ}_{\dot{e}_2}^2\widehat{CZ}_{\dot{e}_1}|122\rangle = \omega_3^{1\times 2+1\times 2^2\times 2+1^2\times 2^2\times 2+1\times 2}|122\rangle$. Similarly, the phases of all the other superposition terms can be obtained in this way.

By mapping the set $S_{\dot{e}} = \{s_{v_0}, s_{v_1}\}$ to a d-base integer $(s_{v_0}s_{v_1})_d$, which is used as a parameter for the thickness of edges, we can create an intuitive diagram of the multigraph $\hat{G}$ in Fig. 3(a).

C. Multihypergraph state

In the multihypergraph state, the hyperedges are delineated through the operation

$$\widehat{\widehat{CZ}}_{(v_0,\ldots,v_{t-1}|s_0,\ldots,s_{t-1})} = {}^{\Lambda}C^tZ_{\{v_0,\ldots,v_{t-1}\}} \otimes I_{N-t} = \sum_{i_0,\ldots,i_{N-1}=0}^{d-1} \left(\omega_d^{\prod_{j=0}^{t-1}(i_{v_j})^{s_j}} |i_0,\ldots,i_{N-1}\rangle\langle i_0,\ldots,i_{N-1}|\right), \tag{18}$$

where $\Lambda = \{s_{v_0},\ldots,s_{v_{t-1}}\}$ and I_{N-t} represents the $(N-t)$-particle identity operation. The multihypergraph state, cor-

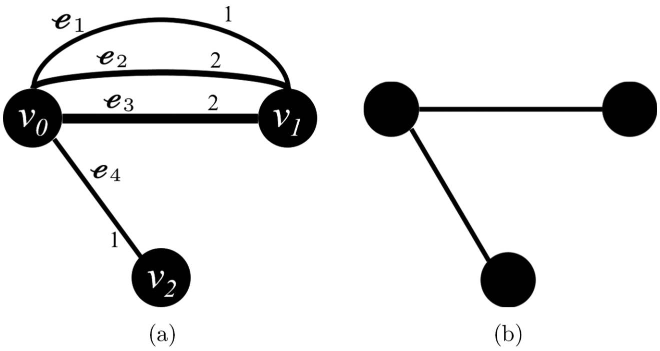


FIG. 3. (a) Three-vertex multigraph with four edges $\dot e_1$, $\dot e_2$, $\dot e_3$, and $\dot e_4$, labeled with different thickness values of $11_3 = 4_{10}$, $12_3 = 5_{10}$, $22_3 = 8_{10}$, and $11_3 = 4_{10}$, respectively, where the subscripts 3 and 10 represent the numbers in ternary and decimal notation, respectively. The number on each edge represents the weight of that edge. (b) Three-vertex graph with a uniform thickness parameter of the edge.

responding to the structure of multihypergraph $\hat{\hat{G}} = (V, \hat{\hat{E}})$, is characterized as

$$|\hat{\hat{G}}\rangle = \left(\prod_{\dot e\in\hat{\hat{E}}} \widehat{\widehat{CZ}}_{\dot e}^{m_{\dot e}}\right)|+_d\rangle^{\otimes N}. \tag{19}$$

The multihypergraph contains $\sum_{t=1}^{N}\binom{N}{t}(d-1)^t = d^N - 1$ varieties of edges or hyperedges and each one represents a unique quantum operation that can be executed d times. Therefore, the number of multihypergraph states is d^{d^N-1}. The stabilizer operator on vertex k for a multihypergraph state can be represented as

$$\begin{aligned}\hat{\hat{g}}_k &= \left(\prod_{\dot e\in\hat{\hat{E}}} \widehat{\widehat{CZ}}_{\dot e}^{m_{\dot e}}\right) X_k \left(\prod_{\dot e'\in\hat{\hat{E}}} \widehat{\widehat{CZ}}_{\dot e'}^{d-m_{\dot e'}}\right)\\ &= X_k \prod_{\dot e\in\hat{\hat{E}}, k\in V_{\dot e}} \left(\sum_{i_k=0}^{d-1} |i_k\rangle\langle i_k| \otimes \widehat{\widehat{CZ}}_{(V_{\dot e}\backslash\{k\}|S_{\dot e}\backslash\{s_k\})}^{m_{\dot e}[\sum_{l=0}^{s_k-1}\binom{s_k}{l}(i_k)^l]}\right),\end{aligned} \tag{20}$$

where $\dot e = (v_0, \dots, v_{t-1}|s_{v_0}, \dots, s_{v_{t-1}})$ and $\dot e'$ are any two hyperedges in $\hat{\hat{E}}$, and $(V_{\dot e}\backslash\{k\}|S_{\dot e}\backslash\{s_k\})$ represents the edge from $\dot e$ by only deleting the vertex k. The operator $\hat{\hat{g}}_k$ stabilizes the multigraph state $|\hat{\hat{G}}\rangle$, as $|\hat{\hat{G}}\rangle = \hat{\hat{g}}_k|\hat{\hat{G}}\rangle$. The proof of Eq. (20) is delineated in Appendix B. We consider an example of a multihypergraph state with $N = 3$, $d = 3$, and the edge set $\hat{\hat{E}} = \{\dot e_1, \dot e_2, \dot e_3, \dot e_4\}$. Here $\dot e_1 = (0, 1|1, 2)$, $\dot e_2 = (0, 1, 2|1, 2, 1)$, $\dot e_3 = (0, 1, 2|2, 1, 2)$, and $\dot e_4 = (2|2)$, with the weights of edges $m_{\dot e_1} = 1$, $m_{\dot e_2} = 2$, $m_{\dot e_3} = 2$, and $m_{\dot e_4} = 2$. Then the quantum operations to which the edges respond are

$$\begin{aligned}\widehat{\widehat{CZ}}_{\dot e_1} &= \sum_{i_0,i_1,i_2=0}^{2} \omega_3^{i_0\times i_1^2}|i_0, i_1, i_2\rangle\langle i_0, i_1, i_2|,\\ \widehat{\widehat{CZ}}_{\dot e_2}^2 &= \sum_{i_0,i_1,i_2=0}^{2} \omega_3^{i_0\times i_1^2\times i_2\times 2}|i_0, i_1, i_2\rangle\langle i_0, i_1, i_2|,\\ \widehat{\widehat{CZ}}_{\dot e_3}^2 &= \sum_{i_0,i_1,i_2=0}^{2} \omega_3^{i_0^2\times i_1\times i_2^2\times 2}|i_0, i_1, i_2\rangle\langle i_0, i_1, i_2|,\\ \widehat{\widehat{CZ}}_{\dot e_4}^2 &= \sum_{i_0,i_1,i_2=0}^{2} \omega_3^{i_2^2\times 2}|i_0, i_1, i_2\rangle\langle i_0, i_1, i_2|.\end{aligned} \tag{21}$$

