Robustness of higher-dimensional nonlocality against dual noise and sequential measurements

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The robustness of the violation of the Collins-Linden-Gisin-Masser-Popescu (CGLMP) inequality is investigated from the dual perspectives of noise in measurements and in states. To quantify it, we introduce a quantity called the area of the nonlocal region which reveals a dimensional advantage. Specifically, we report that with an increase in dimension, the maximally violating states show greater enhancement in the area of the nonlocal region in comparison to the maximally entangled states and the scaling of the increment in this case grows faster than visibility. Moreover, we examine the robustness of the sequential violation of the CGLMP inequality using weak measurements and find that even for higher dimensions, the simultaneous violation of the CGLMP inequalities of two observers as obtained for two-qubit states persists. We notice that the complementarity between information gain and disturbance from measurements is manifested by the decrease in the visibility in the first round and the increase in the same in the second round with dimensions. Furthermore, the amount of white noise that can be added to a maximally entangled state so that it has two rounds of the violation decreases with the dimension, while it does not appreciably change for the maximally violating states.

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I. INTRODUCTION

The journey from the Einstein-Podolsky-Rosen paradox [1] to Bell's theorem [2] via Bohmian mechanics [3] is a fascinating story that contributed to our present outlook on a physical theory. It asserts that a satisfactory description of nature cannot assume both locality and reality simultaneously. Jointly, these two assumptions, known as local realism, were recently refuted experimentally by the loophole-free Bell test [4–6]. In addition to its foundational significance, the Bell-Clauser-Horne-Shimony-Holt (CHSH) inequality [7] enables the device-independent certification of randomness [8], secure key distribution [9–11], detection of entanglement [12], etc.

Going beyond the much-studied simplest Bell scenario involving two settings of measurements for two parties with two outcomes, denoted by $(2-2-2)$, new insightful and qualitatively different results have been derived which would otherwise be impossible if we were restricted to the simplest scenario. In particular, violation of local realism manifests more sharply than in the $(2-2-2)$ case via the Greenberger-Horne-Zeilinger argument [13], which requires at least a three-qubit system. It has been shown that with a suitable choice of binary observables, maximal violation of Bell's inequality persists for a singlet state of arbitrary spins [14], refuting the belief that systems may lose their quantumness with increasing system size, thereby leading to a decrease in the violation of Bell's inequality [15]. Later, the dimensional advantage in the violation of local realism was established considering a more general choice of observables [16–18] since dichotomic measurements cannot exploit the higher-dimensional system with full generality. In a bipartite system of arbitrary local dimension with two choices of nondegenerate measurements, called the $(2-2-d)$ situation, corresponding Bell inequalities were derived by Collins, Gisin, Linden, Massar, and Popescu (CGLMP) [19] that were violated maximally by a nonmaximally entangled state [20] for the specific choices of observables [16–18]. These tight higher-dimensional Bell inequalities [21] exhibit enhanced visibility with the increase of dimension, thereby showing more robustness against noise [16,19] Here visibility refers to the noise strength up to which a pure state mixed with white noise exhibits a violation. It was shown explicitly that the performance of many quantum information processing tasks is enhanced by considering higher-dimensional systems. Specifically, compared to qubits, they exhibit greater robustness to noise [19], stronger security in device-dependent quantum key distribution (QKD) [22], and device-independent extraction of random bits [23]. However, the dimensional advantage is not always straightforward, as noted in a recent work [24] in the context of device-independent QKD; the estimated lower bound on the secure key rate does not improve with increasing dimension, whereas the upper bound on the key rate exhibits the opposite trend.

In another direction, the conventional Bell scenario was extended so that half of a bipartite system is possessed by a single observer called Alice while the other half is possessed by a series of observers referred to as Bobs, who can measure sequentially [25]. In this new scheme of the Bell test, it has been shown that no more than two observers can violate the Bell-CHSH inequality if the observers in the series measure independently [25,26]. Such a sequential scenario was also tested experimentally [27,28] and was further extended to several situations which include detecting the steerable correlation [29,30], witnessing entanglement [31,32], testing Bell inequalities other than CHSH [33], identifying genuine entanglement [34], and preparation contextuality [35]. An interesting twist in this situation is that with a slight modification of the independent and unbiased measurement scheme, an unbounded series of observers can be found who can certify nonclassical correlation with a single observer on another side [25,36]. Recently, some interesting applications of the sequential scheme such as self-testing unsharp measurements [37,38], reusing a teleportation channel [39], and generating randomness [40] were proposed, thereby showing its potential for quantum technologies. An interesting observation from the above studies is that if one is restricted to a particular measurement scheme, i.e., independent and unbiased measurement by the series of observers [26], predicting the number of sequential observers that would show nonlocal correlations is not straightforward. As the number of sequential violations depends on the initial strength of the correlation, detection, and measurement process in a nontrivial way, it is not well characterized yet. The number is finite and dictated by the trade-off between the disturbance and information gained from measurements. For example, it was found that for a maximally entangled two-qubit initially shared state, at most 12 Bobs can detect entanglement with a single Alice employing measurement settings pertinent to the optimal witness operator [31], while two Bobs sharing the same state with Alice can violate the CHSH inequality [25,26] based on the optimal measurement settings required for Bell violation. In the sequential measurement, partial information is extracted which is sufficient for the detection scheme, and at the same time, some residual correlation remains for other rounds which gradually diminishes with a longer sequence of Bobs. This reveals that witnessing entanglement possibly disturbs the state less than the situation in which the Bell-CHSH test is performed, thereby admitting more robustness of the former scheme against noise. Similarly, a measurement-device-independent entanglement witness [41] turns out to be more suitable in the sequential situation than a standard entanglement witness [42], as shown through the increased number of Bobs [32].

In the present work, we first investigate the robustness of the CGLMP inequality by going beyond the visibility measure of "nonlocality" [16,19]. Specifically, in addition to white noise in the state, we consider a noisy measurement (which we call a weak or unsharp measurement) on the maximally entangled state $\mathcal{M}_E$ as well as on the maximally CGLMP violating states $\mathcal{M}_V$. This consideration of dual noise leads to a measure of robustness, dubbed the "area of the nonlocal region" (where nonlocality means the violation of the CGLMP inequality), which scales with dimension more sharply than the visibility one. The introduction of noise to the measurement enables the possibility of sequential violation of the CGLMP inequality. In particular, we find that the violation by two Bobs persists even with the increase in dimension, as found in the two-qubit case with the CHSH inequality. In this respect, the pertinent question is how the robustness of CGLMP is reflected in the sequential scenario. It was noted that in the context of a violation of the CGLMP inequality, the visibility decreases with an increase in dimension [16,19]. However, we observe that if we demand the violation of the CGLMP inequality in two rounds of a sequential scheme, the required visibility increases with the dimension for maximally entangled states, while surprisingly, it remains constant for maximally violating states. This demonstrates that the sequential scenario can reveal a kind of robustness which is qualitatively different from the visibility and area of the nonlocal region obtained for a single round. This is due to the trade-off present in the disturbance by the weak measurements and the information gain via measurements in a sequential scheme.

