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Relating completely positive divisibility of dynamical maps with compatibility of channels

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The role of completely positive–indivisibility (CP-indivisibility) and incompatibility as valuable resources for various information-theoretic tasks is widely acknowledged. This study delves into the intricate relationship between CP-divisibility and channel compatibility. Our investigation focuses on the behavior of incompatibility robustness of quantum channels for a pair of generic dynamical maps. We show that the incompatibility robustness of channels is monotonically nonincreasing for a pair of generic CP-divisible dynamical maps. Further, our explicit study of the behavior of incompatibility robustness with time for some specific dynamical maps reveals nonmonotonic behavior in the CP-indivisible regime. Additionally, we propose a measure of CP-indivisibility based on the incompatibility robustness of quantum channels. Our investigation provides valuable insights into the nature of quantum dynamical maps and their relevance in information-theoretic applications.

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I. INTRODUCTION

Incompatibility is one of the main features of quantum mechanics that makes it different from classical mechanics [1]. A set of devices is said to be compatible if those devices can be simultaneously implemented on a quantum system. Otherwise, the set is incompatible. These devices can be measurements, channels, instruments, etc. Incompatibility is a resource in several information-theoretic tasks and is necessary to demonstrate nonclassical advantage in such tasks. For example, measurement incompatibility is necessary and sufficient to demonstrate quantum steering [2]. Incompatibility of measurements is also necessary to demonstrate Bell inequality violation [2,3] and any quantum advantage in communication tasks [4]. Measurement incompatibility also provides advantage in some state discrimination tasks [5]. Recently, it has been shown that incompatibility of channels and measurement-channel incompatibility both provide advantages in quantum state discrimination tasks [6].

A general quantum evolution is described by a completely positive trace preserving (CPTP) dynamical map [7–13]. This representation has wide application, since it is almost impossible to keep a quantum system truly isolated. When considering a Markovian evolution [11,12], it is necessary to take the quantum system weakly interacting with much larger stationary environment, and hence, the reduced dynamics of the system can be considered to be memoryless, leading to one-way information flow from the quantum system to the bath degrees of freedom. Therefore, the quantum features of a system subjected to such dynamics vanishes gradually with time [11–13]. However, in practical situations like in an experiment, the coupling between the system and environmental degrees of freedom may not always be sufficiently weak. Moreover, the concerning environment can very well be finite or nonstationary. These situations may lead to the signature of non-Markovian information backflow [14–24]. Though quantum non-Markovianity has been associated with varied physical attributes [14,15,25,26], the focus of this work is based solely on indivisibility of the dynamics exhibiting information backflow from the environment to the system [14].

A divisible quantum operation is the one that can be realized as an arbitrary number of CPTP maps. In other words, such operations can be divided into an arbitrary number of CPTP maps. The precise mathematical definition of such maps is later presented in Eq. (5) for better understanding. Divisible maps do not exhibit information backflow from the environment to the system [14,15] and hence can be understood as Markovian operations. The Born-Markov approximation and stationary bath state approximation are imperative to

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realize such quantum operations, and hence, in the absence of these initial approximations, the dynamics is bound to be CP-indivisible and prone to show information backflow [11]. Adopting this line of reasoning, in this work we take CP-indivisible quantum operations as non-Markovian. Note that CP-indivisible operations are necessary to have information backflow from the environment to the system, enabling recovery of lost information to an extent, and hence they can be considered resourceful operations in information processing scenarios. For example, it has been shown that information backflow allows perfect teleportation with mixed states [27], improvement of capacity for long quantum channels [28], and efficient work extraction from an Otto cycle [29].

From the above discussions, it is clear that (in-) compatibility and CP-(in-) divisibility can play the role of resources in various information-theoretic and thermodynamic tasks. There are several resources in quantum theory that provide an advantage in information-theoretic or thermodynamic tasks [30]. Entanglement [31], coherence [32], nonlocality [33], contextuality [34,35], and incompatibility [36] are examples of some widely studied resources. Evidently, therefore, exploring the interplay among different resources forms an important avenue of research. For example, it is well known that coherence can be measured with entanglement [37]. The relation between incompatibility and Bell nonlocality [2,3], as well as the relation between incompatibility and steerability [2], are well known. Furthermore, it has been recently shown that superposition and entanglement are equivalent concepts in any physical theory [38].

The motivation for the present work is to explore the connection between the two resources of (in-)compatibility and CP-(in-)divisibility, which have been hitherto investigated separately in the literature. It is known that both incompatibility and CP-indivisibility are resources for different information-theoretic tasks. Moreover, to the best of our knowledge, the notion of compatibility has been considered only for devices in the static backdrop. In Ref. [39] the authors have done a qualitative study regarding the relation between CP-divisibility and incompatibility. However, they did not consider dynamical maps that involve time or the robustness measure of incompatibility to draw a quantitative connection between the above-said resources. On the other hand, in this work we introduce and characterize the notion of compatibility of dynamical maps, incorporating their evolution in time. Through our present analysis, we characterize CP-(in-)divisibility with respect to the (in-)compatibility of channels. We study the behavior of incompatibility robustness of quantum channels for some examples of dynamical maps. We further present an example where the non-Markovian advantage manifested in terms of CP-indivisibility and information backflow is clearly seen to act as a quantum resource in the task of teleportation. Moreover, we define a measure of CP-indivisibility based on incompatibility robustness of channels.

The rest of the paper is organized as follows. In Sec. II we provide definitions of various quantities required for the subsequent analysis. Our main results are presented from Sec. III onwards. In Sec. III A we show that the incompatibility robustness of quantum channels for a pair of CP-divisible dynamical maps is monotonically nonincreasing with respect to time. In Sec. III B we show that for any pair of dynamical maps, incompatibility robustness of measurements is upper bounded by incompatibility of channels for an arbitrary time. In Sec. III C we discuss the notion of compatibility of dynamical maps and its connection to channel compatibility. In Sec. IV we study the behavior of incompatibility robustness of quantum channels for certain specific dynamical maps and show its nonmonotonic behavior in a CP-indivisible case. In Sec. V we discuss the usefulness of CP-indivisibility in the context of quantum teleportation and compare the behavior of teleportation fidelity with the incompatibility robustness with respect to time. In Sec. VI we propose a measure of CP-indivisibility based on incompatibility of channels. Finally, in Sec. VII we present our concluding remarks.

II. PRELIMINARIES

A. Compatibility of measurements

A measurement M acting on the Hilbert space $\mathcal{H}$ is a set of positive semidefinite matrices, i.e., $M = \{M(x)\}_{x\in\Omega_M}$, such that $\sum_{x\in\Omega_M} M(x) = \mathbb{1}_{\mathcal{H}}$, where $\mathbb{1}_{\mathcal{H}}$ is the identity matrix on the Hilbert space $\mathcal{H}$ and Ω_M is known as the outcome set of M. The set of all measurements acting on Hilbert space $\mathcal{H}$ and with outcome set Ω is denoted by $\mathbb{M}(\mathcal{H}, \Omega)$. A set of measurements $\mathcal{M} = \{M_i\}_{i=1}^n$ is said to be compatible if there exists a joint measurement $M = \{M(j_1, \ldots, j_n)\} \in \mathbb{M}(\mathcal{H}, \Omega_M)$ with $\Omega_M = \Omega_{M_1} \times \ldots \times \Omega_{M_n}$ such that $M_i(j_i) = \sum_{\{j_k\}\setminus j_i} M(j_1, \ldots, j_n)$ for all $j_i \in \Omega_{M_i}$ and for all $i \in \{1, \ldots n\}$, where the sum over $\{j_k\} \setminus j_i$ denotes the sum over all j_ks and for all ks, except for $k = i$. Otherwise, the set is incompatible [1,40].

A measure of incompatibility of quantum measurements is the incompatibility robustness of quantum measurements, defined below. The incompatibility robustness of two quantum measurements, $M_1 \in \mathbb{M}(\mathcal{H}, \Omega_1)$ and $M_2 \in \mathbb{M}(\mathcal{H}, \Omega_2)$, can be defined as

$$\begin{aligned} R_M(M_1, M_2) &= \min\ r \\ \text{s.t.}\ \frac{M_1(i_1) + r\tilde{M}_1(i_1)}{1+r} &= \sum_{i_2} M(i_1.i_2) \\ \frac{M_2(i_2) + r\tilde{M}_2(i_2)}{1+r} &= \sum_{i_1} M(i_1.i_2) \\ M &\in \mathbb{M}(\mathcal{H}, \Omega_1 \times \Omega_2) \\ \tilde{M}_i &\in \mathbb{M}(\mathcal{H}, \Omega_i) \quad i = 1, 2. \end{aligned} \tag{1}$$

Here, $\tilde{M}_i$s are arbitrary noise measurements, and the optimization is over all variables, other than the given pair of measurements (M_1, M_2). We call the set of all values of r that satisfies the above equalities for different noise measurements *the compatibility range*. Clearly, the incompatibility robustness is the minimum of all values of r that belongs to the compatibility range.

B. Compatibility of quantum channels

A quantum channel $\Gamma : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{K})$ is a CPTP linear map where $\mathcal{L}(\mathcal{H})$ is the bounded linear operator on the Hilbert space $\mathcal{H}$ and $\mathcal{L}(\mathcal{K})$ is the bounded linear operator on the

Hilbert space $\mathcal{K}$. We denote the set of all quantum channels from $\mathcal{L}(\mathcal{H})$ to $\mathcal{L}(\mathcal{K})$ as $\mathbb{Ch}(\mathcal{H},\mathcal{K})$. We also denote the composition (also known as concatenation) of two quantum channels $\Phi_1 : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\bar{\mathcal{K}})$ and $\Phi_2 : \mathcal{L}(\bar{\mathcal{K}}) \to \mathcal{L}(\mathcal{K})$ as $\Phi_2 \circ \Phi_1$, where $\bar{K}$ is another Hilbert space. For two quantum channels $\Gamma_1 \in \mathbb{Ch}(\mathcal{H},\mathcal{K}_1)$ and $\Gamma_2 \in \mathbb{Ch}(\mathcal{H},\mathcal{K}_2)$, if there exists another quantum channel $\Theta \in \mathbb{Ch}(\mathcal{K}_1,\mathcal{K}_2)$ such that $\Gamma_2 = (\Theta \circ \Gamma_1)$, then we say that Γ_2 is a postprocessing of Γ_1 and we denote it as $\Gamma_2 \preceq \Gamma_1$. Let $\Phi \in \mathbb{Ch}(\mathcal{H},\mathcal{K})$ be a quantum channel. Then the $\Phi^* : \mathcal{L}(\mathcal{K}) \to \mathcal{L}(\mathcal{H})$ is called a dual map of Φ if $\mathrm{Tr}[\Phi(T)X] = \mathrm{Tr}[T\Phi^*(X)]$ holds for all $T \in \mathcal{L}(\mathcal{H})$ and $X \in \mathcal{L}(\mathcal{K})$. Clearly, Φ^* is the action of Φ in the Heisenberg picture. As Φ is CP trace preserving, Φ^* is CP unital. Now, consider a measurement $M = \{M(x)\}_{x\in\Omega_M} \in \mathbb{M}(\Omega_M,\mathcal{K})$ and a channel $\Lambda \in \mathbb{Ch}(\mathcal{H},\mathcal{K})$. If Λ^* is applied on the measurement M, the resulting measurement is $\Lambda^*(M) = \{\Lambda^*[M(x)]\}_{x\in\Omega_M} \in \mathbb{M}(\Omega_M,\mathcal{H})$. Implementation of M on an arbitrary quantum system after implementation of Λ is equivalent to implementation of $\Lambda^*[M(x)]$ before implementation of Λ on that quantum system.