Then we can obtain the multihypergraph state

$$\begin{aligned}|\hat{\hat{G}}\rangle =& \widehat{\widehat{CZ}}_{\dot e_4}^2\widehat{\widehat{CZ}}_{\dot e_3}^2\widehat{\widehat{CZ}}_{\dot e_2}^2\widehat{\widehat{CZ}}_{\dot e_1}|+_3\rangle^{\otimes 3}\\ =& \frac{1}{3^{3/2}}\big(|000\rangle + \omega_3^{1^2\times 2}|001\rangle + \omega_3^{2^2\times 2}|002\rangle + |010\rangle + \omega_3^{1^2\times 2}|011\rangle + \omega_3^{2^2\times 2}|012\rangle\\ &+ |020\rangle + \omega_3^{1^2\times 2}|021\rangle + \omega_3^{2^2\times 2}|022\rangle + |100\rangle + \omega_3^{1^2\times 2}|101\rangle + \omega_3^{2^2\times 2}|102\rangle\\ &+ \omega_3^{1\times 1^2}|110\rangle + \omega_3^{1\times 1^2+1\times 1^2\times 1\times 2+1^2\times 1\times 1^2\times 2+1^2\times 2}|111\rangle\\ &+ \omega_3^{1\times 1^2+1\times 1^2\times 2\times 2+1^2\times 1\times 2^2\times 2+2^2\times 2}|112\rangle + \omega_3^{1\times 2^2}|120\rangle\\ &+ \omega_3^{1\times 2^2+1\times 2^2\times 1\times 2+1^2\times 2\times 1^2\times 2+1^2\times 2}|121\rangle\\ &+ \omega_3^{1\times 2^2+1\times 2^2\times 2\times 2+1^2\times 2\times 2^2\times 2+2^2\times 2}|122\rangle + |200\rangle + \omega_3^{1^2\times 2}|201\rangle\\ &+ \omega_3^{2^2\times 2}|202\rangle + \omega_3^{2\times 1^2}|210\rangle + \omega_3^{2\times 1^2+2\times 1^2\times 1\times 2+2^2\times 1\times 1^2\times 2+1^2\times 2}|211\rangle\\ &+ \omega_3^{2\times 1^2+2\times 1^2\times 2\times 2+2^2\times 1\times 2^2\times 2+2^2\times 2}|212\rangle + \omega_3^{2\times 2^2}|220\rangle\\ &+ \omega_3^{2\times 2^2+2\times 2^2\times 1\times 2+2^2\times 2\times 1^2\times 2+1^2\times 2}|221\rangle\\ &+ \omega_3^{2\times 2^2+2\times 2^2\times 2\times 2+2^2\times 2\times 2^2\times 2+2^2\times 2}|222\rangle\big).\end{aligned} \tag{22}$$

To explain this equation more clearly, we take the phase $|222\rangle$ as an example. For the phase $|222\rangle$, $\widehat{\widehat{CZ}}_{\dot e_1}|222\rangle = \omega_3^{2\times 2^2}|222\rangle$, $\widehat{\widehat{CZ}}_{\dot e_2}^2|222\rangle = \omega_3^{2\times 2^2\times 2\times 2}|222\rangle$, $\widehat{\widehat{CZ}}_{\dot e_3}^2|222\rangle = \omega_3^{2^2\times 2\times 2^2\times 2}|222\rangle$, and $\widehat{\widehat{CZ}}_{\dot e_4}^2|222\rangle = \omega_3^{2^2\times 2}|222\rangle$. Consequently, $\widehat{\widehat{CZ}}_{\dot e_4}^2\widehat{\widehat{CZ}}_{\dot e_3}^2\widehat{\widehat{CZ}}_{\dot e_2}^2\widehat{\widehat{CZ}}_{\dot e_1}|222\rangle =$

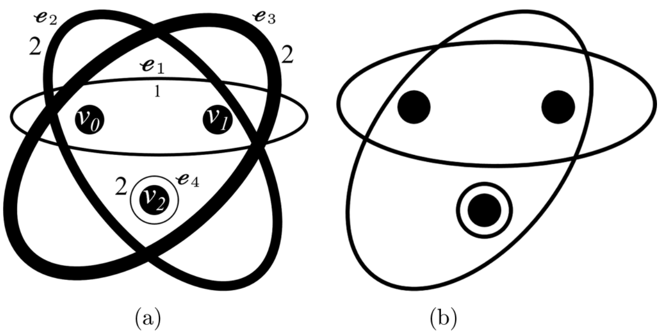


FIG. 4. (a) Three-vertex multihypergraph with four edges $\dot{e}_1$, $\dot{e}_2$, $\dot{e}_3$, and $\dot{e}_4$, labeled with different thickness values of $12_3 = 5_{10}$, $121_3 = 16_{10}$, $212_3 = 23_{10}$, and $2_3 = 2_{10}$, respectively. The number on each edge represents the weight of that edge. (b) Three-vertex hypergraph with a uniform thickness parameter of the edge.

$\omega_3^{2\times 2^2+2\times 2^2\times 2\times 2+2^2\times 2\times 2^2\times 2+2^2\times 2}|222\rangle$. Similarly, the phases of all the other superposition terms can be obtained in this way.

By mapping the set $S_{\dot{e}} = \{s_{v_0}, \ldots, s_{v_{t-1}}\}$ to a d-base integer $(s_{v_0}, \ldots, s_{v_{t-1}})_d$, which is used as a parameter for the thickness of hyperedges, we can create an intuitive diagram of the multihypergraph $\hat{\hat{G}}$ in Fig. 4(a).

Here we use the terms multigraph and multihypergraph states because the multigraph and multihypergraph we use belong to the second kind of multigraph we described in Sec. II A. In Ref. [43], Xiong *et al.* proposed the concept of multihypergraph states and pointed out that the multihypergraph states are equivalent to qudit hypergraph states. However, our proposed multihypergraph states contain more states than qudit hypergraph states and can represent the quantum states increasing from d^{2^N-1} to d^{d^N-1}. Similarly, the proposed multigraph states also contain more states than qudit graph states [39–41] and can represent the quantum states increasing from $d^{\binom{N}{2}}$ to $d^{\binom{N}{2}\times(d-1)^2}$. The difference between our proposed multigraph (multihypergraph) states and qudit graph (hypergraph) states lies in the proposed quantum operations, the proposed quantum states based on these operations, and the proposed stabilizers of the quantum states, which are all different. Additionally, when the values of N integral variables are all the constant 1's, the proposed multigraph and multihypergraph states will degenerate into the qudit graph states in Refs. [39–41] and the qudit hypergraph states in Refs. [42–44], respectively. Finally, when d is an odd prime, the proposed multihypergraph states are equivalent to GREWSSs, while qudit hypergraph states [42–44] form a proper subset of GREWSSs, which we introduce in next section.

IV. RELATIONSHIP BETWEEN GREWSS AND MULTIHYPERGRAPH STATES

Rossi *et al.* [6] highlighted the one-to-one correspondence between qubit hypergraph states and REWSs. Later, Xiong *et al.* [43] indicated that qudit hypergraph states are a subset of GREWSs. In this section we first provide rigorous proof of the one-to-one correspondence between the proposed multihypergraph states and GREWSs. Furthermore, using the same method, we provide proof of the relationship between qudit hypergraph states and GREWSs, as well as the relationship between qubit hypergraph states and REWSs, in Secs. IV B and IV C, respectively.

