Strong local passive state in the minimal quantum energy teleportation model

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Like quantum state teleportation, quantum energy teleportation (QET) can transmit energy through local operation and classical communication. This QET process even can extract energy from a strong local passive (SLP) state, from which any local operation can only inject energy. However, finding a SLP state is an important open question. In this paper, in the Hamiltonian of the minimal QET model, we obtain a set of SLP pure states by correlation matrix analysis. Our method is computable and easier to understand in certain Hamiltonians and states. Through the method of mathematic programming, we find an amazing phenomenon in which the energy teleportation efficiency can be greater than 1 and we reveal the essence of this phenomenon. It can be extended to the mixed state and spin chain model. Our result shows that quantum steering plays an important role in the QET protocol.

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I. INTRODUCTION

As is well known, quantum teleportation is a very important achievement in quantum information processing [1]. A natural question arises whether energy can be transmitted similarly by quantum entanglement. To answer this question, Hotta proposed the theory of quantum energy teleportation (QET) [2] and subsequently the minimal QET protocol [3]. This protocol was realized recently in [4,5].

The QET protocol is important because it can extract energy from the strong local passive (SLP) state [6,7]. The local passive (LP) state is a kind of multipartite state from which we cannot extract energy with local unitary operation. For the SLP state, no local operation, including completely positive and trace-preserving (CPTP) maps, can extract energy from local subsystems.

In the QET protocol, Alice and Bob share a certain entangled state, which cannot extract any energy from Bob's subsystem B. Alice measures her subsystem A and then B will collapse to a corresponding state. Bob's collapsed state is not under the lowest energy, which can be extracted energy by unitary operations, but first Alice should tell Bob the measurement result of her particle. In the real world, the speed of energy transfer must be less than or equal to the vacuum speed of light. This means that when Bob extracts energy from his subsystem B, the energy injected into subsystem A has not yet spread to subsystem B. This protocol is called QET.

Since the QET protocol was proposed, many researchers have shown great interest in it, especially after its realization. Various models have been studied, such as spin chain systems [8–11], trapped ions systems [12], and so on [13–21]. In addition, some researchers considered that the problem of black hole information loss [22–24] and quantum field theory [18,25] would benefit from the research on QET. Yusa *et al.* proposed an experimental protocol based on the quantum Hall effect to realize QET [26]. Rodríguez-Briones *et al.* realized QET in a nuclear magnetic resonance system in [4]. Also Ikeda completed a QET experiment on superconductor quantum hardware [5]. There is no doubt that these experiments have proven the validity of QET, but there is still an important open question about the efficiency of the energy transfer.

Most previous researchers believed that the transmitted energy in the QET protocol is not really transmitted, which is from quantum systems; however, if that were true, the energy transfer efficiency could be greater than 1. Researchers have not yet obtained this result. To resolve this issue, first we should find an appropriate SLP state in the minimal QET model. By the method of correlation matrix analysis, we obtain a set of SLP pure states. Compared with previous methods, our method is computable. In fact, if the Hamiltonian and correlation matrix are given, we can determine whether the quantum state is a SLP state easily. Then, using the methods of mathematical programming, we find the lowest-energy state with a measured subsystem. Combining with the SLP state we find, we get the result that the energy transfer efficiency is greater than 1; even in some special case, we can extract energy without injecting any additional energy. In the QET field, this method can be generalized to mixed states and the spin chain model.

II. THE SLP PURE STATE IN THE MINIMAL QET MODEL

We consider the Hamiltonian of the minimal QET model [3]

$$H_0 = H_1 = h\sigma_z,$$
$$H_V = 2k\sigma_x \otimes \sigma_x, \tag{1}$$

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where $\sigma_{x,y,z}$ are Pauli operators, h and k are real numbers, and $h, k \in (0, +\infty)$. In this case, the quantum state

$$|\psi\rangle = \cos\theta|00\rangle - \sin\theta|11\rangle, \quad \theta \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right), \tag{2}$$

is a state from which any local unitary operation on subsystem B cannot extract energy. The reason is as follows.

Any two-qubit state density matrix can be written as

$$\rho = \tfrac{1}{4} \sum_{i,j=0,1,2,3} t_{ij}\sigma_i \otimes \sigma_j, \tag{3}$$

where $\sigma_{0,1,2,3}$ represent identity and $\sigma_{x,y,z}$. The correlation matrix $T = (t_{ij})$ of $|\psi\rangle$ is

$$T = \begin{pmatrix} 1 & 0 & 0 & \cos 2\theta \\ 0 & -\sin 2\theta & 0 & 0 \\ 0 & 0 & \sin 2\theta & 0 \\ \cos 2\theta & 0 & 0 & 1 \end{pmatrix}. \tag{4}$$

With this form, the total energy is $E = \mathrm{Tr}[(H_0 + H_1 + H_V)\rho]$, where

$$E = ht_{03} + ht_{30} + 2kt_{11}. \tag{5}$$

Note that $\theta \in [\frac{\pi}{4}, \frac{\pi}{2})$ and thus $\cos 2\theta < 0$. In fact, any LP state must be satisfied for $t_{03}, t_{30}, t_{11} < 0$. Otherwise, we can always find some appropriate local unitary operations which make $t_{03} \to -t_{03}$, $t_{30} \to -t_{30}$, or $t_{11} \to -t_{11}$ (the three cases can be achieved at the same time) and extract energy. This means that if any of $t_{03}, t_{30}, t_{11} > 0$, the state cannot be a LP state.

