Generic quartic solitons in optical media

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Our analysis suggests strongly that stationary pulses exist in nonlinear media with second-, third-, and fourth-order dispersion. A theory, based on the variational approach, is developed for finding approximate parameters of such solitons. It is obtained that the soliton velocity in the retarded reference frame can be different from the inverse of the group velocity of linear waves. It is shown that the interaction of the pulse spectrum with that of linear waves can affect the existence of stationary solitons. These theoretical results are supported by numerical simulations. Transformations between solitons of different systems are derived. A generalization for solitons in media with the highest even-order dispersion is suggested.

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I. INTRODUCTION

A balance between second-order dispersion, or group velocity dispersion (GVD), and cubic (Kerr) nonlinearity results in a formation of optical solitons—stable pulses that propagate without a change of parameters [1]. Moreover, these pulses preserve their shapes and parameters after interactions with each other. Additional effects, such as higher-order dispersion, the Raman frequency shift, and self-steepening, change the parameters of solitons (see, e.g., Ref. [1]).

Recently, it was found that stable pulses exist also in media with quartic dispersion only and Kerr nonlinearity [2]. These pulses are called "pure quartic solitons" (PQS). Such solitons have been studied theoretically and experimentally in several papers (see, e.g., Refs. [2–9]). Stationary and dynamical properties of PQS were presented in Ref. [5]. In particular, it was shown that PQS have oscillating tails (see also Ref. [10]). A realization of a laser on PQS was suggested in Ref. [6]. The dynamics of cavity solitons in pure quartic media was considered in Refs. [4,7].

In the present paper, we consider general quartic media described by the second-, third- (TOD), and fourth-order dispersion (FOD) terms. We demonstrate that such media also admits the propagation of stable localized pulses. Since all dispersion terms are involved, we call these pulses "generic quartic solitons" (GQS) to distinguish them from PQS. The parameters of stationary solitons are found approximately, using the variational approach. Regions of the GQS existence in the space of the system parameters are obtained. A relation between moving solitons in general quartic media and pure quartic media is established. A generalization of results to higher-order dispersion is discussed.

II. MODEL AND STATIONARY SOLITONS

The dynamics of optical pulses in nonlinear dispersive media is described by the modified nonlinear Schrödinger (NLS) equation [1],

$$i\psi_z - \frac{\beta_2}{2}\psi_{\tau\tau} - i\frac{\beta_3}{6}\psi_{\tau\tau\tau} + \frac{\beta_4}{24}\psi_{\tau\tau\tau\tau} + \gamma|\psi|^2\psi = 0, \quad (1)$$

where $\psi(\tau, z)$ is the envelope of the electric field, τ is the time in the retarded frame, z is the propagation distance, β_j is the parameter of dispersion of the jth order, $j = 2$, 3, and 4, and γ is the Kerr nonlinearity parameter. We consider dispersion terms of up to the fourth order only. We mention that at $\beta_3 = \beta_4 = 0$, the standard NLS equation is completely integrable [11], and has the soliton solution. At $\beta_3 = 0$, there is also an exact soliton solution [12]. Soliton solutions of Eq. (1) for some sets of parameters β_j are found in Ref. [3]. These solutions have smooth, nonoscillating tails.

The influence of higher-order dispersion on the dynamics of solitons was studied intensively (see, e.g., Refs. [1–10,12–16]). Usually, two extreme cases are investigated. Namely, either TOD and FOD are treated as perturbation to the GVD effect [13–16], or the FOD effect is considered as a dominant one [2–10,12]. The former (latter) approach is valid far from (close to) zero dispersion points (ZDPs). In particular, the consideration of a system as a medium with pure quartic dispersion is only valid near a specific ZDP, where both GVD and TOD are negligible. In contrast to previous works, we make no assumptions on the values of GVD, TOD, and FOD effects. We show also that TOD does not result in the pulse asymmetry, if it acts together

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with FOD. Our results, based on the variational approach, indicate clearly that the joint action of GVD, TOD, and FOD can be balanced by Kerr nonlinearity, giving solitons with symmetric shapes. Numerical simulations of Eq. (1) support this conclusion.

