Evolution of strictly localized states in noninteracting quantum field theories with background fields

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We investigate the construction of spin-1/2 fermionic and spin-0 bosonic wavepackets having compact spatial support in the framework of a computational quantum field theory (QFT) scheme offering space-time solutions of the relativistic wave equations in background fields. To construct perfectly localized wavepackets, we introduce a spatial density operator accounting for particles of both positive and negative charge. We examine properties of the vacuum and single-particle expectation values of this operator and compare them to the standard QFT particle and antiparticle spatial densities. The formalism is illustrated by computing numerically the Klein tunneling dynamics of strictly localized wavepackets impinging on a supercritical electrostatic step. The density operator introduced here could be useful to model situations in which it is desirable to avoid dealing with the infinite spatial tails intrinsic to pure particle or antiparticle wavepackets.

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I. INTRODUCTION

Quantum field theory with a background field has been employed in various studies, beginning with the Schwinger-Sauter effect for pair creation [1], the excitation of a vacuum by intense laser fields [2], and the analysis of scattering problems [3–7]. In particular, a time-dependent nonperturbative formalism of quantum field theory with a background potential has been developed to compute numerically the dynamics of pair creation [4]. This formalism has successfully computed the rates of pair creation for spin-0 bosons and spin-1/2 fermions resulting from the excitation of the vacuum by arbitrary electrostatic potentials [8–12]. Recently, we used this formalism to provide insights into the dynamics of Klein tunneling for both fermions and bosons [13].

One of the main advantages of using this formalism is the ability to account for the time-dependent dynamics of wavepackets propagating in arbitrary electromagnetic potentials. Typically, one considers an approximately spatially localized wavepacket of unit charge representing the initial particle (say an electron) scattering on a supercritical potential. The corresponding (e.g., fermionic) field operator is then used to construct charge density operators that give the evolution of the fermionic or antifermionic states as the wavepacket scatters on the potential producing particle-antiparticle pairs.

It is well known that a wavepacket composed solely of positive energy plane-wave solutions of the Dirac (or Klein-Gordon) equation cannot have a finite support [14–16]. Therefore, the charge density computed from the field operator will also exhibit infinite tails. In most practical scenarios, the infinite tails of the wavepacket can be neglected: they are very small, in particular, for wavepackets that are wide enough (relative to the Compton wavelength). However, in some instances (e.g., when addressing issues related to time interval detection), it might be desirable to avoid dealing with infinite tails and model the initial wavepacket as having compact support.

In this paper, our aim is to extend this computational QFT scheme to take into account wavepackets with initial compact support. The existence of perfectly localized states in relativistic quantum mechanics and QFT remains a controversial issue [17–23] and we will not dwell here into this debate. Our starting point is the pragmatic observation that a state with compact support must contain both positive and negative energy components. Hence, if such a state were measured right after preparation, one would not obtain a particle with certainty, but might find an antiparticle with a small probability. However, in typical situations only the positive energy component propagates towards a given direction of interest, at which point one is dealing with the dynamics of a particle wavepacket without tails interacting with the background field.

We will see that the usual charge density operators cannot account for the dynamics of compact support wavepackets given that a proper density should encompass the positive and negative energy manifolds. We will build instead a density operator reminiscent of the way in which states with compact support are treated in the first quantized theory. By considering the pair creation process that occurs when exciting the vacuum with a supercritical potential, the density operator will allow us to study the propagation of these finite support wavepackets through arbitrary potentials.

The paper is organized as follows. In Sec. II, we briefly recall the propagation of wavepackets with compact support in first quantized relativistic quantum mechanics and give a brief overview of wavepacket treatment within the computational QFT framework we are using. We will see, in particular, why the charge density operators fail to account for the propagation of wavepackets with finite support. In Sec. III we introduce a different density operator that considers modes associated with both positive and negative energies. This "charge-blind" density operator allows us to propagate wavepackets with compact support; we will examine some properties of the expectation values of this operator. In Sec. IV we illustrate both aspects of the density operator, first by computing the free propagation of Dirac and Klein-Gordon wavepackets having compact support and by carrying out numerical calculations for fermionic wavepackets scattering on a supercritical potential step giving rise to Klein tunneling. We close the paper by discussing our findings and drawing our conclusions (Sec. V).

II. PROPAGATION OF WAVE PACKETS

We recall here the propagation of wavepackets in first quantized relativistic quantum mechanics (RQM) and in the computational QFT framework employed primarily to tackle the space-time-resolved dynamics of bosonic or fermionic fields in a background potential. We will more specifically focus on the solutions of the Dirac and Klein-Gordon equations in one spatial dimension with the respective Hamiltonians $H^D = H_0^D + V$ and $H^{\mathrm{KG}} = H_0^{\mathrm{KG}} + V$. In the Dirac case, $H_0^D = -i\hbar c\alpha_x\partial_x + \beta mc^2$ (α and β are the usual Dirac matrices [1], m the electron mass, and c the light velocity) while the free KG Hamiltonian is given by $H_0^{\mathrm{KG}} = -\frac{\hbar^2}{2m}(\tau_3 + i\tau_2)\partial_x^2 + mc^2\tau_3$ (we are using the so-called Hamiltonian form of the KG equation, where τ_i are the Pauli matrices and m now represents the boson mass [24]). $V(x)$ is the background potential. The eigenstates of the free Hamiltonians H_0 will be denoted by $|\phi_p\rangle$ for positive energies and $|\varphi_p\rangle$ for negative energies, where $\pm|E_p| = \pm\sqrt{p^2c^2 + m^2c^4}$. We will use the same notation (including for the scalar product) for the Dirac and Klein-Gordon cases and only specify differences between bosons and fermions when relevant.

A. Wavepackets in the first quantized formalism

In the first-quantized formalism a wavepacket with an arbitrary profile in configuration space contains an expansion over both positive and negative energy components [24]. This is, in particular, the case for a state having compact support. Let us assume $\psi(0, x)$ describes a state with positive charge equal to 1 that is zero outside an interval D,

$$\langle x|\psi\rangle = \psi(0, x) = N\begin{pmatrix} G(x) \\ 0 \end{pmatrix}, \tag{1}$$

where $G(x)$ is a function with compact support equal to zero outside D and N is a normalization constant. We can rewrite this initial wavepacket in terms of the projection over the positive and negative energy components

$$\psi(0, x) = \psi_+(0, x) + \psi_-(0, x), \tag{2}$$

where we define

$$\psi_+(0, x) = \int g_+(p)\langle x|\phi_p\rangle dp,\ \ \psi_-(0, x) = \int g_-(p)\langle x|\varphi_p\rangle dp, \tag{3}$$

and

$$g_+(p) = \int dx\langle\phi_p|x\rangle\sigma\langle x|\psi\rangle,\ \ g_-(p) = -\epsilon\int dx\langle\varphi_p|x\rangle\sigma\langle x|\psi\rangle, \tag{4}$$

where σ is equal to the Pauli matrix τ_3 and $\epsilon = 1$ in the case of bosons, and σ is equal to the identity and $\epsilon = -1$ in the case of fermions.