We now discuss a special type of channel that maps any input state to a fixed output state. These channels are called completely depolarizing (CD) channels [41], which may completely erase the information of input states. If $\Upsilon_\eta(\rho) = \eta$ for all input states ρ, then Υ_η is a completely depolarizing channel. We denote the set of all completely depolarizing channels from $\mathcal{L}(\mathcal{H})$ to $\mathcal{L}(\mathcal{K})$ as $\mathbb{Ch}^{\mathcal{CD}}(\mathcal{H},\mathcal{K})$. We will use this type of channel in the later sections. Now, suppose $\Upsilon_\eta \in \mathbb{Ch}^{\mathcal{CD}}(\mathcal{H},\mathcal{K})$ is an arbitrary completely depolarizing channel and $\Lambda_1 \in \mathbb{Ch}(\mathcal{K},\mathcal{K}')$ is an arbitrary quantum channel. Then $(\Lambda_1 \circ \Upsilon_\eta)$ is also a completely depolarizing channel [41]. Below, we define the compatibility of quantum channels.

Definition 1. Two quantum channels $\Lambda_1 : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{K}_1)$ and $\Lambda_2 : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{K}_2)$ are compatible if there exists a quantum channel $\Lambda : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{K}_1 \otimes \mathcal{K}_2)$ such that for all $T \in \mathcal{L}(\mathcal{H})$,

$$\Lambda_1(T) = \mathrm{Tr}_{\mathcal{K}_2}\Lambda(T);\ \Lambda_2(T) = \mathrm{Tr}_{\mathcal{K}_1}\Lambda(T). \tag{2}$$

Otherwise, Λ_1 and Λ_2 are incompatible [41] .

The quantum channel Λ in Definition 1 is also known as the joint quantum channel. Equation (2) can be rewritten using shorthand notation as

$$\Lambda_1 = \mathrm{Tr}_{\mathcal{K}_2}\Lambda;\ \Lambda_2 = \mathrm{Tr}_{\mathcal{K}_1}\Lambda. \tag{3}$$

We will use the shorthand notations throughout the paper.

Suppose $\bar{\Gamma}_1 \preceq \Gamma_1$ and $\bar{\Gamma}_2 \preceq \Gamma_2$. Then it is proved in Ref. [41, Proposition 3] that $\bar{\Gamma}_1$ and $\bar{\Gamma}_2$ are compatible if Γ_1 and Γ_2 are compatible. We will use this result in the proof of Theorem 1 and Theorem 2.

A measure of incompatibility of quantum channels is the incompatibility robustness of quantum channels that is defined below [6]. The incompatibility robustness of two quantum channels $\Phi_1 : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{K}_1)$ and $\Phi_2 : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{K}_2)$ can be defined as

$$\begin{aligned} &R_C(\Phi_1,\Phi_2) = \min r \\ &\text{s.t. } \frac{\Phi_1 + r\tilde{\Phi}_1}{1+r} = \mathrm{Tr}_{\mathcal{K}_2}\Psi \\ &\frac{\Phi_2 + r\tilde{\Phi}_2}{1+r} = \mathrm{Tr}_{\mathcal{K}_1}\Psi \\ &\Psi \in \mathbb{Ch}(\mathcal{H},\mathcal{K}_1\otimes\mathcal{K}_2) \\ &\tilde{\Phi}_i \in \mathbb{Ch}(\mathcal{H},\mathcal{K}_i) \quad i = 1,2. \end{aligned} \tag{4}$$

Here, $\tilde{\Phi}_i$s are arbitrary noise channels, and the optimization is over all variables, other than the given pair of channels (Φ_1,Φ_2). We call the set of all values of r that satisfy the above equalities *the compatibility range*. Clearly, the incompatibility robustness is the minimum of all values of r that belongs to the compatibility range. Note that the definition of incompatibility robustness of quantum channels does not directly guarantee that the compatibility range is continuous. Such a statement is proved in Lemma 1 below. It is known that for any two given channels, R_C is upper bounded by 1 [41, Example 2]. Now, following Ref. [42], broadcasting quantum channels can be defined.

Definition 2. A channel $\Gamma : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ with $\mathcal{H}_1 = \mathcal{H}_2$ is known as a broadcasting quantum channel [42].

This definition will be used in later sections.

C. Dynamical maps and CP-divisibility

A dynamical map is a family of CPTP linear maps $\{\Lambda_{t,t_0} : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})\}_t$ (where $t \geqslant t_0$). Here, t represents the time and t_0 is the fixed initial time. Without loss of generality, we can take t_0 to be 0 and denote $\Lambda_{t,0}$ as Λ_t. We denote the set of all dynamical maps on $\mathcal{L}(\mathcal{H})$ [i.e., from $\mathcal{L}(\mathcal{H})$ to $\mathcal{L}(\mathcal{H})$] as $\mathbb{DM}(\mathcal{H},\mathcal{H})$. Let $\mathcal{D}_1 = \{\Lambda_t^1\}_t \in \mathbb{DM}(\mathcal{H},\mathcal{H})$ and $\mathcal{D}_2 = \{\Lambda_t^2\}_t \in \mathbb{DM}(\mathcal{H},\mathcal{H})$. Then the convex combination of $\mathcal{D}_1$ and $\mathcal{D}_2$ (with respect to $p \geqslant 0$) is defined as $p\mathcal{D}_1 + (1-p)\mathcal{D}_2 := \{p\Lambda_t^1 + (1-p)\Lambda_t^2\}_t$. Now, we provide the definition of a CP-divisible dynamical map below.

Definition 3. A dynamical map $\mathcal{D} = \{\Lambda_t\}_t$ is called CP-divisible if for all t and all s, it can be written as

$$\Lambda_t = V_{t,s} \circ \Lambda_s,\ (t \geqslant s), \tag{5}$$

where $\circ$ denotes the composition of maps and $V_{t,s} : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ is a CPTP linear map.

A dynamical map that is not CP-divisible is known as CP-indivisible dynamical map. We refer the readers to Refs. [11–13] for more details.

III. RELATING CP-DIVISIBILITY OF DYNAMICAL MAPS WITH COMPATIBILITY OF QUANTUM CHANNELS

A. CP-indivisibility of dynamical maps and incompatibility robustness of quantum channels

In this section we establish a connection of CP-divisibility of dynamical maps with compatibility of quantum channels.

Theorem 1. Suppose that the quantum dynamical maps $\mathcal{D}_1 = \{\Lambda_t^1\}_t \in \mathbb{DM}(\mathcal{H},\mathcal{H})$ and $\mathcal{D}_2 = \{\Lambda_t^2\}_t \in \mathbb{DM}(\mathcal{H},\mathcal{H})$ are both CP-divisible. Then $R_C(\Lambda_t^1,\Lambda_t^2) \geqslant R_C(\Lambda_{t+\delta t}^1,\Lambda_{t+\delta t}^2)$ for any $\delta t \geqslant 0$.

Proof. If both $\mathcal{D}_1$ and $\mathcal{D}_2$ are CP-divisible, then

$$\Lambda_{t+\delta t}^1 = V_{t,t+\delta t}^1 \circ \Lambda_t^1; \tag{6}$$

$$\Lambda_{t+\delta t}^2 = V_{t,t+\delta t}^2 \circ \Lambda_t^2 \tag{7}$$

hold for any $\delta t \geqslant 0$, where $V^1_{t,t+\delta t}$ and $V^2_{t,t+\delta t}$ are quantum channels. Now, suppose that $R_C(\Lambda^1_t, \Lambda^2_t) = l$. Therefore, from the definition of incompatibility robustness of quantum channels, it follows that there exist two quantum channels $\bar{\Lambda}^1_t$ and $\bar{\Lambda}^2_t$ such that the channels $\Sigma^1_t = \frac{\Lambda^1_t + l\bar{\Lambda}^1_t}{1+l}$ and $\Sigma^2_t = \frac{\Lambda^2_t + l\bar{\Lambda}^2_t}{1+l}$ are compatible. Now, consider the quantum channels,

$$V^1_{t,t+\delta t} \circ \Sigma^1_t = \frac{\Lambda^1_{t+\delta t} + l\left(V^1_{t,t+\delta t} \circ \bar{\Lambda}^1_t\right)}{1+l}, \tag{8}$$

$$V^2_{t,t+\delta t} \circ \Sigma^2_t = \frac{\Lambda^2_{t+\delta t} + l\left(V^2_{t,t+\delta t} \circ \bar{\Lambda}^2_t\right)}{1+l}. \tag{9}$$

Next, as mentioned in Sec. II B, for two pairs of channels (Γ_1, Γ_2) and $(\bar{\Gamma}_1, \bar{\Gamma}_2)$, if $\bar{\Gamma}_1 \preceq \Gamma_1$ and $\bar{\Gamma}_2 \preceq \Gamma_2$ hold, then $\bar{\Gamma}_1$ and $\bar{\Gamma}_2$ are compatible if Γ_1 and Γ_2 are compatible [41, Proposition 3]. Therefore, as Σ^1_t and Σ^2_t are compatible, from Eqs. (8) and (9) we get that the quantum channels $V^1_{t,t+\delta t} \circ \Sigma^1_t$ and $V^2_{t,t+\delta t} \circ \Sigma^2_t$ are also compatible. Therefore, from the definition of incompatibility robustness of quantum channels, it follows that $R_C(\Lambda^1_t, \Lambda^2_t) = l \geqslant R_C(\Lambda^1_{t+\delta t}, \Lambda^2_{t+\delta t})$. ■

Note that the incompatibility robustness defined in Eq. (4) has the minimization over all possible noise channels. Incompatibility robustness can be defined with regard to only the set of all completely depolarizing channels as well. The incompatibility robustness of two quantum channels $\Phi_1 : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{K}_1)$ and $\Phi_2 : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{K}_2)$ with regard to completely depolarizing channels can be defined as

$$\begin{aligned} R^{\mathcal{CD}}_C(\Phi_1, \Phi_2) &= \min\ r \\ \text{s.t.}\ \frac{\Phi_1 + r\Theta_1}{1+r} &= \mathrm{Tr}_{\mathcal{K}_2}\Psi \\ \frac{\Phi_2 + r\Theta_2}{1+r} &= \mathrm{Tr}_{\mathcal{K}_1}\Psi \\ \Psi &\in \mathbb{Ch}(\mathcal{H}, \mathcal{K}_1 \otimes \mathcal{K}_2) \\ \Theta_i &\in \mathbb{Ch}^{\mathcal{CD}}(\mathcal{H}, \mathcal{K}_i) \quad i = 1, 2. \end{aligned} \tag{10}$$

A similar measure for the incompatibility of measurements has been studied earlier [1,43].