This paper is organized in the following way. In Sec. II, we briefly discuss the prerequisites of the present work. In Sec. III, the robustness of CGLMP is discussed with a new measure introducing dual noise. For higher-dimensional pure states, the CGLMP inequality is used to certify entanglement sequentially in Sec. IV, and a similar study is carried out for noisy mixed states in Sec. V. We conclude in Sec. VI with a brief discussion.

II. PREREQUISITES: BELL INEQUALITIES IN HIGHER DIMENSIONS AND SEQUENTIAL MEASUREMENT SCHEME

Before we present our results, let us briefly discuss the CGLMP inequality and sequential scenario of the Bell test.

A. CGLMP inequality

Let Alice and Bob be two observers allowed to perform two d outcome measurements. A_1 and A_2 are Alice's measurement settings, and B_1 and B_2 are Bob's; they can take values in the range $[0, d-1]$, i.e., $A_{1(2)}, B_{1(2)} = 0, 1, \ldots, d-1$. The CGLMP inequality reads [19]

$$I_d = \sum_{k=0}^{\lfloor \frac{d}{2} \rfloor -1} \left(1 - \frac{2k}{d-1}\right)[f(k) - f(-k-1)] \leqslant 2, \quad (1)$$

where

$$f(k) = P(A_1 = B_1 + k) + P(B_1 = A_2 + k + 1) + P(A_2 = B_2 + k) + P(B_2 = A_1 + k). \quad (2)$$

The probabilities of the outcomes of Alice's measurement A_a and Bob's measurement B_b $(a, b = 1, 2)$ in $f(k)$ differ by k mod d and can be written as

$$P(A_1 = B_1 + k) = \sum_{j=0}^{d-1} P(A_a = j, B_b = j + k \bmod d).$$

The strongest violation of the CGLMP inequality is obtained for a maximally entangled state and a particular class of nonmaximally entangled states if the measurements performed by

Alice and Bob are in the bases $\{|k\rangle_{A_a}\}$ and $\{|l\rangle_{B_b}\}$, with

$$|k\rangle_{A_a} = \frac{1}{\sqrt{d}}\sum_{j=0}^{d-1}\exp\left[i\frac{2\pi}{d}j(k+\alpha_a)\right]|j\rangle_A \tag{3}$$

and

$$|l\rangle_{B_b} = \frac{1}{\sqrt{d}}\sum_{j=0}^{d-1}\exp\left[i\frac{2\pi}{d}j(-l+\beta_b)\right]|j\rangle_B, \tag{4}$$

where

$$\alpha_1 = 0, \quad \alpha_2 = 1/2, \quad \beta_1 = 1/4, \quad \beta_2 = -1/4. \tag{5}$$

The special thing about the above inequality is that its quantum violation increases with dimension d.

B. Sequential measurement scenario

The sequential measurement scenario considers an entangled state of two d-dimensional systems shared in such a way that half of the system is in the possession of the observer (say, Alice) and the other half is possessed by several observers (say, n Bobs, referred to as $\text{Bob}_1, \text{Bob}_2, \text{Bob}_3, \dots, \text{Bob}_n$). The task of Bob_1 is to pass the system to Bob_2 after performing an unsharp measurement on his part. Similarly, Bob_2 passes the system to Bob_3 after the measurement and so on. In other words, several Bobs measure their part sequentially, hence the name "sequential measurement scheme." Note that the measurement of each Bob is independent and all the measurement settings of each Bob are equally probable. In the scenario described above, if the measurement statistics between Alice and any Bob, say, Bob_k ($k > 1$), exhibits a CGLMP violation, then we call this "sharing the nonlocality" between them in the sequential scenario.

To determine the number of Bobs sharing the nonlocality of a shared entangled state (say, ρ) between Alice and n Bobs, we have to assume that the measurements of Alice and Bob_n are sharp (i.e., they perform projection measurements on their parts). In contrast, $1, \dots, n-1$ Bobs perform unsharp measurements represented by positive operator-valued measurements (POVMs). If Alice's measurement settings are denoted by $\{|k\rangle_A\langle k|\}$, the measurement settings of Bob_m are represented by

$$E^l_{B_m} = \lambda_m|l\rangle_B\langle l| + \frac{1-\lambda_m}{d}\mathbb{I}_d, \tag{6}$$

where $k, l = 0, 1, 2, d-1$; $m = 1, 2, 3, 4, \dots, n-1$; λ_m ($0 < \lambda_m \leqslant 1$) is the sharpness parameter of Bob_m; and $\mathbb{I}_d$ is the d-dimensional identity matrix. The state after the measurements of the $(m-1)$th Bob without any measurement on Alice's end transforms as

$$\rho_m = \frac{1}{d}\sum_{l=0}^{d-1}\left(\mathbb{I}_d \otimes \sqrt{E^l_{B_{m-1}}}\right)\rho_{m-1}\left(\mathbb{I}_d \otimes \sqrt{E^l_{B_{m-1}}}\right), \tag{7}$$

where ρ_{m-1} is the state before the unsharp measurement performed by Bob_{m-1}. We will use the postmeasured state ρ_m and POVM in Eqs. (6) and (7), respectively, when we certify nonlocality via the CGLMP inequality in this scenario.

III. ROBUSTNESS IN THE CGLMP VIOLATION: AREA OF THE NONLOCAL REGION

The study of the violation of Bell-type inequalities is a major endeavor in studies of nonlocality. Another important aspect is the investigation of robustness in the obtained violation. Typical studies of robustness consist of the addition of noise to a state and tracking the response of violation due to the amount of noise added to the state. However, for the violation of Bell-type inequalities, measurements play as crucial a role as states. Therefore, robustness analysis should also be carried out when noise is added to the measurements.