At time t, this wavepacket will evolve to

$$\psi(t, x) = \int g_+(p)\langle x|\phi_p\rangle e^{-iE_pt}dp + \int g_-(p)\langle x|\varphi_p\rangle e^{+iE_pt}dp = \psi_+(t, x) + \psi_-(t, x). \tag{5}$$

If the mean momentum of the initial wavepacket is positive, the positive energy part will propagate in the positive direction (towards the right) while the negative energy one will propagate to the left (see, e.g., Ref. [26] for an illustration). The density, which satisfies the continuity equation, is given by

$$r(t, x) = \psi^\dagger(t, x)\sigma\psi(t, x) = \psi_+^\dagger(t, x)\sigma\psi_+(t, x) + \psi_-^\dagger(t, x)\sigma\psi_-(t, x^\dagger) + 2\mathrm{Re}(\psi_+^\dagger(t, x)\sigma\psi_-(t, x)). \tag{6}$$

It is important to notice this density can be interpreted as a probability density in the case of fermions, but for bosons it is a charge density that can be positive or negative. Recall finally that the full propagator (including positive as well as negative energy components) is causal [14], so that a wavepacket with initial compact support will always remain within the light cone emanating from the bounds of the initial support of the wave function.

B. Computational QFT formalism

1. Basic expressions

We start from the familiar QFT expressions [27] for the creation operator of particles and the annihilation operator of antiparticles ("an") at some position x,

$$\hat{\Psi}_{\mathrm{pa}} = \int dp\hat{b}_p\langle x|\phi_p\rangle,\qquad \hat{\Psi}_{\mathrm{an}}^\dagger = \int dp\hat{d}_p^\dagger\langle x|\varphi_p\rangle, \tag{7}$$

where b_p and d_p are the annihilation operators for a particle and an antiparticle, respectively (they yield a vanishing result when applied to the vacuum). $b_p^\dagger$ and $d_p^\dagger$ are the corresponding creation operators. These operators obey the usual commutation relations, i.e., for fermions the only nonzero equal time anticommutators are $[b_p, b_k^\dagger]_+ = [d_p, d_k^\dagger]_+ = \delta(p-k)$ (and similar commutators for bosons). We define as usual the conjugate of these field operators acting on the dual Fock space

$$\hat{\Psi}_{pa}^\dagger = \int dp\hat{b}_p^\dagger\langle\phi_p|x\rangle,\qquad \hat{\Psi}_{an} = \int dp\hat{d}_p\langle\varphi_p|x\rangle. \tag{8}$$

[1] We will consider the usual one effective spatial dimension approximation, neglecting spin-flip [25] and replacing α_x and β by the Pauli matrices σ_1 and σ_3, respectively.

The time evolution of these operators is obtained in the Heisenberg picture by calculating the time evolution of each creation and annihilation operator [4] (see also the Supplemental Material of Ref. [13])

$$b_p(t) = \int dp'(U_{\phi_p\phi_{p'}}(t)\hat{b}_{p'} + U_{\phi_p\varphi_{p'}}(t)\hat{d}^\dagger_{p'}),$$
$$d^\dagger_p(t) = \int dp'(U_{\varphi_p\phi_{p'}}(t)\hat{b}_{p'} + U_{\varphi_p\varphi_{p'}}(t)\hat{d}^\dagger_{p'}), \qquad (9)$$

where

$$U_{\zeta_p\xi_q}(t) = \langle\zeta_p|e^{-iHt}|\xi_q\rangle \qquad (10)$$

are the evolution amplitudes generated by the full Hamiltonian (hence including the background field). The field operators (7), therefore, become in the Heisenberg picture

$$\hat{\Psi}_{\rm pa}(t,x) = \int dp dp'(U_{\phi_p\phi_{p'}}(t)\hat{b}_{p'} + U_{\phi_p\varphi_{p'}}(t)\hat{d}^\dagger_{p'})\langle x|\phi_p\rangle,$$
$$\hat{\Psi}^\dagger_{\rm an}(t,x) = \int dp dp'(U_{\varphi_p\phi_{p'}}(t)\hat{b}_{p'} + U_{\varphi_p\varphi_{p'}}(t)\hat{d}^\dagger_{p'})\langle x|\varphi_p\rangle, \qquad (11)$$

with analog expressions for their conjugate.

The field operator obtained by quantizing the "classical" fields defined from the Klein-Gordon or Dirac Lagrangians is related to the charge structure of the field [22]. It is obtained by combining the operators given by Eqs. (7) and (8) as

$$\hat{\Psi}(t,x) = \hat{\Psi}_{\rm pa}(t,x) + \hat{\Psi}^\dagger_{\rm an}(t,x), \qquad (12)$$

while the expression

$$\hat{\rho}_{\rm ch}(t,x) = \hat{\Psi}^\dagger(t,x)\sigma\hat{\Psi}(t,x), \qquad (13)$$

represents the total charge density operator. Usually, one is interested in the dynamics of the particle (or antiparticle) density. The particle density operator is given by

$$\hat{\rho}_{\rm pa}(t,x) = \hat{\Psi}^\dagger_{\rm pa}(t,x)\sigma\hat{\Psi}_{\rm pa}(t,x). \qquad (14)$$

The particle density is then obtained as usual from the expectation value of such operators. For instance, denoting the vacuum by $\|0\rangle\rangle$ (where the symbol $\|\rangle\rangle$ refers to a state in Fock space) the vacuum expectation value $\langle\langle 0\|\hat{\rho}_{\rm pa}(t,x)\|0\rangle\rangle$ gives the space-time particle density (created by the background field) when there are initially no particles or antiparticles. In the present computational QFT framework such densities are obtained by computing numerically the evolution operator amplitudes given by Eq. (10) over a basis of solutions of the free (Klein-Gordon or Dirac) Hamiltonian (see, e.g., Ref. [13]). The total number of positively charged particles is obtained as usual by integrating the density

$$N_{\rm pa}(t) = \int dx\langle\langle 0\|\hat{\rho}_{\rm pa}(t,x)\|0\rangle\rangle. \qquad (15)$$

The antiparticle density operator is defined from Eqs. (7) and (8) as

$$\hat{\rho}_{\rm an}(t,x) = \hat{\Psi}^\dagger_{\rm an}(t,x)\sigma\hat{\Psi}_{\rm an}(t,x). \qquad (16)$$

Note that the expectation values of this operator involves the scalar product between the basis expansion functions $\varphi_j(t,x)$, which is positive for Dirac fields but negative in the Klein-Gordon case. Hence the antiparticle number is now given by

$$N_{\rm an}(t) = -\epsilon\int dx\langle\langle 0\|\hat{\rho}_{\rm an}(t,x)\|0\rangle\rangle, \qquad (17)$$

where $\epsilon = 1$ for spin-0 bosons and $\epsilon = -1$ for spin-1/2 fermions.

