Conserved photon current

Margaret Hawton *

Lakehead University, Thunder Bay, Ontario, Canada P7B 5E1

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A conserved photon current is derived from the commutation relations satisfied by the electromagnetic four-potential and field tensor operators. The density is found to be a sum over positive and negative frequency terms, both of which contribute a positive number density and propagate in a common direction. Discrete positive and negative frequency excitations are both identified as photons. The photon number, equal to the spatial integral of the photon density, is conserved in the absence of sources and sinks.

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I. INTRODUCTION

In quantum field theory (QFT) and quantum optics (QO) photons are indivisible excitations of the electromagnetic (EM) field with integral helicity $\lambda = \pm 1$, making them bosons described by second quantized fields satisfying commutation relations. Since there is no exclusion principle for bosons, an EM state is an arbitrary linear combination of n-photon states where n is any positive integer or 0; it is a whole number. We will refer to both positive and negative frequency EM excitations as photons.

Physical one-photon pulses coupled to transmission lines and optical circuits are now routinely prepared in the laboratory [1]. A one-photon pulse with bandwidth equal to an appreciable fraction of its center frequency has a definite photon number but not definite energy. Photon density is fundamental and the scalar product that normalizes a one-photon state should be conserved in the absence of sources and sinks.

The model described here can be applied to the essential components of an optical quantum computer. In 2000, Knill, Laflamme, and Milburn proved that it is possible to create a universal quantum computer solely with single-photon sources, optical gates consisting of beam splitters, phase shifters, and photodetectors (KLM protocol) [2]. Photons are ideal quantum devices [3] and a photonic integrated circuit (PIC) can be implemented using established foundry-based technology [4]. These devices can incorporate photon sources, low-loss dielectric transmission lines, optical gates, and photon counting detectors [5,6]. The transmission lines and optical gates will be referred to here as the optical circuit.

The source, optical circuit, and detector should be, at least approximately, confined to separable finite regions of space. This degree of localization cannot be described by a positive frequency field alone since any positive frequency function initially localized in a finite region spreads instantaneously throughout space [7] and there are no local annihilation or creation operators [8]. The early work on approximate photon localization is reviewed by Mandel and Wolf [9] and the use of Hermitian operators to describe electromagnetic excitation in QFT is discussed in Ref. [10].

Motivated by these no-go theorems, previous work on the quantization of positive and negative frequency photon states [11] and the need to model localized devices such as beam splitters, researchers at the University of Leeds quantized both positive and negative frequency solutions of Maxwell's equation in position space to give real localizable photon pulses [12]. Consistent with these requirements and previous work, the conserved photon current described here is also a real localizable sum of positive and negative frequency fields.

The conserved electric current is well known and serves as a model for the quantitative description of a conserved photon current. Since $\mathcal{F}^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ is a tensor, its contraction with A_μ,

$$A_\mu \mathcal{F}^{\mu\nu} = \left(\frac{1}{c}\mathbf{A}\cdot\mathbf{E}, \mathbf{A}\times\mathbf{B} + \frac{1}{c^2}\mathbf{E}\phi\right), \tag{1}$$

is a four-vector, suggesting a candidate for the conserved photon current. In the next section we will calculate the commutator $\frac{i\varepsilon_0}{2\hbar}[\widehat{A}_\mu, \widehat{\mathcal{F}}^{\mu\nu}]$ and show that it describes a real localizable four-current operator satisfying a continuity equation with a material source and sink current $\widehat{J}_e$. In the final section its relationship to one-photon quantum mechanics and experiment will be discussed.