The dispersion relation of linear waves, $\psi(\tau, z) \sim \exp\{i[\eta(\omega)z - \omega\tau]\}$, of Eq. (1) at $\gamma = 0$ has the following form:

$$\eta(\omega) = \frac{\beta_2}{2}\omega^2 + \frac{\beta_3}{6}\omega^3 + \frac{\beta_4}{24}\omega^4. \quad (2)$$

Then $\eta_1(\omega) \equiv d\eta(\omega)/d\omega = \beta_2\omega + \beta_3\omega^2/2 + \beta_4\omega^3/6$ and $\eta_2(\omega) \equiv d^2\eta(\omega)/d\omega^2 = \beta_2 + \beta_3\omega + \beta_4\omega^2/2$ are the inverse of the group velocity and the second dispersion parameter of linear waves at ω, respectively.

It is known [1] that in media with GVD only, $\beta_3 = \beta_4 = 0$, bright solitons do not exist when $\beta_2\gamma > 0$. Following this relation for dispersive media, when all β_j, $j = 2, 3$, and 4, are involved, one would expect that solitons do not exist when $\eta_2(b)\gamma > 0$, where b is the soliton frequency. Our theory gives different conditions (see below).

Equation (1) has the following Lagrangian density:

$$\mathcal{L} = \frac{i}{2}(\psi^*\psi_z - \psi\psi_z^*) + \frac{\beta_2}{2}|\psi_\tau|^2 + i\frac{\beta_3}{12}(\psi_\tau^*\psi_{\tau\tau} - \psi_\tau\psi_{\tau\tau}^*) + \frac{\beta_4}{24}|\psi_{\tau\tau}|^2 + \frac{\gamma}{2}|\psi|^4, \quad (3)$$

where the asterisk means the complex conjugation.

We use a trial function in the form of the Gaussian function:

$$\psi(\tau, z) = A\exp[-(\tau - \tau_c)^2/(2a^2)]e^{i[\phi - b(\tau-\tau_c) + c(\tau-\tau_c)^2]}. \quad (4)$$

Here, the parameters A, a, τ_c, b, c, and ϕ are the soliton amplitude, width, position of the center, linear phase parameter (the soliton frequency), chirp parameter, and phase parameter, respectively. All these parameters are assumed to be functions of z. The minus sign of a term proportional to b is taken for convenience. The actual form of solitons differs from Eq. (4). In particular, a soliton can have oscillating tails [5,10]. However, numerical simulations show that trial function (4) captures well the overall soliton shape, so the parameter values predicted are close to the actual ones.

The Lagrangian $L = \int_{-\infty}^{\infty} \mathcal{L} d\tau$ is expressed in terms of the pulse parameters, using trial function (4). The Euler-Lagrange equations for L give the following equations:

$$a' = -c\left[2\eta_2(b)a + \frac{\beta_4}{2}(a^{-1} + 4c^2a^3)\right], \quad (5)$$

$$c' = \frac{1}{2}\eta_2(b)(4c^2 - a^{-4}) - \frac{E_0\gamma}{2\sqrt{2\pi}}a^{-3} + \frac{\beta_4}{8}(-a^{-6} + 16c^4a^2), \quad (6)$$

$$\tau_c' = \eta_1(b) + \frac{\beta_3 + \beta_4 b}{4}(a^{-2} + 4c^2a^2), \quad (7)$$

$$\phi' = \frac{\beta_2}{2}(a^{-2} - b^2) + \frac{\beta_3}{12}(3ba^{-2} - 4b^3 - 12bc^2a^2) + \frac{\beta_4}{32}(3a^{-4} - 4b^4 + 8c^2 - 32b^2c^2a^2 - 16c^4a^4) + \frac{5\gamma E_0}{4\sqrt{2\pi}\,a}, \quad (8)$$

and $b' = 0$, where the prime denotes d/dz. The parameter $E_0 = \sqrt{\pi}A^2a = \sqrt{\pi}A^2(0)a(0)$ does not depend on z, and represents the initial energy of the pulse. Equations, similar to Eqs. (5)–(8), have been obtained previously (see, e.g., Refs. [15,16]). However, these equations were used mainly to analyze the influence of higher-order effects on the soliton of the unperturbed NLS equation. Pure quartic solitons have been studied by the same method in Ref. [9], but only for zero soliton frequency. Here, we are interested in the existence of stationary solitons in the presence of higher-order dispersion.