Theorem 2. Suppose that the quantum dynamical maps $\mathcal{D}_1 = \{\Lambda^1_t\}_t \in \mathbb{DM}(\mathcal{H}, \mathcal{H})$ and $\mathcal{D}_2 = \{\Lambda^2_t\}_t \in \mathbb{DM}(\mathcal{H}, \mathcal{H})$ are both CP-divisible. Then $R^{\mathcal{CD}}_C(\Lambda^1_t, \Lambda^2_t) \geqslant R^{\mathcal{CD}}_C(\Lambda^1_{t+\delta t}, \Lambda^2_{t+\delta t})$ for any $\delta t \geqslant 0$.

Proof. If both $\mathcal{D}_1$ and $\mathcal{D}_2$ are CP-divisible, then

$$\Lambda^1_{t+\delta t} = V^1_{t,t+\delta t} \circ \Lambda^1_t; \tag{11}$$

$$\Lambda^2_{t+\delta t} = V^2_{t,t+\delta t} \circ \Lambda^2_t \tag{12}$$

hold for any $\delta t \geqslant 0$, where $V^1_{t,t+\delta t}$ and $V^2_{t,t+\delta t}$ are quantum channels. Now, suppose that $R^{\mathcal{CD}}_C(\Lambda^1_t, \Lambda^2_t) = l$. Therefore, from the definition of incompatibility robustness of quantum channels with respect to only the set of completely depolarizing channels, we get that there exist two completely depolarizing channels $\Theta_1 \in \mathbb{Ch}^{\mathcal{CD}}(\mathcal{H}, \mathcal{K}_1)$ and $\Theta_2 \in \mathbb{Ch}^{\mathcal{CD}}(\mathcal{H}, \mathcal{K}_2)$ such that the channels $\Sigma^1_t = \frac{\Lambda^1_t + l\Theta_1}{1+l}$ and $\Sigma^2_t = \frac{\Lambda^2_t + l\Theta_2}{1+l}$ are compatible. Now, consider the quantum channels $V^1_{t,t+\delta t} \circ \Sigma^1_t = \frac{\Lambda^1_{t+\delta t} + l(V^1_{t,t+\delta t} \circ \Theta_1)}{1+l}$ and $V^2_{t,t+\delta t} \circ \Sigma^2_t = \frac{\Lambda^2_{t+\delta t} + l(V^2_{t,t+\delta t} \circ \Theta_2)}{1+l}$. Now, as mentioned in Sec. II B, for two pairs of channels (Γ_1, Γ_2) and $(\bar{\Gamma}_1, \bar{\Gamma}_2)$, if $\bar{\Gamma}_1 \preceq \Gamma_1$ and $\bar{\Gamma}_2 \preceq \Gamma_2$ hold, then $\bar{\Gamma}_1$ and $\bar{\Gamma}_2$ are compatible if Γ_1 and Γ_2 are compatible [41, Proposition 3]. Therefore, as Σ^1_t and Σ^2_t are compatible, we get that the quantum channels $V^1_{t,t+\delta t} \circ \Sigma^1_t = \frac{\Lambda^1_{t+\delta t} + l(V^1_{t,t+\delta t} \circ \Theta_1)}{1+l}$ and $V^2_{t,t+\delta t} \circ \Sigma^2_t = \frac{\Lambda^2_{t+\delta t} + l(V^2_{t,t+\delta t} \circ \Theta_2)}{1+l}$ are also compatible. Therefore, from the definition of incompatibility robustness of quantum channels with respect to only the set of completely depolarizing channels and the fact that the channels $(V^1_{t,t+\delta t} \circ \Theta_1)$ and $(V^2_{t,t+\delta t} \circ \Theta_2)$ are completely depolarizing, it follows that $R^{\mathcal{CD}}_C(\Lambda^1_t, \Lambda^2_t) = l \geqslant R^{\mathcal{CD}}_C(\Lambda^1_{t+\delta t}, \Lambda^2_{t+\delta t})$. ■

Note that Theorems 1 and 2 do not directly imply each other.

B. CP-divisibility and compatibility of quantum measurements

The relation between compatibility of measurements and CP-divisibility of dynamical maps has been studied in detail in Refs. [40,43]. Here, we further study it in the context of our present analysis. In Ref. [43], the authors studied the behavior of incompatibility robustness of quantum measurements with respect to fixed noise measurements. Here, we show that incompatibility robustness of measurements (with respect to generic noise) for any pair of dynamical maps is upper bounded by incompatibility robustness of channels for that pair of dynamical maps.

Theorem 3. Consider an arbitrary pair of measurements $\mathcal{M} = \{M_i \in \mathbb{M}(\Omega_{M_i}, \mathcal{H})\}_{i\in 1,2}$ and an arbitrary pair of dynamical maps $\mathcal{D}_1 = \{\Lambda^1_t\}_t \in \mathbb{DM}(\mathcal{H}, \mathcal{H})$ and $\mathcal{D}_2 = \{\Lambda^2_t\}_t \in \mathbb{DM}(\mathcal{H}, \mathcal{H})$. Then

$$\max_{M_1, M_2} R_M\left[\Lambda^{1*}_t(M_1), \Lambda^{2*}_t(M_2)\right] \leqslant R_C\left(\Lambda^1_t, \Lambda^2_t\right). \tag{13}$$

Proof. Let $R_C(\Lambda^1_t, \Lambda^2_t) = l(t)$. Then there exist two noise channels $\tilde{\Lambda}^1_t$ and $\tilde{\Lambda}^2_t$ such that

$$\frac{\Lambda^1_t + l(t)\tilde{\Lambda}^1_t}{1 + l(t)} = \mathrm{Tr}_{\mathcal{H}_2}\Lambda, \tag{14}$$

$$\frac{\Lambda^2_t + l(t)\tilde{\Lambda}^2_t}{1 + l(t)} = \mathrm{Tr}_{\mathcal{H}_1}\Lambda, \tag{15}$$

where $\Lambda \in \mathbb{Ch}(\mathcal{H}, \mathcal{H}_1 \otimes \mathcal{H}_2)$ with $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$.

Using the definition of dual maps, Eqs. (14) and (15), we obtain

$$\frac{\Lambda^{1*}_t[M_1(i)] + l(t)\tilde{\Lambda}^{1*}_t[M_1(i)]}{1 + l(t)} = \Lambda^*[M_1(i) \otimes \mathbb{1}_{\mathcal{H}_2}], \tag{16}$$

$$\frac{\Lambda^{2*}_t[M_2(j)] + l(t)\tilde{\Lambda}^{2*}_t[M_2(j)]}{1 + l(t)} = \Lambda^*[\mathbb{1}_{\mathcal{H}_1} \otimes M_2(j)], \tag{17}$$

for all $i \in \Omega_{M_1}$ and $j \in \Omega_{M_2}$. Let the measurement $M := \{M(i, j) = \Lambda^*[M_1(i) \otimes M_2(j)]\}$. Then clearly,

$$\sum_j M(i, j) = \frac{\Lambda^{1*}_t[M_1(i)] + l(t)\tilde{\Lambda}^{1*}_t[M_1(i)]}{1 + l(t)}, \tag{18}$$

$$\sum_i M(i, j) = \frac{\Lambda^{2*}_t[M_2(j)] + l(t)\tilde{\Lambda}^{1*}_t[M_2(j)]}{1 + l(t)}. \tag{19}$$

Hence, the measurements $M'_1 = \{\frac{\Lambda^{1*}_t[M_1(i)] + l(t)\tilde{\Lambda}^{1*}_t[M_1(i)]}{1+l(t)}\}_{i\in\Omega_{M_1}}$ and $M'_2 =$

$\{\frac{\Lambda_t^{2*}[M_2(j)]+l(t)\tilde{\Lambda}_t^{2*}[M_2(j)]}{1+l(t)}\}_{i\in\Omega_{M_2}}$ are compatible. Thus, from the definition of incompatibility robustness of quantum measurements, we get $R_M[\Lambda_t^{1*}(M_1), \Lambda_t^{2*}(M_2)] \leqslant l(t) = R_C(\Lambda_t^1, \Lambda_t^2)$. ■

Clearly, from Theorems 1 and 3, it follows that the upper bound of $R_M[\Lambda_t^{1*}(M_1), \Lambda_t^{2*}(M_2)]$ is monotonically nonincreasing if both $\mathcal{D}_1$ and $\mathcal{D}_2$ are CP-divisible.

C. CP-divisibility and compatibility of dynamical maps

First, we define broadcasting quantum dynamical maps, as follows.

Definition 4. A broadcasting quantum dynamical map is a family of CPTP linear maps $\{\Lambda_{t,t_0} : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_2)\}_t$ (where $t \geqslant t_0$) with $\mathcal{H}_1 = \mathcal{H}_2$.

We denote the set of all broadcasting dynamical maps on $\mathcal{L}(\mathcal{H})$ [i.e., from $\mathcal{L}(\mathcal{H})$ to $\mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_2)$] with $\mathcal{H}_1 = \mathcal{H}_2$ as $\mathbb{BD}(\mathcal{H}, \mathcal{H}_1 \otimes \mathcal{H}_2 \mid \mathcal{H}_1 = \mathcal{H}_2)$. Again, without the loss of generality, we can take t_0 to 0. We now define the compatibility of dynamical maps.

Definition 5. Two dynamical maps $\mathcal{D}_1 = \{\Lambda_t^1\}_t \in \mathbb{DM}(\mathcal{H}, \mathcal{H})$ and $\mathcal{D}_2 = \{\Lambda_t^2\}_t \in \mathbb{DM}(\mathcal{H}, \mathcal{H})$ are said to be compatible if a joint broadcasting quantum dynamical map $\mathcal{J} = \{\Theta : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_2)\}_t$ with $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$ exists such that

$$\mathcal{D}_1 = \mathrm{Tr}_{\mathcal{H}_2}\mathcal{J};\ \mathcal{D}_2 = \mathrm{Tr}_{\mathcal{H}_1}\mathcal{J}. \tag{20}$$

where $\mathrm{Tr}_{\mathcal{H}_i}\mathcal{J} := \{\mathrm{Tr}_{\mathcal{H}_i}\Theta\}_t$ for $i \in \{1, 2\}$.

Definition 5 is similar to Definition 1, but now compatibility relations should hold for all t.