As is well known, we can transform SU(2) operators to SO(3) operators as

$$\begin{aligned}
U_x &= \begin{pmatrix} \cos\varphi & -i\sin\varphi \\ -i\sin\varphi & \cos\varphi \end{pmatrix} \mapsto \\
O_x &= \begin{pmatrix} 1 & & \\ & \cos 2\varphi & -\sin 2\varphi \\ & \sin 2\varphi & \cos 2\varphi \end{pmatrix}, \\
U_y &= \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \mapsto \\
O_y &= \begin{pmatrix} \cos 2\varphi & & -\sin 2\varphi \\ & 1 & \\ \sin 2\varphi & & \cos 2\varphi \end{pmatrix}, \\
U_z &= \begin{pmatrix} 1 & \\ & e^{i\varphi} \end{pmatrix} \mapsto \\
O_z &= \begin{pmatrix} \cos\varphi & -\sin\varphi & \\ \sin\varphi & \cos\varphi & \\ & & 1 \end{pmatrix}.
\end{aligned} \tag{6}$$

With this process, SU(2) operators acting on the density matrix can be written as SO(3) operators acting on the correlation matrix. The correlation matrix is a 4×4 matrix. Note that $t_{00} \equiv 1$ preserves the trace of the density matrix as 1. We can direct sum 1 to SO(3) operators, which keeps t_{00} the same; in addition, the local operators acting on the two subsystems do not affect each other. In this form, the local unitary operation in the first subsystem can be seen as row transformation of the correlation matrix T with $i = 1, 2, 3$ and in the second subsystem it can be seen as column transformation with $j = 1, 2, 3$. The i and j are rank and column indicators [see Eq. (3)]. Thus the local operation can be written as

$$(U_1 \otimes U_2)\rho(U_1 \otimes U_2)^\dagger \mapsto (1 \oplus O_1)T(1 \oplus O_2)^\dagger. \tag{7}$$

From Eq. (5) we can find that any unitary operation on subsystem B in the σ_x and σ_z directions makes $t_{03} \to t_{03}\cos\varphi$ and $t_{11} \to t_{11}\cos\varphi$, respectively. Both unitary operations can only input energy into subsystem B. The unitary operators in the σ_y direction will also input energy. In this case,

$$E' = ht_{03}\cos\varphi + ht_{30} + 2kt_{11}\cos\varphi \geqslant E, \tag{8}$$

where the greater than or equal to sign is because t_{03}, t_{30}, and t_{11} are negative.

Then any unitary operation can be written as

$$U(\phi, \alpha_1, \alpha_2, \alpha_3) = e^{i\phi}U_z(\alpha_1)U_y(\alpha_2)U_z(\alpha_3). \tag{9}$$

When the global phase $e^{i\phi}$ does not change the quantum state, we will generally omit it. According to Eqs. (6) and (7) and the correlation matrix (4), we can get any unitary operator acting on subsystem B, which results in

$$\begin{aligned}
t_{30} &\mapsto t'_{30} = t_{30}, \\
t_{03} &\mapsto t'_{03} = \cos(2\alpha_2)t_{03}, \\
t_{11} &\mapsto t'_{11} = [\cos^2\alpha_2\cos(\alpha_1 + \alpha_3) \\
&\qquad - \sin^2\alpha_2\cos(\alpha_1 - \alpha_3)]t_{11} \\
&\qquad = Ct_{11},
\end{aligned} \tag{10}$$

where $-1 \leqslant C \leqslant 1$. Thus $t'_{03} \geqslant t_{03}$, $t'_{11} \geqslant t_{11}$, and

$$E' = ht'_{03} + ht'_{30} + 2kt'_{11} \geqslant E. \tag{11}$$

This means that the state $|\psi\rangle$ is the state from which any local unitary operation on subsystem B cannot extract energy. Similarly, any local unitary operation on subsystem A also cannot extract energy.

It is worth noting that even if any local unitary operation on subsystem A (or B) could not extract energy, we cannot say that the state is a LP state. Using Eqs. (6) and (7) and the correlation matrix (4), we find that the local operation $U_y(\frac{\pi}{4}) \otimes U_y(-\frac{\pi}{4})$ may change the system energy,

$$\Delta E = -2k + 2k\sin 2\theta - 2h\cos 2\theta. \tag{12}$$

If $\Delta E \geqslant 0$, the state will be a LP state. Solving Eq. (12) results in two situations. First, if $h \geqslant k$, the quantum state $|\psi\rangle$ is always a LP state. Second, if $h < k$, the quantum state $|\psi\rangle$ is a LP state when $\theta \in [\frac{\pi}{4}, \arctan\frac{k+h}{k-h})$.

Furthermore, any local operation on subsystem B also cannot extract energy if this state is a SLP state. The local operations can be written as local CPTP maps, which can be represented as the combination of rotation, scaling, and translation in the Bloch sphere. For rotation corresponding to a unitary operation, now we consider scaling and translation.

Obviously, to make sure one map is positive, this scaling operation must be shrunken. In this case, any scaling operation on subsystem B can only inject energy; it cannot also extract energy.

For translation operation, we must shrink the Bloch sphere. With the reduced density matrix of subsystem A,

$$\rho_A = \mathrm{Tr}_B(|\psi\rangle\langle\psi|) = \begin{pmatrix} \cos^2\theta & 0 \\ 0 & \sin^2\theta \end{pmatrix}, \tag{13}$$

the Bloch vector has no σ_x-direction component. Thus if we compress the Bloch sphere of subsystem B to one point, because the σ_x-direction component in subsystem A is 0, any σ_x-direction component in B will not impact the energy of H_V; it will make $\langle H_V\rangle \to 0$. So if we compress the Bloch sphere to state $|1\rangle$, perhaps we can maximize the energy extraction. Note that, in this process, the curvature radius of the compressed Bloch sphere at $|1\rangle$ is always equal to the radius of the Bloch sphere. Then we can obtain an amplitude damping class channel compressed towards $|1\rangle$. The details are given in Appendix A. If we make the noisy strength of the amplitude damping class channel be p, in this case, $t'_{11} = -(\sqrt{1-p})\sin 2\theta$ and $t'_{03} = -p + (1-p)\cos 2\theta$, resulting in

$$\begin{aligned} E_V &= \langle H_V\rangle = -2k\sin 2\theta, \\ E'_V &= \langle H'_V\rangle = -2k(\sqrt{1-p})\sin 2\theta, \\ \Delta E_V &= E'_V - E_V = 2k\sin 2\theta(1-\sqrt{1-p}) \end{aligned} \tag{14}$$

and

$$\begin{aligned} E_1 &= \langle H_1\rangle = h\cos 2\theta, \\ E'_1 &= \langle H'_1\rangle = h[-p + (1-p)\cos 2\theta], \\ \Delta E_1 &= E'_1 - E_1 = -hp - hp\cos 2\theta. \end{aligned} \tag{15}$$