A. Relationship between multihypergraph states and GREWSs

In this section we first state the theorem establishing the bijective relationship between the proposed multihypergraph states and GREWSs. Subsequently, linear equations and mathematical induction are developed to demonstrate this theorem. An arbitrary GREWS is defined as

$$|f_d\rangle = \frac{1}{2^{n/2}} \sum_{i_0,\ldots,i_{N-1}=0}^{d-1} \omega_d{}^{f(i_0,\ldots,i_{N-1})}|i_0, \ldots, i_{N-1}\rangle, \tag{23}$$

where $f(i_0, \ldots, i_{N-1})$ is an integer function of N independent variables $i_0, \ldots, i_{N-1}$ in the integer residual ring $\mathbb{Z}_d$. By isolating the coefficient of the term $|0, \ldots, 0\rangle$ as the global phase and setting $f(0, \ldots, 0) = 0$, the number of GREWSs is d^{d^N-1}. Notably, given that $i_0, \ldots, i_{N-1}$ are not simultaneously zero, we define the condition $i_{l_0}, \ldots, i_{l_{t'-1}} > 0$ to contain all nonzero terms in $i_0, \ldots, i_{N-1}$, where $l_0 < l_1 < \cdots < l_{t'-1} \in \mathbb{Z}_N$, $t' \in \mathbb{Z}_{N+1}$.

Theorem 1. Any given multihypergraph state $|\hat{\hat{G}}\rangle = (\prod_{\dot{e}\in\hat{\hat{E}}} \widehat{\widetilde{CZ}}_{\dot{e}}^{m_{\dot{e}}})|+\rangle^{\otimes N}$ that corresponds to a multihypergraph $\hat{\hat{G}} = (V, \hat{\hat{E}})$ must be a GREWS. Furthermore, for odd prime $d = q$, every GREWS $|f_d\rangle$ can be associated with a specific multihypergraph state $|\hat{\hat{G}}\rangle$ such that $|f_d\rangle = |\hat{\hat{G}}\rangle$.

Proof. Based on the definitions of multihypergraph states and GREWSs, it is obvious that a multihypergraph state is invariably a GREWS. Therefore, to prove Theorem 1 it is sufficient to demonstrate that each GREWS corresponds to a multihypergraph state. Consider a multihypergraph state $|\hat{\hat{G}}\rangle = (\prod_{\dot{e}\in\hat{\hat{E}}} \widehat{\widetilde{CZ}}_{\dot{e}}^{m_{\dot{e}}})|+_d\rangle^{\otimes N}$, which is constructed by $\{m_{\dot{e}\in\hat{\hat{E}}}\}\{\widetilde{CZ}_{\dot{e}\in\hat{\hat{E}}}\}$ operations such that $|f_d\rangle = |\hat{\hat{G}}\rangle$. Then the coefficients $\{\omega_d{}^{f(i_0,\ldots,i_{N-1})}\}$ of all superposition terms in $|f_d\rangle$ must satisfy

$$\left\{ \sum_{\substack{\{v_0,v_1,\ldots,v_{t-1}\}\in\wp(V)\setminus\varnothing,\\ \{v_0,v_1,\ldots,v_{t-1}\}\subseteq\{l_0,l_1,\ldots,l_{t'-1}\}}} \sum_{s_0,\ldots,s_{t-1}=1}^{d-1} \left(\prod_{j=0}^{t-1} (i_{v_j})^{s_j}\right) m_{(v_0,\ldots,v_{t-1}|s_0,\ldots,s_{t-1})} = f(i_0,\ldots,i_{N-1}) \;\middle|\; t, t' \in \mathbb{Z}_{N+1}^*; \; i_0,\ldots,i_{N-1} \in \mathbb{Z}_d; (i_0,\ldots,i_{N-1}) \neq (0,\ldots,0) \right\}, \tag{24}$$

where the operations are conducted modulo q. Consequently, the equations above formulate nonhomogeneous linear equations with d^N-1 independent variables and d^N-1 equations in the finite field GF_q. Consistent with the aforementioned approach, all nonzero terms $i_0,\ldots,i_{N-1}$ are collectively denoted by $i_{l_0},\ldots,i_{l_{l'-1}}$. The equations are segregated into N groups based on the number of nonzero terms in $i_0,\ldots,i_{N-1}$. Each group, denoted by n ($n\in\mathbb{Z}^*_{N+1}$), comprises $\binom{N}{n}(d-1)^n$ independent variables and $\binom{N}{n}(d-1)^n$ equations. Specifically, in the nth set of equations, the independent variables of $f(i_0,\ldots,i_{N-1})$, $i_0,\ldots,i_{N-1}$, with n nonzero values, are represented as $i_{l_0},i_{l_1},\ldots,i_{l_{n-1}}$. Equation (24) is solvable if each of the N groups of equations has a solution. First, for the case of $n=1$, the initial set of equations is

$$\left\{\sum_{s_{v_0}=1}^{d-1}(i_{v_0})^{s_{v_0}}m_{(v_0|s_0)}=f(i_0,\ldots,i_{N-1})\,\middle|\,v_0\in V,\; i_{v_0}\in\mathbb{Z}^*_d; i_j=0, j\in V\backslash\{v_0\}\right\}, \tag{25}$$

whose coefficient matrix is

$$A=I_{\binom{N}{1}\times\binom{N}{1}}\otimes\begin{bmatrix}1&1&\cdots&1\\2&2^2&\cdots&2^{d-1}\\\vdots&\vdots&\ddots&\vdots\\(d-1)&(d-1)^2&\cdots&(d-1)^{d-1}\end{bmatrix}, \tag{26}$$

where $I_{\binom{N}{1}\times\binom{N}{1}}$ represents the $\binom{N}{1}\times\binom{N}{1}$ identity matrix. The coefficient matrix possesses full rank, $\text{rank}(A)=N\times(d-1)$, ensuring the solvability of these equations. Then, if the first $n-1$ sets of equations can be solved, the nth set can be simplified to

$$\left\{\sum_{s_0,\ldots,s_{n-1}=1}^{d-1}\left(\prod_{j=0}^{n-1}(i_{v_j})^{s_j}\right)m_{(v_0,\ldots,v_{n-1}|s_0,\ldots,s_{n-1})}=b_{v_0,\ldots,v_{n-1},i_{v_0},\ldots,i_{v_{n-1}}}\,\middle|\,n\in\mathbb{Z}^*_{N+1};\{v_0,\ldots,v_{n-1}\}\in\wp(V)\backslash\varnothing;\; i_{v_0},\ldots,i_{v_{n-1}}\in\mathbb{Z}^*_d; i_j=0, j\in V\backslash\{v_0,\ldots,v_{n-1}\}\right\}, \tag{27}$$

whose coefficient matrix can be derived as

$$B=I_{\binom{N}{n}\times\binom{N}{n}}\otimes\begin{bmatrix}1&1&\cdots&1\\2&2^2&\cdots&2^{d-1}\\\vdots&\vdots&\ddots&\vdots\\(d-1)&(d-1)^2&\cdots&(d-1)^{d-1}\end{bmatrix}^{\otimes n}, \tag{28}$$

where $b_{v_0,\ldots,v_{n-1},i_{v_0},\ldots i_{v_{n-1}}}$ represents the simplified value to the right of each equation and $I_{\binom{N}{n}\times\binom{N}{n}}$ represents the $\binom{N}{n}\times\binom{N}{n}$ identity matrix. The coefficient matrix exhibits full rank, $\text{rank}(B)=\binom{N}{n}\times(d-1)^n$. Therefore, if solutions exist for the first $n-1$ sets of equations, the nth set also possesses a solution. Finally, together with the solvability of the first set of equations, all the linear equations are demonstrated to be solvable. This means that Eq. (24) is resolvable using mathematical induction. Hence, when d is an odd prime, every GREWS $|f_d\rangle$ corresponds to a specific multihypergraph state $|\hat{\tilde{G}}\rangle$, satisfying $|f_d\rangle=|\hat{\tilde{G}}\rangle$. Given that a multihypergraph state is inherently a GREWS, the validity of Theorem 1 is confirmed. ■