Equations (5) and (6) constitute a closed set because their right-hand sides, $f_a(a, b, c)$ and $f_c(a, b, c, E_0)$, do not depend on τ_c and ϕ. The right-hand sides of Eqs. (7) and (8) correspond to the soliton velocity $1/v$ in the retarded frame and the phase coefficient δ, respectively. Equations (5) and (6) have the invariant, which is the effective Hamiltonian of these equations:

$$H(a, c) = 16\beta_4a^4c^4 + 8c^2[\beta_4 + 4\eta_2(b)a^2] + 32\eta(b) + \beta_4a^{-4} + 8\eta_2(b)a^{-2} + \frac{16\gamma E_0}{\sqrt{2\pi}}a^{-1}. \quad (9)$$

Using $H(a, c) = H[a(0), c(0)]$, one can express variable c in terms of a, and substitute it into the equation for a', or a''. In the latter case, the equation for the soliton width describes the motion of a particle with coordinate a in an effective potential.

Stationary solutions are found from conditions $f_a(a, b, c) = 0$ and $f_c(a, b, c, E_0) = 0$. From Eqs. (5) and (6), it follows that stationary states exist only when $c = 0$. Then, the stationary soliton width $a_s > 0$ is determined from the following equation,

$$a^3 + s_1a^2 + s_2 = 0, \quad (10)$$

where $s_1 = \sqrt{2\pi}\,\eta_2(b)/(\gamma E_0)$, and $s_2 = \sqrt{2\pi}\,\beta_4/(4\gamma E_0)$. Applying the Sturm's theorem for the number of positive roots to Eq. (10), we obtain the following result:

(i) If ($s_1 > 0$ and $s_2 > 0$), or ($s_1 < 0$ and $s_2 > s_{2,\mathrm{th}}$), then Eq. (10) does not have positive roots, where $s_{2,\mathrm{th}} = -4s_1^3/27$.

(ii) For any s_1, if $s_2 < 0$, then Eq. (10) has one positive root.

(iii) If $s_1 < 0$ and $0 < s_2 < s_{2,\mathrm{th}}$, then Eq. (10) has two positive roots.

The first condition of case (i) is reduced to [$\eta_2(b)\gamma > 0$ and $\beta_4\gamma > 0$], cf. with the standard NLS equation. The second condition of case (i) indicates that solitons do not exist also for negative $\eta_2(b)\gamma$ and corresponding β_4. Solitons for parameters from case (iii) are mostly nonstationary due to the interaction with linear waves (see the corresponding discussion below). Also, notice that solitons exist for any sign of γ.

Though Eq. (10) can be solved analytically, this gives a complicated dependence of a_s on the system parameters. Therefore, it is useful to consider some limiting cases. First, we consider the case of small β_4, namely, if $\beta_4, \beta_4b^2a^2 \ll \hat{\beta} \equiv \beta_2 + \beta_3b$, then

$$a_s \approx -4\hat{\beta}/p - (a^2 + 32\hat{\beta}^2b^2)\beta_4/(16\hat{\beta}^2p), \quad (11)$$

where $p = 4\gamma E_0/\sqrt{2\pi}$. The second case is for small β_2, β_3, and b, namely, if $\beta_2 a^2$, $\beta_3 b a^2$, $\beta_4 b^2 a^2 \ll \beta_4$, then

$$a_s \approx (-\beta_4/p)^{1/3} - 4\eta_2(b)/(3p). \tag{12}$$

Therefore, for large $|\beta_4|$, solitons exist when $\beta_4\gamma < 0$. Having root a_s of Eq. (10), the stationary amplitude is found as $A_s = [E_0/(\sqrt{\pi}\, a_s)]^{1/2}$. Then (A_s, a_s, b) and $c = 0$, together with $1/v_s$ and δ_s, correspond to stationary parameters of a GQS.