We perform a general robustness analysis in which both the state and the measurements are simultaneously noisy. In particular, we explore the role of the dimension of a bipartite state whose nonlocal characteristics in terms of violation of the CGLMP inequality in Eq. (1) are under investigation. Before that, we briefly discuss the scenario in which white noise is mixed with the state, given by

$$\rho = p|\psi\rangle\langle\psi| + \frac{1-p}{d^2}\mathbb{I}_{d^2}, \tag{8}$$

where $|\psi\rangle$ is a bipartite pure state with each party having dimension d and $\mathbb{I}_{d^2}$ is the $d \otimes d$ maximally mixed state (white noise). It was observed [19] that when $|\psi\rangle$ is a maximally entangled state in $d \otimes d$, given by $|\psi^d_{\mathcal{M}_E}\rangle = \frac{1}{\sqrt{d}}\sum_i |ii\rangle$, the robustness to noise, which can be called visibility and is measured as p, increases with the increase in d. This is in the sense that the maximal white noise that can be added to $|\psi^d_{\mathcal{M}_E}\rangle$ such that the resultant mixed state ρ violates the CGLMP inequality which increases with dimension d. For a given d, this maximal amount of white noise is denoted by $1 - p_{\min}$. In other words, for $|\psi^d_{\mathcal{M}_E}\rangle$, as $p_{\min}$ decreases, $1 - p_{\min}$ increases with d. This dimensional advantage of robustness is enhanced when, instead of $\mathcal{M}_E$, $|\psi\rangle$ is chosen to be the nonmaximally entangled state, which violates the CGLMP inequality maximally [20]. We denote such maximally violating states as $\mathcal{M}_V$. The exact form of $\mathcal{M}_V$ up to $d = 10$ can be found in Ref. [43]. $\mathcal{M}_V$ offer greater robustness with d in comparison to $\mathcal{M}_E$. For the convenience of readers, we present the exact form of the maximally violating states in Appendix A.

We now consider the opposite situation in which the shared state is noise free and the measurements are taken to be noisy. The effect operators for the noisy measurements are described by the POVMs given in Eq. (6). Note that here we are interested in the first-round violation [$m = 1$; see Eq. (6)]. Considering $|\psi\rangle$ to be noiseless $\mathcal{M}_E$ and $\mathcal{M}_V$, the amount by which measurements can be made noisy while preserving the violation of the CGLMP inequality is denoted by $1 - \lambda_{\min}$, which also increases with increasing d. We observe exactly the same dimensional dependence on noise in the measurements denoted by λ as obtained in the case of noisy states caused by adding white noise to the state. Mathematically, for any pure state $|\psi\rangle$,

$$I^Q_d(p = 1, \lambda = x) = I^Q_d(p = x, \lambda = 1). \tag{9}$$

Here the superscript Q in implies that the probabilities associated with the CGLMP expression are calculated for the quantum states and measurements performed by Alice and Bob, as stated earlier. For brevity, we do not henceforth use the

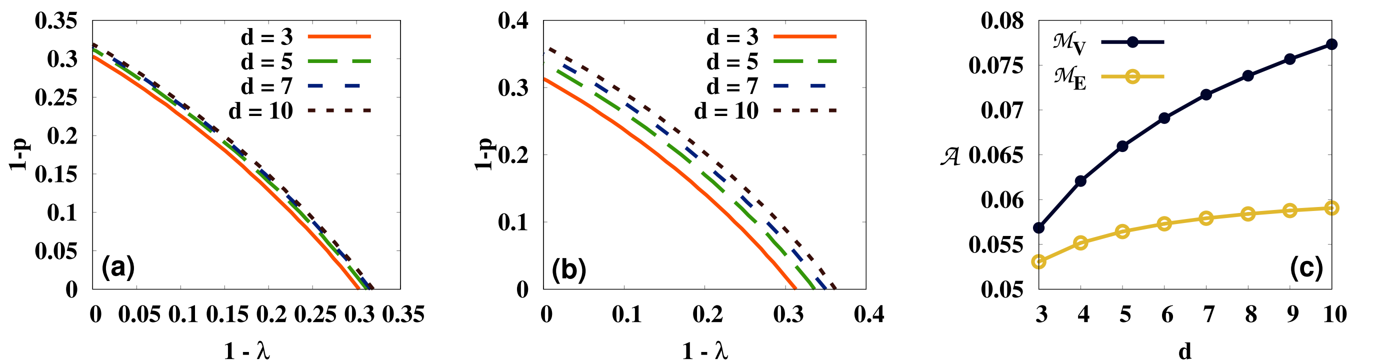

FIG. 1. (a) For a fixed d, each point in the curve just crosses the local realist value of 2, i.e., when the value of I_d in Eq. (1) is just above 2 by taking the maximally entangled state $\mathcal{M}_E$ in the $(1-\lambda, 1-p)$ plane. Different d values are considered. (b) A similar plot when the shared state is the maximally CGLMP violating state $\mathcal{M}_V$. (c) $\mathcal{A}$ (ordinate) defined in Eq. (10) vs d (abscissa) for $\mathcal{M}_E$ and $\mathcal{M}_V$.

superscript Q as we always work with probabilities generated by quantum states and measurements. Things become more interesting and complex when both the state and measurements suffer from noise simultaneously, which we will discuss in the next section.

Complementarity of robustness

We will now study the robustness obtained from the violation of the CGLMP inequality by considering both the state and the measurements to be noisy. We again start with a d-dimensional maximally entangled state $|\psi^d_{\mathcal{M}_E}\rangle$ as well as $\mathcal{M}_V$ and $|\psi_{\mathcal{M}_V}\rangle$ (for a two-qutrit state, $\mathcal{M}_V$ has the form $\frac{1}{\sqrt{t}}(|00\rangle + \gamma|11\rangle + |22\rangle)$, where $t = 2 + \gamma^2$ and $\gamma = 0.7923$ [20,43]). In this general framework, $p_{\min}$ is a function of the noise in the measurements, which we denote as $p_{\min}(\lambda)$, and naturally, $\lambda_{\min}$, in turn, becomes a function of the noise added to the state, which is referred to as $\lambda_{\min}(p)$. For convenience, we drop the min and functional labels, thereby indicating $1-p_{\min}(\lambda)$ and $1-\lambda_{\min}(p)$ as $1-p$ and $1-\lambda$, respectively. We investigate the dual version of robustness by tracking the locus of all the points in the $(1-\lambda, 1-p)$ plane that just crosses the local realist value of 2 by considering $\mathcal{M}_E$ and $\mathcal{M}_V$ [see Figs. 1(a) and 1(b)]. Note that all noise configurations that fall below the curve lead to the violation of the CGLMP inequality. Therefore, in the $(1-\lambda, 1-p)$ plane, the ratio of the areas under the curve can be considered to be a measure of robustness when both the state and the measurement are affected by noise. Motivated by this observation, we introduce a generalized robustness measure as the area under this curve, which we call the "area of the nonlocal region" $\mathcal{A}$. Mathematically, the area of the nonlocal region $\mathcal{A}$ in the noise plane can be defined as

$$\mathcal{A} = \int_0^{1-p_{\min}(\lambda=1)} [1 - \lambda_{\min}(1-p)]\, dp. \tag{10}$$