If $\Delta E_V + \Delta E_1 \geqslant 0$, no CPTP map on subsystem B can extract energy. Solving the inequality

$$\begin{aligned} \Delta E &= \Delta E_V + \Delta E_1 \\ &= 2k\sin 2\theta(1-\sqrt{1-p}) - hp - hp\cos 2\theta \geqslant 0, \end{aligned} \tag{16}$$

we can get that $\theta \geqslant \arctan\frac{hp}{2k(1-\sqrt{1-p})}$ and $\frac{\partial \Delta E}{\partial p} \geqslant 0$. When $p = 1$, we have $\theta \geqslant \arctan\frac{h}{2k}$. In other words, if $\theta \in [\arctan\frac{h}{2k}, \frac{\pi}{2}) \bigcap [\frac{\pi}{4}, \frac{\pi}{2})$, the state $|\psi\rangle$ will be a SLP state in the minimal QET model for subsystem B.

In fact, for the case of a certain Hamiltonian and quantum state, we can always use Eqs. (6) and (7) to analyze whether the state is a LP state or a SLP state. If we compress the Bloch sphere to an arbitrary point, we can also convert this problem to the above form by using appropriate unitary operations. If it is a multiqubit question, we can use the correlation tensor instead of the correlation matrix.

III. MAXIMAL EXTRACTED ENERGY IN THE QET MODEL

In the minimal QET model, the state $|\psi\rangle$ is the ground state under the Hamiltonian. In this process, energy will be injected into subsystem A and extracted from B at a long distance; the injected energy is greater than the extracted energy. However, if we use the SLP state instead of the ground state, the extracted energy may be greater than the injected energy.

For the pure state $|\psi\rangle$, if we act a projector on subsystem A, subsystem B will collapse to a pure state. This system will be a product pure state.

From Eqs. (1) and (4) we know that quantum energy teleportation must be measured on subsystem A in the σ_x direction because only in the σ_x direction could $\langle H_1\rangle$ and $\langle H_V\rangle$ remain unchanged. In this case, subsystem A will collapse to $\frac{|0\rangle+|1\rangle}{\sqrt{2}}$ and $\frac{|0\rangle-|1\rangle}{\sqrt{2}}$ with Bloch vectors $(1,0,0)^T$ and $(-1,0,0)^T$.

At the same time, any two-qubit product state with Bloch vectors $(a_1,a_2,a_3)^T$ and $(b_1,b_2,b_3)^T$ can be written as

$$\rho = \frac{1}{4}\left(\mathbf{1} + \sum_i a_i\sigma_i\otimes\mathbf{1} + \sum_j b_j\mathbf{1}\otimes\sigma_j + \sum_{i,j} a_i b_j \sigma_i\otimes\sigma_j\right). \tag{17}$$

In the QET protocol, after the measurement in subsystem A, the state of A is $\rho_A = \frac{1}{2}(\mathbf{1}\pm\sigma_x)$, and subsystem B will be $\rho_B = \frac{1}{2}(\mathbf{1}\mp\sigma_x\sin 2\theta + \sigma_z\cos 2\theta)$ with $b_1 = \mp\sin 2\theta$ and $b_3 = \cos 2\theta$. From Eqs. (1) and (17) (note that $a_1 = \pm 1$) we can easily find that only b_1 and b_3 can affect H_V and H_1. Furthermore, because $|\psi\rangle$ is a pure state, subsystem B with measured A must be a pure state, which satisfies $b_1^2 + b_3^2 = 1$. Acting a unitary operation on B to the minimal energy state, we can extract energy from the SLP state $|\psi\rangle$.

Measuring on subsystem A in the σ_x direction, if we get the positive result $+1$, the state of B will be

$$\rho_B = \begin{pmatrix} \frac{1+\cos 2\theta}{2} & -\frac{\sin 2\theta}{2} \\ -\frac{\sin 2\theta}{2} & \frac{1-\cos 2\theta}{2} \end{pmatrix}, \tag{18}$$

with $b_1 = -\sin 2\theta$ and $b_3 = \cos 2\theta$. This means that we can adjust θ to obtain an appropriate ρ_B. This process is called quantum steering [27–29]. In this case, because $a_1 = +1$, $E_V = 2ka_1b_1 = 2kb_1$. Thus

$$\begin{aligned} \langle H_V\rangle &= -2k\sin 2\theta, \quad \langle H_1\rangle = h\cos 2\theta, \\ E_B &= -2k\sin 2\theta + h\cos 2\theta. \end{aligned} \tag{19}$$

Now we should find the minimal energy in the Hamiltonian. Letting $b_1 = X$ and $b_3 = Z$, we have the following mathematic programming:

$$\begin{aligned} &\text{find } E'_B = \min q = 2kX + hZ \\ &\text{s.t. } X^2 + Z^2 = 1 \end{aligned} \tag{20}$$

(see Fig. 1). This mathematic programming method can be easily extended to other two-qubit mixed states (see Appendix B). We use it to calculate the spin model of hyperbolic quantum networks [30] in Appendix C.

Obviously, when the line is tangent to the circle, we can get E'_B. According to the distance relationship between the line and the center of the circle, we have

$$\frac{|E'_B|}{\sqrt{(2k)^2 + h^2}} = 1, \tag{21}$$

with $E'_B = -\sqrt{4k^2 + h^2}$. Solving the equations

$$\begin{aligned} 2kX + hZ - E'_B &= 0, \\ X^2 + Z^2 &= 1, \end{aligned} \tag{22}$$