2. Wavepacket densities

It is useful when considering a particle scattering on a potential to model the particle as a wavepacket An initial particle wavepacket $\chi(0,x)$ is written in terms of creation operators $b^\dagger_p$ as [4]

$$\|\chi_+\rangle\rangle = \int dp g_+(p)b^\dagger_p\|0\rangle\rangle. \qquad (18)$$

The corresponding particle density is then given by the expectation value

$$\rho_{\rm pa}(t,x) = \langle\langle\chi_+\|\hat{\rho}_{\rm pa}(t,x)\|\chi_+\rangle\rangle, \qquad (19)$$

representing the density due to the background field and the evolved wavepacket. The amplitudes $g_+(p)$ determine the spatial profile of the wavepacket, as is obvious by recalling that a first quantized single-particle wave function $\chi(t,x)$ is obtained from the second quantized states as [27]

$$\chi(t,x) = \langle\langle 0\|\hat{\Psi}(t,x)\|\chi\rangle\rangle. \qquad (20)$$

Note that if $\|\chi_+\rangle\rangle$ of Eq. (18) is inserted into Eq. (20), the resulting wavepacket $\chi_+(t,x)$ contains only positive energy modes, and can therefore only account for a particle wave function presenting infinite tails. This remains true if we replace Eq. (18) by

$$\|\chi\rangle\rangle = \int dp(g_+(p)\hat{b}^\dagger_p + g_-(p)\hat{d}^\dagger_p)\|0\rangle\rangle, \qquad (21)$$

which would be the analog of the first quantized wavepacket given by Eq. (3) since the negative energy sector components vanish when inserted into Eq. (20).

Similarly, the single-particle wave function generated from the field operator $\hat{\Psi}^\dagger(t,x)$ only keeps the negative energy modes, yielding an antiparticle wave function with infinite tails that can only be approximately localized. We therefore see that we cannot represent a wavepacket with compact support within the computational QFT framework. Of course, all the densities that can be computed also present infinite tails: $\rho_{\rm pa}(t,x)$ projects to the particle sector only and $\rho_{\rm an}(t,x) = \langle\langle\chi\|\hat{\rho}_{\rm an}(t,x)\|\chi\rangle\rangle$ to the antiparticle sector, while the charge density operator of Eq. (13) can be seen to yield (see Appendix A for details) the charge density $\rho_{\rm ch}(t,x) = \rho_{\rm pa}(t,x) - \epsilon\rho_{\rm an}(t,x)$ (the tails in $\rho_{\rm pa}$ and $\rho_{\rm an}$ are different and do not cancel out).

III. DENSITY OPERATOR AND LOCALIZED WAVEPACKETS

To use the computational QFT formalism with wavepackets having compact spatial support, we need to define a density operator that does not project to the particle or antiparticle sectors. This is done by introducing

$$\hat{\rho}(t,x)=\hat{\mathfrak{v}}^{\dagger}(t,x)\sigma\hat{\mathfrak{v}}(t,x), \tag{22}$$

where

$$\hat{\mathfrak{v}}^{\dagger}(t,x)=\int dp(\hat{b}_p^{\dagger}(t)\langle\phi_p|x\rangle+(\hat{d}_p^{\dagger}(t))^{*}\langle\varphi_p|x\rangle),$$
$$\hat{\mathfrak{v}}(t,x)=\int dp(\hat{b}_p(t)\langle x|\phi_p\rangle+(\hat{d}_p(t))^{*}\langle x|\varphi_p\rangle). \tag{23}$$

One can express the new operator $\hat{\mathfrak{v}}$ in terms of the familiar QFT expressions, Eqs. (7) and (11), as

$$\hat{\mathfrak{v}}^{\dagger}(t,x)=\hat{\Psi}_{\rm pa}^{\dagger}(t,x)+(\hat{\Psi}_{\rm an}^{\dagger}(t,x))^{*T},$$
$$\hat{\mathfrak{v}}(t,x)=\hat{\Psi}_{\rm pa}(t,x)+(\hat{\Psi}_{\rm an}(t,x))^{*T}, \tag{24}$$

where the congujation applies to the c-numbers and the transpose to the spinors in the expressions defined in Eqs. (7) and (8). The operator $\hat{\rho}(t,x)$ accounts for the density of particles and antiparticles regardless of their charge. The rationale for taking this combination of the field operators (7) is that rather than creating (or annihilating) a charge, we are now creating (or destroying) a particle and an antiparticle without changing the charge of a state. Indeed, the standard field operator is well known to be related to the charge structure in the sense that $\Psi^{\dagger}$ raises the charge by 1, i.e., if $\hat{Q}=\int dx\hat{\rho}_{\rm ch}(x)$ is the total charge operator [see Eq. (13)] and $\|q\rangle\rangle$ is a state of charge q, i.e., $\hat{Q}\|q\rangle\rangle=q\|q\rangle\rangle$, then $\hat{Q}[\Psi^{\dagger}\|q\rangle\rangle]=(q+1)[\Psi^{\dagger}\|q\rangle\rangle]$ so that $\Psi^{\dagger}\|q\rangle\rangle$ appears as a state of charge $q+1$. Similarly, we can show that $\hat{\mathfrak{v}}^{\dagger}$ increases the number of any n particle $\|n\rangle\rangle$ by 1: if $\hat{N}$ is the number operator, $\hat{N}=\int dx(\hat{\rho}_{\rm pa}-\epsilon\hat{\rho}_{\rm an})=\int dp(b_p^{\dagger}b_p+d_p^{\dagger}d_p)$ and $\hat{N}\|n\rangle\rangle=n\|n\rangle\rangle$, then we have (see Appendix B)

$$\hat{N}[\hat{\mathfrak{v}}^{\dagger}\|n\rangle\rangle]=(n+1)[\hat{\mathfrak{v}}^{\dagger}\|n\rangle\rangle]. \tag{25}$$

This implies, in particular, that $\hat{\rho}$ must contain terms allowing for the conversion of a particle into an antiparticle (and vice versa). These are the cross terms obtained when plugging-in Eq. (23) into Eq. (22).

As a consequence we can now accommodate a compact support wavepacket as given by Eq. (21) through

$$\chi(t,x)=\langle\langle 0\|\hat{\mathfrak{v}}(t,x)\|\chi\rangle\rangle \tag{26}$$

[compare with Eq. (20)]. In the presence of such a wavepacket, $\hat{\rho}(t,x)$ may be used to define a density that remains localized over a compact support. Such a space-time-resolved density is obtained from the expectation value $\rho(t,x)=\langle\langle\chi\|\hat{\rho}_n(t,x)\|\chi\rangle\rangle$ which becomes

$$\rho(x,t)=\langle\langle 0\|\int dp(g_+^{*}(p)\hat{b}_p+g_-^{*}(p)\hat{d}_p)\hat{\rho}(t,x)$$
$$\times\int dp(g_+(p)\hat{b}_p^{\dagger}+g_-(p)\hat{d}_p^{\dagger})\|0\rangle\rangle. \tag{27}$$

To highlight the localized character of this density, we will parse $\rho(t,x)$ as the density of particles and antiparticles created by the background field, on the one hand, and a wavepacket density identical to the first quantized single-particle wavepacket given by Eq. (6), on the other, a wavepacket that is known to be supported on a compact support. After some algebra (see Appendix C) we obtain

$$\rho(t,x)=\rho_1(t,x)+\rho_2(t,x)+\rho_3(t,x). \tag{28}$$

The term

$$\rho_1(t,x)=\int dp\left(\int dqU_{\phi_p\varphi_q}\langle x|\phi_p\rangle\right)^{\dagger}\sigma\left(\int dqU_{\phi_p\varphi_q}\langle x|\phi_p\rangle\right)+\left(\int dpdqg_+(q)U_{\phi_p\phi_q}\langle x|\phi_p\rangle\right)^{\dagger}\sigma\left(\int dpdqg_+(q)U_{\phi_p\phi_q}\langle x|\phi_p\rangle\right)$$
$$+\epsilon\left(\int dpdqg_-^{*}(q)U_{\phi_p\varphi_q}\langle x|\phi_p\rangle\right)^{\dagger}\sigma\left(\int dpdqg_-^{*}(q)U_{\phi_p\varphi_q}\langle x|\phi_p\rangle\right) \tag{29}$$

represents the density due to the presence of the background potential (first line), the density corresponding to the incoming particle (second line), and the modulation in the number density of the created particles due to the incident particle wavepacket. The structure of ρ_1 is identical to the particle density defined by taking the expectation value of Eq. (14).