We then compute $\mathcal{A}$ values for both $\mathcal{M}_E$ and $\mathcal{M}_V$ and perform a comparative analysis of their respective scalings with d [see Figs. 1(a)–1(c)]. The $\mathcal{A}$ values for $\mathcal{M}_E$ and $\mathcal{M}_V$ are listed in Table I. Our findings are as follows:

(1) The $\mathcal{A}$ values for $\mathcal{M}_V$ are strictly greater than those obtained for $\mathcal{M}_E$. Furthermore, the gap in $\mathcal{A}$ values for $\mathcal{M}_V$ and $\mathcal{M}_E$ grows with d, which is clearly discernible from Table I and Figs. 1(a)–1(c).

(2) $\mathcal{A}$ scales much faster with d for $\mathcal{M}_V$ in comparison to $\mathcal{M}_E$ [see Fig. 1(c)].

(3) The gap in the growth between $\mathcal{M}_V$ and $\mathcal{M}_E$ in the case of $\mathcal{A}$ grows much faster than that of the visibility.

Because the value of the CGLMP expression is the same when either the state or the measurement becomes noisy with the same amount of white noise, $I_d(p=1, \lambda=x) = I_d(p=x, \lambda=1)$ as in Eq. (9), it seems reasonable to assume that robustness can be completely characterized by looking at either p or λ. However, this symmetry does not hold in the general case where both noises are nonvanishing, i.e., $I_d(p=x, \lambda=y) \neq I_d(p=y, \lambda=x)$. The lack of symmetry is reflected by the fact that the curves in Figs. 1(a) and 1(b) are nonlinear. Therefore, in the general case, the CGLMP nonlocality is both a function of the unsharp parameter and the amount of noise in the state in a nontrivial way. This leads us to introduce $\mathcal{A}$ as a single-letter formula to analyze robustness that incorporates the effects of both.

Furthermore, note that the white-noise paradigm can be motivated by interpreting the noise arising from a depolarizing noisy channel that is independent of the source state as a systematic noise involved in the setup. This, as we understand, gives a fair platform for comparing the performance of various states with respect to their robustness against noise.

TABLE I. The $\mathcal{A}$ values for maximally entangled ($\mathcal{M}_E$) and maximally violating ($\mathcal{M}_V$) states and their percentage differences (Diff.) from $d=3$ to $d=10$ are given in different columns. The difference grows with d since $\mathcal{A}$ for $\mathcal{M}_V$ scales much faster with an increase in d than that for $\mathcal{M}_E$.

d	$\mathcal{A}(\mathcal{M}_E)$	$\mathcal{A}(\mathcal{M}_V)$	Diff.
3	0.05307	0.05685	7.14%
4	0.05517	0.06207	12.51%
5	0.05644	0.06595	16.85%
6	0.0573	0.06909	23.81%
7	0.05792	0.07171	27.45%
8	0.0584	0.07382	26.40%
9	0.05878	0.07567	28.73%
10	0.05906	0.07733	30.93%

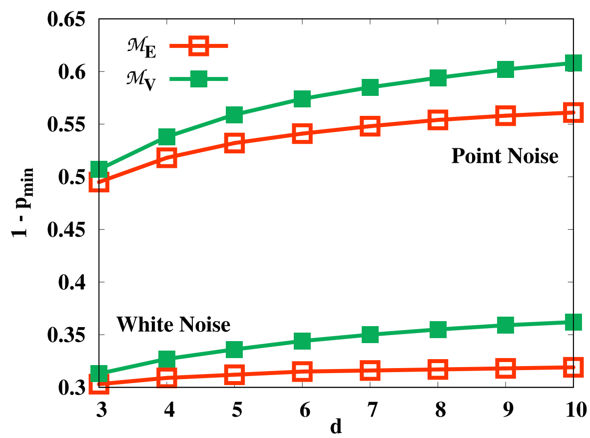

FIG. 2. Comparison of the robustness to the point and white noise of the maximally entangled states $\mathcal{M}_E$ and the maximally violating ones $\mathcal{M}_V$ for $d = 3$. Both $\mathcal{M}_E$ and $\mathcal{M}_V$ are more robust to point noise, as indicated by the higher value of $1 - p_{\min}$.

For colored noise, there is no unique way to ascribe the nonuniformity of the noise, and it can alter the nonlocal properties of various states in completely different ways, which would make our comparative studies hard. That is why we focus on the systematic white noise in both cases. In the general case, one intends to consider an arbitrary separable state as noise. Such noise will depend on the initial shared entangled state. To create a meaningful platform for comparison of various initial shared states ($\mathcal{M}_E$, $\mathcal{M}_V$, etc.), for each initial state, one has to perform an optimization over the entire set of separable states, pinning down the least disturbing noise for each initial state. This, in general, is a very hard problem. Nevertheless, we undertake a particular colored-noise scheme that might be worthwhile to consider, which we call the point noise, where any state $|\psi\rangle$ under consideration is made noisy by

$$\rho = p|\psi\rangle\langle\psi| + (1-p)|k=0\rangle\langle k=0| \otimes |l=0\rangle\langle l=0|. \quad (11)$$

For the expressions of $|k(l) = 0\rangle$, see Eqs. (3) and (4). Compared to white noise (mixing a maximally mixed state), both $\mathcal{M}_E$ and $\mathcal{M}_V$ show enhanced robustness features with respect to the point noise which can be easily observed in Fig. 2 for $d = 3$. However, the scaling of $1 - p_{\min}$ with d remains qualitatively similar, with the hierarchies between $\mathcal{M}_V$ and $\mathcal{M}_E$ being preserved. For the $d = 3$ case, see Fig. 2.

Typically, noise in the system has an adverse effect on the system in the form of lowering the visibility. As shown in this section, the bane can turn out to be a boon in disguise if we look at the situation from a different point of view. In the context of sequential measurements, the "white noise" in the measurement actually constitutes a POVM strategy which allows multiple Bobs to share nonlocality, thereby manifesting the robustness from a different perspective, as will be shown in the next section.