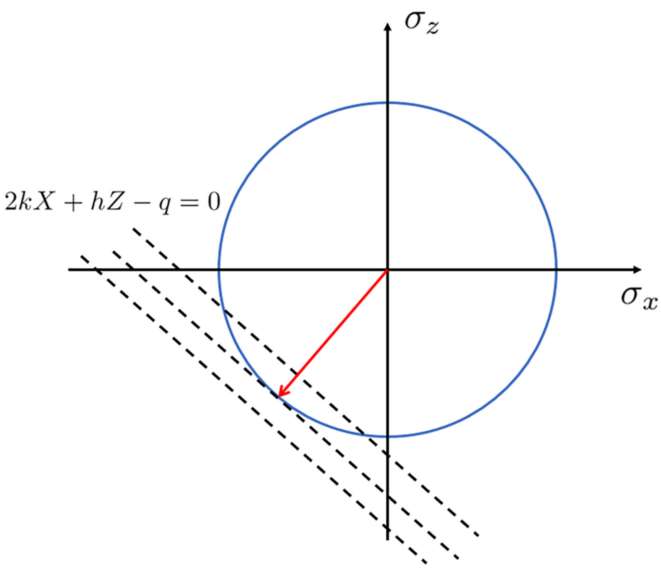

FIG. 1. Profile of the Bloch sphere on the xOz plane, with the y axis perpendicular to the plane facing inward. Here $q = 2kX + hZ$ can be represented as a straight line with slope $-\frac{2k}{h}$ (black dashed line). This problem can be transformed into a mathematic programming problem. The red vector is the state vector corresponding to the minimal energy.

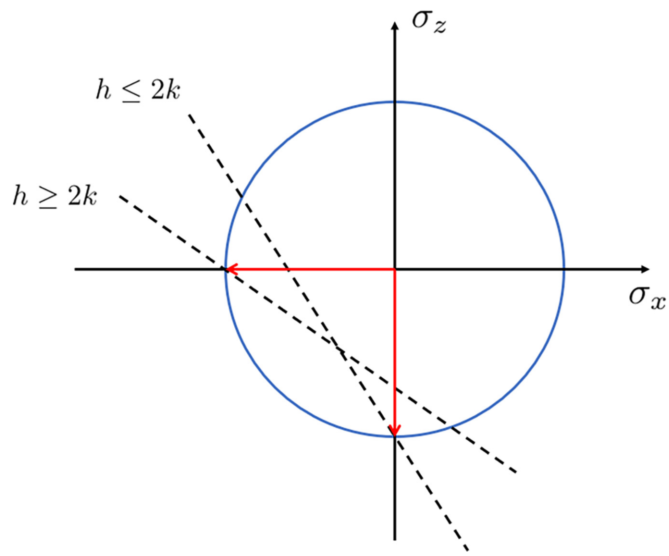

FIG. 2. Profile of the Bloch sphere on the xOz plane, with the y axis perpendicular to the plane facing inward. The Hamiltonian limits the state vector of subsystem B to be in the third quadrant of the xOz plane. The maximal E_B is divided into two situations with red state vectors.

we have

$$X = -\frac{2k}{\sqrt{4k^2 + h^2}},$$
$$Z = -\frac{h}{\sqrt{4k^2 + h^2}}. \tag{23}$$

So the minimal-energy state of subsystem B is

$$\rho'_B = \begin{pmatrix} \frac{1}{2} - \frac{1}{2\sqrt{4k^2+h^2}} & -\frac{k}{\sqrt{4k^2+h^2}} \\ -\frac{k}{\sqrt{4k^2+h^2}} & \frac{1}{2} + \frac{1}{2\sqrt{4k^2+h^2}} \end{pmatrix}. \tag{24}$$

Similarly, if subsystem A gets the negative result -1 in the σ_x direction, the above X will become $-X$ and the minimal energy E'_B is still unchanged.

Now we can obtain the extracted energy

$$-\Delta E_B = E_B - E'_B$$
$$= -2k \sin 2\theta + h \cos 2\theta + \sqrt{4k^2 + h^2}. \tag{25}$$

When $\theta = \arctan \frac{-h+\sqrt{h^2+8k^2}}{4k}$, the state $|\psi\rangle$ cannot teleport any energy.

For the maximal $-\Delta E_B$, because the state vector of subsystem B must be in the third quadrant of the xOz plane, the maximal E_B could be

$$\max E_B = \max(-h, -2k) \tag{26}$$

(see Fig. 2). Under certain h and k, we can use appropriate ρ_B to achieve the above results by quantum steering. Thus the maximal $-\Delta E_B$ should be

$$\max(-\Delta E_B) = \begin{cases} -h + \sqrt{4k^2 + h^2}, & h \leqslant 2k \\ -2k + \sqrt{4k^2 + h^2}, & h \geqslant 2k \end{cases} \tag{27}$$

and the maximal energy teleportation efficiency is $\eta = \frac{\max(-\Delta E_B)}{-h \cos 2\theta}$.

We find that this efficiency can be greater than 1. This means that in some special case, the injected energy by measuring subsystem A will be less than the energy extracted from subsystem B. In addition, the energy is not really transferred. We believe that, in the QET protocol, the measurement destroys quantum entanglement, so the system could release energy. After that we can use the quantum steering of the prepared state to allocate energy and then achieve quantum energy teleportation.

IV. EXAMPLE

We prepare an example for the minimal QET model. In this example, we use a certain SLP state to show that the energy transfer efficiency is greater than 1 and even approaches infinity. For some special SLP state, from the minimal QET protocol, we cannot extract any energy at all.

As previously mentioned, the maximal-energy teleportation efficiency can be greater than 1. We provide an example to illustrate this problem.

In the minimal QET model, we set $h = 2k = 1$. We can prove that the state $|\psi_0\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$ is a SLP state, in which $\theta = \arctan 1 \in [\arctan \frac{h}{2k}, \frac{\pi}{2})$, that is to say, no local operation can extract energy from subsystem B.