The second term, given by

$$\rho_2(t,x)=\int dp\left(\int dqU_{\varphi_p\phi_q}\langle x|\varphi_p\rangle\right)^{\dagger}\sigma\left(\int dqU_{\varphi_p\phi_q}\langle x|\varphi_p\rangle\right)+\left(\int dpdqg_-(q)U_{\varphi_p\varphi_q}\langle x|\varphi_p\rangle\right)^{\dagger}\sigma\left(\int dpdqg_-(q)U_{\varphi_p\varphi_q}\langle x|\varphi_p\rangle\right)$$
$$+\epsilon\left(\int dpdqg_+(q)U_{\varphi_q\varphi_p}\langle x|\varphi_p\rangle\right)^{\dagger}\sigma\left(\int dpdqg_+(q)U_{\varphi_q\phi_q}\langle x|\varphi_p\rangle\right) \tag{30}$$

is the counterpart of $\rho_1(t,x)$ for the antiparticle density and is hence identical to the density obtained from the operator given by Eq. (16).

Finally, the third term in Eq. (28)

$$\rho_3(t,x)=2{\rm Re}\left(\int dpdqg_-^{*}(q)U_{\varphi_p\varphi_q}^{*}g_+(q)U_{\phi_p\phi_q}\langle\varphi_q|x\rangle\sigma\langle x|\phi_p\rangle\right)+2{\rm Re}\left(\int dpdqg_-^{*}(q)U_{\varphi_p\phi_q}^{*}g_+(q)U_{\phi_p\varphi_q}\langle\varphi_q|x\rangle\sigma\langle x|\phi_p\rangle\right) \tag{31}$$

involves cross terms between positive and negative energy modes of the initial wavepacket. This term accounts for the cancellation of the infinite spatial tails intrinsic to ρ_1 and ρ_2. When integrated over the entire space this term vanishes, ensuring that ρ obeys

$$\int dx\rho(t,x) = \int dx\rho_{pa}(t,x) + \int dx\rho_{\rm an}(t,x). \tag{32}$$

This is to be compared to the total number of particles N given by the expectation value of the number operator [cf. Eq. (B1)] as

$$N(t) = \int dx\rho_{\rm pa}(t,x) - \epsilon\int dx\rho_{\rm an}(t,x). \tag{33}$$

We therefore see that for fermions ($\epsilon = -1$) $\rho(t,x)$ can be interpreted as a number density, whereas for bosons $\rho(t,x)$ appears as a charge density.

Note that in the absence of a background field $\rho(t,x)$ represents the evolution of the sole wavepacket. In this case, the density simplifies to [put $U_{\phi_p\varphi_q} = 0$ in Eq. (28)]

$$\begin{aligned}\rho^{(0)}(t,x) = &\left(\iint dpdqg_+(q)U_{\phi_p\phi_q}\langle x|\phi_p\rangle\right)^\dagger\sigma\left(\iint dpdqg_+(q)U_{\phi_p\phi_q}\langle x|\phi_p\rangle\right)\\ &+\left(\iint dpdqg_-(q)U_{\varphi_p\varphi_q}\langle x|\phi_p\rangle\right)^\dagger\sigma\left(\iint dpdqg_-(q)U_{\varphi_p\varphi_q}\langle x|\phi_p\rangle\right)\\ &+2\mathrm{Re}\left(\iint dpdqg_-(q)U^*_{\varphi_p\varphi_q}g_+(q')U_{\phi_p\phi_q}\langle\varphi_p|x\rangle\sigma\langle x|\phi'_p\rangle\right),\end{aligned} \tag{34}$$

which is equal to the density calculated in the first quantized theory, given by Eq. (6). This demonstrates that $\rho^{(0)}$ evolves within a compact support inside the light cone. Moreover, in the absence of a wavepacket, the density of created matter is equal to the sum of the two number densities of particles and antiparticles since the cross terms ρ_3 and all the terms with $g_\pm$ vanish in Eqs. (29) and (30). This will be illustrated below.

IV. ILLUSTRATIONS

A. Free propagation of compact support wavepackets

We study numerically the time evolution of perfectly localized wavepackets of a spin-0 boson or a spin-1/2 fermion using the bosonic or fermionic field operators introduced above. The computational techniques employed, based on accurate numerical computations of the densities on a finite space-time grid, were detailed elsewhere (see Supplementary Material of Ref. [13]) for the case of the usual computational QFT framework; here we simply need to arrange the terms differently when computing $\rho(t,x)$.

Let us take the initial wavepacket

$$\psi(x,0) = \mathcal{N}\begin{pmatrix} G(x)\\ 0\end{pmatrix}, \tag{35}$$

where $\mathcal{N}$ is a normalization constant and $G(x)$ is defined on the compact support $x \in [x_0 - D\pi/2, x_0 + D\pi/2]$ as

$$G(x) = \cos^8\left(\frac{x-x_0}{D}\right)e^{ip_0x}. \tag{36}$$

This wavepacket has a mean momentum p_0 and is centered around x_0 in real space with a width D. We then determine the particle and antiparticle densities employing the usual computational QFT framework introduced in Sec. II B, as well as the density operator proposed in Sec. III; for free propagation of interest in this subsection, the corresponding expression is given by Eq. (34).

We show in Fig. 1 the fermionic (Dirac field) particle and antiparticle densities $\rho_{\rm pa}$ and $\rho_{\rm an}$ as well as the density ρ for a freely propagating wavepacket whose initial state is given by Eq. (35) at two instants $t = 0$ a.u. and $t = 8\times10^{-4}$ a.u.. The particle as well as the antiparticle densities have infinite tails, while the density ρ reproduces the compact support of the initial wave function and remains inside the light cone at later times.

Figure 2 shows similar calculations for a bosonic (charged Klein-Gordon) field. Only the evolved densities are shown (the initial wavepacket, centered at $x_0 = 0$, is qualitatively similar to the one shown in Fig. 1). It can be seen that the particle density in the dotted black line (blue) has moved towards the right while the antiparticle density in dashed gray line (red) has moved towards the left. The density in the gray solid line (cyan) takes into account both charges and remains localized within the light-cone emanating from the initial density, while the charged densities have infinite tails leaking from the light cone.