II. CONSERVED PHOTON CURRENT

We first define the notation: SI units will be used throughout. The contravariant space-time, wave vector, and momentum four-vectors are $x = x^\mu = (ct, \mathbf{x})$, $k = (\omega_k/c, \mathbf{k})$, and $p = \hbar k$, where $kx = \omega_k t - \mathbf{k}\cdot\mathbf{x}$ is invariant, the four-gradient is $\partial = (\partial_{ct}, -\nabla)$, $\Box \equiv \partial_\mu \partial^\mu = \partial_{ct}^2 - \nabla^2$, the four-potential is $A(t,\mathbf{x}) = A^\mu = (\frac{\phi}{c}, \mathbf{A})$, and a four-current is $J^\mu = (\rho c, \mathbf{J})$. The covariant four-vector corresponding to $U^\mu = (U_0, \mathbf{U})$ is $U_\mu = g_{\mu\nu}U^\nu = (U_0, -\mathbf{U})$ where $g_{\mu\nu} = g^{\mu\nu}$ is a 4×4 diagonal matrix with diagonal $(1, -1, -1, -1)$ and $U_\mu U^\mu = U^\mu U_\mu$ is an invariant. The mutually orthogonal unit vectors e^μ are defined such that $e_0 = n^\mu = (1, 0, 0, 0)$ is timelike, $\mathbf{e_k} = \mathbf{k}/|\mathbf{k}|$ is longitudinal, and the definite helicity transverse Lorentz invariant unit vectors are $\mathbf{e}_\lambda(\mathbf{k})$ with $\lambda = \pm 1$.

*Contact author: mhawton@lakeheadu.ca

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Based on the Maxwell equations $\nabla\cdot\mathbf{B}=0$, $\nabla\times\mathbf{E}+\partial_t\mathbf{B}=0$ the EM four-potential $(\phi/c,\mathbf{A})$ can be defined such that

$$\mathbf{E}=-\partial_t\mathbf{A}-\nabla\phi,\quad \mathbf{B}=\nabla\times\mathbf{A}. \tag{2}$$

The covariant Faraday tensor is then

$$\mathcal{F}^{\mu\nu}=\partial^\mu A^\nu-\partial^\nu A^\mu=\frac{1}{c}\begin{pmatrix}0 & -E_x & -E_y & -E_z\\ E_x & 0 & -cB_z & cB_y\\ E_y & cB_z & 0 & -cB_x\\ E_z & -cB_y & cB_x & 0\end{pmatrix}, \tag{3}$$

where

$$\mathcal{F}^{\mu\nu}=-\mathcal{F}^{\nu\mu}. \tag{4}$$

The standard Lagrangian density is

$$\mathcal{L}=-\tfrac{1}{4}\varepsilon_0c^2\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}-J_e^\mu A_\mu, \tag{5}$$

where $-\frac{1}{4}\varepsilon_0c^2\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}=\frac{1}{2}\varepsilon_0(\mathbf{E}\cdot\mathbf{E}-c^2\mathbf{B}\cdot\mathbf{B})$ and J_e^μ is the conserved electric four-current. The Lagrangian is $L=\int d\mathbf{x}\mathcal{L}(t,\mathbf{x})$. The analog of the position coordinate in this Lagrangian is $A^\mu(x)$ so the four-momentum conjugate to A^ν is $\varepsilon_0c\mathcal{F}^{0\nu}$. After second quantization the form of the classical equations of motion, $\varepsilon_0c^2\partial_\mu\mathcal{F}^{\mu\nu}=J_e^\mu$, is preserved so the photon field operators satisfy

$$\varepsilon_0c^2\partial_\mu\widehat{\mathcal{F}}^{\mu\nu}=\widehat{J}_e^\mu. \tag{6}$$

The continuity equation describing propagation of the photon four-current density $[\widehat{A}_\mu,\widehat{\mathcal{F}}^{\mu\nu}]$ is

$$\begin{aligned}\partial_\nu[\widehat{A}_\mu,\widehat{\mathcal{F}}^{\mu\nu}]&=[(\partial_\nu\widehat{A}_\mu),\widehat{\mathcal{F}}^{\mu\nu}]+[\widehat{A}_\mu,(\partial_\nu\widehat{\mathcal{F}}^{\mu\nu})],\\ [(\partial_\nu\widehat{A}_\mu),\widehat{\mathcal{F}}^{\mu\nu}]&=(\partial_\nu\widehat{A}_\mu)(\partial^\mu\widehat{A}^\nu)-(\partial_\nu\widehat{A}_\mu)(\partial^\nu\widehat{A}^\mu)\\ &\quad-(\partial^\mu\widehat{A}^\nu)(\partial_\nu\widehat{A}_\mu)+(\partial^\nu\widehat{A}^\mu)(\partial_\nu\widehat{A}_\mu)\\ &=0,\\ \partial_\nu[\widehat{A}_\mu,\widehat{\mathcal{F}}^{\mu\nu}]&=\frac{1}{\varepsilon_0c^2}[\widehat{A}_\mu,\widehat{J}_e^\mu],\end{aligned} \tag{7}$$

where covariant and contravariant indices were exchanged to prove the second equation and (6) was used to give the final result (7). The equations of motion (6) are second quantized versions of the MEs $\nabla\cdot\mathbf{E}=\rho_e/\varepsilon_0$, $\partial_t\mathbf{E}-c^2\nabla\times\mathbf{B}=-\varepsilon_0^{-1}\mathbf{J}_e$.