IV. SHARING OF NONLOCALITY IN HIGHER DIMENSIONS

In the sharing scenario considered in this section, we deal with the maximally entangled and maximally violating states shared by Alice and Bob$_1$ in an arbitrary dimension. We will start our discussion with $d = 3$, and a detailed analysis is presented for $\mathcal{M}_E$ in $d = 3$ to $d = 5$. We then repeat the investigation for the maximally violating states.

After substituting $d = 3$ in Eq. (1), the CGLMP inequality reads

$$\begin{aligned} I_3 = {} & P(A_1 = B_1) + P(B_1 = A_2 + 1) + P(A_2 = B_2) \\ & + P(B_2 = A_1) - [P(A_1 = B_1 - 1) + P(B_1 = A_2) \\ & + P(A_2 = B_2 - 1) + P(B_2 = A_1 - 1)] \leqslant 2. \end{aligned} \quad (12)$$

If the shared state is the two-qutrit $\mathcal{M}_E$, given by

$$\left|\psi^3_{\mathcal{M}_E}\right\rangle = \tfrac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle), \quad (13)$$

by performing the POVM on Bob$'_1$s side and by considering the measurement settings for the CGLMP test given in Eqs. (3), (4), and (5) for Alice and Bob$_1$, the quantum expression for the CGLMP inequality I_3 [Eq. (12)} for the Alice-Bob$_1$ pair reduces to

$$I_3^1 = \tfrac{4}{9}(3 + 2\sqrt{3})\lambda_1, \quad (14)$$

where the superscript 1 represents the number of rounds in the sequential scenario. Hence, the nonlocality can be demonstrated by showing the violation of the CGLMP inequality between Alice and Bob$_1$ if $\lambda_1 > 2/[\frac{4}{9}(3 + 2\sqrt{3})] = 0.69615$, while the optimal quantum value for Alice and Bob$_1$ is 2.87293, obtained at $\lambda_1 = 1$. In a similar fashion, we can find the quantum expressions for the Alice-Bob$_2$ and Alice-Bob$_3$ pairs, which are, respectively,

$$\begin{aligned} I_3^2 = {} & \frac{4\lambda_2}{81}[-2(\sqrt{3}+3)\lambda_1 + 12\sqrt{1-\lambda_1}\sqrt{2\lambda_1+1} \\ & + 4\sqrt{2\lambda_1+1}\sqrt{3-3\lambda_1} + 14,\ \sqrt{3} + 15] \end{aligned} \quad (15)$$

and

$$\begin{aligned} I_3^3 = {} & \frac{4\lambda_3}{729}\{4(\sqrt{3}+6)(2\sqrt{1-\lambda_2}\sqrt{2\lambda_2+1} - \lambda_2) \\ & \times \sqrt{1-\lambda_1}\sqrt{2\lambda_1+1} - 2\lambda_1[7\sqrt{3} + 15 - (\sqrt{3}+6)\lambda_2 \\ & + 2(\sqrt{3}+6)\sqrt{1-\lambda_2}\sqrt{2\lambda_2+1}] - 2(7\sqrt{3}+15)\lambda_2 \\ & + 4(7\sqrt{3}+15)(\sqrt{1-\lambda_1}\sqrt{2\lambda_1+1} \\ & + \sqrt{1-\lambda_2}\sqrt{2\lambda_2+1} + 75 + 98\sqrt{3})\}. \end{aligned} \quad (16)$$

Considering the situation of minimum violation of I_3^1 by Alice and Bob$_1$, the quantum expression for I_3^2 reduces to $2.40856\lambda_2$. In this case, the violation of the CGLMP inequality for Alice and Bob$_2$ is possible if $\lambda_2 > 0.830372$, while the optimal quantum value is 2.40856 with $\lambda_2 = 1$. Substituting the conditions for λ_1 and λ_2, we find that two Bobs surely violate the CGLMP inequality. Let us now check whether the third Bob, Bob$_3$, can also violate the CGLMP inequality or not. In this case, the optimal quantum value of I_3^3 turns out to be $1.83798 < 2$ if we take the minimum violation condition for Bob$_2$ and Bob$_3$. Since the optimal quantum value of I_3^3 is strictly less than 2, we can claim that only two Bobs, Bob$_1$ and Bob$_2$, can exhibit nonlocality with Alice by using the CGLMP inequality for $d = 3$. Notice here that only two Bobs can violate the CHSH inequality with Alice if they initially share a two-qubit maximally entangled state [25].

TABLE II. Optimal quantum values for Bob_1, Bob_2, and Bob_3 are obtained using the CGLMP inequality for a maximally entangled state for $d = 3$ to $d = 10$ dimensions.

	Optimal quantum value of the CGLMP inequality		
Dimension	Bob_1	Bob_2	Bob_3
3	2.8729	2.4086	1.8380
4	2.8962	2.3963	1.7994
5	2.9105	2.3819	1.7650
6	2.9202	2.3699	1.7382
7	2.9272	2.3570	1.7122
8	2.9324	2.3458	1.6910
9	2.9365	2.3360	1.6722
10	2.9398	2.3274	1.6568

TABLE III. I_d^i ($i = 1, 2, 3$) for Bob_1, Bob_2, and Bob_3 are listed for the maximally violating state $\mathcal{M}_V$ as the initial state from $d = 3$ to $d = 10$.

	Optimal quantum value of the CGLMP inequality		
Dimension	Bob_1	Bob_2	Bob_3
3	2.9150	2.4402	1.8578
4	2.9729	2.4526	1.8307
5	3.0158	2.4564	1.8015
6	3.0495	2.4522	1.7702
7	3.0771	2.4418	1.7342
8	3.1012	2.4324	1.7041
9	3.1215	2.4231	1.6768
10	3.1393	2.4142	1.6517

Let us now move to $d = 4$ and $d = 5$. In these cases, I_d in Eq. (1) reduces to

$$\begin{aligned}
I_4 = {} & P(A_1 = B_1) + P(B_1 = A_2 + 1) + P(A_2 = B_2) \\
& + P(B_2 = A_1) - [P(A_1 = B_1 - 1) + P(B_1 = A_2) \\
& + P(A_2 = B_2 - 1) + P(B_2 = A_1 - 1)] \\
& + \tfrac{1}{3}\{P(A_1 = B_1 + 1) + P(B_1 = A_2 + 2) \\
& + P(A_2 = B_2 + 1) + P(B_2 = A_1 + 1) \\
& - [P(A_1 = B_1 - 2)P(B_1 = A_2 - 1) \\
& + P(A_2 = B_2 - 2) + P(B_2 = A_1 - 2)]\} \leqslant 2. \qquad (17)
\end{aligned}$$