The theory predicts that in the absence of β_3 and β_4, the stationary soliton velocity $1/v_s$ coincides with the inverse of the group velocity $\eta_1(b)$ of linear waves. The inclusion of higher-order dispersion breaks this relation [see Eq. (7)]. In particular, even at the extremum of the dispersion relation, at $b = 0$, we have solitons, moving due to β_3. This result is supported by numerical simulations of Eq. (1). A related observation is that a static soliton with $1/v_s = 0$ can have a phase dependence on time, $b \neq 0$. The difference of the soliton velocity $1/v_s$ from $\eta_1(b)$ can be used for slow light and fast light applications of solitons.

Equations (5)–(8) describe the adiabatic dynamics of a soliton. These equations do not take into account the interaction of the soliton with linear waves. However, this interaction can be accounted for qualitatively, using the following arguments. One can distinguish two different ways of generation of linear waves by solitons. When an initial pulse differs slightly from the stationary profile, the pulse adjusts its form to the stationary one, radiating the excess as linear waves. This adjustment is observed as damped oscillations of the soliton width and amplitude. Such a type of interaction is accounted for, to some extent, in Eqs. (5)–(8) by the inclusion of the chirp parameter c. Namely, if the chirp is absent in trial function (4), $c \equiv 0$, width a is constant on z, even when $a(0)$ is different from the stationary value. The variational approach with the chirp included treats the interaction with linear waves as a modulation of the soliton phase [17]. In contrast to the actual dynamics, the method gives undamped oscillations of the soliton shape, but predicts reasonably well the frequency.

The second type is due to the resonance interaction of a soliton with linear waves [13]. It occurs at frequencies where the soliton dispersion relation intersects with the dispersion relation of linear waves. The soliton dispersion relation is usually a straight line obtained from the following procedure. Let the stationary soliton has the form $\psi_s(\tau, z) = A_s f(\tau - z/v_s) \exp[i(\delta_s z - b\tau)]$ [see Eq. (4)], where real $f(\tau)$ describes the soliton profile. Then, the soliton spectrum $\Psi_s(\omega, z)$, obtained from the Fourier transform, is written as $\Psi_s(\omega, z) = A_s F(\omega - b) \exp\{i[\delta_s + (\omega - b)/v_s]z\}$, where $F(\omega)$ is the Fourier transform of $f(\tau)$. This expression indicates that the soliton dispersion relation, or the dependence of the soliton propagation constant $\eta_{\rm sol}(\omega)$ on frequency, is determined as the following:

$$\eta_{\rm sol}(\omega) = \delta_s + (\omega - b)/v_s. \tag{13}$$

The linear dependence (13) means that a soliton propagates without dispersion, $d^2\eta_{\rm sol}(\omega)/d\omega^2 = 0$, since it is balanced by nonlinearity. At frequencies ω_r, defined by

$$\eta_{\rm sol}(\omega_r) = \eta(\omega_r), \tag{14}$$

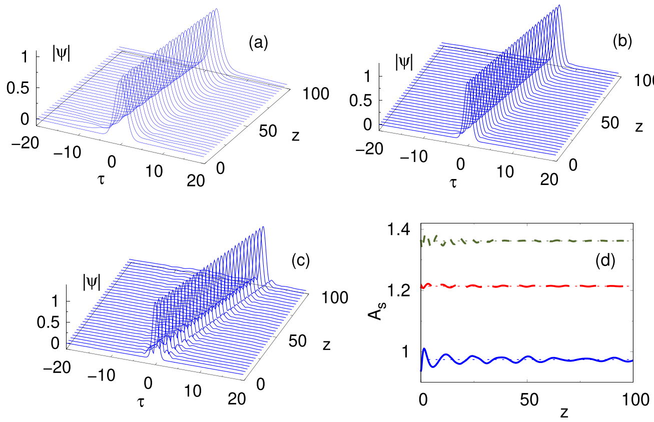

FIG. 1. (a)–(c) The dynamics of solitons for (a) $(\beta_2, \beta_3, \beta_4) = (-1, 0.2, -0.2)$, (b) $(-0.2, 0.2, -1)$, and (c) $(0.2, 0.2, -1)$. Other parameters are $\gamma = 1$ and $E_0 = 2$. (d) The dynamics of A_s on z for parameters from (a) (solid line), (b) (long-dashed line), and (c) (short-dashed line). Horizontal dotted lines correspond to values of stationary amplitudes predicted by the variational approach.

resonance linear waves are generated due to the phase-matching condition (see Refs. [13–16]). The rate at which the soliton energy goes to linear waves depends on the values of the soliton spectrum at these resonant frequencies. The arguments presented above are valid for media with an arbitrary order of dispersion. Resonance condition (14) can be obtained rigorously from the analysis of small modulations of the soliton (see, e.g., Refs. [13,15]).