It is sufficient for the applications to be discussed here to consider only transverse electric fields $\mathbf{E}=\mathbf{E}_\perp$ that propagate in free space and remain transverse when transmitted into a dielectric and consider only $A_\mu\mathcal{F}_\perp^{\mu\nu}$. The four-potential A_μ in (3) is gauge dependent but $\mathbf{E}_\parallel=-\partial_t\mathbf{A}_\parallel-\nabla\phi=0$ gives $\mathbf{A}_\parallel\times\mathbf{B}=-\mathbf{E}_\perp\phi/c^2$ so

$$\begin{aligned}A_\mu\mathcal{F}_\perp^{\mu\nu}&=\left(\frac{1}{c}\mathbf{A}\cdot\mathbf{E}_\perp,\mathbf{A}\times\mathbf{B}+\frac{1}{c^2}\mathbf{E}_\perp\phi\right)\\ &=\left(\frac{1}{c}\mathbf{A}_\perp\cdot\mathbf{E}_\perp,\mathbf{A}_\perp\times\mathbf{B}\right)\end{aligned} \tag{8}$$

is gauge independent and remains gauge independent when second quantized. Although (2) and (3) depend explicitly on gauge, the continuity Eq. (7) is gauge independent for transverse fields.

Starting with the positive frequency annihilation operator $\widehat{\mathbf{A}}_\lambda^+(x)$ the remaining EM creation and annihilation operators can be calculated. The positive frequency operators are second quantized versions of the classical analytic signal. These operators provide a convenient mathematical description of propagation in the optical circuit with phase shifters $e^{i\phi}$ and beam splitters with complex reflectivity and transmissivity r and t that satisfy $|r|^2+|t|^2=1$ and hence conserve photon number. The remaining field operators are

$$\widehat{\mathbf{A}}^-(x)=\widehat{\mathbf{A}}^{+\dagger}(x), \tag{9}$$

$$\widehat{\mathbf{A}}(x)=\sum_{\lambda=\pm1}[\widehat{\mathbf{A}}_\lambda^+(x)+\widehat{\mathbf{A}}_\lambda^-(x)], \tag{10}$$

$$\widehat{\mathbf{E}}_\perp(x)=-\partial_t\widehat{\mathbf{A}}_\perp(x),\ \widehat{\mathbf{B}}(x)=\nabla\times\widehat{\mathbf{A}}(x). \tag{11}$$

Since annihilation and creation operators commute amongst themselves,

$$[\widehat{A}_\mu,\widehat{\mathcal{F}}_\perp^{\mu\nu}]=[\widehat{A}_\mu^+,\widehat{\mathcal{F}}_\perp^{\mu\nu-}]+[\widehat{A}_\mu^-,\widehat{\mathcal{F}}_\perp^{\mu\nu+}]. \tag{12}$$

The photon current density operator array

$$\begin{aligned}\widehat{J}_{p12}^{\lambda\lambda'}(x,x')=\frac{-i\varepsilon_0}{2\hbar}\big[&\widehat{\mathbf{A}}_2^{\lambda+}(x)\cdot\widehat{\mathbf{E}}_1^{\lambda'-}(x')\\ &-\widehat{\mathbf{E}}_1^{\lambda'+}(x')\cdot\widehat{\mathbf{A}}_2^{\lambda-}(x),\widehat{\mathbf{A}}_2^{\lambda+}(x)\times c\widehat{\mathbf{B}}_1^{\lambda'-}(x')\\ &-c\widehat{\mathbf{B}}_1^{\lambda'+}(x)\times\widehat{\mathbf{A}}_2^{\lambda+}(x')\big]\end{aligned} \tag{13}$$

generalizes (7) to describe the creation and annihilation of photons at different space-time points for different, possibly orthogonal, states. The generalization to modes 1 and 2 is only included for convenience in defining the scalar product. The current density operator (13) describes the addition of one photon to any Fock state.