The two-particle, three-dimensional GREWS

$$|f_3\rangle=\tfrac{1}{3}(|00\rangle+e^{2\pi\mathbf{i}/3}|01\rangle+|02\rangle+e^{2\pi\mathbf{i}/3}|10\rangle+e^{2\pi\mathbf{i}/3}|11\rangle+|12\rangle+|20\rangle+e^{2\pi\mathbf{i}/3}|21\rangle+|22\rangle) \tag{29}$$

is taken as an example. Then the coefficients $\{\omega_3{}^{f(i_0,i_1)}\}$ of all superposition terms in $|f_3\rangle$ satisfy $f(1,0)=1$, $f(2,0)=0$, $f(0,1)=1$, $f(0,2)=0$, $f(1,1)=1$, $f(1,2)=0$, $f(2,1)=1$, and $f(2,2)=0$. Here we let $\dot{e}_1=(0|1)$, $\dot{e}_2=(0|2)$, $\dot{e}_3=(1|1)$, $\dot{e}_4=(1|2)$, $\dot{e}_5=(0,1|1,1)$, $\dot{e}_6=(0,1|1,2)$, $\dot{e}_7=(0,1|2,1)$, and $\dot{e}_8=(0,1|2,2)$. Since $|f_3\rangle=(\prod_{j=1}^{8}\hat{\widetilde{CZ}}{}_{\dot{e}_j}^{m_{\dot{e}_j}})|+_3\rangle^{\otimes 2}$ and

$$\hat{\widetilde{CZ}}_{\dot{e}_1}=\sum_{i_0,i_1=0}^{2}\omega_3^{i_0}|i_0,i_1\rangle\langle i_0,i_1|,$$

$$\hat{\widetilde{CZ}}_{\dot{e}_2}=\sum_{i_0,i_1=0}^{2}\omega_3^{i_0^2}|i_0,i_1\rangle\langle i_0,i_1|,$$

$$\hat{\widetilde{CZ}}_{\dot{e}_3}=\sum_{i_0,i_1=0}^{2}\omega_3^{i_1}|i_0,i_1\rangle\langle i_0,i_1|,$$

$$\hat{\widetilde{CZ}}_{\dot{e}_4}=\sum_{i_0,i_1=0}^{2}\omega_3^{i_1^2}|i_0,i_1\rangle\langle i_0,i_1|,$$

$$\hat{\widetilde{CZ}}_{\dot{e}_5}=\sum_{i_0,i_1=0}^{2}\omega_3^{i_0\times i_1}|i_0,i_1\rangle\langle i_0,i_1|,$$

$$\hat{\widetilde{CZ}}_{\dot{e}_6}=\sum_{i_0,i_1=0}^{2}\omega_3^{i_0\times i_1^2}|i_0,i_1\rangle\langle i_0,i_1|,$$

$$\widehat{\widehat{CZ}}_{\dot{e}_7}=\sum_{i_0,i_1=0}^{2}\omega_3^{i_0^2\times i_1}|i_0,i_1\rangle\langle i_0,i_1|,\qquad \widehat{\widehat{CZ}}_{\dot{e}_8}=\sum_{i_0,i_1=0}^{2}\omega_3^{i_0^2\times i_1^2}|i_0,i_1\rangle\langle i_0,i_1|, \tag{30}$$

we can obtain the linear equations

$$\begin{gathered}1^1\times m_{\dot{e}_1}+1^2\times m_{\dot{e}_2}=1,\quad 2^1\times m_{\dot{e}_1}+2^2\times m_{\dot{e}_2}=0,\quad 1^1\times m_{\dot{e}_3}+1^2\times m_{\dot{e}_4}=1,\quad 2^1\times m_{\dot{e}_3}+2^2\times m_{\dot{e}_4}=0,\\ 1^1\times m_{\dot{e}_1}+1^2\times m_{\dot{e}_2}+1^1\times m_{\dot{e}_3}+1^2\times m_{\dot{e}_4}+1^1\times 1^1\times m_{\dot{e}_5}+1^1\times 1^2\times m_{\dot{e}_6}+1^2\times 1^1\times m_{\dot{e}_7}+1^2\times 1^2\times m_{\dot{e}_8}=1,\\ 1^1\times m_{\dot{e}_1}+1^2\times m_{\dot{e}_2}+2^1\times m_{\dot{e}_3}+2^2\times m_{\dot{e}_4}+1^1\times 2^1\times m_{\dot{e}_5}+1^1\times 2^2\times m_{\dot{e}_6}+1^2\times 2^1\times m_{\dot{e}_7}+1^2\times 2^2\times m_{\dot{e}_8}=0,\\ 2^1\times m_{\dot{e}_1}+2^2\times m_{\dot{e}_2}+1^1\times m_{\dot{e}_3}+1^2\times m_{\dot{e}_4}+2^1\times 1^1\times m_{\dot{e}_5}+2^1\times 1^2\times m_{\dot{e}_6}+2^2\times 1^1\times m_{\dot{e}_7}+2^2\times 1^2\times m_{\dot{e}_8}=1,\\ 2^1\times m_{\dot{e}_1}+2^2\times m_{\dot{e}_2}+2^1\times m_{\dot{e}_3}+2^2\times m_{\dot{e}_4}+2^1\times 2^1\times m_{\dot{e}_5}+2^1\times 2^2\times m_{\dot{e}_6}+2^2\times 2^1\times m_{\dot{e}_7}+2^2\times 2^2\times m_{\dot{e}_8}=0.\end{gathered} \tag{31}$$