Clearly, the implementation of $\mathcal{J}$ is equivalent to the simultaneous implementation of $\mathcal{D}_1$ and $\mathcal{D}_2$. Note that the set of all compatible dynamical maps is convex. Therefore, a measure of incompatibility of quantum dynamical maps is the incompatibility robustness of quantum dynamical maps that we define below. The incompatibility robustness of two quantum dynamical maps $\mathcal{D}_1 = \{\Lambda_t^1\}_t \in \mathbb{DM}(\mathcal{H}, \mathcal{H})$ and $\mathcal{D}_2 = \{\Lambda_t^2\}_t \in \mathbb{DM}(\mathcal{H}, \mathcal{H})$ can be defined as

$$\begin{aligned} &R_D(\mathcal{D}_1, \mathcal{D}_2) = \min\ r \\ &\text{s.t.}\ \frac{\mathcal{D}_1 + r\tilde{\mathcal{D}}_1}{1+r} = \mathrm{Tr}_{\mathcal{H}_2}\mathcal{J} \\ &\frac{\mathcal{D}_2 + r\tilde{\mathcal{D}}_2}{1+r} = \mathrm{Tr}_{\mathcal{H}_1}\mathcal{J} \\ &\mathcal{J} \in \mathbb{BD}(\mathcal{H}, \mathcal{H}_1 \otimes \mathcal{H}_2 \mid \mathcal{H}_1 = \mathcal{H}_2) \\ &\tilde{\mathcal{D}}_i \in \mathbb{DM}(\mathcal{H}, \mathcal{H}) \quad i = 1, 2. \end{aligned} \tag{21}$$

Here $\tilde{\mathcal{D}}_i$s are arbitrary noise dynamical maps, and the optimization is over all variables, other than the given pair of dynamical maps $(\mathcal{D}_1, \mathcal{D}_2)$.

Let us now present the following Lemma:

Lemma 1. Consider two quantum channels $\Phi_1 \in \mathbb{Ch}(\mathcal{H}, \mathcal{K}_1)$ and $\Phi_2 \in \mathbb{Ch}(\mathcal{H}, \mathcal{K}_2)$ with $R_C(\Phi_1, \Phi_2) = l$. Then for all $\epsilon \geqslant 0$, there exists two quantum channels $\tilde{\Phi}_1 \in \mathbb{Ch}(\mathcal{H}, \mathcal{K}_1)$ and $\tilde{\Phi}_2 \in \mathbb{Ch}(\mathcal{H}, \mathcal{K}_2)$ such that the quantum channels $\frac{\Phi_1+l'\tilde{\Phi}_1}{1+l'}$ and $\frac{\Phi_2+l'\tilde{\Phi}_2}{1+l'}$ are compatible where $l' = l + \epsilon \geqslant l$.

Proof. As $R_C(\Phi_1, \Phi_2) = l$, there exist $\hat{\Phi}_1 \in \mathbb{Ch}(\mathcal{H}, \mathcal{K}_1)$, $\hat{\Phi}_2 \in \mathbb{Ch}(\mathcal{H}, \mathcal{K}_2)$, and $\Phi \in \mathbb{Ch}(\mathcal{H}, \mathcal{K}_1 \otimes \mathcal{K}_2)$ such that

$$\bar{\Phi}_1 = \frac{\Phi_1 + l\hat{\Phi}_1}{1+l} = \mathrm{Tr}_{\mathcal{K}_2}\Phi, \tag{22}$$

$$\bar{\Phi}_2 = \frac{\Phi_2 + l\hat{\Phi}_2}{1+l} = \mathrm{Tr}_{\mathcal{K}_1}\Phi. \tag{23}$$

Now consider two completely depolarizing channels (i.e., channels with fixed output states) $\hat{\Upsilon}_{\eta_1} \in \mathbb{Ch}(\mathcal{H}, \mathcal{K}_1)$ and $\hat{\Upsilon}_{\eta_2} \in \mathbb{Ch}(\mathcal{H}, \mathcal{K}_2)$ [where $\eta_1 \in \mathcal{S}(\mathcal{K}_1)$ and $\eta_2 \in \mathcal{S}(\mathcal{K}_2)$] such that for all $\rho \in \mathcal{S}(\mathcal{H})$,

$$\hat{\Upsilon}_{\eta_1}(\rho) = \eta_1, \tag{24}$$

$$\hat{\Upsilon}_{\eta_2}(\rho) = \eta_2. \tag{25}$$

In Ref. [41, Proposition 10], it is proved that completely depolarizing channels are compatible with any quantum channel. Therefore, $\hat{\Upsilon}_{\eta_1}$ and $\hat{\Upsilon}_{\eta_2}$ are compatible. The corresponding joint channel is $\hat{\Upsilon}_{\eta_1\otimes\eta_2} \in \mathbb{Ch}(\mathcal{H}, \mathcal{K}_1 \otimes \mathcal{K}_2)$ such that for all $\rho \in \mathcal{S}(\mathcal{H})$,

$$\hat{\Upsilon}_{\eta_1\otimes\eta_2}(\rho) = \eta_1 \otimes \eta_2. \tag{26}$$

Let $\gamma = \frac{\epsilon}{1+l}$. Clearly, $\gamma \geqslant 0$.

Now, consider the quantum channels,

$$\Gamma_1 = \frac{\bar{\Phi}_1 + \gamma\hat{\Upsilon}_{\eta_1}}{1+\gamma} = \mathrm{Tr}_{\mathcal{K}_2}\left[\frac{\Phi + \gamma\hat{\Upsilon}_{\eta_1\otimes\eta_2}}{1+\gamma}\right], \tag{27}$$

$$\Gamma_2 = \frac{\bar{\Phi}_2 + \gamma\hat{\Upsilon}_{\eta_2}}{1+\gamma} = \mathrm{Tr}_{\mathcal{K}_1}\left[\frac{\Phi + \gamma\hat{\Upsilon}_{\eta_1\otimes\eta_2}}{1+\gamma}\right]. \tag{28}$$

Clearly, Γ_1 and Γ_2 are compatible. Recall that $l' = l + \epsilon$. Now, it can be easily shown that

$$\Gamma_1 = \frac{\Phi_1 + l'\tilde{\Phi}_1}{1+l'}, \tag{29}$$

$$\Gamma_2 = \frac{\Phi_2 + l'\tilde{\Phi}_2}{1+l'}, \tag{30}$$

where $\tilde{\Phi}_1 = \frac{l\hat{\Phi}_1+\epsilon\hat{\Upsilon}_{\eta_1}}{l+\epsilon}$ and $\tilde{\Phi}_2 = \frac{l\hat{\Phi}_2+\epsilon\hat{\Upsilon}_{\eta_2}}{l+\epsilon}$ are valid quantum channels. Hence, the lemma is proved. ■

Now, defining a quantity $R_C^{\max}(\mathcal{D}_1, \mathcal{D}_2) := \max_t R_C(\Lambda_t^1, \Lambda_t^2)$, we state the following result.

Theorem 4. For two arbitrary quantum dynamical maps $\mathcal{D}_1 = \{\Lambda_t^1\}_t \in \mathbb{DM}(\mathcal{H}, \mathcal{H})$ and $\mathcal{D}_2 = \{\Lambda_t^2\}_t \in \mathbb{DM}(\mathcal{H}, \mathcal{H})$, the equality $R_D(\mathcal{D}_1, \mathcal{D}_2) = R_C^{\max}(\mathcal{D}_1, \mathcal{D}_2)$ holds.

Proof. Suppose that $R_D(\mathcal{D}_1, \mathcal{D}_2) = l$. Therefore, from Definition 5 and the definition of incompatibility robustness of dynamical maps, there exist two dynamical maps $\bar{\mathcal{D}}_1 = \{\bar{\Lambda}_t^1\}_t$ and $\bar{\mathcal{D}}_2 = \{\bar{\Lambda}_t^2\}_t$ such that the channels $\frac{\Lambda_t^1+l\bar{\Lambda}_t^1}{1+l}$ and $\frac{\Lambda_t^1+l\bar{\Lambda}_t^1}{1+l}$ are compatible for all t. We know that $R_C^{\max}(\mathcal{D}_1, \mathcal{D}_2) := \max_t R_C(\Lambda_t^1, \Lambda_t^2)$ and suppose the maximum occurs for $t = t'$. Therefore, $R_C^{\max}(\mathcal{D}_1, \mathcal{D}_2) = R_C(\Lambda_{t'}^1, \Lambda_{t'}^2)$, which we will denote by h. Now, as discussed above, the channels $\frac{\Lambda_{t'}^1+l\bar{\Lambda}_{t'}^1}{1+l}$ and $\frac{\Lambda_{t'}^1+l\bar{\Lambda}_{t'}^1}{1+l}$ are compatible. Then, from the definition of incompatibility robustness for quantum channels, we get

$$R_D(\mathcal{D}_1, \mathcal{D}_2) = l \geqslant R_C\left(\Lambda_{t'}^1, \Lambda_{t'}^2\right) = R_C^{\max}(\mathcal{D}_1, \mathcal{D}_2). \tag{31}$$

Now, suppose $R_C(\Lambda_t^1, \Lambda_t^2) = k_t$ for an arbitrary t. Then $k_t \leqslant h$. Next, from Lemma 1 it follows that there exist quantum channels $\tilde{\Lambda}_t^1$ and $\tilde{\Lambda}_t^2$ such that such that $\frac{\Lambda_t^1 + h\tilde{\Lambda}_t^1}{1+h}$ and $\frac{\Lambda_t^2 + h\tilde{\Lambda}_t^2}{1+h}$ are compatible for all t. Therefore, from the definition of incompatibility robustness of the dynamical maps, one obtains

$$R_D(\mathcal{D}_1, \mathcal{D}_2) \leqslant h = R_C^{\max}(\mathcal{D}_1, \mathcal{D}_2). \tag{32}$$

Therefore, from inequalities (31) and (32), it follows that

$$R_D(\mathcal{D}_1, \mathcal{D}_2) = h = R_C^{\max}(\mathcal{D}_1, \mathcal{D}_2). \tag{33}$$

■

From Theorems 1 and 4, one can obtain the following result.

Corollary 5. For two arbitrary CP-divisible quantum dynamical maps $\mathcal{D}_1 = \{\Lambda_t^1\}_t \in \mathbb{DM}(\mathcal{H}, \mathcal{H})$ and $\mathcal{D}_2 = \{\Lambda_t^2\}_t \in \mathbb{DM}(\mathcal{H}, \mathcal{H})$, the equality $R_D(\mathcal{D}_1, \mathcal{D}_2) = R_C(\Lambda_0^1, \Lambda_0^2)$ holds.

Clearly, this corollary relates CP-indivisibility with incompatibility of channels.