The source-free MEs are space-time reversal invariant but emission and detection of a photon in the laboratory is not. It will be assumed that $t>t'$ in the laboratory frame so the creation of a photon at x with annihilation at x' will be interpreted as propagation of an antiphoton from x' to x. The $\mathbf{A}^+(x)\cdot\mathbf{E}^-(x')$ term describes propagation of a photon from space-time point x' to x, while the $\mathbf{E}^+(x')\cdot\mathbf{A}^-(x)$ term is equivalent to an antiphoton propagating from x' to x. Both photons and antiphotons that propagate from x' to x in the laboratory frame will be counted as photons. Since the minus sign in the commutation relation is canceled by the sign of the space-time derivatives of $\mathbf{A}$ in $\mathbf{E}$ and $\mathbf{B}$ in (13), density is positive and propagation is in a common direction for both its terms.

The generalized photon number density operator is

$$\begin{aligned}\widehat{\rho}_{p12}^{\lambda\lambda'}(x,x')=\frac{i\varepsilon_0}{2\hbar}\big[&\widehat{\mathbf{A}}_2^{\lambda+}(x)\cdot\widehat{\mathbf{E}}_1^{\lambda'-}(x')\\ &-\widehat{\mathbf{E}}_1^{\lambda'+}(x')\cdot\widehat{\mathbf{A}}_2^{\lambda-}(x)\big].\end{aligned} \tag{14}$$

Defining the scalar product in the normalized zero-photon state as $\langle0|0\rangle=1$, the one-photon scalar product on the t

hyperplane is

$$\rho_{p12}^{\lambda\lambda'} = \langle 0|\widehat{\rho}_{p12}^{\lambda\lambda'}(x,x)|0\rangle = \frac{i\varepsilon_0}{2\hbar}\int d\mathbf{x}\big[\mathbf{A}_2^{\lambda+}(x)\cdot\mathbf{E}_1^{\lambda'-}(x) - \widehat{\mathbf{E}}_1^{\lambda'+}(x)\cdot\mathbf{A}_2^{\lambda-}(x)\big], \tag{15}$$

where $\mathbf{A}_2^{\lambda}(x)$ and $\mathbf{E}_1^{\lambda'}(x)$ are one-photon EM fields with helicities λ and λ', respectively. Since $i\times$ the integrand in (15) is real, its localizability is not limited by the Hegerfeldt theorem.

In QFT it is conventional to define a plane-wave basis localized in $\mathbf{k}$ space and a space-time basis that is localized in position space. Here, we follow the derivation of Fock space in Ref. [13] by starting with the periodic boundary conditions $k_i L = 2\pi l_i$ for integral l_i with $i = x, y, z$ in volume $V = L^3$ and then taking the $V \to \infty$ limit. The commutation relations will be written as

$$[\widehat{a}_{\lambda\mathbf{k}}, \widehat{a}^{\dagger}_{\lambda'\mathbf{k}'}] = \delta_{\lambda\lambda'}\delta_{\mathbf{k},\mathbf{k}'}. \tag{16}$$

Defining the n-photon annihilation operator

$$\widehat{a}_{\lambda\mathbf{k}n} \equiv \frac{(\widehat{a}_{\lambda\mathbf{k}})^n}{\sqrt{n!}}, \tag{17}$$

it can be verified using the commutation relations (16) that

$$|a_{\lambda\mathbf{k}n}\rangle = \widehat{a}_{\lambda\mathbf{k}n}|0\rangle \tag{18}$$

are the normalized n-photon Fock states. The number of states per unit volume for a photon with definite helicity is

$$\lim_{V\to\infty}\frac{\Delta\mathbf{n}}{V} = \frac{d\mathbf{k}}{(2\pi)^3}, \tag{19}$$

so $V^{-1}\sum_{\mathbf{k}} \to (2\pi)^{-3}\int_t d\mathbf{k}$ and the scalar product

$$[\widehat{a}_{\lambda}(\mathbf{k}), \widehat{a}^{\dagger}_{\lambda'}(\mathbf{k}')] = \delta_{\lambda\lambda'}\delta(\mathbf{k}-\mathbf{k}') \tag{20}$$