With a simple calculation, we can derive $m_{\dot{e}_1}=2$, $m_{\dot{e}_2}=2$, $m_{\dot{e}_3}=2$, $m_{\dot{e}_4}=2$, $m_{\dot{e}_5}=0$, $m_{\dot{e}_6}=1$, $m_{\dot{e}_7}=0$, and $m_{\dot{e}_8}=1$. Consequently, the two-particle, three-dimensional GREWS $|f_3\rangle$ can be constructed as

$$\begin{aligned}\widehat{\widehat{CZ}}_{\dot{e}_8}\widehat{\widehat{CZ}}_{\dot{e}_6}\widehat{\widehat{CZ}}^2_{\dot{e}_4}\widehat{\widehat{CZ}}^2_{\dot{e}_3}\widehat{\widehat{CZ}}^2_{\dot{e}_2}\widehat{\widehat{CZ}}^2_{\dot{e}_1}|+_3\rangle^{\otimes 2}=&\tfrac{1}{3}\big(|00\rangle+\omega_3^{1\times 2+1^2\times 2}|01\rangle+\omega_3^{2\times 2+2^2\times 2}|02\rangle+\omega_3^{1\times 2+1^2\times 2}|10\rangle\\ &+\omega_3^{1\times 2+1^2\times 2+1\times 2+1^2\times 2+1\times 1^2+1^2\times 1^2}|11\rangle\\ &+\omega_3^{1\times 2+1^2\times 2+2\times 2+2^2\times 2+1\times 2^2+1^2\times 2^2}|12\rangle+\omega_3^{2\times 2+2^2\times 2}|20\rangle\\ &+\omega_3^{2\times 2+2^2\times 2+1\times 2+1^2\times 2+2\times 1^2+2^2\times 1^2}|21\rangle\\ &+\omega_3^{2\times 2+2^2\times 2+2\times 2+2^2\times 2+2\times 2^2+2^2\times 2^2}|22\rangle\big)\\ =&\tfrac{1}{3}(|00\rangle+\omega_3|01\rangle+|02\rangle+\omega_3|10\rangle+\omega_3|11\rangle+|12\rangle+|20\rangle\omega_3|21\rangle+|22\rangle)=|f_3\rangle.\end{aligned} \tag{32}$$

Next we examine the relationship between the GREWSs and the proposed multihypergraph states in scenarios where d is a composite. By selecting $d=4$ and $d=6$ as examples, we can derive the coefficient matrices of Eq. (27) as

$$I_{\binom{N}{n}\times\binom{N}{n}}\otimes\begin{bmatrix}1&1&1\\2&0&0\\3&1&3\end{bmatrix}^{\otimes n},\quad I_{\binom{N}{n}\times\binom{N}{n}}\otimes\begin{bmatrix}1&1&1&1&1\\2&4&2&4&2\\3&3&3&3&3\\4&4&4&4&4\\5&1&5&1&5\end{bmatrix}^{\otimes n}. \tag{33}$$

In the case of $d=4$, element 2 lacks a multiplicative inverse within the integer residual ring $\mathbb{Z}_4$. Similarly, when $d=6$, elements 2, 3, and 4 are devoid of multiplicative inverses in the integer residual ring $\mathbb{Z}_6$. Under these conditions, the system of equations (27) is inevitably unsolvable, implying that some of the GREWSs cannot be realized through multihypergraph states. For instance, with $d=4$ and $N=1$, it is infeasible to construct the specified GREWS $|f_4\rangle=\frac{1}{2}[|0\rangle+e^{2\pi\mathbf{i}/4}|1\rangle+e^{2\pi\mathbf{i}/4}|2\rangle+(e^{2\pi\mathbf{i}/4})^2|3\rangle]$ by a multihypergraph state because the corresponding linear equations

$$\begin{aligned}1\times m_{(0|1)}+1\times m_{(0|2)}+1\times m_{(0|3)}&=1,\\ 2\times m_{(0|1)}+0\times m_{(0|2)}+0\times m_{(0|3)}&=1,\\ 3\times m_{(0|1)}+1\times m_{(0|2)}+3\times m_{(0|3)}&=2\end{aligned} \tag{34}$$

are unsolvable. Based on the preceding analysis, it is discerned that the number of multihypergraph states is equivalent to the number of GREWSs, both being d^{d^N-1}. Furthermore, multihypergraph states are invariably GREWSs. Consequently, some of these multihypergraph states form identical GREWSs. For instance, by considering the scenario where $d=4$ and $N=1$, we observe

$$\begin{aligned}1\times m_{(0|1)}+1\times m_{(0|2)}+1\times m_{(0|3)}&=1,\\ 2\times m_{(0|1)}+0\times m_{(0|2)}+0\times m_{(0|3)}&=2,\\ 3\times m_{(0|1)}+1\times m_{(0|2)}+3\times m_{(0|3)}&=1,\end{aligned} \tag{35}$$

which admits two distinct solutions (1,3,1) and (3,1,1), indicating that two separate multihypergraph states correspond to an identical GREWS $|f_4\rangle=\frac{1}{2}[|0\rangle+e^{2\pi\mathbf{i}/4}|1\rangle+(e^{2\pi\mathbf{i}/4})^2|2\rangle+e^{2\pi\mathbf{i}/4}|3\rangle]=\widehat{\widehat{CZ}}_{(0|1)}\widehat{\widehat{CZ}}^3_{(0|2)}\widehat{\widehat{CZ}}_{(0|3)}|+_4\rangle=\widehat{\widehat{CZ}}^3_{(0|1)}\widehat{\widehat{CZ}}_{(0|2)}\widehat{\widehat{CZ}}_{(0|3)}|+_4\rangle$. This observation leads to the inference that when d is a composite number, the proposed multihypergraph states form a subset of GREWSs, with certain multihypergraph states corresponding to the same GREWS.

B. Relationship between qudit hypergraph states and GREWSs

In this section the relationship between GREWSs and qudit hypergraph states is examined. The reason why GREWSs are more numerous and which GREWSs cannot be generated from qudit hypergraph states are also revealed. Consistent with the aforementioned approach, all nonzero terms in $i_0, \ldots, i_{N-1}$ are collectively denoted by $i_{l_0}, \ldots, i_{l_{t'-1}}$. Considering a qudit hypergraph state $|\tilde{G}\rangle = (\prod_{e\in\tilde{E}} \widetilde{CZ}_e^{m_e})|+_d\rangle^{\otimes N}$, constructed by $\{m_{e\in\tilde{E}}\}\{\widetilde{CZ}_{e\in\tilde{E}}\}$ operations such that $|f_d\rangle = |\tilde{G}\rangle$, the coefficients $\{\omega_d{}^{f(i_0,\ldots,i_{N-1})}\}$ in all superposition terms of $|f_d\rangle$ must satisfy

$$\left\{ \sum_{\substack{\{v_0,v_1,\ldots,v_{t-1}\}\in\wp(V)\setminus\varnothing,\\ \{v_0,v_1,\ldots,v_{t-1}\}\subseteq\{l_0,l_1,\ldots,l_{t'-1}\}}} \left(\prod_j^t i_{v_j}\right) m_{\{v_0,v_1,\ldots,v_{t-1}\}} = f(i_0,\ldots,i_{N-1}) \,\middle|\, t,t'\in\mathbb{Z}_{N+1}^*; i_0,\ldots,i_{N-1}\in\mathbb{Z}_d; (i_0,\ldots,i_{N-1})\neq(0,\ldots,0)\right\}, \tag{36}$$

where the operations are conducted modulo d. Consequently, the equations above formulate a system of linear equations with $2^N - 1$ independent variables and $d^N - 1$ equations in the integer residual ring $\mathbb{Z}_d$. A fundamental requirement for the solvability of the system is that the rank of its coefficient matrix must equal the rank of its augmented matrix. Thus, for the linear equations in Eq. (36) to be solvable, $\{f(i_0, \ldots, i_{N-1})\}$ must adhere to a specific relational criterion. Given that the augmented matrix of Eq. (36) has a maximum rank of $2^N - 1$, qudit hypergraph states can construct at most d^{2^N-1} of the GREWSs (the total number of GREWSs is d^{d^N-1}). For instance, with $N = 2$ and $d = 3$, the linear equations over the finite field GF_3 can be derived as

