Quantum metrology with a squeezed Kerr oscillator

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We study the squeezing dynamics in a Kerr-nonlinear oscillator, and quantify the metrological usefulness of the resulting states. Even if the nonlinearity limits the attainable squeezing by making the evolution non-Gaussian, the states obtained still have a high quantum Fisher information for sensing displacements. However, contrary to the Gaussian case, the amplitude of the displacement cannot be estimated by simple quadrature measurements. Therefore, we propose the use of a measurement-after-interaction protocol where a linear quadrature measurement is preceded by an additional nonlinear evolution, and show the significant sensitivity enhancement that can be obtained. Our results are robust when considering realistic imperfections such as energy relaxation, and can be implemented in state-of-the-art experimental setups.

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I. INTRODUCTION

Continuous variable (CV) quantum systems, such as optical fields or mechanical oscillators, constitute a platform of primary importance for quantum metrology applications. Examples include gravitational wave detection [1–4], force sensing [5,6], and the measurement of electric and magnetic fields [7]. Largely explored in this context are squeezed states, namely Gaussian states that show along some phase-space quadrature an uncertainty that is below the quantum noise of the vacuum. Besides being relatively easy to prepare experimentally, for example through a parametric process [8,9], such states are also easy to measure, as they can be fully characterized by linear quadrature (i.e., homodyne) measurements.

One of the experimental factors limiting squeezing are the inevitable nonlinearities present in the system. In fact, highly squeezed states have a large average number of excitations (i.e., energy), as well as wave functions significantly distributed in phase space, which is manifested by the antisqueezed quadrature. When these go beyond the linear regime of the considered experimental platform, the state evolution become non-Gaussian, and squeezing gets degraded by a "wrapping around" of the state [10,11]. Similar results are found in the context of "crescent states," namely coherent states undergoing Kerr evolution [12–15]. This hinders metrological applications of the resulting state, despite the fact that non-Gaussian states can still have high sensitivity to perturbations [16–21].

One of the main difficulties in doing quantum-enhanced metrology with non-Gaussian states lies in the fact that the parameter to be retrieved is encoded in high-order correlators [22]. This requires us to access high moments of the measurement's probability distribution, or equivalently to perform non-Gaussian (e.g., number-resolving) measurements, which is of challenging experimental implementation [23–27]. In addition, noise constraints for these observables become also very stringent. Therefore, nonlinearities are typically seen as limitations, and in experiments one tries to reduce them as much as possible.

A paradigmatic model in which to study the interplay between squeezing and nonlinear interactions is given by the Hamiltonian of a Kerr oscillator with a squeezing drive:

$$\hat{H} = \Delta \hat{a}^\dagger \hat{a} + \epsilon(\hat{a}^{\dagger 2} + \hat{a}^2) - K\hat{a}^{\dagger 2}\hat{a}^2. \tag{1}$$

Here Δ is the detuning between oscillator and drive, ϵ is the squeezing rate, and K is the Kerr nonlinearity. Besides

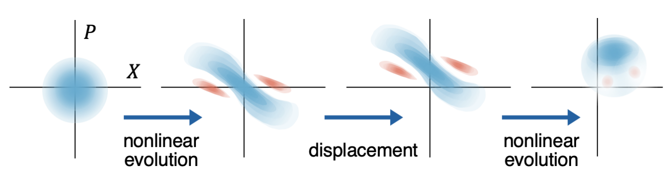

FIG. 1. Illustration of the proposed metrological protocol. Squeezing the ground state of a nonlinear oscillator results in a non-Gaussian state that might not have a quadrature with reduced quantum noise. Despite this, the state can still have high sensitivity to displacements, even if the information is hidden in high-order moments of quadrature operators. The method we propose to access the displacement amplitude consists instead in undergoing a second nonlinear evolution, which allows us to keep the measurement linear.

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being interesting for the study of interesting processes such as chaotic dynamics [28], tunneling [29], coherent superpositions [30,31], and phase transitions and blockade effects [32,33], Hamiltonian (1) also attracted significant attention in the context of quantum information processing, since its ground state is a Schrödinger cat state that can be exploited for error-protected qubit encoding [34–36]. This observation motivated the recent experimental implementation of Eq. (1) for electromagnetic modes with superconductive devices [37,38], and in mechanical modes with acoustic resonators [39].

Here we study the use of a squeezed Kerr oscillator for quantum metrology, and show that when sufficient control over the system's parameters is available, the presence of a nonlinearity can significantly improve the metrological performances even for simple quadrature measurements. The idea relies on preparing non-Gaussian states that, even when not showing reduced uncertainty compared to vacuum, can have high sensitivity to displacements. This can be then accessed by preceding the measurement step by an additional nonlinear evolution of the state (see Fig. 1), which we show to result in an effective measurement of higher-order moments of the quadratures. To conclude, we show that our results are robust to noise, and of immediate implementation in electrical and mechanical systems.

II. SQUEEZING LIMITS

Let us begin with considering the task of preparing squeezed vacuum states with Eq. (1), and investigate the limitations posed by the nonlinear term.

We imagine a protocol where a system that is initially in the ground state of the harmonic oscillator $|0\rangle$ evolves for $t \geqslant 0$ according to Eq. (1), due to the application of a parametric drive. During this dynamics, we are interested in studying the evolution of the state's minimum uncertainty quadrature, namely of $V_{\min}(t) \equiv \min_\theta \mathrm{Var}[(\hat{a}e^{-i\theta} + \hat{a}^\dagger e^{i\theta})/\sqrt{2}]$, with $\mathrm{Var}[\hat{A}] = \langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2$ the variance of A. Since for coherent states $V_{\min} = 1/2$, observing $V_{\min} < 1/2$ implies a reduction of the quantum noise below the classical limit, and thus implies that the state is squeezed.