defines a basis of orthonormal states. Since $\int \frac{d\mathbf{k}}{\omega_k}$ is an invariant, $\omega_k\delta(\mathbf{k}-\mathbf{k}')$ and $\omega_k^{1/2}\widehat{a}_{\lambda}(\mathbf{k})$ are invariant. The operator

$$\widehat{n}_{\lambda}(\mathbf{k}) = \widehat{a}^{\dagger}_{\lambda}(\mathbf{k})\widehat{a}_{\lambda}(\mathbf{k}) \tag{21}$$

counts photons with wave vector $\mathbf{k}$ and helicity λ. This is the Schrödinger picture. In the Heisenberg picture the positive frequency plane-wave annihilation operator $\widehat{a}_{\lambda}(\mathbf{k},t) = \widehat{a}_{\lambda}(\mathbf{k})e^{-i\omega_k t}$ satisfies $i\partial_t\widehat{a}_{\lambda}(\mathbf{k},t) = \omega_k\widehat{a}_{\lambda}(\mathbf{k},t)$.

In the plane-wave basis a general positive frequency vector potential operator is

$$\widehat{\mathbf{A}}^{+}_{\lambda}(x) = i\sqrt{\frac{\hbar}{2\varepsilon_0}}\int\frac{d\mathbf{k}}{(2\pi)^{3/2}\omega_k^{1/2}}c_{\lambda}(\mathbf{k})\widehat{a}_{\lambda}(\mathbf{k})\mathbf{e}_{\lambda}(\mathbf{k})e^{-ikx}. \tag{22}$$

The photon number (23) for a state with helicity λ is

$$n_p^{\lambda} = \frac{i\varepsilon_0}{2\hbar}\int d\mathbf{x}\big[\mathbf{A}_2^{\lambda+}(x)\cdot\mathbf{E}_1^{\lambda-}(x) - \widehat{\mathbf{E}}_1^{\lambda+}(x)\cdot\mathbf{A}_2^{\lambda-}(x)\big] \tag{23}$$

$$= \int\frac{d\mathbf{k}}{(2\pi)^3}|c_{\lambda}(\mathbf{k})|^2, \tag{24}$$

with mode 2 equal to mode 1 omitted from the notation now that the scalar product (15) has been defined. For a one-photon state $n_p^{\lambda} = 1$.

If $c_{\lambda}(\mathbf{k}) = e^{ikx'}$, expressions (11) and (16) to (20) substituted into (13) give

$$J_p^{\lambda\lambda'}(x,x') = \frac{1}{2}\int\frac{d\mathbf{k}}{(2\pi)^3}\delta_{\lambda\lambda'}(1, \mathbf{e}_{\lambda}(\mathbf{k}))e^{-ik(x-x')} + \text{c.c.}, \tag{25}$$

where c.c. is the complex conjugate. Its zero component is the photon density

$$\rho_p^{\lambda\lambda'}(x,x') = \frac{1}{2}\int\frac{d\mathbf{k}}{(2\pi)^3}\delta_{\lambda\lambda'}e^{ik(x-x')} + \text{c.c.} \tag{26}$$

On the $t = t'$ hyperplane

$$\rho_p^{\lambda\lambda'}(x,x') = \delta_{\lambda\lambda'}\delta(\mathbf{x}'-\mathbf{x}), \tag{27}$$

so (22) describes an orthonormal basis of localized states. The operators $\widehat{\mathbf{A}}(t,\mathbf{x})$ and $\widehat{\mathbf{E}}(t,\mathbf{x}')$ commute for spacelike separated points $\mathbf{x}'\neq\mathbf{x}$, so a measurement at $\mathbf{x}'$ does not change the outcome at $\mathbf{x}$. This enforces causality in QED. Using (19), the zero component of (25) can be written as

$$\rho_p^{\lambda\lambda'}(x,x') = \frac{1}{2}\sum_{\mathbf{k}}\delta_{\lambda\lambda'}\frac{e^{ik(x-x')}}{V} + \text{c.c.}, \tag{28}$$

verifying that at $x = x'$ it is a density equal to a sum over plane-wave number densities.