B. Klein tunneling with localized wavepackets

Let us now examine the propagation of the densities in the presence of a background field. For definiteness, let us take a supercritical potential step rising at $x = d$ given by

$$V(x) = V_0\{1+\tanh[(x-d)/\alpha]\}/2, \tag{37}$$

where V_0 is the step height and α the smoothness parameter of the background field. In the absence of any wavepacket, the background potential creates particle-antiparticle pairs. This is illustrated in Fig. 3 where the space-time-resolved densities $\rho_{\rm pa}$, $\rho_{\rm an}$, and ρ are plotted in the fermionic case.

The more interesting case is that of Klein tunneling in which an electron wavepacket scatters on the supercritical step and propagates undamped in the potential region. The field operators now account for pair creation and wavepacket propagation. In the standard computational QFT treatment [4,13],

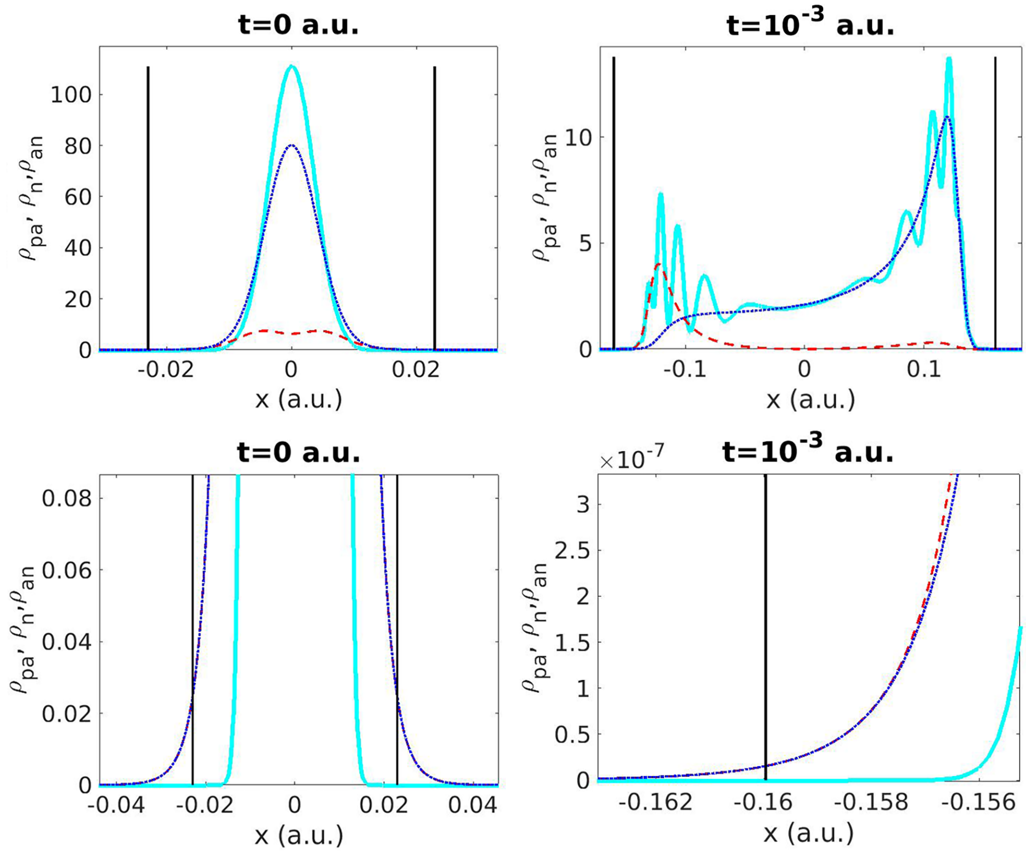

FIG. 1. The spatial number densities for spin-1/2 fermions of unit mass (a.u.) are shown at $t = 0$ (left panels) and $t = 10^{-3}$ a.u. (right panels). The bottom row zooms the top row densities to contrast the tails of the particle and antiparticle densities $\rho_{\rm pa}$ (dotted black, online color blue) and $\rho_{\rm an}$ (dashed gray, online color red) from the localized character of the proposed density ρ (solid gray, online color cyan). In the left panel, the two vertical black lines indicate the bounds of the compact support density. In the right panel, the vertical black lines are the positions of the light-cone emanating from these bounds. Note that a small fraction of the particle and antiparticle densities present tails outside the light cone. The freely propagating wave packet is given by Eq. (35) with $D = 2/c$ and $p_0 = 100$ a.u. (atomic units are used throughout).

the wavepacket displays infinite tails at any time, whereas the densities proposed in Sec. III remain within the light cone emanating from the compact support region over which the initial wavepacket is localized. This is illustrated in Fig. 4. We plot there, for the fermionic case, only the terms in ρ_1, ρ_2, and ρ_3 [cf. Eqs. (28) to (31)] containing terms relevant to the wavepacket (which is tantamount to subtracting the terms in the total density that account for pair creation).

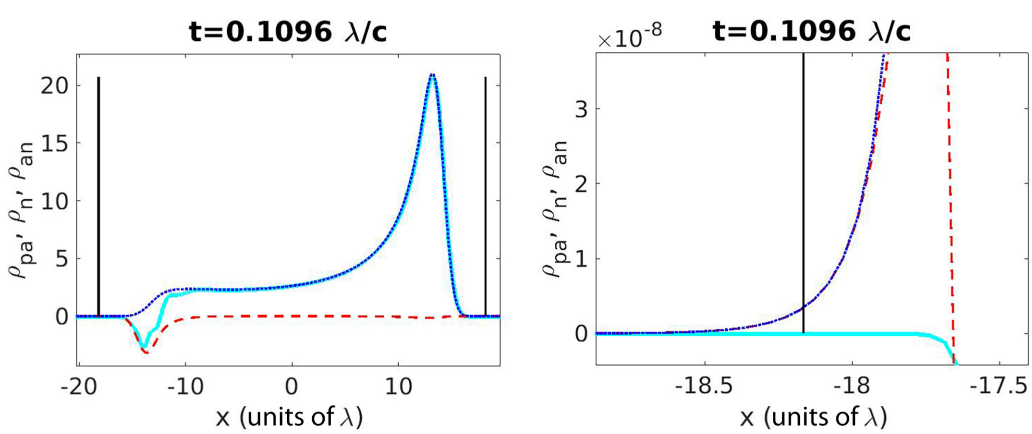

FIG. 2. Same as Fig. 1 but for a Klein-Gordon wavepacket. Only the propagated wavepacket at $t = 8 \times 10^{-4}$ is shown. The plot on the right panel is a zoom of the left panel figure near the position of the light-cone originating from the left bound of the $t = 0$ compact support density. The color coding for the densities is the same as in Fig. 1. Units are given in terms of the Compton wavelength λ. The freely propagating wavepacket is given by Eq. (35) with $D = 2\lambda$ and $p_0 = 100\hbar/\lambda$.