IV. ILLUSTRATION OF INCOMPATIBILITY ROBUSTNESS OF QUANTUM CHANNELS FOR SEVERAL DYNAMICAL MAPS

In this section we study the behavior of incompatibility robustness of quantum channels for some specific dynamical maps, for both the CP-divisible and CP-indivisible regime. Our goal is to study if the information backflow induced by CP-indivisibility can be witnessed from nonmonotonic behavior of incompatibility robustness of quantum channels with respect to time.

To obtain the incompatibility robustness for two quantum channels $\Lambda_1(t)$ and $\Lambda_2(t)$, we implement the following algorithm, incorporating semidefinite optimization techniques. For the examples we consider here, the input Hilbert space ($\mathcal{H}_{\text{in}}$) and the output Hilbert spaces ($\mathcal{H}_{i,\text{out}}$) of both channels are the same as $\mathbb{C}^d$ (i.e., d-dimensional complex Hilbert space for finite d), where $d = 2$.

(1) Fix a value of time $t = 0$, and we obtain the Choi matrices, $C_{\Lambda_1(t=0)}, C_{\Lambda_2(t=0)}$, of the channels, where $C_{\Lambda_i(t=0)} \in \mathcal{L}(\mathcal{H}_{\text{in}} \otimes \mathcal{H}_{i,\text{out}})$. We denote $C_{\Lambda_i(t=0)}$ as C_{Λ_i}.

(2) We start from the value of $r = 0$ and execute the following optimization:

$$\begin{aligned}
&\max q \\
&C_{\overline{\Lambda}_1} \in \mathcal{L}(\mathcal{H}_{\text{in}} \otimes \mathcal{H}_{1,\text{out}}),\ C_{\overline{\Lambda}_1} \geqslant 0, \quad \text{Tr}_{\mathcal{H}_{1,\text{out}}}(C_{\overline{\Lambda}_1}) = \mathbb{1}_{\mathcal{H}_{\text{in}}}, \\
&C_{\overline{\Lambda}_2} \in \mathcal{L}(\mathcal{H}_{\text{in}} \otimes \mathcal{H}_{2,\text{out}}),\ C_{\overline{\Lambda}_2} \geqslant 0, \quad \text{Tr}_{\mathcal{H}_{2,\text{out}}}(C_{\overline{\Lambda}_2}) = \mathbb{1}_{\mathcal{H}_{\text{in}}}, \\
&C_\Gamma \in \mathcal{L}(\mathcal{H}_{\text{in}} \otimes \mathcal{H}_{1,\text{out}} \otimes \mathcal{H}_{2,\text{out}}), \quad C_\Gamma \geqslant q\mathbb{1}_{\mathcal{H}_{\text{in}} \otimes \mathcal{H}_{1,\text{out}} \otimes \mathcal{H}_{2,\text{out}}}, \\
&\text{Tr}_{\mathcal{H}_{2,\text{out}}}(C_\Gamma) = \frac{C_{\Lambda_1} + rC_{\overline{\Lambda}_1}}{1+r}, \\
&\text{Tr}_{\mathcal{H}_{1,\text{out}}}(C_\Gamma) = \frac{C_{\Lambda_2} + rC_{\overline{\Lambda}_2}}{1+r}.
\end{aligned} \tag{34}$$

(i) If the value of q is negative, it indicates that $\Lambda_1(t = 0)$ and $\Lambda_2(t = 0)$ are incompatible for that specific value of r. In this case, we proceed by repeating step (II) with an updated value of r, i.e., $r = r + \delta r$.

(ii) If the value of q is greater than or equal to zero, it signifies that $\Lambda_1(t = 0)$ and $\Lambda_2(t = 0)$ have become compatible. We store the current value of r as the incompatibility robustness. Subsequently, we return to step (I) and increment the parameter t by δt.

The incompatibility robustness with respect to completely depolarizing (CD) noise can be determined using the same method. In this case, instead of the constraint $C_{\overline{\Lambda}_i} \in \mathcal{L}(\mathcal{H}_{\text{in}} \otimes \mathcal{H}_{\text{i,out}})$, $C_{\overline{\Lambda}_i} \geqslant 0$, $\text{Tr}_{\mathcal{H}_{i,\text{out}}}(C_{\overline{\Lambda}_i}) = \mathbb{1}_{\mathcal{H}_{\text{in}}}$, the constraint $C_{\overline{\Lambda}_i} = \mathbb{1}_{\mathcal{H}_{\text{in}}} \otimes \eta_i$, $\eta_i \in \mathcal{L}(\mathcal{H}_{i,\text{out}})$, $\eta_i \geqslant 0$, $\text{Tr}[\eta_i] = 1$ needs to be imposed for each $i = 1, 2$.

A. Qubit-depolarizing dynamical maps

Consider a qubit-depolarizing dynamical map $\mathcal{D} = \{\Lambda_t\}_t$ of the form

$$\Lambda_t(\rho) = w(t)\rho + [1 - w(t)]\frac{\mathbb{1}}{2}. \tag{35}$$

Here, $\mathbb{1} = \mathbb{1}_{2\times 2}$. The Choi matrix of Λ_t can be written as

$$C_{\Lambda_t} = \begin{bmatrix} \frac{1+w(t)}{2} & 0 & 0 & w(t) \\ 0 & \frac{1-w(t)}{2} & 0 & 0 \\ 0 & 0 & \frac{1-w(t)}{2} & 0 \\ w(t) & 0 & 0 & \frac{1+w(t)}{2} \end{bmatrix}. \tag{36}$$

Now, let us first consider the CP-divisible scenario, where we can take $w(t) = e^{-\lambda t}$ with λ to be some positive real constant and we call the dynamical map $\mathcal{D}_1 = \{\Lambda_t^1\}_t$ [i.e., $\Lambda_t^1(\rho) = e^{-\lambda t}\rho + (1 - e^{-\lambda t})\frac{\mathbb{1}}{2}$]. This is a divisible depolarizing dynamical map, which can be shown very easily [14]. (For our purpose of study, we take λ to be 0.5. The step size δr is taken to be 0.005. The values of t are taken from 0 to 1 (in units of $1/\lambda$), and the interval of t is taken to be 0.01 in all the cases.)

Note that it can be directly shown from the Lindblad evolution of trace distance that the necessary condition for nonmonotonic behavior of the trace distance is the breaking down of divisibility [15]. The increment of trace distance between any two possible states of a system (evolving through a dynamical map) with respect to time is an indication of information backflow from the environment to the system. In the following examples studied by us, we use the trace distance curve to show that nonmonotonicity of it has similarity with nonmonotonicity of the incompatibility robustness curve. The incompatibility robustness of two copies of Λ_t^1 [i.e., $R_C(\Lambda_t^1, \Lambda_t^1)$ and $R_C^{C\mathcal{D}}(\Lambda_t^1, \Lambda_t^1)$] is plotted with respect to time t in Fig. 1 and we observe the monotonic behavior. Incompatibility of channels becomes permanently zero at approximately $t = 0.81$.

Now, let us take $w(t) = e^{-\lambda t}\cos^2 \omega t$, for the example of a CP-indivisible dynamical map. The inclusion of the cosine function imparts oscillation in the term, allowing information backflow from the environment to the system. For our purpose, we take $\lambda = 0.5$, $\omega = 5\pi$ and denote the dynamical map as $\mathcal{D}_2 = \{\Lambda_t^2\}_t$ [i.e., $\Lambda_t^2(\rho) = e^{-0.5t}\cos^2 5\pi t\,\rho + (1 - e^{-0.5t}\cos^2 5\pi t)\frac{\mathbb{1}}{2}$]. In this case the incompatibility robustness of two copies of Λ_t^2 [i.e., $R_C(\Lambda_t^2, \Lambda_t^2)$ and $R_C^{C\mathcal{D}}(\Lambda_t^2, \Lambda_t^2)$] is plotted with regard to time t in Fig. 2. Note that the behavior of

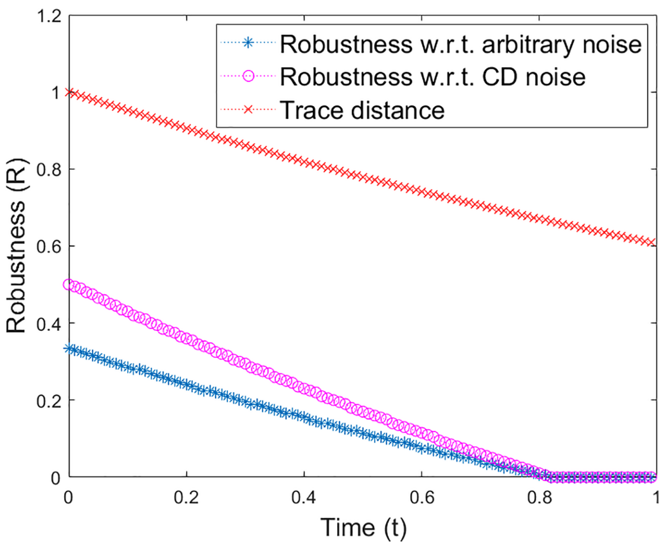

FIG. 1. This plot shows the behavior of incompatibility robustness of quantum channels [with respect to completely depolarizing (CD) noise and generic noise] for two copies of the depolarizing dynamical map $\mathcal{D}_1$ [with $w(t) = e^{-\lambda t}$ where $\lambda = 0.5$] with respect to time t. It also shows the trace distance between $\Lambda_t^1(|0\rangle\langle 0|)$ and $\Lambda_t^1(|1\rangle\langle 1|)$. Clearly, there is no information backflow and the behavior of incompatibility robustness is monotonically nonincreasing.

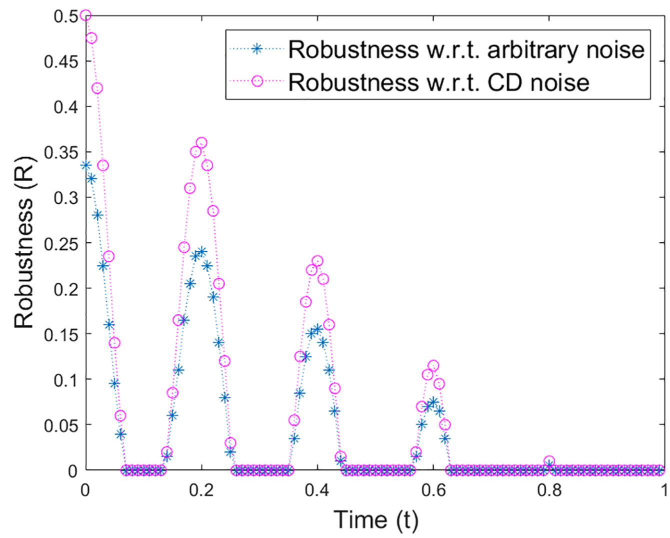

FIG. 3. This plot shows the nonmonotonic behavior of incompatibility robustness of channels for dynamical maps $\mathcal{D}_1$ and $\mathcal{D}_2$ [i.e., $R_C(\Lambda_t^1, \Lambda_t^2)$ and $R_C^{\mathcal{CD}}(\Lambda_t^1, \Lambda_t^2)$] with respect to time t. Although $\mathcal{D}_1$ is CP-divisible, $\mathcal{D}_2$ is CP-indivisible and exhibits information backflow, which is the cause of nonmonotonic behavior of incompatibility robustness.

incompatibility of channels is nonmonotonic regarding time t but permanently becomes zero at approximately $t = 0.81$.