In the initial situation, from Eq. (1) we can get

$$E_A = \langle H_0 \rangle = \mathrm{tr}(H_0 \rho) = 0,$$
$$E_B = \langle H_1 \rangle + \langle H_V \rangle = -2k = -1, \tag{28}$$

where ρ is the density matrix of the quantum state. After subsystem A has been measured in the σ_x direction, suppose the measurement result is $+1$ with probability $p_{x|+1} = \frac{1}{2}$. Then state $|\psi_0\rangle$ becomes

$$|\psi_+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \tag{29}$$

and its correlation matrix is

$$T_+ = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{30}$$

Unitarily operating subsystem B in the σ_y direction with $\varphi = -\frac{\pi}{8}$, we can get the minimal-energy state $|\psi'\rangle$ as

$$|\psi'\rangle = \Big[\mathbf{1} \otimes U_y\Big(-\frac{\pi}{8}\Big)\Big]|\psi\rangle, \tag{31}$$

where

$$U_y\Big(-\frac{\pi}{8}\Big) = \begin{pmatrix} \cos(-\frac{\pi}{8}) & -\sin(-\frac{\pi}{8}) \\ \sin(-\frac{\pi}{8}) & \cos(-\frac{\pi}{8}) \end{pmatrix} \tag{32}$$

and the correlation matrix of $|\psi'\rangle$ is

$$T' = \begin{pmatrix} 1 & -\frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ 1 & -\frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{33}$$

Then we can easily calculate that

$$E'_A = \mathrm{tr}(H_0\rho') = 0,$$
$$E'_B = -\frac{\sqrt{2}}{2} \times 2k - \frac{\sqrt{2}}{2} \times h = -\sqrt{2}. \tag{34}$$

Thus we can find that

$$\Delta E_A = E'_A - E_A = 0,$$
$$\Delta E_B = E'_B - E_B = -\sqrt{2} + 1. \tag{35}$$

In this process, the energy of subsystem A remains unchanged, but $-\Delta E_B = \sqrt{2} - 1$ is extracted from subsystem B. The energy teleportation efficiency $\eta = \frac{-\Delta E_B}{\Delta E_A}$ is infinite.

Similarly, if the measurement result in subsystem A is -1 with probability $p_{x|-1} = \frac{1}{2}$, this process is equal to making subsystem B of $|\psi_+\rangle$ [see Eq. (29)] rotate π on the σ_z axis, and the ΔE_A, ΔE_B, and η in energy teleportation are unchanged. That is why we consider the transmitted energy to be from the entangled system itself.

Additionally, we give in Fig. 3 a plot of $-\Delta E$ as a function of θ, where $\theta \in [\frac{\pi}{4}, \frac{\pi}{2})$, $h = 2k = 1$, and $\Delta E_A = 0$. When $\theta = \frac{3\pi}{8}$, we cannot teleport any energy through the QET protocol because the collapsed subsystem B is in the lowest-energy state. This is caused by quantum steering; any entangled pure state has the maximal quantum steering [29].

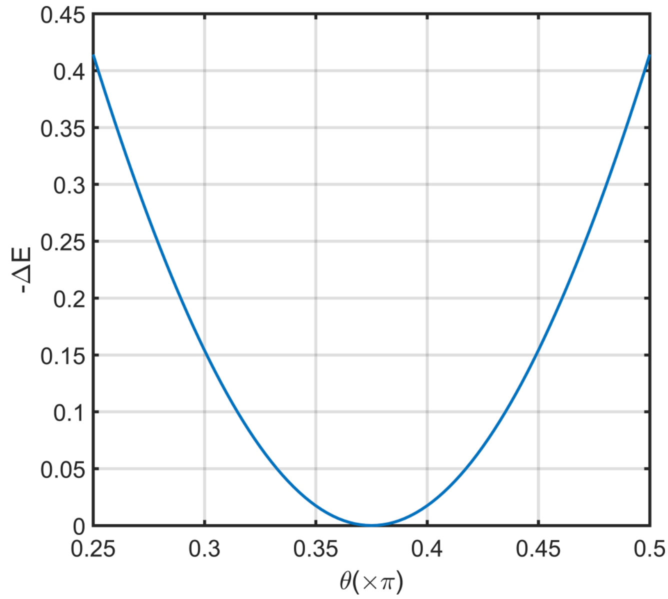

FIG. 3. Plot of the maximal extractable energy $-\Delta E$ as a function of θ, where $h = 2k = 1$. In this case, the state $|\psi\rangle = \cos\theta|00\rangle - \sin\theta|11\rangle$ is the SLP state with $\theta \in [\frac{\pi}{4}, \frac{\pi}{2})$. The maximal energy is $\sqrt{2} - 1$ for $\theta = \frac{\pi}{4}$ and $\theta \to \frac{\pi}{2}$ and the minimal energy is 0 for $\theta = \frac{3\pi}{8}$.

V. CONCLUSION

In this article we obtained a set of LP pure states and SLP pure states in the minimal QET model by correlation matrix analysis. With a certain Hamiltonian and quantum state, our method is computable and relatively easy to understand, which can be generalized to other Hamiltonian systems.

In the initial concept of the minimal QET model, to ensure that local operation cannot extract any energy from subsystem B, the quantum state must be the ground state in the Hamiltonian. However, the ground state is only a part of the SLP state. We found a set of SLP pure states and confirmed they can be used in the QET protocol. Furthermore, we found that the energy teleportation efficiency can be greater than 1. In some cases the injected energy in subsystem A is less than the energy extracted from subsystem B. Considering the law of conservation of energy, this is evidence that the extracted energy is from the quantum system itself. This amount of energy cannot be extracted directly unless the entanglement is broken by measuring in subsystem A. In addition, some special quantum states do not need to inject additional energy in subsystem A but can extract energy from subsystem B. In contrast, some special quantum states cannot teleport any energy. This is because, after measuring subsystem A, subsystem B will collapse to the minimal-energy state. This characteristic shows that quantum steering plays an important role in the QET protocol.

In addition, our method is computable and easier to understand than previous works. It can be used in the QET protocol with mixed states or the spin chain model. We believe that our work can inform extensive studies of the passive state and QET protocol.

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APPENDIX A: CPTP MAPS IN THE TWO-QUBIT SYSTEM

Any CPTP maps Λ can be written as a set of Kraus operators K_k with

$$\Lambda(\rho) = \sum_k K_k \rho K_k^\dagger. \tag{A1}$$

The CPTP maps (or Kraus operators) have four characteristics: They must be (i) linear, (ii) Hermite preserving, (iii) trace preserving, and (iv) completely positive. By completely positive we mean that in a two-qubit system ρ_{12}, $\Lambda \otimes \mathbf{1}(\rho_{12})$ must be positive. The positive maps just require positive $\Lambda(\rho)$. For example, the partial transpose map is positive but not completely positive.