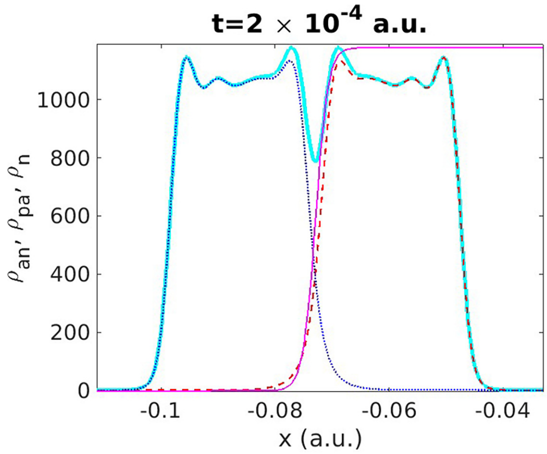

FIG. 3. Vacuum expectation number for fermionic particles (electrons) and antiparticles (positrons) created by a supercritical background field. The position of the step is represented in light thin solid gray line (online color magenta), while $\rho_{\rm pa}(t,x)$ and $\rho_{\rm an}(t,x)$ are pictured in dotted black (blue) and dashed gray (red), respectively. The vacuum expectation value of $\hat{\rho}(t,x)$ in solid gray (cyan) appears as the algebraic sum $\rho_{\rm pa}+\rho_{\rm an}$ since ρ_3 [Eq. (31)] vanishes. The parameters of the step are $V_0=9mc^2$ and $d=-10\lambda$, with $\alpha=0.3/\lambda$.

V. DISCUSSION AND CONCLUSION

We proposed in this paper a way to work with states defined over a compact spatial support in the framework of computational QFT. Our approach hinged on introducing a density operator $\hat{\rho}(t,x)$ [Eq. (22)] from which expectation values yield densities lacking the infinite tails characterizing the standard computational QFT formalism. Our starting point was the remark that a compact support KG or Dirac wavepacket must have both positive and negative energy components: by relying on the connection between QFT states and first quantized ones, we have seen how such a density operator could be constructed. We analyzed some properties of the expectation values of the proposed operator, and computed numerical results for specific illustrations for Klein-Gordon and Dirac fields (free propagation of a wavepacket and a wavepacket impinging on a Klein step).

We emphasize that the approach developed here should be regarded as a practical recipe to manipulate compactly localized states when studying space-time-resolved wavepacket dynamical problems in situations in which it is awkward to use the "essentially localized states" [22] of standard quantum field theory intrinsically displaying infinite tails. Such states could be useful, for example, in certain detector models aiming at measuring arrival times. We are not claiming that the operators $\nu(x,t)$ and $\nu^\dagger(x,t)$ of Eq. (23) can be promoted to fundamental quantities from which a full-fledged quantum field approach can be defined. Previous attempts to construct strictly localized states from vacuum excitations of a quantized field have run into difficulties, such as field-theoretic Hamiltonians unbounded from below [20]. Nevertheless, besides the practical usage alluded to above, the present results could also be useful in investigating problems such as optimizing the localization properties of a single particle state, or conversely, finding how close a strictly localized wavepacket can be to a genuine single-particle state. These problems will be investigated in the future.

APPENDIX A: DENSITIES WITH INFINITE TAILS IN THE STANDARD COMPUTATIONAL QFT FRAMEWORK

Let us determine the particle density $\rho_{\rm pa}(t,x)$ in the absence of a background potential. $\rho_{\rm pa}$, defined by Eq. (19), becomes

$$\rho_{\rm pa}(t,x)=\langle\langle 0\|\int dp(g_+^*(p)b_p+g_-^*(p)d_p)\hat{\rho}_{\rm pa}(t,x)\times\int dp(g_+(p)b_p^\dagger+g_-(p)d_p^\dagger)\|0\rangle\rangle. \quad \text{(A1)}$$

Writing $\hat{\rho}_{pa}(t,x)$ in terms of the field operators, Eq. (14), using Eq. (9), and noticing that $U_{\phi_p\varphi_{p'}}=0$ in the case of free propagation, one obtains

$$\begin{aligned}\rho_{\rm pa}(t,x)=\langle\langle 0\|&\int dp(g_+^*(p)b_p+g_-^*(p)d_p)\\&\times\iint dpdp'U^*_{\phi_p\phi_{p'}}b^\dagger_{p'}\langle\phi_p|x\rangle\sigma\\&\times\iint dpdp'U_{\phi_p\phi_{p'}}b_{p'}\langle x|\phi_p\rangle\\&\times\int dp(g_+(p)b_p^\dagger+g_-(p)d_p^\dagger)\|0\rangle\rangle.\end{aligned} \quad \text{(A2)}$$

The only nonvanishing term in this case gives

$$\rho_{\rm pa}(t,x)=\left(\int dpg_+(p)U_{\phi_{p'}\phi_p}\langle x|\phi_p\rangle\right)^\dagger\times\sigma\left(\int dpg_+(p)U_{\phi_{p'}\phi_p}\langle x|\phi_p\rangle\right). \quad \text{(A3)}$$

Similarly, the density of antiparticles is computed as

$$\rho_{\rm an}(t,x)=\left(\int dpg_-(p)U_{\varphi_{p'}\varphi_p}\langle x|\varphi_p\rangle\right)^\dagger\times\sigma\left(\int dpg_-(p)U_{\varphi_{p'}\varphi_p}\langle x|\varphi_p\rangle\right). \quad \text{(A4)}$$

Now using

$$\begin{aligned}U_{\phi_p\phi_q}&=\int dx\langle\phi_p|x\rangle\sigma\langle x|e^{-iE_qt}|\phi_q\rangle\\&=e^{-iE_qt}\int dx\langle\phi_p|x\rangle\sigma\langle x|\phi_q\rangle,,\\U_{\varphi_p\varphi_q}&=\int dx\langle\varphi_p|x\rangle\sigma\langle x|e^{iE_qt}|\varphi_q\rangle\\&=e^{iE_qt}\int dx\langle\varphi_p|x\rangle\sigma\langle x|\varphi_q\rangle,\end{aligned} \quad \text{(A5)}$$

and

$$\begin{aligned}&\int dx\langle\phi_p|x\rangle\sigma\langle x|\phi_q\rangle=\delta_{pq},\\&\int dx\langle\varphi_p|x\rangle\sigma\langle x|\varphi_q\rangle=-\epsilon\delta_{pq},\end{aligned} \quad \text{(A6)}$$