With $|\Delta\mathbf{x}| \equiv |\mathbf{x}-\mathbf{x}'|$ and $\Delta t \equiv t - t'$ (26) gives

$$\rho_p^{\lambda\lambda'}(x,x') = \rho_p^{\lambda\lambda'+}(x,x') + \rho_p^{\lambda\lambda'+*}(x,x'), \tag{29}$$

where the positive frequency part of $\rho_p^{\lambda\lambda'}$ is

$$\rho_p^{\lambda\lambda'+}(x,x') = \int\frac{d\mathbf{k}}{2(2\pi)^3}e^{-ik(x-x')}\delta_{\lambda\lambda'} = \frac{-1}{8\pi^2 r}\frac{\partial}{\partial r}\Big[\pi\delta(|\Delta\mathbf{x}| - c\Delta t) + iP\Big(\frac{1}{|\Delta\mathbf{x}| - c\Delta t}\Big)\Big]\delta_{\lambda\lambda'}, \tag{30}$$

and P is the principal value. The positive and negative frequency contributions to the photon number density are separately nonlocal, but their sum is real and localized on a spherical shell. The three-dimensional case can be used to model emission from a localized atom or quantum dot initially in an excited state.

For a one-dimensional photon pulse with a spatially uniform area A propagating in the $+k_x$ direction, the positive frequency part of the photon density is

$$\rho_p^{\lambda\lambda'+}(x,x') = \int_0^{\infty}\frac{dk_x}{2\pi A}e^{ik_x(x-x')}\delta_{\lambda\lambda'} = \frac{1}{2A}\Big[\delta(\Delta x - c\Delta t) - \frac{i}{\pi}P\Big(\frac{1}{\Delta x - c\Delta t}\Big)\Big]\delta_{\lambda\lambda'}, \tag{31}$$

where $\Delta t \equiv t - t'$, $\Delta x \equiv x - x'$, and

$$\rho_p^{\lambda\lambda'+} + \rho_p^{\lambda\lambda'+*} = \frac{1}{A}\delta(\Delta x - c\Delta t)\delta_{\lambda\lambda'} \tag{32}$$

is δ localized and propagates at speed c. This localized basis can be integrated over to describe localization in a finite region. An example of instantaneous localization in a square well is given in Ref. [14].

In classical EM the macroscopic description of transmission and reflection at a dielectric interface is know to work for visible and infrared light. This is based on averaging over domains with dimensions of order 10^{-8} m that include many molecules [15]. This classical macroscopic model will be second quantized to model photon propagation in a dielectric such as a transmission line. The photon source and photon counting detectors are separate devices whose details are not considered here, so it is only assumed that the sources emit single photons and the detectors are photon counting devices.

Only whole numbers of EM excitations exist—there are no fractional photons. This was verified experimentally in Ref. [16] where a detector was placed in two paths and, within experimental error, no coincident photon detection events were observed. The photon number density in free space is $\varepsilon_0\mathbf{E}\cdot\mathbf{A}$. When a one-photon pulse passes from free space into a dielectric material or optical circuit, the photon number must remain $n=1$ until the photon is absorbed in a lossy material or counted in an optical detector and reduced to the $n=0$ state. To preserve the normalization $\int d\mathbf{x}\varepsilon_0\mathbf{E}\cdot\mathbf{A}=1$, the photon density must be $\varepsilon_0\mathbf{E}\cdot\mathbf{A}$ in the dielectric.

In a polarizable dielectric the transverse positive frequency second quantized operators satisfy the MEs

$$\partial_t(\varepsilon_0\widehat{\mathbf{E}}^+ + \widehat{\mathbf{P}}^+) - \nabla\times\widehat{\mathbf{H}}^+ = -\widehat{\mathbf{J}}^{+s}, \tag{33}$$

where $\widehat{\mathbf{H}}^+ = \mu_0^{-1}\widehat{\mathbf{B}}^+ = \varepsilon_0 c^2\widehat{\mathbf{B}}^+$ and, for simplicity, the material has been assumed to be nonmagnetic. The current $\widehat{J_e}$ in (7) is driven by the electric field operator so it is also operator valued. A single-photon pulse transmitted into a transparent medium must remain normalized so its number density remains $\varepsilon_0\mathbf{E}\cdot\mathbf{A}$ as in free space.