We know that a CP-divisible map does not exhibit information backflow. Therefore, to demonstrate the connection between nonmonotonic behavior of incompatibility robustness and information backflow, we plot incompatibility robustness of quantum channels for the dynamical maps $\mathcal{D}_1$ (CP-divisible) and $\mathcal{D}_2$ (CP-indivisible), i.e., we plot $R_C(\Lambda_t^1, \Lambda_t^2)$ and $R_C^{\mathcal{CD}}(\Lambda_t^1, \Lambda_t^2)$ for time t in Fig. 3. Here, $\mathcal{D}_1$ can be considered as a *reference CP-divisible dynamical map*, and information backflow of $\mathcal{D}_2$ can be witnessed through the nonmonotonic behavior of $R_C(\Lambda_t^1, \Lambda_t^2)$ and $R_C^{\mathcal{CD}}(\Lambda_t^1, \Lambda_t^2)$ with respect to t. This is one of the plots that help us to define a measure of CP-indivisibility based on incompatibility of channels in Sec. VI.

From Figs. 2 and 3, we observe that there are some values of t (between two ripples) where incompatibility robustness remains zero (i.e., nonincreasing), but there exists nonmonotonic behavior in trace distance that indicates the information backflow. Therefore, the information backflow cannot be witnessed from the graph of the incompatibility robustness of quantum channels for all those values of t. But it is possible to eliminate such a limitation if we carefully choose the reference CP-divisible dynamical map. For example, consider $\mathcal{D}_I = \{\mathbb{I}_t\}_t$ be the identity dynamical map, i.e., $\mathbb{I}_t = \mathbb{I}_{\mathcal{H}_q}$ is an identity channel for all at t where $\mathcal{H}_q$ is the qubit Hilbert space. Clearly, $\mathcal{D}_I$ is a CP-divisible dynamical map. Let us choose $\mathcal{D}_I$ as the reference CP-divisible dynamical map and plot the incompatibility robustness of channels for dynamical maps $\mathcal{D}_I$ and $\mathcal{D}_2$ [i.e., $R_C(\mathbb{I}_t, \Lambda_t^2)$ and $R_C^{\mathcal{CD}}(\mathbb{I}_t, \Lambda_t^2)$] with respect to time t in Fig. 4. From Fig. 4 we observe that for all values of t (although, displayed only for finite range of time), there is nonmonotonic behavior of both trace distance and incompatibility robustness of channels, and for an arbitrary time t, if the information backflow is nonzero (observed from the trace distance graph), then the incompatibility robustness of quantum channels is strictly increasing. Therefore, for an arbitrary time t, if the information backflow is nonzero then it can be witnessed from the graph of incompatibility of

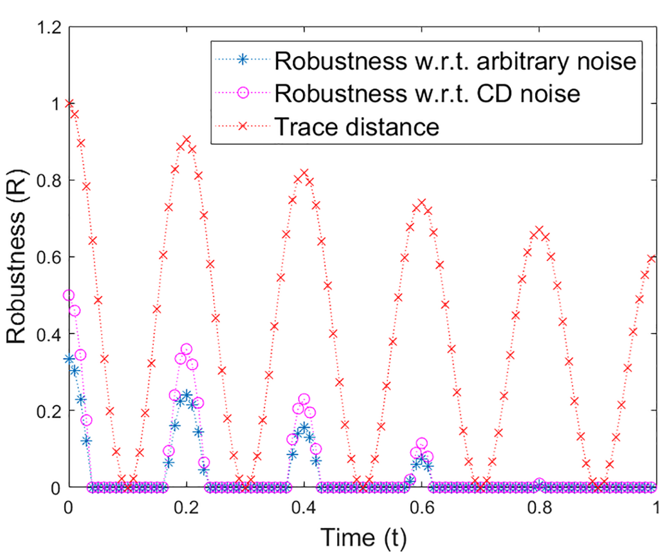

FIG. 2. This plot shows the behavior of incompatibility robustness of quantum channels [with respect to completely depolarizing (CD) noise and generic noise] for two copies of the depolarizing dynamical map $\mathcal{D}_2$ with $w(t) = e^{-\lambda t}\cos^2\omega t$, where $\lambda = 0.5, \omega = 5\pi$ with respect to time t. It also shows the trace distance between $\Lambda_t^2(|0\rangle\langle 0|)$ and $\Lambda_t^2(|1\rangle\langle 1|)$. Clearly, there is information backflow that can be witnessed from nonmonotonicity of trace distance, and also, the behavior of incompatibility robustness is nonmonotonic.

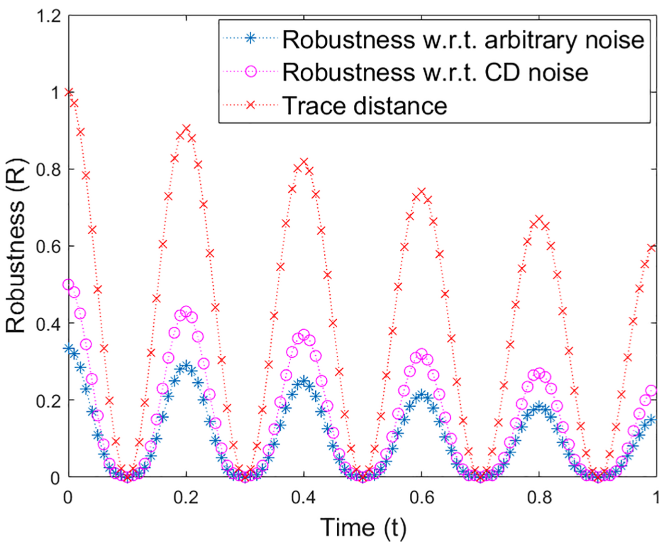

FIG. 4. This plot shows the nonmonotonic behavior of incompatibility robustness of channels for dynamical maps $\mathcal{D}_I$ and $\mathcal{D}_2$ [i.e., $R_C(\mathbb{I}_t, \Lambda_t^2)$ and $R_C^{\mathcal{CD}}(\mathbb{I}_t, \Lambda_t^2)$] with respect to time t. It also shows the trace distance between $\Lambda_t^2(|0\rangle\langle 0|)$ and $\Lambda_t^2(|1\rangle\langle 1|)$. Clearly, for an arbitrary time t, if the information backflow (observed from the trace distance graph) is nonzero then the incompatibility robustness of quantum channels is also strictly increasing. Therefore, for an arbitrary time t, if the information backflow is nonzero, it can be witnessed from the graph of incompatibility of quantum channels.

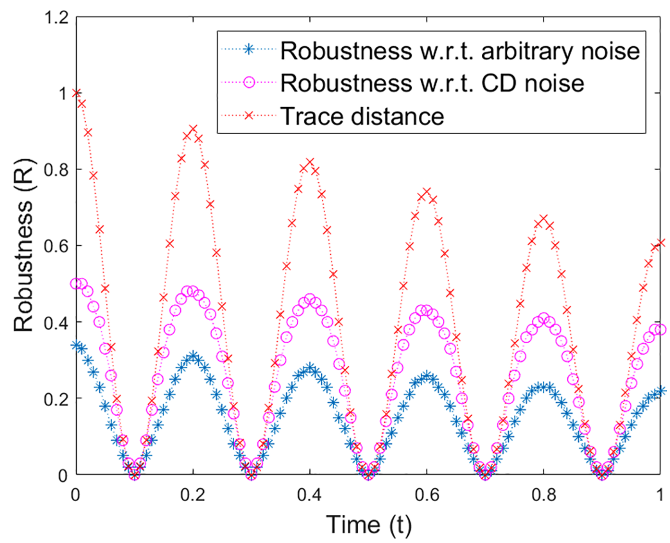

FIG. 5. This plot shows the nonmonotonic behavior of incompatibility robustness of channels for dynamical maps $\mathcal{D}_I$ and $\mathcal{D}_{ad}$ [i.e., $R_C(\mathbb{I}_t, \Gamma_t^{ad})$ and $R_C^{\mathcal{CD}}(\mathbb{I}_t, \Gamma_t^{ad})$] with respect to time t. It also shows the trace distance between $\Gamma_t^{ad}(|0\rangle\langle 0|)$ and $\Gamma_t^{ad}(|1\rangle\langle 1|)$. Clearly, for an arbitrary time t, if the information backflow (witnessed from nonmonotonicity of trace distance) is nonzero then the incompatibility robustness of quantum channels is also strictly increasing, and therefore, if the information backflow is nonzero for any time t, it can be witnessed from the graph of incompatibility of quantum channels.

quantum channels. Therefore, the above-said limitation has been removed by choosing D_I as the reference CP-divisible dynamical map.

B. Qubit amplitude damping dynamical maps

In this section we consider a qubit-amplitude-damping dynamical map $\mathcal{D}_{ad} = \{\Gamma_t^{ad}\}_t$, where the Choi matrix of Γ_t^{ad} can be written as

$$C_{\Gamma_t^{ad}} = \begin{bmatrix} 1 & 0 & 0 & \sqrt{1-w(t)} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1-w(t) & 0 \\ \sqrt{1-w(t)} & 0 & 0 & w(t) \end{bmatrix}, \quad (37)$$

where $0 \leqslant w(t) \leqslant 1$. We choose $w(t)$ in such a way that the dynamics is CP-indivisible and exhibits information backflow. Taking $w(t) = 1 - e^{-\alpha t}\cos^2 \omega t$, we set the value of $\alpha = 0.5$ and $\omega = 5\pi$. As discussed in a previous section (mainly from Fig. 4), we observed that the identity dynamical map $\mathcal{D}_I$ is possibly a suitable reference CP-divisible dynamical map. Therefore, we plot the incompatibility robustness of channels for dynamical maps $\mathcal{D}_I$ and $\mathcal{D}_{ad}$ [i.e., $R_C(\mathbb{I}_t, \Gamma_t^{ad})$ and $R_C^{\mathcal{CD}}(\mathbb{I}_t, \Gamma_t^{ad})$] with respect to time t in Fig. 5, and we observe information backflow as well as nonmonotonic behavior of incompatibility robustness of channels.

From the above discussion, through examples of depolarizing as well as amplitude damping channels (mainly from Figs. 3–5), we observe the *simultaneous presence* of both information backflow (a signature of CP-indivisibility) and nonmonotonicity of incompatibility robustness of channels. Therefore, this observation motivates us to define a measure of CP-indivisibility based on incompatibility robustness of channels. We will define such a measure in Sec. VI.