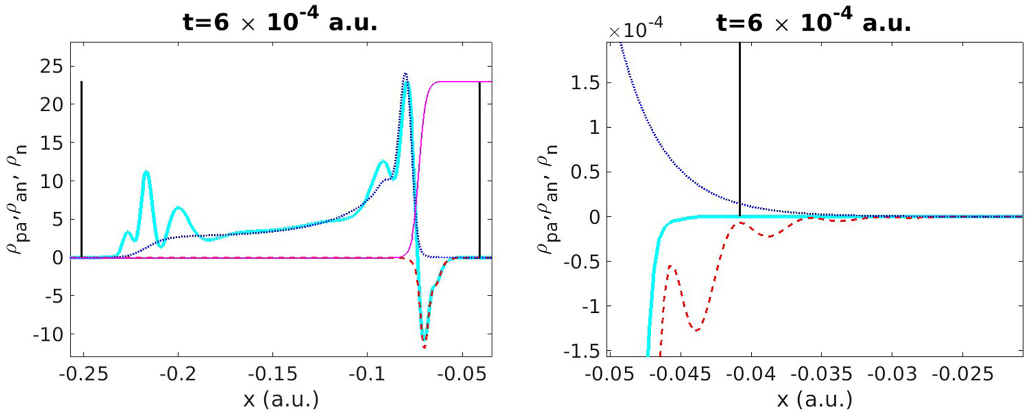

FIG. 4. Wavepacket densities (obtained by subtracting the terms corresponding to pair creation) for an electron packet scattering on a supercritical potential step of height $V_0 = 9mc^2$, smoothness parameter $\alpha = 0.3/c$, and centered around $d = -10\lambda$. The position of the step is represented in light thin solid gray line (magenta), the particle (electron) density is shown in blue, the antiparticle (positron) density in dashed gray line (red) and the compact support density in solid light gray (cyan). The plot on the right zooms on the region around the light cone originating at the $t = 0$ right boundary of the spatial support of the initial wavepacket: the compact-support density is seen to vanish outside the light cone while this is not the case for the particle and antiparticle densities.

where $\epsilon = +1(-1)$ in the bosonic (fermionic) case, one obtains

$$\rho_{\rm pa}(t,x) = \left(\int dp g_+(q) e^{-iE_p}\langle x|\phi_p\rangle\right)^\dagger \times \sigma\left(\int dp g_+(q) e^{-iE_p}\langle x|\phi_p\rangle\right), \tag{A7}$$

$$\rho_{\rm an}(t,x) = \left(\int dp g_-(q) e^{iE_p}\langle x|\varphi_p\rangle\right)^\dagger \times \sigma\left(\int dp g_-(q) e^{iE_p}\langle x|\varphi_p\rangle\right). \tag{A8}$$

It is straightforward to see that these densities are exactly equal to the first quantized densities $\psi_+(t,x)^\dagger\sigma\psi_+(t,x)$ and $\psi_-(t,x)^\dagger\sigma\psi_-(t,x)$, respectively [see Eq. (3)]. They can thus not be localized within a region of compact support, given that the cross term in Eq. (6) is necessary to suppress the tails.

The total charge density $\rho_{\rm ch}(t,x)$, defined by Eq. (13) and whose integration over all space defines the usual charge operator [27–29], also presents tails. This can be seen immediately by relying on the textbook result

$$\rho_{\rm ch}(x,t) = \rho_{\rm pa}(t,x) - \epsilon\rho_{\rm an}(t,x), \tag{A9}$$

and by using the reasoning below Eqs. (A7) and (A8). Alternatively, it can be directly derived within the present framework, starting from the field operator (12) in the Heisenberg picture

$$\hat\Psi(t,x) = \int dp(b_p^\dagger(t)\langle x|\phi_p\rangle + d_p(t)\langle x|\varphi_p\rangle). \tag{A10}$$

Equation (13) leads to

$$\begin{aligned}\rho_{\rm ch}(t,x) = \langle\langle 0\| \int dp(g_+^*(p)b_p + g_-^*(p)d_p) \\ \times \iint dp dp'(U^*_{\phi_p\phi_{p'}} b^\dagger_{p'}\langle\phi_p|x\rangle + U^*_{\varphi_p\varphi_{p'}} d_{p'}\langle\varphi_p|x\rangle)\sigma \\ \times \iint dp dp'(U_{\phi_p\phi_{p'}} b_{p'}\langle x|\phi_p\rangle + U_{\varphi_p\varphi_{p'}} d^\dagger_{p'}\langle x|\varphi_p\rangle) \\ \times \int dp(g_+(p)b_p^\dagger + g_-(p)d_p^\dagger)\|0\rangle\rangle,\end{aligned} \tag{A11}$$

which can be simplified to

$$\begin{aligned}\rho_{\rm ch}(x,t) = -\epsilon\left(\iint dp dq g_-(q)U_{\varphi_p\varphi_q}\langle x|\varphi_p\rangle\right)^\dagger \\ \times\sigma\left(\iint dp dq g_-(q)U_{\varphi_p\varphi_q}\langle x|\varphi_p\rangle\right) \\ + \left(\iint dp dq g_+(q)U_{\phi_p\phi_q}\langle x|\phi_p\rangle\right)^\dagger \\ \times\sigma\left(\iint dp dq g_+(q)U_{\phi_p\phi_q}\langle x|\phi_p\rangle\right).\end{aligned} \tag{A12}$$

APPENDIX B: NUMBER RAISING OPERATOR

We provide here a proof of Eq. (25).
Let

$$\hat N = \int dp(\hat b_p^\dagger \hat b_p + d_p^\dagger d_p) \tag{B1}$$

denote the total particle number. From Eq. (23), $\hat\mathfrak{v}^\dagger$ can be written as

$$\hat\mathfrak{v}^\dagger = \int dp'(\hat b^\dagger_{p'}\langle\phi_{p'}| + \hat d^\dagger_{p'}\langle\varphi_{p'}|). \tag{B2}$$

Therefore, we have

$$\hat N\hat\mathfrak{v}^\dagger = \iint dp dp'(\hat b_p^\dagger\hat b_p + \hat d_p^\dagger\hat d_p)(\hat b^\dagger_{p'}\langle\phi_{p'}| + \hat d^\dagger_{p'}\langle\varphi_{p'}|) \tag{B3}$$

and by using the fermionic or bosonic commutation relations in the first and last terms we obtain

$$\hat{N}\hat{\mathfrak{v}}^{\dagger} = \iint dp dp' (\epsilon \hat{b}_p^{\dagger}\hat{b}_{p'}^{\dagger}\hat{b}_p \langle\phi_{p'}| + \delta(p-p')\hat{b}_p^{\dagger}\langle\phi_{p'}| + \hat{d}_p^{\dagger}\hat{d}_p\hat{b}_{p'}^{\dagger}\langle\phi_{p'}|) + \iint dp dp' (\hat{b}_p^{\dagger}\hat{b}_p\hat{d}_{p'}^{\dagger}\langle\varphi_{p'}| + \epsilon\hat{d}_p^{\dagger}\hat{d}_{p'}^{\dagger}\hat{d}_p\langle\varphi_{p'}| + \delta(p-p')\hat{d}_p^{\dagger}\langle\varphi_{p'}|). \tag{B4}$$

Integrating the Dirac deltas and using again the the commutativity (anticommutativity) of bosons (fermions) in the first and fourth terms leads to

$$\hat{N}\hat{\mathfrak{v}}^{\dagger} = \int dp'\hat{b}_{p'}^{\dagger}\langle\phi_{p'}| \int dp(\hat{b}_p^{\dagger}\hat{b}_p + \hat{d}_p^{\dagger}\hat{d}_p) + \int dp'\hat{b}_{p'}^{\dagger}\langle\phi_{p'}| + \int dp'\hat{d}_{p'}^{\dagger}\langle\varphi_{p'}| \int dp(\hat{b}_p^{\dagger}\hat{b}_p + \hat{d}_p^{\dagger}\hat{d}_p) + \int dp'\hat{d}_{p'}^{\dagger}\langle\varphi_{p'}, \tag{B5}$$

and hence

$$\begin{aligned}\hat{N}\hat{\mathfrak{v}}^{\dagger} &= \int dp'\hat{b}_{p'}^{\dagger}\langle\phi_{p'}|(\hat{N}+1) + \int dp'\hat{d}_{p'}^{\dagger}\langle\varphi_{p'}|(\hat{N}+1)\\ &= \hat{\mathfrak{v}}^{\dagger}(\hat{N}+1),\end{aligned} \tag{B6}$$

from which Eq. (25) follows.