When a light pulse with momentum $\mathbf{p}_{\rm em}$ propagating in free space encounters a planar interface of a dielectric with an index of refraction n, $r=\frac{n-1}{n+1}$ and $t=\frac{2n}{n+1}$ are determined by the Fresnel equations. Total momentum is conserved so if this pulse is reflected off an ideal mirror with reflectivity $r=1$ the mirror will gain momentum $2\mathbf{p}_{\rm em}$, and if it is absorbed the dielectric slab will gain momentum $\mathbf{p}_{\rm em}$ [17]. A one-photon pulse incident on this ideal mirror will remain a one-photon pulse when reflected and will be reduced to the $n=0$ state if absorbed.

A one-dimensional plane wave with helicity λ, $A^+_{\lambda k} = \exp[i\omega_k(x/c-t)]/2\pi\omega_k^{1/2}$, incident from free space on a weakly absorbing dielectric medium with an index of refraction n will be transmitted with probability amplitude t. The index of refraction is in general complex with real and imaginary parts $n'(\omega)$ and $n''(\omega)$, so

$$n(\omega)=\sqrt{1+\chi(\omega)}=n'(\omega)+in''(\omega), \tag{34}$$

and, in the dielectric,

$$A^+_{\lambda k}(x,t)=\frac{t}{2\pi\omega_k^{1/2}}\exp\left(-\omega_k x\frac{n''}{c}\right)\exp\left[i\omega_k\left(x\frac{n'}{c}-t\right)\right]. \tag{35}$$

Since the one-photon number density is $\varepsilon_0\mathbf{E}\cdot\mathbf{A}$ in the dielectric, its momentum is of the Abraham form, $\mathbf{p}_A=\mathbf{p}_{\rm em}=\varepsilon_o\mathbf{E}\times\mathbf{B}$ [15,17]. The Minkowski momentum $\mathbf{p}_M=\mathbf{p}_{\rm em}+\chi\mathbf{p}_{\rm em}$ includes the momentum due to polarization of the dielectric medium. This acts as a drag force on the single photon, reducing its speed from c to c/n'. In a PIC photons propagate in multiple transmission lines with essentially identical characteristics and common dielectric susceptibility with complex reflectivity r and transmissivity t of the light pulses determined by the angles of their intersections [6].

III. DISCUSSION

In Sec. II a conserved photon four-current was derived from the potential-field commutation relations. The photon probability density was used to define a scalar product that can form a basis for a first quantized theory of single photons. Here, this scalar product is derived from fundamental principles according to which the minus from space-time differentiation is cancelled out by minus from the commutation relations to give a positive photon number density for both positive and negative frequency fields. Previous definitions of the one-photon scalar product were limited to use of nonlocalizable positive frequency fields [18] or, motivated by the observation that experimental one-photon pulses can be modeled classically [19], use of the Mostafazadeh [20] sign of the frequency operator [21]. In the latter case, the scalar product (15), derived here from fundamental principles, was constructed on an *ad hoc* basis. The propagation of highly localized wave packets that remain localized at all times is discussed in Refs. [22,23].

In a quantum optical circuit single photons are injected into input modes of a linear interferometer described by a unitary operator $\widehat{U}$. Since $\widehat{U}$ is unitary, the photon number is conserved, consistent with the conservation law derived in Sec. II. The propagation of photon pulses in dielectric media and through a beam splitter is discussed in Ref. [24]. Since $\int d\mathbf{x}\varepsilon_0\mathbf{E}\cdot\mathbf{A}=1$ for a one-photon state in free space, the free-space form of the photon number density must be preserved in a dielectric. The polarization induced in the medium by $\mathbf{E}$ does not contribute to the photon number density, and instead it acts as a drag force that reduces the propagation speed of the single-photon pulse. Single-photon momentum density is of the Abraham form, $\mathbf{p}_{\rm em}=\varepsilon_0\mathbf{E}\times\mathbf{A}$.

There are no photons in classical electromagnetic theory—the discrete excitations that we call photons are created and annihilated by second quantized operators. The localizable causally propagating photon number density derived here determines the probability that the photon will be counted. The photon density must necessarily be interpreted as a probability density since photons are indivisible and can be counted only once as verified experimentally in Ref. [16]. We have identified a conserved photon four-current operator that describes both positive and negative frequency EM excitations. Their sum is positive, real, localizable in a finite region, and propagates causally. The photon number is conserved in the absence of sources and sinks. Single-photon states are represented by normalized fields that collapse to the zero-photon state when the photon is counted. This is a purely quantum effect described by second quantization—it has no counterpart in classical EM. The one-photon density must be interpreted as a probability density.

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