By following a similar prescription, for a maximally entangled state $|\psi^4_{\mathcal{M}_E}\rangle = \frac{1}{2}(|00\rangle + |11\rangle + |22\rangle + |33\rangle)$, we find that Bob_1 starts sharing nonlocality with Alice through the violation of the CGLMP inequality when $\lambda_1 > 0.690551$ and that $\max I_4^1 = 2.89624$ for $\lambda_1 = 1$. Again, if we restrict the situation such that the Alice-Bob_1 duo just shows violation, Alice and Bob_2 violate the CGLMP inequality when $\lambda_2 > 0.834603$, and in the second round, the maximal quantum value is reduced to 2.39635 ($\lambda_2 = 1$). By taking the minimum violation condition of the sharpness parameter for Bob_2 and Bob_3, the optimal quantum value of I_3^3, given in Table II, again turns out to be less than 2. For $d = 5$,

$$\begin{aligned}
I_5 = {} & P(A_1 = B_1) + P(B_1 = A_2 + 1) + P(A_2 = B_2) \\
& + P(B_2 = A_1) - [P(A_1 = B_1 - 1) + P(B_1 = A_2) \\
& + P(A_2 = B_2 - 1) + P(B_2 = A_1 - 1)] \\
& + \tfrac{1}{2}\{P(A_1 = B_1 + 1) + P(B_1 = A_2 + 2) \\
& + P(A_2 = B_2 + 1) + P(B_2 = A_1 + 1) \\
& - [P(A_1 = B_1 - 2)P(B_1 = A_2 - 1) \\
& + P(A_2 = B_2 - 2) + P(B_2 = A_1 - 2)]\} \leqslant 2 \qquad (18)
\end{aligned}$$

can be used to obtain violations of the CGLMP inequality in a sequential situation with $|\psi^5_{\mathcal{M}_E}\rangle = \frac{1}{\sqrt{5}}(|00\rangle + |11\rangle + |22\rangle + |33\rangle + |44\rangle)$. The optimal quantum violations of the CGLMP inequality for Bob_1, Bob_2, and Bob_3 are given in Table II up to $d = 10$. From Table II, we can see that as the dimension increases, there is an increment of the optimal quantum value for Bob_1, although it decreases with the increase of dimension for Bob_2 and Bob_3. Table II also indicates that the trade-off between the information gained by the measurement and the disturbance created by the measurement plays a crucial role in this enterprise.

Since the CGLMP inequality gives the maximum violation for a nonmaximally entangled state, let us examine whether the initial shared state in a sequential scenario is $\mathcal{M}_V$ and whether the situation improves or not. We observe that although the first round of violation is greater and increases faster with d in comparison to that for $\mathcal{M}_E$, the measurements disturb the state to such an extent that violation for more than two Bobs remains an impossibility. Also, note that the third-round value of the CGLMP expression decreases with increasing dimension, so the possibility of getting a simultaneous violation for three rounds is unlikely even if d is increased beyond 10. See Table III for details. Comparing Tables II and III, we observe that the gap between the I_d^3 values obtained for $\mathcal{M}_V$ and $\mathcal{M}_E$ decreases with the increase in dimension. This possibly indicates that the unsharp measurements disturb $\mathcal{M}_V$ more drastically than $\mathcal{M}_E$ in higher dimensions.

Optimality analysis of measurements in the sequential scenario

An important question is whether the POVMs considered by weakening the optimal single-round measurement strategy with white noise might not be the optimal ones to get the maximal number of Bobs that can sequentially violate the CGLMP inequality with a single Alice. The optimality in the sequential case might be measured by the strategy that maximizes the number of Bobs that violate the CGLMP inequality. However, as one can clearly see, such a definition of optimality leaves a lot of degeneracy in the number of measurement choices that achieve the optimal number of Bobs since the indicator of optimality increases in integer steps.

Ideally, one would like to perform a "full optimization" to maximize n. The full optimization refers to a maximization over measurement settings of the CGLMP inequality and unsharp parameters of all n rounds collectively. Since the measurement settings of the mth round adaptively depend on the settings of all the previous rounds, the full optimization process becomes realistically intractable. Even if each round is treated independently with the objective being to obtain an infinitesimal amount of violation with minimal disturbance to the state, we again run into the domain of infeasibility. This is because if the optimization at every round is carried over arbitrary POVMs, the dimension of the parameter space in the optimization will become very large even if one considers POVMs with a fixed number of outcomes. This again renders the problem intractable. Therefore, we resort to optimization over a simplified setting in which we are left with a five-parameter optimization in every round. Specifically, we numerically scan the measurement strategies used for the CGLMP inequality made unsharp with white noise, denoted as (CGLMP + white), to evaluate the optimal setting. In particular, we perform a five-dimensional optimization over $\alpha_1, \alpha_2, \beta_1, \beta_2$, and λ to find the violation and the postmeasurement state in one round and then perform optimal projective CGLMP measurements to check whether we get any violation in the next round. If not, we stop, and if we do, we repeat the same drill of numerical optimization. Our analysis using the dividing rectangles algorithm [44] for global optimization reveals that for the measurement strategies of the form (CGLMP + white), the best setting is when $\alpha_1, \alpha_2, \beta_1$, and β_2 are chosen to be in the optimal setting as in the projective (single-round) case of the CGLMP inequality.

Last, we believe that our choice of weak measurements is one of the most "intuitive" options one may consider. If the full optimization, in principle, yields some other strategy, we suspect that this approach might constitute some exotic measurements that might be difficult to implement operationally. In such a situation, we argue that our choice of unsharp measurements is the most practically motivated one.

Now we present an analysis of the level of optimality provided by our choice of the set of weak measurements over which our optimization runs. Given a weak-measurement scheme, the optimality of such a strategy involves the study of the trade-off between the information gain by the measurement and the amount of disturbance it imparts to the state. The amount of disturbance quantifies the quality of measurement F and can be obtained from the postmeasurement state ρ',

$$\rho' = F\rho + (1-F)\sum_{i=0}^{d-1} P_i\rho P_i, \tag{19}$$

where ρ is the initial state and P_i are the projectors that have been weakened by the strategy

$$E_i = \lambda P_i + \frac{1-\lambda}{d}\mathbb{I}_d, \tag{20}$$

with $\mathbb{I}$ being the d-dimensional identity, the same strategy we have used in our work [see Eq. (6)]. Again, from Eq. (6), we can make the following identification: $P_i = |l=i\rangle\langle l=i|$. The second quantity of interest measures the amount of information gained in the experiment, which can be interpreted as the precision of the experiment given by G, where