C. CP-indivisible dynamical maps without information backflow

Although information backflow that may be measured using trace distance is a signature of CP-indivisibility, it is not equivalent to CP-indivisibility [44,45]. In order to illustrate this point, let us consider the Choi matrix of a dynamical map $\mathcal{D}_{et} = \{\Lambda_t^{et}\}_t$, given by

$$C_{\Lambda_t^{et}} = \begin{bmatrix} A(t) & 0 & 0 & B(t) \\ 0 & 1-A(t) & 0 & 0 \\ 0 & 0 & 1-A(t) & 0 \\ B(t) & 0 & 0 & A(t) \end{bmatrix}, \quad (38)$$

where $A(t) = \frac{1+e^{-2t}}{2}$ and $B(t) = e^{-\int_0^t (1-\tanh x)dx}$. Such a dynamical map is CP-indivisible but does not show information backflow (i.e., the trace distance is monotonically nonincreasing with respect to time t) [45].

For the above dynamics we plot the robustness with respect to both arbitrary and depolarizing noise in Fig. 6. No nonmonotonicity indicative of information backflow is displayed. Although we did not find any nonmonotonic behavior of incompatibility robustness of quantum channels with regard to time t, it is a matter of further investigation to conclude whether it is possible for nonmonotonic behavior to be revealed for any other choice of the reference CP-divisible map instead of the identity dynamical map.

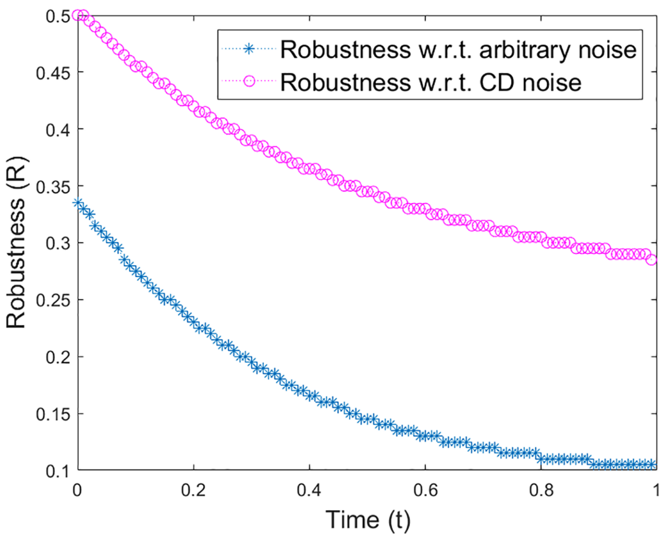

FIG. 6. This plot shows monotonic behavior of incompatibility robustness of channels for dynamical maps $\mathcal{D}_I$ and $\mathcal{D}_{et}$ [i.e., $R_C(\mathbb{I}_t, \Lambda_t^{et})$ and $R_C^{\mathcal{CD}}(\mathbb{I}_t, \Lambda_t^{et})$] with respect to time t.

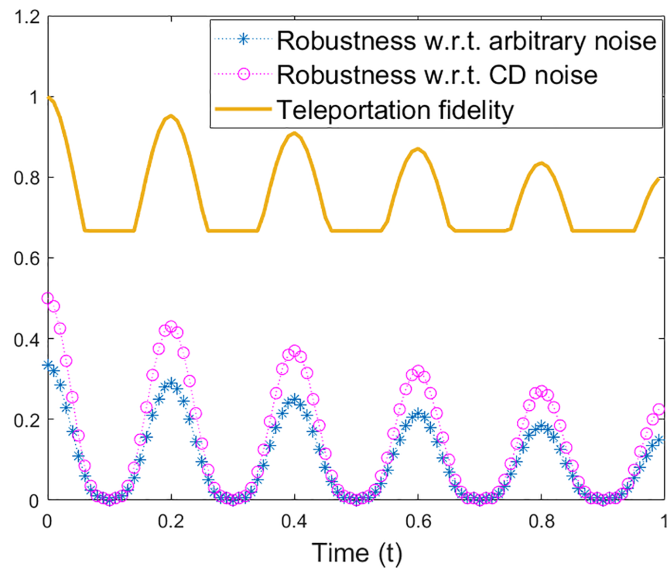

FIG. 7. This plot shows the maximum fidelity for teleportation using the state $\rho'_{AB} = (\mathcal{I}_A \otimes \Lambda_t^2)(\rho_{AB})$ and the nonmonotonic behavior of incompatibility robustness of channels for dynamical maps $\mathcal{D}_I$ and $\mathcal{D}_2$ [i.e., $R_C(\mathbb{I}_t, \Lambda_t^2)$ and $R_C^{\mathcal{CD}}(\mathbb{I}_t, \Lambda_t^2)$] with respect to time t. Clearly, the teleportation fidelity increases with a corresponding increase in incompatibility robustness.

V. CP-INDIVISIBILITY AND INCOMPATIBILITY OF CHANNELS AS A RESOURCE FOR QUANTUM TELEPORTATION

Quantum teleportation is a very-well-known quantum communication protocol [46,47]. Although perfect quantum teleportation can be performed using maximally entangled states, it can be performed imperfectly with mixed nonmaximal entangled states, in general. A criterion for a two-qubit state to be useful in quantum teleportation is provided in Ref. [48]. Consider a two-qubit state ρ_{AB}. Define a 3×3 matrix $S_{\rho_{AB}} = [s(\rho_{AB})_{ij}]$ such that $s(\rho_{AB})_{ij} = \mathrm{Tr}[\rho_{AB}(\sigma_i \otimes \sigma_j)]\,\forall i, j \in \{x, y, z\}$, where σ_js are Pauli matrices. Let $N(\rho_{AB}) = \mathrm{Tr}[\sqrt{S_{\rho_{AB}}^\dagger S_{\rho_{AB}}}]$. A quantum state is useful for teleportation if and only if $N(\rho_{AB}) > 1$, and in this case, the maximum fidelity is $F_{\max} \equiv \frac{1}{2}[1 + \frac{1}{3}N(\rho)] > 2/3$ [48].

Now, consider the two-qubit maximally entangled state $|\Psi_-\rangle = \frac{1}{\sqrt{2}}[|01\rangle - |10\rangle]$, where $\{|0\rangle, |1\rangle\}$ are the eigenbasis of σ_z. Let us take the dynamical map $\mathcal{D}_2 = \{\Lambda_t^2\}_t$ from Sec. IV. On application of Λ_t^2 on the state $|\Psi_-\rangle\langle\Psi_-|$, we obtain

$$\begin{aligned}\rho'_{AB} &= \left(\mathcal{I}_A \otimes \Lambda_t^2\right)(\rho_{AB})\\ &= w(t)|\Psi_-\rangle\langle\Psi_-| + [1 - w(t)]\frac{\mathbb{1}_{4\times 4}}{4},\end{aligned} \tag{39}$$

where $w(t) = e^{-\lambda t}\cos^2\omega t$ with $\lambda = 0.5$ and $\omega = 5\pi$. Now,

$$S_{\rho'_{AB}} = \begin{bmatrix} -w(t) & 0 & 0\\ 0 & -w(t) & 0\\ 0 & 0 & -w(t)\end{bmatrix}. \tag{40}$$

It follows that $N(\rho'_{AB}) = 3w(t)$ and therefore, $F_{\max} = \frac{1}{2}[1 + \frac{1}{3}w(t)]$ for $N(\rho'_{AB}) > 1$ and otherwise, $F_{\max} = \frac{2}{3}$.

In Fig. 7 we plot the robustness measure and teleportation fidelity versus time. From the figure we observe that the teleportation fidelity rises with increase of the incompatibility robustness, which is a signature of CP-indivisibility. This clearly establishes CP-indivisibility is a resource for quantum teleportation.

VI. MEASURING CP-INDIVISIBILITY USING INCOMPATIBILITY ROBUSTNESS OF QUANTUM CHANNELS

In this section we propose a measure of CP-indivisibility based on incompatibility robustness of quantum channels. As we can see from the study in Sec. IV, the incompatibility robustness of channels shows the signature of CP-indivisibility–induced information backflow through nonmonotonic behavior. Therefore, it is evident that an information backflow measure–based measure of CP-indivisibility [15] can be constructed using the incompatibility robustness of channels. Here, we follow the procedure proposed by Laine *et al.* [15] to construct such a measure of CP-indivisibility.

Consider a dynamical map $\mathcal{D} = \{\Lambda_t\}_t \in \mathbb{DM}(\mathcal{H}, \mathcal{H})$. We have to construct a CP-indivisibility measure of $\mathcal{D}$. For this, take an arbitrary CP-divisible dynamical map $\bar{\mathcal{D}} = \{\bar{\Lambda}_t\}_t \in \mathbb{DM}(\mathcal{H}, \mathcal{H})$ that acts as a reference CP-divisible dynamical map. Let $\theta(t) := \frac{dR_C(\Lambda_t, \bar{\Lambda}_t)}{dt}$. Then, we define the CP-indivisibility measure $\mathcal{N}(\mathcal{D})$ as

$$\mathcal{N}(\mathcal{D}) = \sup_{\bar{\mathcal{D}}\in\mathbb{DM}(\mathcal{H},\mathcal{H})_{\mathcal{CP}}} \int_{\theta(t)>0} R_C(\Lambda_t, \bar{\Lambda}_t)dt, \tag{41}$$

where $\mathbb{DM}(\mathcal{H}, \mathcal{H})_{\mathcal{CP}}$ is the set of all CP-divisible dynamical maps acting from $\mathcal{L}(\mathcal{H})$ to $\mathcal{L}(\mathcal{H})$. Clearly, $\mathcal{N}(\mathcal{D}) \geqslant 0$. We obtain from Theorem 1 that this measure is always *zero* for CP-divisible dynamical maps [as $\theta(t) \leqslant 0$ for those dynamical maps]. But nonmonotonic behavior of incompatibility robustness of channels in Figs. 3 and 4 suggests that that the proposed measure is nonzero for the CP-indivisible depolarizing dynamical map $\mathcal{D}_2$. Similarly, nonmonotonic behavior of incompatibility robustness of channels in Fig. 5 suggests that this measure is nonzero for the CP-indivisible amplitude damping dynamical map $\mathcal{D}_{ad}$. Therefore, in short, there exists CP-indivisible dynamical maps for which this measure is nonzero. It is evident that as the expression of $\mathcal{N}(\mathcal{D})$ consists of integration with respect to time t over the range from 0 to ∞, this measure $\mathcal{N}(\mathcal{D})$ can take an arbitrary value and it is not normalized. For the sake of compactness, we can also propose a normalized measure of the form

$$\mathfrak{N}(\mathcal{D}) = \frac{\mathcal{N}(\mathcal{D})}{1+\mathcal{N}(\mathcal{D})}. \tag{42}$$

Clearly, for any dynamical map $\mathcal{D}$, $0 \leqslant \mathfrak{N}(\mathcal{D}) \leqslant 1$ and $\mathfrak{N}(\mathcal{D}) = 0$ for CP-divisible dynamical maps [as $\mathcal{N}(\mathcal{D}) = 0$ for CP-divisible dynamical maps] and $\mathfrak{N}(\mathcal{D})$ is *nonzero* whenever $\mathcal{N}(\mathcal{D})$ is nonzero. Our proposed measure is on a similar footing of the information-backflow-based non-Markovianity measure [15].