In a linear space, we can always map an area to another area by rotation, scaling, and translation. Since the Bloch space is a linear space, the CPTP maps have the same forms. Any single qubit can be written in the form of a Bloch vector with

$$\rho = \tfrac{1}{2}(a_0\sigma_0 + a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3), \tag{A2}$$

where $\vec{a} = (a_1, a_2, a_3)^T$ is the Bloch vector and $a_0 \equiv 1$. The rotation in the quantum state is equal to unitary operation U and in the Bloch space it is equal to special orthogonal operation O. The CPTP maps acting on $\vec{a}$ will be

$$\Lambda(\vec{a}) = O\begin{pmatrix} \mu_1 + \eta_1 a_1 \\ \mu_2 + \eta_2 a_2 \\ \mu_3 + \eta_3 a_3 \end{pmatrix}, \tag{A3}$$

where $\eta_{1,2,3}$ are scaling factors and $\mu_{1,2,3}$ are translation factors.

In the real world, there are three kinds of basic noisy channels (or decoherent processing), which compress the Bloch space to a spherical center (depolarizing channel), any spherical diameter (phase-flip channel, bit-flip channel, phase-bit-flip channel, etc.), or any point on the Bloch sphere (amplitude damping channel). Except for the depolarizing channel, which compresses the Bloch space to a spherical center, we can always take an appropriate unitary operator to compress the Bloch space to any diameter or point on the sphere. Any feasible noisy channel can be represented as a composition of the above basic noisy channels. Generally, the μ and η in Eq. (A3) are related to the noisy strength p, i.e., the greater p is, the less μ and η are. In addition, some CPTP maps only have mathematical forms; we cannot achieve them by physical processing. Since the strong local passive state should extract energy by physical processing, we only consider the feasible noisy channel forms, which can be combined by the basic noisy channels.

This scaling operation of the linear map is from $\Lambda(\sigma_i) = \eta_i\sigma_i$, with $i \neq 0$. Because a_1, a_2, and a_3 are arbitrary, the translation factors are only from σ_0, with $\Lambda(\sigma_0) = \sigma_0 + \sum_i \mu_i\sigma_i$ and $i \neq 0$. Because the CPTP maps always map a quantum state to another quantum state, the η are non-negative and $0 \leqslant \eta \leqslant 1$. The translation factors μ are constrained by η to ensure that the compressed Bloch sphere is still inside the Bloch space, so $\sum_i \mu_i^2 \leqslant 1$ is necessary. This means that the compressed Bloch sphere is translated the distance of $|\vec{\mu}|$ in the direction of $\vec{\mu}$.

In a two-qubit system, the Bloch vector can be extended to the correlation matrix $T = (t_{ij})$, where

$$t_{ij} = \mathrm{tr}(\sigma_i \otimes \sigma_j \rho), \quad i, j = 0, 1, 2, 3. \tag{A4}$$

Then any local operation can be represented as the row and column transformations of the correlation matrix. The operation on the first qubit is row transformation and on the second qubit it is column transformation. So for any CPTP map acting on the second qubit (subsystem B), the correlation matrix will be

$$\mathbf{1} \otimes \Lambda(T) = (1 \oplus O_1)\begin{pmatrix} 1 & \mu_1 + \eta_1 t_{01} & \mu_2 + \eta_2 t_{02} & \mu_3 + \eta_3 t_{03} \\ t_{10} & \mu_1 t_{10} + \eta_1 t_{11} & \mu_2 t_{10} + \eta_2 t_{12} & \mu_3 t_{10} + \eta_3 t_{13} \\ t_{20} & \mu_1 t_{20} + \eta_1 t_{21} & \mu_2 t_{20} + \eta_2 t_{22} & \mu_3 t_{20} + \eta_3 t_{23} \\ t_{30} & \mu_1 t_{30} + \eta_1 t_{31} & \mu_2 t_{30} + \eta_2 t_{32} & \mu_3 t_{30} + \eta_3 t_{33} \end{pmatrix}(1 \oplus O_2)^\dagger, \tag{A5}$$

where O_1 and O_2 are SO(3) operators and $O_1 = \mathbf{1}$. For the sign of the direct sum $\oplus$ see Eq. (7).

Now let us look at the minimal QET model. The Hamiltonian is as in (1). With the relation of the Pauli operator

$$\mathrm{tr}(\sigma_i\sigma_j) = 0, \quad \forall\, i \neq j, \tag{A6}$$

the energy can be easily calculated as

$$\begin{aligned} E_0 &= ht_{30}, \\ E_1 &= ht_{03}, \\ E_V &= 2kt_{11}, \end{aligned} \tag{A7}$$

that is to say, the energy of the quantum system is only related to t_{30}, t_{03}, and t_{11}. Therefore, in the minimal QET model, for any strong local passive state, t_{30}, t_{03}, and t_{11} should be small enough. We can always find an appropriate rotation operator that makes t_{01}, t_{02}, t_{10}, and t_{20} become 0. The state $|\psi\rangle = \cos\theta|00\rangle - \sin\theta|11\rangle$ with $\theta \in [\frac{\pi}{4}, \frac{\pi}{2})$ is a perfect choice.