$$p(E_i) = G\,\mathrm{tr}(P_i\rho) + \frac{1-G}{d}, \tag{21}$$

where $p(E_i)$ is the clicking probability of E_i. With F and G, we have the following information gain vs disturbance inequality [25]:

$$F^2 + G^2 \leqslant 1. \tag{22}$$

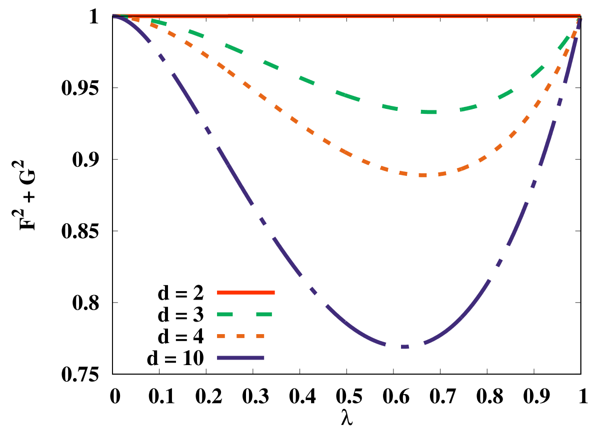

FIG. 3. Variation of $F^2 + G^2$ versus the sharpness parameter of measurement λ for various d. As d increases, the weakening strategy via white noise becomes suboptimal.

The optimal measurement strategy saturates the above inequality. In the $d = 2$ case, it was proven that weakening the optimal projective measurements via white noise can saturate the $F^2 + G^2 \leqslant 1$ inequality, thereby demonstrating its optimality. In the absence of the solution for the full optimization, we extend the weakening strategy via white noise in accordance with the optimal setting for $d = 2$ to higher dimensions. To test the optimality of our ansatz, for $d > 2$, we test how close to unity $F^2 + G^2$ gets for our choice of weak measurements. For a general d, we compute

$$F = \frac{2}{d}\sqrt{[1+(d-1)\lambda](1-\lambda)} + \frac{(d-2)(1-\lambda)}{d}. \tag{23}$$

See Appendix B for a detailed calculation. Furthermore, from Eq. (21), we get $G = \lambda$. We find that for $d > 2$, $F^2 + G^2 < 1$, thereby suggesting that our measurement choices are not optimal. However, to our advantage, our analysis reveals that for low d, the inequality reaches near saturation. For example, for $d = 3$ and 4, we get almost 90% saturation of the inequality for the relevant choice of system parameters (see Fig. 3).

V. ROBUSTNESS IN SEQUENTIAL EXHIBITION OF NONLOCALITY

In Sec. III, we analyzed how much noise we could add to the state as well as measurements so that it continued to violate the CGLMP inequality. However, the option of using sequential measurements to obtain violations for multiple Bobs with a single Alice opens up the possibility to examine robustness from a new point of view. In this context, we define robustness as the maximal amount of noise that can be added to a state such that the CGLMP inequality can be violated for multiple rounds, which we claim to be two since, from the previous section, we observed that for both $\mathcal{M}_E$ and $\mathcal{M}_V$, the maximum number of Bobs that can violate the CGLMP inequality with Alice remains two.

Let us consider the pure state $|\psi\rangle$ admixed with white noise, given in Eq. (8), with visibility q as an initial state in the sequential scenario. We now demand that if two Bobs have to show a violation of local realism with Alice, both I_d^1 and I_d^2 have to be greater than 2. We define $q_{\min}$ as the

TABLE IV. The $q_{\min}$ values for maximally entangled states $\mathcal{M}_E$ and maximally violating states $\mathcal{M}_V$ for $d = 3$ to 10 are reported for the violation of the CGLMP inequality by two Bobs sequentially with Alice.

Dimension	$q_{\min}^{\mathcal{M}_V}$	$q_{\min}^{\mathcal{M}_E}$
3	0.8773	0.8845
4	0.8748	0.8872
5	0.8737	0.8900
6	0.8736	0.8933
7	0.8738	0.8963
8	0.8741	0.8987
9	0.8748	0.9012
10	0.8752	0.9034

minimum value of q above which both $I_d^1 > 2$ and $I_d^2 > 2$. We now compute how $q_{\min}$ scales with d and compare the results with the scaling obtained for $p_{\min}$ discussed in Sec. III for both $\mathcal{M}_E$ and $\mathcal{M}_V$.

Recall that in the CGLMP test, we observed an enhanced amount of robustness (as defined in terms of the persistence of the violation with the addition of white noise) on increasing d, as indicated by lower values of $p_{\min}$. The maximal amount of white noise that the state can absorb such that the violation persists is simply given by $1 - p_{\min}$. For both $\mathcal{M}_E$ and $\mathcal{M}_V$, $p_{\min}$ decreases with d [16,19,20]. Furthermore, note that we expectedly find $p_{\min} < q_{\min} < 1$.

When robustness is analyzed in the context of sustaining dual-round violation via the use of sequential measurements, we observe a qualitatively different trend. For $\mathcal{M}_E$, $q_{\min}$ actually increases with d. This implies that robustness actually decreases with d when $\mathcal{M}_E$ are employed and we demand CGLMP violations by two Bobs. However, for $\mathcal{M}_V$, $q_{\min}$ values do not change significantly on increasing d. See Table IV for the details of the $q_{\min}$ values for both $\mathcal{M}_E$ and $\mathcal{M}_V$. However, in both cases, the gap between $q_{\min}$ and $p_{\min}$ increases with d. For a pictorial representation of the situation, see Fig. 4.

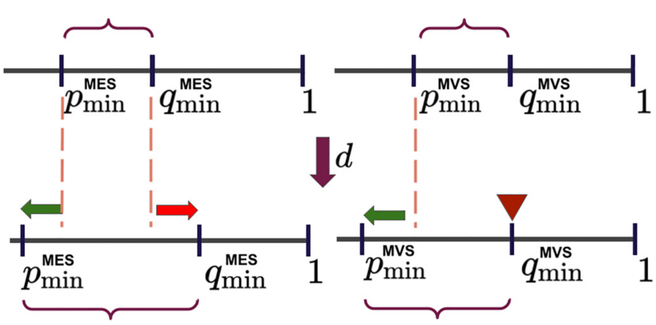

FIG. 4. Schematic depiction of the dynamics of $q_{\min}$ and $p_{\min}$ for both $\mathcal{M}_E$ and $\mathcal{M}_V$ with d. $p_{\min}$ denotes the visibility of the state, while $q_{\min}$ is the minimum value of the visibility above which the CGLMP inequality in the second round starts violating. The superscripts represent the states considered. The green and red arrows respectively indicate the advantages and disadvantages of robustness with dimensions.