In the absence of higher-order dispersion ($\beta_3 = \beta_4 = 0$), the parameter $1/v_s = \eta_1(b) = d\eta(b)/db$ [see Eq. (7)]. Therefore, for media with quadratic dependence only, $\eta_{\rm sol}(\omega)$ is a straight line that is parallel to the tangent to the dispersion relation of linear waves at frequency b, and shifted up by the amount depending on peak power A_s^2 [13,15,16]. In presence of higher-order terms ($\beta_3 \neq 0$, $\beta_4 \neq 0$), the soliton velocity $1/v_s$ differs from $\eta_1(b)$, therefore $\eta_{\rm sol}(\omega)$ is not parallel to the tangent [see Eq. (7)].

In media with quadratic and cubic dispersion terms ($\beta_4 = 0$), resonant linear waves are always generated because $\eta_{\rm sol}(\omega)$ intersects $\eta(\omega)$. Though the theory developed predicts the presence of solitons in media with cubic dispersion, these solitons are not stationary due to the continuous transfer of energy from solitons to linear waves. The lifetime of such solitons can be large if the resonant frequency is far from center b of the soliton spectrum. In contrast, when the quartic term is included, one can find a range of frequencies b, for which $\eta_{\rm sol}(\omega)$ does not intersect with $\eta(\omega)$. For example, such frequencies can be found near the extrema of $\eta(\omega)$.

The discussion above can be generalized with the following statement. Localized pulses are possible in nonlinear media with any order of dispersion. If the highest-order dispersion term is odd, then these pulses are nonstationary (quasistationary) due to the continuous radiation of linear waves. If the highest-order dispersion term is even, stationary stable pulses may exist. A necessary condition in the latter case is that the soliton dispersion relation does not intersect with the dispersion relation of linear waves.

We mention also the embedded solitons. The spectrum of these localized waves is located within the spectrum of linear waves (see, e.g., Refs. [18,19]). Embedded solitons appear mainly in multicomponent systems [19], though they also exist in scalar systems with cubic-quintic nonlinearity [18]. However, these solitons exist for particular relations of the system parameters. To the best of our knowledge, embedded solitons are not found for a system with cubic nonlinearity only. Extensive numerical simulations of Eq. (1) show that when the intersection of spectra occurs (with the soliton parameters found from the variational approach), no stationary solitons exist for $\beta_4\gamma > 0$.

To summarize, Eqs. (5)–(8) are valid when $\eta_{\rm sol}(\omega)$ does not intersect with $\eta(\omega)$. As it follows from Eq. (8), δ_s has a contribution $\sim\gamma E_0$. Line $\eta_{\rm sol}(\omega)$ is shifted up with the increase of E_0 for $\gamma > 0$. Then, for $\beta_4 > 0$ and large E_0, $\eta_{\rm sol}(\omega)$ most likely intersects with $\eta(\omega)$. Therefore, case $\beta_4 < 0$ is more favorable for the existence of stationary solitons, when $\gamma > 0$.