From Eq. (A5) we can find that t_{30} remains unchanged. Because $t_{10} = 0$, $t_{12} = 0$, and $t_{13} = 0$, any CPTP map can only make t_{11} increase [note that t_{11} is negative for the state $|\psi\rangle$ and the effect of the unitary operator O_2 is similar to Eqs. (10) and (11)]. In other words, we can only extract energy from H_1 by decreasing t_{03}. At the same time, we should make t_{11} increase slowly enough to minimize energy injection as much as possible. Thus, we can translate the compressed Bloch sphere to the point of state $|1\rangle$. In this process, the compressed Bloch sphere and Bloch space are inscribed at the point of state $|1\rangle$. Also, the curvature radius of the compressed Bloch sphere at state $|1\rangle$ is always equal to the radius of the Bloch space (see Fig. 4). According to the law that the density matrix must be semipositive, we can get that for a definite Bloch vector $\vec{b} = (b_1, b_2, b_3)^T$, the CPTP map will

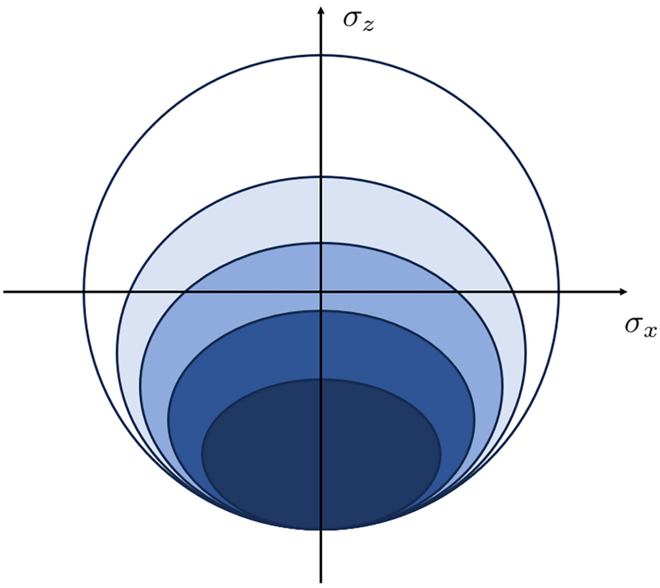

FIG. 4. Profile of the Bloch sphere on the xOz plane. This noisy channel form, which may extract energy from subsystem B, will compress the Bloch sphere to the state $|1\rangle$, whose Bloch vector is $(0, 0, -1)^T$. The color depth of the ellipsoids (compressed Bloch sphere) represents the noisy strength; the darker the color, the stronger the noisy strength. All the ellipsoids are inscribed on the point of $|1\rangle$, and on this point, all the ellipsoids have the same curvature radius as the Bloch space.

map $\vec{b}$ to $\vec{b}' = (\sqrt{\eta}b_1, \sqrt{\eta}b_2, -(1-\eta)+\eta b_3)^T$. This process can be represented as an amplitude damping class channel with Kraus operators

$$K_1 = \begin{pmatrix} \sqrt{1-p} & 0 \\ 0 & 1 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 0 & 0 \\ \sqrt{p} & 0 \end{pmatrix}, \tag{A8}$$

where $p = 1-\eta$ is the noisy strength. With this CPTP map, we can calculate the energy change of the system ΔE. If $\Delta E \geqslant 0$ in $p \in [0, 1]$, the state $|\psi\rangle$ will be a strong local passive state with local operation on subsystem B. Note that $\Delta E = 0$ when $p = 0$; thus we can calculate $\frac{\partial \Delta E}{\partial p}$ to find the minimal ΔE in different θ. Therefore, in the minimal QET model, as illustrated in the main text, if $\theta \in [\arctan\frac{h}{2k}, \frac{\pi}{2}) \bigcap [\frac{\pi}{4}, \frac{\pi}{2})$, the state $|\psi\rangle$ will be a strong local passive state.

APPENDIX B: MINIMAL-ENERGY STATE WITH MEASURED SUBSYSTEM *A*

From the above, we can get that in the minimal QET model, the correlation matrix of the strong local passive state has the form

$$T = \begin{pmatrix} 1 & 0 & 0 & b \\ 0 & v & 0 & 0 \\ 0 & 0 & m & 0 \\ a & 0 & 0 & n \end{pmatrix}. \tag{B1}$$

As is well known, the projector in the σ_i direction will erase the information in other directions. With $t_{10} = 0$, we have $p_{x|+1} = \frac{1}{2}$ and $p_{x|-1} = \frac{1}{2}$. The $p_{x|\pm 1}$ is the probability of projection results in the σ_x direction with positive and negative results.

Making the correlation matrix be T_+ and T_- in different results, we will have

$$T_+ = \begin{pmatrix} 1 & v & 0 & b \\ +1 & v & 0 & b \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad T_- = \begin{pmatrix} 1 & -v & 0 & b \\ -1 & v & 0 & -b \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{B2}$$

Noting that $t_{10}t_{01} = v$ after measuring subsystem A, the Bloch vector of subsystem B will be $\vec{\beta} = (\pm v, 0, b)^T$. Let the Bloch vector of the minimal energy state be $\vec{\beta}' = (X, 0, Z)$. The process of finding the minimal energy of the system can be seen as a mathematic programming as

$$\begin{aligned} &\text{find } E_B' = \min q = \pm 2kX + hZ \\ &\text{s.t. } X^2 + Z^2 = |\vec{\beta}|^2 \end{aligned} \tag{B3}$$

and

$$\frac{|E_B'|}{\sqrt{(\pm 2k)^2 + h^2}} = |\vec{\beta}| \tag{B4}$$

so that

$$E_B' = -|\vec{\beta}|\sqrt{4k^2 + h^2}. \tag{B5}$$

Here and in the following the signs $\pm$ depend on the measurement results ± 1.

Solving the equations

$$\begin{aligned} \pm 2kX + hZ - E_B' &= 0, \\ X^2 + Z^2 &= |\vec{\beta}|^2, \end{aligned} \tag{B6}$$

we have

$$\begin{aligned} X &= \mp\frac{2k|\vec{\beta}|}{\sqrt{4k^2 + h^2}}, \\ Z &= -\frac{h|\vec{\beta}|}{\sqrt{4k^2 + h^2}}. \end{aligned} \tag{B7}$$

So we can get the minimal-energy state

$$\rho_B = \tfrac{1}{2}(\sigma_0 + X\sigma_1 + Z\sigma_3). \tag{B8}$$

APPENDIX C: SPIN MODEL OF HYPERBOLIC QUANTUM NETWORKS

Ikeda studied the QET in the hyperbolic quantum network [30]. In quantum networks, people can deliver quantum resources instantaneously to the vast number of nodes. This hyperbolic quantum network is tiled with a $\{3, q\}$ tiling, which is tessellations of the plane consisting of regular triangles and each vertex is connected to q vertices. Generally, for the $\{p, q\}$ tiling, when $(p-2)(q-2) > 4$, the network is hyperbolic and otherwise Euclidean (see Fig. 5).