$$\begin{aligned}
1\times m_{\{0\}} &= f(1,0), \quad 2\times m_{\{0\}} = f(2,0),\\
1\times m_{\{1\}} &= f(0,1), \quad 2\times m_{\{1\}} = f(0,2),\\
1\times m_{\{0\}} + 1\times m_{\{1\}} + 1\times 1\times m_{\{0,1\}} &= f(1,1),\\
1\times m_{\{0\}} + 2\times m_{\{1\}} + 1\times 2\times m_{\{0,1\}} &= f(1,2),\\
2\times m_{\{0\}} + 1\times m_{\{1\}} + 2\times 1\times m_{\{0,1\}} &= f(2,1),\\
2\times m_{\{0\}} + 2\times m_{\{1\}} + 2\times 2\times m_{\{0,1\}} &= f(2,2),
\end{aligned} \tag{37}$$

which comprises $2^2 - 1 = 3$ independent variables and $3^2 - 1 = 8$ equations. The system is solvable if and only if $f(i_0, i_1)$ fulfills

$$\begin{aligned}
f(1,0) + f(2,0) &= 0,\\
f(0,1) + f(0,2) &= 0,\\
f(1,0) + f(1,1) + f(1,2) &= 0,\\
f(0,1) + f(1,1) + f(2,1) &= 0,\\
2f(1,0) + f(2,1) + f(2,2) &= 0.
\end{aligned} \tag{38}$$

For example, the probability amplitude of the two-particle, three-dimensional GREWS $|f_3\rangle$ in Eq. (29) fails to conform with Eq. (38). Consequently, it cannot be achieved via the qudit hypergraph state.

This study acknowledges that the resolution of Eq. (36) necessitates consideration of the variable d. Specifically, when d equals an odd prime number q, the solution of Eq. (36) can be determined with relative ease within the finite field GF_q. Conversely, if d is a composite number, the solution becomes more complex within the integer residual ring $\mathbb{Z}_d$. This intricacy is explained in Sec. IV A. Anyway, this analysis has led to a formalization of the GREWSs that are unattainable through qudit hypergraph states. This limitation stems from the discrepancy in the linear system associated with the hyperedge count, which consists of $d^N - 1$ equations but is constrained by only $2^N - 1$ independent variables.

C. Relationship between qubit hypergraph states and REWSs

In this section we provide proof of the relationship between qubit hypergraph states and REWSs using linear equations and mathematical induction from Sec. IV A. Similarly, we first state the theorem establishing the bijective relationship. Consider an arbitrary REWS, defined as

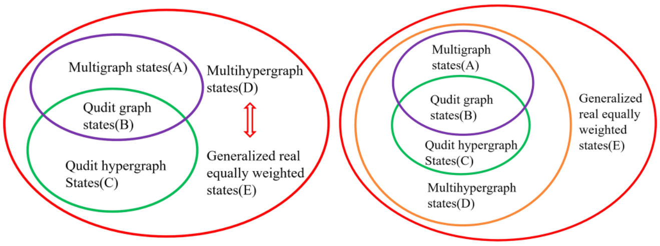
$$|f\rangle = \frac{1}{2^{N/2}} \sum_{i_0,\ldots,i_{N-1}=0}^{1} (-1)^{f(i_0,\ldots,i_{N-1})} |i_0,\ldots,i_{N-1}\rangle, \tag{39}$$

where $f(i_0, \ldots, i_{N-1})$ is a Boolean function of N independent variables $i_0, \ldots, i_{N-1} \in \mathbb{Z}_2$. By setting the coefficient of the term $|0, \ldots, 0\rangle$ as the global phase and letting $f(0, \ldots, 0) = 0$, the total number of REWSs is reduced to 2^{2^N-1}.

Theorem 2. Any given qudit hypergraph state $|\tilde{G}\rangle$ that corresponds to a hypergraph $\tilde{G} = (V, \tilde{E})$ must be a REWS. Furthermore, every REWS $|f\rangle$ can be associated with a specific qudit hypergraph state $|\tilde{G}\rangle$ such that $|f\rangle = |\tilde{G}\rangle$.

Proof. Based on the definitions of hypergraph states and REWSs, it is obvious that a hypergraph state is invariably a REWS [6]. Therefore, to prove Theorem 2 it is sufficient to demonstrate that each REWS corresponds to a hypergraph state. Considering a hypergraph state $|\tilde{G}\rangle = (\prod_{e\in\tilde{E}} \widetilde{CZ}_e^{m_e})|+_d\rangle^{\otimes N}$ constructed by $\{m_{e\in\tilde{E}}\}\{\widetilde{CZ}_{e\in\tilde{E}}\}$ operations such that $|f\rangle = |\tilde{G}\rangle$, the coefficients $\{(-1)^{f(i_0,\ldots,i_{N-1})}\}$ of all superposition terms in $|f\rangle$ must satisfy

$$\left\{ \sum_{\substack{\{v_0,v_1,\ldots,v_{t-1}\}\in\wp(V)\setminus\varnothing,\\ \{v_0,v_1,\ldots,v_{t-1}\}\subseteq\{l_0,l_1,\ldots,l_{t'-1}\}}} m_{\{v_0,v_1,\ldots,v_{t-1}\}} = f(i_0,\ldots,i_{N-1}) \,\middle|\, t,t'\in\mathbb{Z}_{N+1}^*; i_0,\ldots,i_{N-1}\in\mathbb{Z}_2; (i_0,\ldots,i_{N-1})\neq(0,\ldots,0)\right\}, \tag{40}$$



(a) (b)

FIG. 5. Relationships among proposed multigraph states A, qudit graph states B [39–41], qudit hypergraph states C [42–44], proposed multihypergraph states D, and GREWSs E [43], where $B = A \cap C$ and $(A \cup C) \subset D$, when (a) d is an odd prime number, $D = E$, and (b) d is a composite number, $D \subset E$.