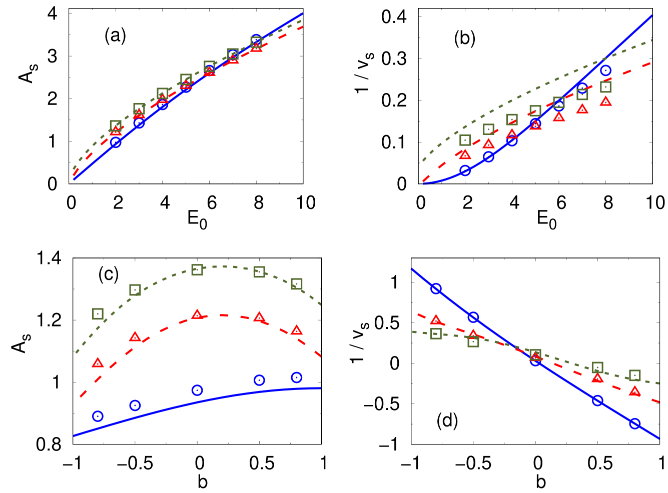

FIG. 2. Soliton parameters as functions of (a) and (b) E_0 at $b = 0$, and (c) and (d) b at $E_0 = 2$. Solid lines (circles): $(\beta_2, \beta_3, \beta_4) = (-1, 0.2, -0.2)$; dashed lines (triangles): $(-0.2, 0.2, -1)$; and dotted lines (squares): $(0.2, 0.2, -1)$. Points corresponds to parameters found from numerical simulations of Eq. (1).

III. NUMERICAL SIMULATIONS AND TRANSFORMATIONS

In order to check theoretical predictions, we perform numerical simulations of Eq. (1). For this purpose, we take all variables as dimensionless. We consider three cases: (i) small $|\beta_4|$, (ii) small $|\beta_2|$, and (iii) $\beta_2 > 0$. We take such values of parameters that there is no intersection of $\eta_{\rm sol}(\omega)$ and $\eta(\omega)$. The split-step Fourier method [1] is used. The size of the computational window is $T_{\rm num} = 30$–50, and the number of discretization points is 512–1024. Absorbing boundary conditions are used to prevent the reflection of linear waves from edges. Initial conditions are in the form of Eq. (4). A relatively small value of $\beta_3 = 0.2$ is taken for convenience to restrict the size of the computational window because $1/v$ grows with an increase of β_3 [see Eq. (7)]. Theoretical predictions have a similar accuracy for larger β_3 as well.

Figures 1(a)–1(c) show the dynamics of solitons for the three sets of parameters, and $E_0 = 2$. Since initial profiles are approximate, solitons adjust their shapes, emitting linear waves. In Fig. 1(c), the field at large z has oscillating tails. Though trial function (4) is different from this form, the theory gives acceptable values for stationary parameters with a deviation of 10%–20%, even for larger values of β_2 ($\geqslant 0.5$). In Fig. 1(d), variations of the soliton amplitudes for the dynamics in Figs. 1(a)–1(c) are presented. The amplitudes tend to stationary values via damped oscillations. Also, Fig. 1 demonstrates the stability of solitons to small modulations.

Dependencies of A_s and $1/v_s$ on E_0 and b are presented in Fig. 2. Soliton velocity $1/v_s$ is found as the average velocity over range $z \sim 20$–50 after the adjustment process. There are small deviations of the predicted values from those found from numerical simulations. However, the theory gives correctly the overall trend of all dependencies in Fig. 2. The soliton amplitude increases on E_0, and correspondingly, the soliton width a_s decreases on E_0. Contributions of the dispersion terms can be compared using the characteristic lengths [1] $L_{\rm GVD} = a_s^2/|\beta_2|$, $L_{\rm TOD} = a_s^3/|\beta_3|$, and $L_{\rm FOD} = a_s^4/|\beta_4|$. The smaller the length is, the more important is the contribution of the corresponding effect. Since a_s varies on E_0 and b, relative contributions of the dispersion terms are changed as well. For example, for $(\beta_2, \beta_3, \beta_4) = (-1, 0.2, -0.2)$, the corresponding lengths are $L_{\rm GVD} = 1.67$, $L_{\rm TOD} = 10.8$, and $L_{\rm FOD} = 13.9$ at $E_0 = 2$, while at $E_0 = 10$, $L_{\rm GVD} = 0.124$, $L_{\rm TOD} = 0.218$, and $L_{\rm FOD} = 0.0767$. Therefore, as E_0 increases, the influence of TOD and FOD increases as well.