For the $\{3, q\}$ ($q \geqslant 6$) tiling, the Hamiltonian is

$$\begin{aligned} H_{Z,i} &= h\sigma_i^z \quad (i = 0, 1, \ldots, q), \\ H_{X,j} &= k\sigma_0^x \otimes \sigma_j^x \quad (j = 1, 2, \ldots, q), \\ H_{\text{hyp}} &= \sum_{i=0}^{q} H_{Z,i} + \sum_{j=1}^{q} H_{X,j}. \end{aligned} \tag{C1}$$

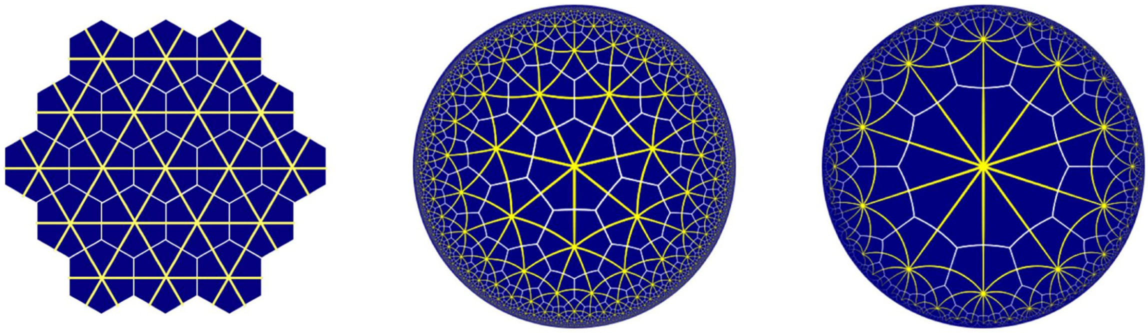

FIG. 5. Euclidean lattice with {3, 6} tiling and the hyperbolic lattice with {3, 7} and {3, 10} tiling [30]. We calculate the {3, 7} lattice unit (spin 8) as an example.

The ground state of the total Hamiltonian $H_{\rm hyp}$ is $|g\rangle$. In Ref. [30] the Hamiltonian is written as $H'_{Z,i} = H_{Z,i} + \epsilon_i$, $H'_{X,j} = H_{X,j} + \varepsilon_j$, and $H'_{\rm hyp} = \sum_{i=0}^{q} H'_{Z,i} + \sum_{j=1}^{q} H'_{X,j}$. Each of ϵ_i and ε_j makes

$$\langle g|H'_{\rm hyp}|g\rangle = \langle g|H'_i|g\rangle = \langle g|H'_j|g\rangle = 0. \tag{C2}$$

These ϵ_i and ε_j are constants and they do not influence the results of calculations.

Here we choose $q = 7$, $h = 9$, and $k = 2$. In this case, the density matrix is a 256×256 matrix. We will show the calculation process with the numerical method (keeping four significant digits after the decimal point), where $\epsilon_0 = -8.6079$, $\epsilon_j = -8.9398$, and $\varepsilon_j = -0.2290$. Projective measurement of the central particle ($i = 0$) in the σ_x direction with $P_0(x|\pm 1) = \frac{1}{2}(\mathbf{1} \pm \sigma_x)$, we can get the state of

$$\rho = \sum_{m=\{\pm 1\}} P_0(x|m)|g\rangle\langle g|P_0^\dagger(x|m). \tag{C3}$$

Then, from Ref. [30], the receivers at $i = 1, 2, \ldots, 6$ can extract energy by unitary operators $U_j(m)$ with the form

$$U_j(m) = \cos\theta\mathbf{1} - {\rm i}m\sin\theta\sigma^y \tag{C4}$$

and θ obey

$$\cos(2\theta) = \frac{\xi}{\sqrt{\xi^2 + \lambda^2}} = 0.9946,$$
$$\sin(2\theta) = -\frac{\lambda}{\sqrt{\xi^2 + \lambda^2}} = -0.1037, \tag{C5}$$

where

$$\xi = \langle g|\sigma_j^y H'_{\rm hyp}\sigma_j^y|g\rangle = 18.3377,$$
$$\lambda = \langle g|\sigma_i^x \dot{\sigma}_j^y|g\rangle = 1.9119, \tag{C6}$$

with $\dot{\sigma}_j^y = {\rm i}[H'_{\rm hyp}, \sigma_j^y] = {\rm i}[H'_j, \sigma_j^y]$. So the maximal average energy that a receiver gains is

$$-\Delta E_j = {\rm tr}(\rho_{\rm QET} H'_j) = 0.0497, \tag{C7}$$

where

$$\rho_{\rm QET} = \sum_{m=\{\pm 1\}} U_j(m)P_0(\sigma^x|m)|g\rangle\langle g|P_0^\dagger(\sigma^x|m)U_j^\dagger(m). \tag{C8}$$

For our method, ρ_{0j} is defined as the reduced density matrix of ρ with central and jth particles; we can get its Bloch vector $\vec{\beta}$ with length $|\vec{\beta}| = 0.999\,889\,415\,0$. So the minimal energy will be

$$E'_j = -|\vec{\beta}|\sqrt{k^2 + h^2} = -9.2185. \tag{C9}$$

The energy of ρ_j is

$$E_j = {\rm tr}(h\mathbf{1} \otimes \sigma^z\rho_{0j}) + {\rm tr}(k\sigma^x \otimes \sigma^x\rho_{0j}) = -9.1688 \tag{C10}$$

and the maximal average energy that a receiver gains is

$$-\Delta\hat{E}_j = E_j - E'_j = 0.0497. \tag{C11}$$

If we need U_j, we can calculate $\vec{\beta}$ and the minimal-energy state $\vec{\beta}'$ (see Appendix B) and then get the angle φ between them. Substituting this φ into Eq. (6), we obtain U_j. So we can find that $\Delta E_j = \Delta\hat{E}_j$, i.e., the two methods get the same theoretical result. Our method is easier to calculate and understand.

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