which are nonhomogeneous linear equations with $2^N - 1$ independent variables and $2^N - 1$ equations in the finite field GF_2. The equations in Eq. (40) are categorized based on the number of nonzero terms in $i_0, \ldots, i_{N-1}$. The nth ($n \in \mathbb{Z}^*_{N+1}$) group of equations contains $\binom{N}{n}$ independent variables and $\binom{N}{n}$ linear equations. Within the nth subset, all nonzero elements among the independent variables $i_0, \ldots, i_{N-1}$ of $f(i_0, \ldots, i_{N-1})$ are exclusively $i_{l_0} i_{l_1}, \ldots, i_{l_{n-1}}$. The solvability of Eq. (40) depends on each set within the N sets of equations possessing a solution. First, for $n = 1$, the set of equations is formulated as

$$\{m_{\{v_0\}} = f(i_0, \ldots, i_{N-1}) \mid v_0 \in V, i_{v_0} \in \mathbb{Z}^*_2; i_j = 0, \\ j \in V \backslash \{v_0\}\}, \tag{41}$$

whose coefficient matrix is the identity matrix $I_{\binom{N}{1} \times \binom{N}{1}}$, indicating that the equations are solvable. Then, if the first $n - 1$ sets of equations are solvable, we can simplify the nth set to

$$\{m_{\{v_0, v_1, \ldots, v_{n-1}\}} = b_{v_0, v_1, \ldots, v_{n-1}} \mid n \in \mathbb{Z}^*_{N+1}; \\ \{v_0, v_1, \ldots, v_{n-1}\} \in \wp(V) \backslash \varnothing; i_{v_0}, \ldots, i_{v_{n-1}} \in \mathbb{Z}^*_2; \\ i_j = 0, j \in V \backslash \{v_0, v_1, \ldots, v_{n-1}\}\}, \tag{42}$$

whose coefficient matrix is the identity matrix $I_{\binom{N}{n} \times \binom{N}{n}}$, where $b_{v_0, v_1, \ldots, v_{n-1}}$ represents the value on the right-hand side of each equation postsimplification (referred to as $b_{v_0, \ldots, v_{n-1}, i_{v_0}, \ldots, i_{v_{n-1}}}$ in the proof presented in Sec. IV A, here owing to all nonzero terms in $\mathbb{Z}_2$ being 1). Therefore, if solutions exist for the initial $n - 1$ sets of equations, then a solution is also ensured for the nth set. Finally, together with the solvability of the first set of equations, all the linear equations are demonstrated to be solvable. This means that Eq. (15) is resolvable using mathematical induction. Therefore, any REWS $|f\rangle$ can be equated to a hypergraph state $|\tilde{G}\rangle$, constructed through a specified sequence of $\{m_{e \in \tilde{E}}\}\{\widetilde{CZ}_{e \in \tilde{E}}\}$ operations such that $|f\rangle = |\tilde{G}\rangle$. Given that hypergraph states are inherently REWSs [6], Theorem 2 is proven. ■

Finally, we can derive the comprehensive relationships among qudit graph states, multigraph states, hypergraph states, multihypergraph states, and GREWSs in Fig. 5.

V. CONCLUSION

In this paper we proposed two classes of multiparticle entangled states, the multigraph states and the multihypergraph states, each corresponding to their respective constructs in graph theory. We employed linear equations and mathematical induction to demonstrate the one-to-one correspondence between the proposed multihypergraph states and the GREWSs when the dimension of the states is an odd prime. In the same way, we also offered proof of the one-to-one correspondence between the qubit hypergraph states and REWSs. Additionally, we identified the GREWS for which there are no equivalent qudit hypergraph states. The proposed quantum states were constructed by a series of quantum gates that represent the edges in the graph. We provided three examples of the gates that can be derived from the existing qudit Z, S, and T gates. In other words, the implementation of the proposed states is feasible. Given the extensive use of existing graph and hypergraph states in quantum information and computing, our work suggests that the properties of multigraph and multihypergraph states may surpass those of traditional graph and hypergraph states. Consequently, the potential application of the proposed multigraph and multihypergraph states in corresponding fields merits further exploration. For example, we previously used graph states to construct the quantum secret reconstruction protocol [55]. Currently, we are attempting to design the quantum secret reconstruction protocol with the proposed multigraph and multihypergraph states, in which stronger security and higher efficiency are pursued. Given the enhanced information encoding capabilities of the proposed multigraph and multihypergraph states, exploring the design of quantum cryptographic protocols based on these two types of quantum states is worthwhile.

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APPENDIX A: DERIVATION OF EQ. (6)

Proof. If $k \notin e$, $\widetilde{CZ}_e^{m_e} X_k \widetilde{CZ}_e^{d-m_e} = X_k$. If $k \in e$, $\widetilde{CZ}_e^{m_e} X_k \widetilde{CZ}_e^{d-m_e} = X_k X_k^{d-1} \widetilde{CZ}_e^{m_e} X_k \widetilde{CZ}_e^{d-m_e}$. Since

$$\widetilde{CZ}_e = \sum_{i_0, \ldots, i_{N-1}=0}^{d-1} \omega_d^{\prod_{j=0}^{t-1} i_{v_j}} |i_0, \ldots, i_{N-1}\rangle\langle i_0, \ldots, i_{N-1}|, \tag{A1}$$

$$X_k{}^{d-1}\widetilde{CZ}_e^{m_e}X_k=\sum_{i_0,\ldots,i_{N-1}=0}^{d-1}\omega_d^{m_e(\prod_{j=0}^{t-1}i_{v_j})}|i_0,\ldots,i_{k-1},i_k+d-1,i_{k+1},\ldots,i_{N-1}\rangle\langle i_0,\ldots,i_{k-1},i_k-1,i_{k+1},\ldots,i_{N-1}|, \tag{A2}$$

$$\begin{aligned}X_k{}^{d-1}\widetilde{CZ}_e^{m_e}X_k\widetilde{CZ}_e^{d-m_e}&=\sum_{i_0,\ldots,i_{N-1}=0}^{d-1}\omega_d^{m_e(\prod_{j=0}^{t-1}i_{v_j})}|i_0,\ldots,i_{k-1},i_k+d-1,i_{k+1},\ldots,i_{N-1}\rangle\langle i_0,\ldots,i_{k-1},i_k-1,i_{k+1},\ldots,i_{N-1}|\\
&\quad\times\sum_{i'_0,\ldots,i'_{N-1}=0}^{d-1}\omega_d^{(d-m_e)(\prod_{j=0}^{t-1}i'_{v_j})}|i'_0,\ldots,i'_{N-1}\rangle\langle i'_0,\ldots,i'_{N-1}|\\
&=\sum_{i_0,\ldots,i_{N-1}=0}^{d-1}\omega_d^{m_e(\prod_{j=0,j\neq r}^{t-1}i_{v_j})(i_{v_r}+1)+(d-m_e)\prod_{j=0}^{t-1}i_{v_j}}|i_0,\ldots,i_{N-1}\rangle\langle i_0,\ldots,i_{N-1}|\\
&=\sum_{i_0,\ldots,i_{N-1}=0}^{d-1}\omega_d^{m_e(\prod_{j=0,j\neq r}^{t-1}i_{v_j})(i_{v_r}+1-i_{v_r})+d\prod_{j=0}^{t-1}i_{v_j}}|i_0,\ldots,i_{N-1}\rangle\langle i_0,\ldots,i_{N-1}|\\
&=\sum_{i_0,\ldots,i_{N-1}=0}^{d-1}\omega_d^{m_e(\prod_{j=0,j\neq r}^{t-1}i_{v_j})}|i_0,\ldots,i_{N-1}\rangle\langle i_0,\ldots,i_{N-1}|=\widetilde{CZ}_{e\setminus\{k\}}^{m_e},\end{aligned} \tag{A3}$$

where $v_r=k$. Then we obtain $X_kX_k{}^{d-1}\widetilde{CZ}_e^{m_e}X_k\widetilde{CZ}_e^{d-m_e}=X_k\widetilde{CZ}_{e\setminus\{k\}}^{m_e}$ and $\tilde{g}_k=(\prod_{e\in\tilde{E}}\widetilde{CZ}_e^{m_e})X_k(\prod_{e'\in\tilde{E}}\widetilde{CZ}_{e'}^{d-m_{e'}})=X_k\prod_{e\in\tilde{E},k\in e}\widetilde{CZ}_{e\setminus\{k\}}^{m_e}$. ■