Ratios of $|\beta_2|$ and $|\beta_4|$, as in Figs. 1 and 2, can be obtained in optical media at wavelengths close to zero dispersion points. We consider, as an example, the structure in Ref. [2]. To find values of β_3 and β_4, we fit the dependence $\beta_2(\lambda)$ in Fig. 1(d) of Ref. [2], therefore obtained values of β_3 and β_4 can be slightly different from the original values. At $\lambda = 1547.7$ nm, we get $\beta_2 = -2.7\ {\rm ps}^2\,{\rm mm}^{-1}$, $\beta_3 = -3.4\ {\rm ps}^3\,{\rm mm}^{-1}$, $\beta_4 = -0.40\ {\rm ps}^4\,{\rm mm}^{-1}$. Then $|\beta_2/\beta_4|a_s^2 = L_{\rm FOD}/L_{\rm GVD} \approx 5$ [see Fig. 1(a)], where $a_s = 0.85$ ps is found from Eq. (10) for $E_0 = 2$ pJ and $\gamma = 4.1 \times 10^3\ ({\rm W\,m})^{-1}$. At $\lambda = 1548.5$ nm, we get $\beta_2 = -0.79\ {\rm ps}^2\,{\rm mm}^{-1}$, $\beta_3 = -2.8\ {\rm ps}^3\,{\rm mm}^{-1}$, $\beta_4 = -1.5\ {\rm ps}^4\,{\rm mm}^{-1}$. Then $|\beta_2/\beta_4|a_s^2 \approx 0.2$ [see Fig. 1(b)], where $a_s = 0.58$ ps is found for the same E_0 and γ.

The standard NLS equation, i.e., Eq. (1) with $\beta_3 = \beta_4 = 0$, is invariant under the Galilean transformation. It means that a moving solution of the NLS equation can be obtained from a static solution by a corresponding change of variables. In contrast, the full Eq. (1) is not Galilean invariant. The shape of the soliton can be altered as the velocity changes. This property is ignored in trial function (4). Nevertheless, this function gives a reasonable approximation for solitons.

It is possible to establish relations between exact solutions with different velocities of the two related models. For a particular choice of β_2, namely $\beta_2 = \beta_3^2/(2\beta_4)$, solutions of Eq. (1) are related to solutions of the pure quartic NLS equation. Let $\psi(\tau, z)$ be a solution of Eq. (1), then $u(T, Z)$, defined from $\psi(\tau, z) = [u(T, Z)/g^2]\exp[i(KZ - \Omega T)]$, is a solution of the pure quartic NLS equation,

$$iu_Z + \frac{\beta_4}{24}u_{TTTT} + \gamma|u|^2u = 0, \tag{15}$$

where $T = \tau/g - z/(Vg^4)$ and $Z = z/g^4$, g is a free parameter, and

$$V = -6\beta_4^2/(\beta_3^3 g^3), \quad \beta_2 = \beta_3^2/(2\beta_4),$$
$$\Omega = -\beta_3 g/\beta_4, \quad K = -\beta_3^4 g^4/(24\beta_4^3). \tag{16}$$

Alternatively, if $u(T, Z)$ is a solution of Eq. (15), then $\psi(\tau, z)$, defined from $u(T, Z) = [\psi(\tau, z)/g^2]\exp[i(Kz - \Omega\tau)]$, is a solution of Eq. (1), provided that $\tau = T/g - Z/(Vg^4)$, $z = Z/g^4$, and

$$\beta_2 = \beta_4\Omega^2/2, \quad \beta_3 = \beta_4\Omega,$$
$$V = 6/(\beta_4\Omega^3), \quad K = -\beta_4\Omega^4/8, \tag{17}$$

where g and Ω are free parameters. Transformations (17) have been obtained also in Ref. [8]. Therefore, static and moving (in the retarded reference frame) solutions of Eq. (1) can be obtained from solutions of Eq. (15) that move, in general, with different velocities, and vice versa.

IV. CONCLUSIONS

In conclusion, we have demonstrated that stationary pulses, generic quartic solitons, can propagate in media with GVD, TOD, and FOD. Numerical simulations of Eq. (1) show that these pulses are stable for sufficiently long distances. Conditions in terms of the system parameters have been identified for the existence of GQS. In particular, these solitons exist both for the positive GVD and negative GVD parameters. The parameters of stationary solitons for different energies and soliton frequencies have been found approximately. The values of these parameters are close to those found numerically. It has been demonstrated that the soliton velocity in general quartic media differs, in principle, from the inverse of the group velocity of linear waves. It has been shown that the resonance interaction of a pulse with linear waves can prevent the existence of stationary solitons. Transformations, that connect solutions of Eq. (1) with those of Eq. (15), have been obtained. Our analysis provides strong support for a conjecture that stable solitons can exist in media with a general form of dispersion if the highest-order dispersion term is even.

