Erratum: Family of oscillatory electromagnetic pulses [Phys. Rev. A 108, 063502 (2023)]

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I have been made aware of an error in Sec. IV of my paper, relating to backflow.

I attempted to prove the absence of backflow in transverse electric (TE) and transverse magnetic (TM) pulses based on a general waveform. The electric and magnetic fields are calculated from a vector potential, which is the curl of the vector $[0, 0, \psi]$, where ψ is a solution of the wave equation $(\nabla^2 - \partial_{ct}^2)\psi = 0$.

In cylindrical coordinates ρ, ϕ, z the solutions of the wave equation may be expressed in the form

$$\psi(\rho, \phi, z, t) = e^{im\phi} \int_0^\infty dk\, e^{-ikct} \int_0^k dq\, w(k, q) e^{iqz} J_m(\kappa\rho), \qquad k^2 = q^2 + \kappa^2.$$

These scalar solutions are manifestly forward propagating since the longitudinal component of the wave vector q is non-negative. However, the vector electromagnetic fields are obtained from ψ by operations that include curl, and the curl of a vector has, in general, a different direction from the that of the original vector. It is not true in general that the momentum density $\boldsymbol{p}$, proportional to $\boldsymbol{E} \times \boldsymbol{B}$, has no regions with negative p_z (assuming net propagation in the positive z direction).

My attempted method of proof of positive-definite p_z was to examine the sign of the integrand in the z component of the momentum. The integral is over four wave-number variables k, q, k', q', and two angle variables $\chi,\ \chi'$ (which appear when the Bessel function is expressed as an integral over angle), but my "proof" considered the sign only in the dominant diagonal case $k' = k,\ \ q' = q,\ \ \chi' = \chi$.

In fact, there is a small amount of backflow. Figure 1 below shows the energy density and momentum density at $t = 0$ for a TE or TM pulse based on the imaginary part of the simplest waveform, $G = \frac{a^2}{R(R-iz)}$, $\ R^2 = \rho^2 + (a + ict)^2$. This solution of the wave equation is of the form given above, with $m = 0$ and $w = a^2 e^{-ka}$. The regions of negative p_z are those outside the hyperboloids of revolution $\rho^2 + a^2 = (1 + 2/\sqrt{3})z^2$, shown in the figure as red curves. The momentum density is largely transverse in these regions of backflow. The maximum negative p_z is located at the points indicated by the diamonds. It is about 2.4% of the maximum positive value located at $\rho = a\sqrt{2}/3, z = 0$. More details may be found in Appendix 3C of [1]; see also [2–4], for other examples of backflow.

Electromagnetic pulses localized in space-time may be Lorentz transformed, as discussed in Chapter 5 of [1]. Pulses with net propagation along the z direction always have $cP_z < U$ (P_z and U are the total pulse momentum along z and the total pulse energy) and a Lorentz boost along z with speed $c^2 P_z/U$ will bring the pulses to their zero-momentum frame, in which $P_z' = \int d^3r' p_z' = 0$. In the zero-momentum frame there are equal amounts of positive and negative momentum density, and net backflow equals net

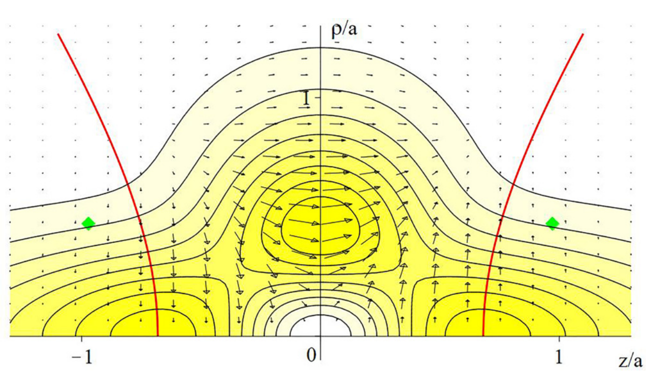

FIG. 1. Energy density (contours) and momentum density (arrows) for a TE or TM pulse based on the imaginary part of the waveform G, at $t = 0$. The red curves (hyperboloids of revolution, see text) are the boundaries of the backflow regions where $p_z < 0$. The location of the most negative p_z is indicated by the diamonds.

forward flow. In general, the amounts of forward and backward flow of momentum and energy (the momentum density and energy flux vectors $\boldsymbol{p}$ and $\boldsymbol{S}$ are related by $\boldsymbol{S} = c^2\boldsymbol{p}$) depends on which inertial frame the electromagnetic pulse is observed.

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