APPENDIX B: DERIVATION OF EQ. (20)

Proof. If $k\notin V_{\dot{e}}$, $\widehat{\widetilde{CZ}}_{\dot{e}}^{m_{\dot{e}}}X_k\widehat{\widetilde{CZ}}_{\dot{e}}^{d-m_{\dot{e}}}=X_k$. If $k\in V_{\dot{e}}$, $\widehat{\widetilde{CZ}}_{\dot{e}}^{m_{\dot{e}}}X_k\widehat{\widetilde{CZ}}_{\dot{e}}^{d-m_{\dot{e}}}=X_kX_k{}^{d-1}\widehat{\widetilde{CZ}}_{\dot{e}}^{m_e}X_k\widehat{\widetilde{CZ}}_{\dot{e}}^{d-m_{\dot{e}}}$. Since

$$\widehat{\widetilde{CZ}}_{\dot{e}}=\sum_{i_0,\ldots,i_{N-1}=0}^{d-1}\omega_d^{\prod_{j=0}^{t-1}(i_{v_j})^{s_{v_j}}}|i_0,\ldots,i_{N-1}\rangle\langle i_0,\ldots,i_{N-1}|, \tag{B1}$$

$$X_k^{d-1}\widehat{\widetilde{CZ}}_{\dot{e}}^{m_{\dot{e}}}X_k=\sum_{i_0,i_1,\ldots,i_{N-1}=0}^{d-1}\omega_d^{m_{\dot{e}}[\prod_{j=0}^{j=t-1}(i_{v_j})^{s_{v_j}}]}|i_0,\ldots,i_{k-1},i_k+d-1,i_{k+1},\ldots,i_{N-1}\rangle\langle i_0,\ldots,i_{k-1},i_k-1,i_{k+1},\ldots,i_{N-1}|, \tag{B2}$$

$$\begin{aligned}X_k^{d-1}\widehat{\widetilde{CZ}}_{\dot{e}}^{m_{\dot{e}}}X_k\widehat{\widetilde{CZ}}_{\dot{e}}^{d-m_{\dot{e}}}&=\sum_{i_0,\ldots,i_{N-1}=0}^{d-1}\omega_d^{m_{\dot{e}}[\prod_{j=0}^{t-1}(i_{v_j})^{s_j}]}|i_0,i_{k-1},\ldots,i_k+d-1,i_{k+1},\ldots,i_{N-1}\rangle\langle i_0,\ldots,i_{k-1},i_k-1,i_{k+1},\ldots,i_{N-1}|\\
&\quad\times\sum_{i'_0,i'_1,\ldots,i'_{N-1}=0}^{d-1}\omega_d^{(d-m_{\dot{e}})[\prod_{j=0}^{t-1}(i'_{v_j})^{s_j}]}|i'_0,\ldots,i'_{N-1}\rangle\langle i'_0,\ldots,i'_{N-1}|\\
&=\sum_{i_0,i_1,\ldots,i_{N-1}=0}^{d-1}\omega_d^{m_{\dot{e}}[\prod_{j=0,j\neq r}^{t-1}(i_{v_j})^{s_j}](i_{v_r}+1)^{s_{v_r}}+(d-m_{\dot{e}})[\prod_{j=0}^{t-1}(i_{v_j})^{s_{v_j}}]}|i_0,\ldots,i_{N-1}\rangle\langle i_0,\ldots,i_{N-1}|\\
&=\sum_{i_0,i_1,\ldots,i_{N-1}=0}^{d-1}\omega_d^{m_{\dot{e}}[\prod_{j=0,j\neq r}^{t-1}(i_{v_j})^{s_j}][\sum_{l=0}^{s_{v_r}}\binom{s_{v_r}}{l}(i_{v_r})^l-(i_{v_r})^{s_{v_r}}]+d[\prod_{j=0}^{t-1}(i_{v_j})^{s_{v_j}}]}|i_0,\ldots,i_{N-1}\rangle\langle i_0,\ldots,i_{N-1}|\\
&=\sum_{i_0,i_1,\ldots,i_{N-1}=0}^{d-1}\omega_d^{m_{\dot{e}}[\prod_{j=0,j\neq r}^{t-1}(i_{v_j})^{s_j}][\sum_{l=0}^{s_{v_r}-1}\binom{s_{v_r}}{l}(i_{v_r})^l]}|i_0,\ldots,i_{N-1}\rangle\langle i_0,\ldots,i_{N-1}|\\
&=\sum_{i_0,i_1,\ldots,i_{N-1}=0}^{d-1}\left(\omega_d^{\prod_{j=0,j\neq r}^{t-1}(i_{v_j})^{s_{v_j}}}\right)^{m_e[\sum_{l=0}^{s_{v_r}-1}\binom{s_{v_r}}{l}(i_{v_r})^l]}|i_0,\ldots,i_{N-1}\rangle\langle i_0,\ldots,i_{N-1}|\\
&=\sum_{i_k=0}^{d-1}|i_k\rangle\langle i_k|\otimes\widehat{\widetilde{CZ}}_{(V_{\dot{e}}\setminus\{k\}|S_{\dot{e}}\setminus\{s_k\})}^{m_{\dot{e}}[\sum_{l=0}^{s_k-1}\binom{s_k}{l}i_k{}^l]},\end{aligned} \tag{B3}$$

where $v_r = k$. Then we obtain

$$X_k X_k^{d-1} \widehat{\widehat{\mathrm{CZ}}}_{\dot{e}}^{m_{\dot{e}}} X_k \widehat{\widehat{\mathrm{CZ}}}_{\dot{e}}^{d-m_{\dot{e}}} = X_k \sum_{i_k=0}^{d-1} |i_k\rangle\langle i_k| \otimes \widehat{\widehat{\mathrm{CZ}}}_{(V_{\dot{e}}\setminus\{k\}|S_{\dot{e}}\setminus\{s_k\})}^{m_{\dot{e}}[\sum_{l=0}^{s_k-1}\binom{s_k}{l}(i_k)^l]}$$

and

$$\hat{\hat{g}}_k = \left(\prod_{\dot{e}\in\hat{\hat{E}}} \widehat{\widehat{\mathrm{CZ}}}_{\dot{e}}^{m_{\dot{e}}}\right) X_k \left(\prod_{\dot{e}'\in\hat{\hat{E}}} \widehat{\widehat{\mathrm{CZ}}}_{\dot{e}'}^{d-m_{\dot{e}'}}\right) = X_k \prod_{\dot{e}\in\hat{\hat{E}}, k\in V_{\dot{e}}} \left(\sum_{i_k=0}^{d-1} |i_k\rangle\langle i_k| \otimes \widehat{\widehat{\mathrm{CZ}}}_{(V_{\dot{e}}\setminus\{k\}|S_{\dot{e}}\setminus\{s_k\})}^{m_{\dot{e}}[\sum_{l=0}^{s_k-1}\binom{s_k}{l}(i_k)^l]}\right).$$

■

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