APPENDIX C: DENSITIES WITH COMPACT SUPPORT

Let us compute the expectation value of the density defined by Eq. (22) in the presence of an initial wavepacket with compact support. The resulting expression [Eq. (27)] becomes

$$\begin{aligned}\rho(t,x) = \langle\langle 0\| \int dp(g_+^*(p)\hat{b}_p + g_-^*(p)\hat{d}_p)\bigg\{ &\iint dp_1 dp_2 \langle\phi_{p_1}|x\rangle\sigma\langle x|\phi_{p_2}\rangle\hat{b}_{p_1}^{\dagger}(t)\hat{b}_{p_2}(t)\\ &+ \iint dp_1 dp_2\langle\varphi_{p_1}|x\rangle\sigma\langle x|\varphi_{p_2}\rangle\hat{d}_{p_1}^{\dagger}(t)\hat{d}_{p_2}(t) + \bigg(\iint dp_1 dp_2\langle\phi_{p_1}|x\rangle\sigma\langle x|\varphi_{p_2}\rangle\hat{b}_p^{\dagger}(t)\hat{d}_p(t) + \text{H.c.}\bigg)\bigg\}\\ &\times\int dp(g_+(p)\hat{b}_p^{\dagger} + g_-(p)\hat{d}_p^{\dagger})\|0\rangle\rangle.\end{aligned} \tag{C1}$$

We then insert Eq. (9) and parse these terms as per Eq. (28), where

$$\begin{aligned}\rho_1(t,x) = \langle\langle 0\| \int dp(g_+^*(p)\hat{b}_p + g_-^*(p)\hat{d}_p)\bigg\{ &\iint dp_1 dp_2\langle\phi_{p_1}|x\rangle\sigma\langle x|\phi_{p_2}\rangle \int dp'\big(U_{\phi_{p_1}\phi_{p'}}^*(t)\hat{b}_{p'}^{\dagger} + U_{\phi_{p_1}\varphi_{p'}}^*(t)\hat{d}_{p'}\big)\\ &\times\int dp'\big(U_{\phi_{p_2}\phi_{p'}}(t)\hat{b}_{p'} + U_{\phi_{p_2}\varphi_{p'}}(t)\hat{d}_{p'}^{\dagger}\big)\bigg\}\int dp(g_+(p)\hat{b}_p^{\dagger} + g_-(p)\hat{d}_p^{\dagger})\|0\rangle\rangle,\end{aligned} \tag{C2}$$

which expands to

$$\begin{aligned}\rho_1(t,x) &= \langle\langle 0\|\int\cdots\int dq_1 dq_1' dq_2 dq_2' dp_1 dp_2 g_-^*(q_1)g_-(q_2)U_{\phi_{p_1}\varphi_{q_1'}}^*(t)U_{\phi_{p_2}\varphi_{q_2'}}(t)\langle\phi_{p_1}|x\rangle\langle x|\phi_{p_2}\rangle\hat{d}_{q_1}\hat{d}_{q_1'}\hat{d}_{q_2'}^{\dagger}\hat{d}_{q_2}^{\dagger}\|0\rangle\rangle\\ &+ \langle\langle 0\|\int\cdots\int dq_1 dq_1' dq_2 dq_2' dp_1 dp_2 g_+^*(q_1)g_+(q_2)U_{\phi_{p_1}\varphi_{q_1'}}^*(t)U_{\phi_{p_2}\varphi_{q_2'}}(t)\langle\phi_{p_1}|x\rangle\langle x|\phi_{p_2}\rangle\hat{b}_{q_1}\hat{d}_{q_1'}\hat{d}_{q_2'}^{\dagger}\hat{b}_{q_2}^{\dagger}\|0\rangle\rangle\\ &+ \langle\langle 0\|\int\cdots\int dq_1 dq_1' dq_2 dq_2' dp_1 dp_2 g_+^*(q_1)g_+(q_2)U_{\phi_{p_1}\phi_{q_1'}}^*(t)U_{\phi_{p_2}\phi_{q_2'}}(t)\langle\phi_{p_1}|x\rangle\langle x|\phi_{p_2}\rangle\hat{b}_{q_1}\hat{b}_{q_1'}^{\dagger}\hat{b}_{q_2'}\hat{b}_{q_2}^{\dagger}\|0\rangle\rangle.\end{aligned} \tag{C3}$$

By applying the creation and annihilation operators to the vacuum state and using

$$\begin{aligned}\langle\langle 0\|\hat{d}_{q_1}\hat{d}_{q_1'}\hat{d}_{q_2'}^{\dagger}\hat{d}_{q_2}^{\dagger}\|0\rangle\rangle &= \delta_{q_1'q_2'}\delta_{q_1q_2} + \epsilon\delta_{q_1q_2'}\delta_{q_1'q_2},\\ \langle\langle 0\|\hat{b}_{q_1}\hat{d}_{q_1'}\hat{d}_{q_2'}^{\dagger}\hat{b}_{q_2}^{\dagger}\|0\rangle\rangle &= \delta_{q_1q_2}\delta_{q_1'q_2'},\\ \langle\langle 0\|\hat{b}_{q_1}\hat{b}_{q_1'}^{\dagger}\hat{b}_{q_2'}\hat{b}_{q_2}^{\dagger}\|0\rangle\rangle &= \delta_{q_1q_2'}\delta_{q_2q_2'},\end{aligned} \tag{C4}$$

one obtains

$$\begin{aligned}\rho_1(t,x) = &\int dq\left|g_-(q)\right|^2 \int dq\left(\int U_{\phi_p\varphi_q}(t)\langle x|\phi_p\rangle\right)^\dagger \sigma\left(\int U_{\phi_p\varphi_q}(t)\langle x|\phi_p\rangle\right)\\ &+ \int dq\left|g_+(q)\right|^2 \int dq\left(\int U_{\phi_p\varphi_q}(t)\langle x|\phi_p\rangle\right)^\dagger \sigma\left(\int U_{\phi_p\varphi_q}(t)\langle x|\phi_p\rangle\right)\\ &+ \left(\int dpdq g_+(p)U_{\phi_p\phi_q}\langle x|\phi_p\rangle\right)^\dagger \sigma\left(\int dpdq g_+(p)U_{\phi_p\phi_q}\langle x|\phi_p\rangle\right)\\ &+ \epsilon\left(\int dpdq g_-(p)U_{\phi_p\varphi_q}\langle x|\phi_p\rangle\right)^\dagger \sigma\left(\int dpdq g_-(p)U_{\phi_p\varphi_q}\langle x|\phi_p\rangle\right).\end{aligned} \quad \text{(C5)}$$

Using the normalization of the QFT state yields Eq. (29). The terms $\rho_2(t,x)$ and $\rho_3(t,x)$ are obtained similarly.

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