The above results explain in part why, despite an increase in the first-round violation with d, one does not get a higher number of Bobs which sequentially violate the CGLMP inequality, i.e., $I_d^k > 2$, with $k > 2$ for higher-dimensional systems. Although the amount of the maximal first-round violation grows, the disturbance induced by the measurements is high enough to actually bring down the violation in the second round with d, which ultimately leads to the third round becoming nonviolating.

VI. DISCUSSION

To achieve quantum advantage, manipulating and analyzing higher-dimensional quantum systems are essential since, for several quantum information processing tasks, higher-dimensional quantum systems turn out to be more beneficial than qubit pairs. The CGLMP inequality is a family of tight Bell inequalities for bipartite systems of arbitrary dimension which is known to exhibit more robustness against noise with increasing dimension. Therefore, it is interesting to investigate how the CGLMP inequality responds if noise is present not only in the state but also in the measurement.

We introduced a new measure of robustness which we referred to as the "area of the nonlocal region" while considering two noises in the states and measurements. In particular, this area indicates the region in the noise parameter space where violation of the CGLMP inequality can be observed. We found that this region grows more rapidly with the increase in dimension with respect to the increase in visibility associated solely with states or measurements.

The introduction of noise in measurements facilitates the use of a sequential violation of the CGLMP inequality as it retains some residual correlation after obtaining a violation in the first round. Interestingly, we found that the violation of the CGLMP inequality by two sequential observers on one side and another observer on the other end persists with dimension. Moreover, the minimum visibility required to achieve double violation in the sequential case increases with the increase in dimension, thereby exhibiting the opposite behavior compared to the violation obtained for the shared state without unsharp measurement. This indicates that robustness in the sequential measurement scenario is qualitatively distinct from that of the typical Bell test since it involves the disturbance of the state introduced via the measurement. It would be interesting to probe further how the double violation obtained in the CGLMP inequality enables applications in the context of information processing tasks involving higher-dimensional quantum systems.

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APPENDIX A: THE MAXIMALLY VIOLATING STATES

The expressions for the maximally violating states were given in [43]. We state them here once more for the convenience of the readers.

$$\left|\psi^{d=3}_{\mathcal{M}_V}\right\rangle = 0.6169\,|00\rangle + 0.4888\,|11\rangle + 0.6169\,|22\rangle\,, \tag{A1}$$

$$\left|\psi^{d=4}_{\mathcal{M}_V}\right\rangle = 0.5686\,|00\rangle + 0.4204\,|11\rangle + 0.4204\,|22\rangle + 0.5686\,|33\rangle\,, \tag{A2}$$

$$\left|\psi^{d=5}_{\mathcal{M}_V}\right\rangle = 0.5368\,|00\rangle + 0.3859\,|11\rangle + 0.3548\,|22\rangle + 0.3859\,|33\rangle + 0.5368\,|44\rangle\,, \tag{A3}$$

$$\left|\psi^{d=6}_{\mathcal{M}_V}\right\rangle = 0.5137\,|00\rangle + 0.3644\,|11\rangle + 0.3214\,|22\rangle + 0.3214\,|33\rangle + 0.3644\,|44\rangle + 0.5137\,|55\rangle\,, \tag{A4}$$

$$\left|\psi^{d=7}_{\mathcal{M}_V}\right\rangle = 0.4957\,|00\rangle + 0.3493\,|11\rangle + 0.3011\,|22\rangle + 0.2882\,|33\rangle + 0.3011\,|44\rangle + 0.3493\,|55\rangle + 0.4957\,|66\rangle\,, \tag{A5}$$

$$\begin{aligned}\left|\psi^{d=8}_{\mathcal{M}_V}\right\rangle = {} & 0.4812\,|00\rangle + 0.3379\,|11\rangle + 0.2872\,|22\rangle + 0.2679\,|33\rangle + 0.2679\,|44\rangle + 0.2872\,|55\rangle + 0.3379\,|66\rangle \\ & +0.4812\,|77\rangle\,,\end{aligned} \tag{A6}$$

$$\begin{aligned}\left|\psi^{d=9}_{\mathcal{M}_V}\right\rangle = {} & 0.4690\,|00\rangle + 0.3288\,|11\rangle + 0.2770\,|22\rangle + 0.2541\,|33\rangle + 0.2474\,|44\rangle + 0.2541\,|55\rangle + 0.2770\,|66\rangle \\ & +0.3288\,|77\rangle + 0.4690\,|88\rangle\,,\end{aligned} \tag{A7}$$

$$\begin{aligned}\left|\psi^{d=10}_{\mathcal{M}_V}\right\rangle = {} & 0.4587\,|00\rangle + 0.3212\,|11\rangle + 0.2690\,|22\rangle + 0.2440\,|33\rangle + 0.2334\,|44\rangle + 0.2334\,|55\rangle + 0.2440\,|66\rangle \\ & +0.2690\,|77\rangle + 0.3212\,|88\rangle + 0.4587\,|99\rangle\,.\end{aligned} \tag{A8}$$

APPENDIX B: COMPUTATION OF THE MEASUREMENT QUALITY INDEX *F*

Following the weak-measurement strategy given in Eq. (20), the postmeasurement state reads

$$\rho' = \sum_{i=0}^{d-1} M_i \rho M_i^\dagger, \tag{B1}$$

where M_i are the update operators given by

$$M_i = \sqrt{E_i} = xP_i + y(\mathbb{I} - P_i), \tag{B2}$$

with

$$x = \sqrt{\frac{1+(d-1)\lambda}{d}},\ y = \sqrt{\frac{1-\lambda}{d}}. \tag{B3}$$

See Eq. (20) for the form of E_i in terms of P_i. Now ρ' can be written as

$$\begin{aligned}\rho' &= \sum_{i=0}^{d-1} P_i\rho P_i + [2xy + (d-2)y^2] \sum_{\substack{i,j=0\\ i\neq j}}^{d-1} P_i\rho P_j \\ &= [2xy + (d-2)y^2]\rho \\ &\quad + [1 - 2xy - (d-2)y^2] \sum_{i=0}^{d-1} P_i\rho P_i.\end{aligned} \tag{B4}$$

Now following Eq. (19), we make the following identification for the quality factor of the measurement:

$$\begin{aligned}F &= 2xy + (d-2)y^2, \\ &= \frac{2}{d}\sqrt{[1+(d-1)\lambda](1-\lambda)} + \frac{(d-2)(1-\lambda)}{d}.\end{aligned} \tag{B5}$$

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