Our results suggest an alternative view on the dynamics of pulses in dispersive nonlinear media, in particular, during supercontinuum generation. Usually, the dynamics is considered as a perturbation of solitons of the standard (with GVD only) NLS model. However, the dynamics can also be treated as an adjustment of pulses to stationary solitons associated with higher-order dispersion.

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[1] G. P. Agrawal, *Nonlinear Fiber Optics*, 4th ed. (Academic Press, 2007).
[2] A. Blanco-Redondo, C. M. de Sterke, J. E. Sipe, T. F. Krauss, B. J. Eggleton, and C. Husko, Pure-quartic solitons, Nat. Commun. **7**, 10427 (2016).
[3] V. I. Kruglov and J. D. Harvey, Solitary waves in optical fibers governed by higher-order dispersion, Phys. Rev. A **98**, 063811 (2018).
[4] H. Taheri and A. B. Matsko, Quartic dissipative solitons in optical Kerr cavities, Opt. Lett. **44**, 3086 (2019).
[5] K. K. K. Tam, T. J. Alexander, A. Blanco-Redondo, and C. M. de Sterke, Stationary and dynamical properties of pure-quartic solitons, Opt. Lett. **44**, 3306 (2019).
[6] A. F. J. Runge, D. D. Hudson, K. K. K. Tam, C. M. de Sterke, and A. Blanco-Redondo, The pure-quartic soliton laser, Nat. Photonics **14**, 492 (2020).
[7] O. Melchert, A. Yulin, and A. Demircan, Dynamics of localized dissipative structures in a generalized Lugiato-Lefever model with negative quartic group-velocity dispersion, Opt. Lett. **45**, 2764 (2020).
[8] J. Widjaja, E. Kobakhidze, T. R. Cartwright, J. P. Lourdesamy, A. F. J. Runge, T. J. Alexander, and C. M. de Sterke, Absence of Galilean invariance for pure-quartic solitons, Phys. Rev. A **104**, 043526 (2021).
[9] Y. Li, J. Dai, J. Liu, Z. Weng, W. Hu, and D. Lu, Evolution of pure-quartic solitons studied by the variational approach, Opt. Commun. **524**, 128790 (2022).
[10] N. Akhmediev, A. Buryak, and M. Karlsson, Radiationless optical solitons with oscillating tails, Opt. Commun. **110**, 540 (1994).
[11] V. E. Zakharov and A. B. Shabat, Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media, Sov. Phys. JETP **34**, 62 (1972).
[12] M. Karlsson and A. Hook, Soliton-like pulses governed by fourth order dispersion in optical fibers, Opt. Commun. **104**, 303 (1994).
[13] N. Akhmediev and M. Karlsson, Cherenkov radiation emitted by solitons in optical fibers, Phys. Rev. A **51**, 2602 (1995).
[14] V. E. Zakharov and E. A. Kuznetsov, Optical solitons and quasisolitons, J. Exp. Theor. Phys. **86**, 1035 (1998).
[15] F. Biancalana, D. V. Skryabin, and A. V. Yulin, Theory of the soliton self-frequency shift compensation by the resonant radiation in photonic crystal fibers, Phys. Rev. E **70**, 016615 (2004).
[16] E. N. Tsoy and C. M. de Sterke, Theoretical analysis of the self-frequency shift near zero-dispersion points: Soliton spectral tunneling, Phys. Rev. A **76**, 043804 (2007).
[17] D. Anderson, Variational approach to nonlinear pulse propagation in optical fibers, Phys. Rev. A **27**, 3135 (1983).
[18] J. Fujioka and A. Espinosa, Soliton-like solution of an extended NLS equation existing in resonance with linear dispersive waves, J. Phys. Soc. Jpn. **66**, 2601 (1997).
[19] J. Yang, B. A. Malomed, D. J. Kaup, and A. R. Champneys, Embedded solitons: A new type of solitary wave, Math. Comput. Simul. **56**, 585 (2001).