VII. CONCLUSIONS

To summarize, in this work we have considered two important features of quantum theory, viz. CP-indivisibility and incompatibility of channels, that arise naturally in several practical situations of quantum dynamics. These two properties have been utilized as resources in several quantum information processing protocols. Our present analysis enables the characterization of CP-indivisibility of dynamical maps using incompatibility of channels. We have shown that incompatibility robustness of channels for two CP-divisible dynamical maps is monotonically nonincreasing with respect to time. We have shown that for two dynamical maps and for a particular time t, the incompatibility robustness of quantum measurements is upper bounded by the incompatibility robustness of the quantum channel.

Furthermore, we have explicitly analyzed the case of qubit-depolarizing dynamical maps and qubit-amplitude-damping dynamical maps as examples. We have shown that in the case of the CP-divisible regime, incompatibility robustness of channels is monotonically nonincreasing with respect to time. But in the case of a CP-indivisible regime, it loses its monotonic behavior in both cases. The examples studied here clearly illustrate the simultaneous presence of information backflow from the environment to the system, as reflected by the nonmonotonic behavior of the trace distance and nonmonotonic behavior of the incompatibility robustness of quantum channels for both generic and completely depolarizing noise models. We have further shown through an example how information backflow acts as a resource for quantum teleportation. Additionally, we have proposed a measure of CP-indivisibility based on incompatibility robustness of quantum channels.

The results obtained from our present study motivate certain directions of future research. It may be worthwhile to explore whether the incompatibility robustness of quantum channels can be used to witness CP-indivisible maps, such as the one studied in Sec. IV C, that do not show information backflow [45]. Moreover, it would also be interesting to investigate whether the CP-indivisibility measure proposed in Sec. VI can be useful to quantify the performance of some specific information-theoretic or thermodynamic tasks.

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[1] T. Heinosaari, T. Miyadera, and M. Ziman, An invitation to quantum incompatibility, J. Phys. A: Math. Theor. **49**, 123001 (2016).
[2] M. T. Quintino, T. Vértesi, and N. Brunner, Joint measurability, Einstein-Podolsky-Rosen steering, and Bell nonlocality, Phys. Rev. Lett. **113**, 160402 (2014).
[3] W. Son, E. Andersson, S. M. Barnett, and M. S. Kim, Joint measurements and Bell inequalities, Phys. Rev. A **72**, 052116 (2005).
[4] D. Saha, D. Das, A. K. Das, B. Bhattacharya, and A. S. Majumdar, Measurement incompatibility and quantum advantage in communication, Phys. Rev. A **107**, 062210 (2023).
[5] P. Skrzypczyk, I. Šupić, and D. Cavalcanti, All sets of incompatible measurements give an advantage in quantum state discrimination, Phys. Rev. Lett. **122**, 130403 (2019).
[6] J. Mori, Operational characterization of incompatibility of quantum channels with quantum state discrimination, Phys. Rev. A **101**, 032331 (2020).
[7] H. P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems* (Oxford University Press, Oxford, UK, 2002).
[8] R. Alicki and K. Lendi, *Quantum Dynamical Semigroups and Applications*, Lecture Notes in Physics (Springer-Verlag Berlin Heidelberg, 2007).
[9] G. Lindblad, On the generators of quantum dynamical semigroups, Commun. Math. Phys. **48**, 119 (1976).
[10] V. Gorini, A. Kossakowski, and E. C. G. Sudarshan, Completely positive dynamical semigroups of *N*-level systems, J. Math. Phys. **17**, 821 (1976).
[11] A. Rivas, S. F. Huelga, and M. B. Plenio, Quantum non-Markovianity: Characterization, quantification and detection, Rep. Prog. Phys. **77**, 094001 (2014).
[12] H.-P. Breuer, E.-M. Laine, J. Piilo, and B. Vacchini, *Colloquium:* Non-Markovian dynamics in open quantum systems, Rev. Mod. Phys. **88**, 021002 (2016).
[13] I. de Vega and D. Alonso, Dynamics of non-Markovian open quantum systems, Rev. Mod. Phys. **89**, 015001 (2017).
[14] A. Rivas, S. F. Huelga, and M. B. Plenio, Entanglement and non-Markovianity of quantum evolutions, Phys. Rev. Lett. **105**, 050403 (2010).

[15] E.-M. Laine, J. Piilo, and H.-P. Breuer, Measure for the non-Markovianity of quantum processes, Phys. Rev. A **81**, 062115 (2010).
[16] B. Bellomo, R. Lo Franco, and G. Compagno, Non-Markovian effects on the dynamics of entanglement, Phys. Rev. Lett. **99**, 160502 (2007).
[17] A. G. Dijkstra and Y. Tanimura, Non-Markovian entanglement dynamics in the presence of system-bath coherence, Phys. Rev. Lett. **104**, 250401 (2010).
[18] P. Kumar, S. Banerjee, R. Srikanth, V. Jagadish, and F. Petruccione, Non-Markovian evolution: A quantum walk perspective, Open Syst. Inf. Dyn. **25**, 1850014 (2018).
[19] N. P. Kumar, S. Banerjee, and C. M. Chandrashekar, Enhanced non-Markovian behavior in quantum walks with Markovian disorder, Sci. Rep. **8**, 8801 (2018).
[20] S. Bhattacharya, A. Misra, C. Mukhopadhyay, and A. K. Pati, Exact master equation for a spin interacting with a spin bath: Non-Markovianity and negative entropy production rate, Phys. Rev. A **95**, 012122 (2017).
[21] C. Mukhopadhyay, S. Bhattacharya, A. Misra, and A. K. Pati, Dynamics and thermodynamics of a central spin immersed in a spin bath, Phys. Rev. A **96**, 052125 (2017).
[22] S. Bhattacharya, B. Bhattacharya, and A. S. Majumdar, Convex resource theory of non-Markovianity, J. Phys. A: Math. Theor. **54**, 035302 (2021).
[23] A. G. Maity, S. Bhattacharya, and A. S. Majumdar, Detecting non-Markovianity via uncertainty relations, J. Phys. A: Math. Theor. **53**, 175301 (2020).
[24] B. Bhattacharya and S. Bhattacharya, Convex geometry of Markovian Lindblad dynamics and witnessing non-Markovianity, Quant. Inf. Process. **20**, 253 (2021).
[25] D. Burgarth, P. Facchi, D. Lonigro, and K. Modi, Quantum non-Markovianity elusive to interventions, Phys. Rev. A **104**, L050404 (2021).
[26] S. Milz, M. S. Kim, F. A. Pollock, and K. Modi, Completely positive divisibility does not mean Markovianity, Phys. Rev. Lett. **123**, 040401 (2019).
[27] E.-M. Laine, H.-P. Breuer, and J. Piilo, Nonlocal memory effects allow perfect teleportation with mixed states, Sci. Rep. **4**, 4620 (2014).
[28] B. Bylicka, D. Chruściński, and S. Maniscalco, Non-Markovianity and reservoir memory of quantum channels: A quantum information theory perspective, Sci. Rep. **4**, 5720 (2014).
[29] G. Thomas, N. Siddharth, S. Banerjee, and S. Ghosh, Thermodynamics of non-Markovian reservoirs and heat engines, Phys. Rev. E **97**, 062108 (2018).
[30] E. Chitambar and G. Gour, Quantum resource theories, Rev. Mod. Phys. **91**, 025001 (2019).
[31] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Quantum entanglement, Rev. Mod. Phys. **81**, 865 (2009).
[32] S. Aimet and H. Kwon, Engineering a heat engine purely driven by quantum coherence, Phys. Rev. A **107**, 012221 (2023).
[33] A. Acín, N. Gisin, and L. Masanes, From Bell's theorem to secure quantum key distribution, Phys. Rev. Lett. **97**, 120405 (2006).
[34] M. Howard, J. Wallman, V. Veitch, and J. Emerson, Contextuality supplies the 'magic' for quantum computation, Nature (London) **510**, 351 (2014).
[35] C. Budroni, A. Cabello, O. Gühne, M. Kleinmann, and J.-A. Larsson, Kochen-Specker contextuality, Rev. Mod. Phys. **94**, 045007 (2022).
[36] F. Buscemi, E. Chitambar, and W. Zhou, Complete resource theory of quantum incompatibility as quantum programmability, Phys. Rev. Lett. **124**, 120401 (2020).
[37] A. Streltsov, U. Singh, H. S. Dhar, M. N. Bera, and G. Adesso, Measuring quantum coherence with entanglement, Phys. Rev. Lett. **115**, 020403 (2015).
[38] G. Aubrun, L. Lami, C. Palazuelos, and M. Plávala, Entanglement and superposition are equivalent concepts in any physical theory, Phys. Rev. Lett. **128**, 160402 (2022).
[39] C. Duarte, L. Catani, and R. C. Drumond, Relating compatibility and divisibility of quantum channels, Int. J. Theor. Phys. **61**, 189 (2022).
[40] J. Kiukas, P. Lahti, and J.-P. Pellonpää, Joint measurability in Lindbladian open quantum systems, Open Syst. Inf. Dyn. **30**, 2350013 (2023).
[41] T. Heinosaari and T. Miyadera, Incompatibility of quantum channels, J. Phys. A: Math. Theor. **50**, 135302 (2017).
[42] T. Heinosaari, Simultaneous measurement of two quantum observables: Compatibility, broadcasting, and in-between, Phys. Rev. A **93**, 042118 (2016).
[43] C. Addis, T. Heinosaari, J. Kiukas, E.-M. Laine, and S. Maniscalco, Dynamics of incompatibility of quantum measurements in open systems, Phys. Rev. A **93**, 022114 (2016).
[44] D. Chruściński, A. Rivas, and E. Størmer, Divisibility and information flow notions of quantum Markovianity for noninvertible dynamical maps, Phys. Rev. Lett. **121**, 080407 (2018).
[45] M. J. W. Hall, J. D. Cresser, L. Li, and E. Andersson, Canonical form of master equations and characterization of non-Markovianity, Phys. Rev. A **89**, 042120 (2014).
[46] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wootters, Teleporting an unknown quantum state via dual classical and Einstein-Podolsky-Rosen channels, Phys. Rev. Lett. **70**, 1895 (1993).
[47] S. Popescu, Bell's inequalities versus teleportation: What is nonlocality? Phys. Rev. Lett. **72**, 797 (1994).
[48] R. Horodecki, M. Horodecki, and P. Horodecki, Teleportation, Bell's inequalities and inseparability, Phys. Lett. A **222**, 21 (1996).