For the dynamics we consider, there is in general no known analytic closed-form expression for $V_{\min}$, which therefore has to be computed numerically. We show in Fig. 2(a) the plot of $V_{\min}(t)$ for $(\Delta, \epsilon) = (0, 2)$ and different values of K. For $K = 0$ it is known that $V_{\min}(t) = \frac{1}{2} e^{-4\epsilon t}$, meaning that an arbitrarily small uncertainty is achievable for sufficiently long times. For $K \neq 0$, however, we observe that $V_{\min}$ attains a minimum value at a finite time t_{opt}. This is expected, as the nonlinearity results in a non-Gaussian evolution of the state which limits the achievable squeezing [10,11]. We thus define the parameter $\chi_{\text{opt}}^{-2} = 1/V_{\min}(t_{\text{opt}})$, which we will later show to be related to the state sensitivity, and show in Figs. 2(b) and 2(c) the dependence of χ_{opt}^{-2} and t_{opt} on ϵ/K, now also for different values of Δ. Note that higher squeezing can be prepared in a shorter time as ϵ/K increases, since the effect of the nonlinearity gets relatively less important.

Since squeezed states have a quadrature with reduced uncertainty, they can provide an advantage in metrological tasks [40–42]. For this reason, it may look as if the best strategy to achieve a larger advantage is to have ϵ/K as large as possible

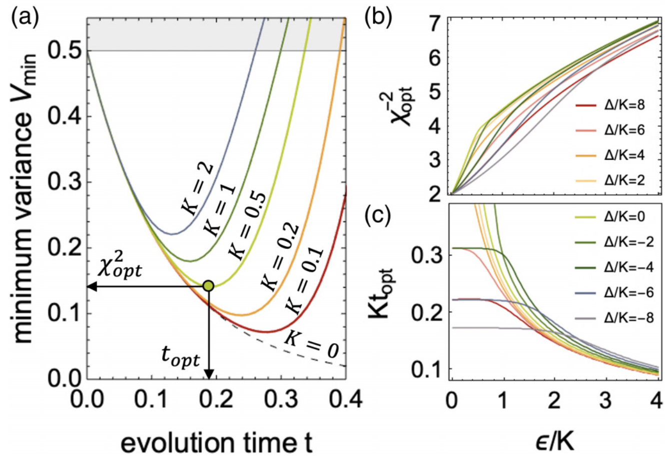

FIG. 2. Optimal squeezing of a Kerr oscillator. (a) Minimum variance of the state $e^{-i\hat{H}t}|0\rangle$, fixing $\Delta = 0$. For $K > 0$ there is an optimal squeezing point, that we further investigate also as a function of Δ. (b) Squeezing level at the optimal point. (c) Optimal squeezing time.

and stop the state preparation at t_{opt}, since longer evolution times degrade $V_{\min}$. However, as we will now show, this is not necessarily true.

III. METROLOGICAL ADVANTAGE OF NON-GAUSSIAN STATES

Let us remember that, in a typical quantum metrology scheme, the task is to estimate a parameter d that is encoded in a probe state ρ by a generator $\hat{G}$ as $\rho_d = e^{-id\hat{G}} \rho e^{id\hat{G}}$. A fundamental limit to the sensitivity is provided by the so-called quantum Cramér-Rao bound $\Delta^2 d \geqslant \Delta^2 d_{\text{QCR}} \equiv (F_Q[\rho, \hat{G}])^{-1}$, where $F_Q[\rho, \hat{G}]$ is the quantum Fisher information (QFI). For a pure state the QFI is calculated from the variance of the generator as $F_Q[\rho, \hat{G}] = 4\mathrm{Var}[\hat{G}]$. Importantly, to achieve the maximum sensitivity $(\Delta^2 d_{\text{QCR}})^{-1}$ it is necessary to optimize the measurement that is performed on ρ_d in order to estimate d. In general, if one measures $\hat{M}$ then the achieved sensitivity is [22]

$$(\Delta^2 d)^{-1} = \chi^{-2}[\rho, \hat{G}, \hat{M}] \equiv \frac{|\langle [\hat{G}, \hat{M}] \rangle|^2}{\mathrm{Var}[\hat{M}]}, \tag{2}$$

which satisfies $\chi^{-2}[\rho, \hat{G}, \hat{M}] \leqslant F_Q[\rho, \hat{G}]$ [43].

To now understand the connection between (2) and squeezing, let us consider the task of sensing the amplitude of a displacement from the measurement of a phase-space quadrature. We thus have $\hat{G}(\phi) = (\hat{a}e^{-i\phi} + \hat{a}^\dagger e^{i\phi})/\sqrt{2}$, the generator of a displacement along direction $\phi + \pi/2$, and $\hat{M}(\theta) = (\hat{a}e^{-i\theta} + \hat{a}^\dagger e^{i\theta})/\sqrt{2}$, the measurement along direction θ. Looking at the numerator of Eq. (2), we note that the sensitivity is highest when $\hat{M}$ is perpendicular to $\hat{G}$, meaning when $\hat{M}$ is along the displacement direction, as we would expect. In this case we obtain $\chi^{-2}[\rho, \hat{G}(\theta + \pi/2), \hat{M}(\theta)] = 1/\mathrm{Var}[(\hat{a}e^{-i\theta} + \hat{a}^\dagger e^{i\theta})/\sqrt{2}]$, and by further optimizing over the measurement direction θ we have $\chi^{-2} \equiv \max_{\phi,\theta} \chi^{-2}[\rho, \hat{G}(\phi), \hat{M}(\theta)] = 1/V_{\min}$.

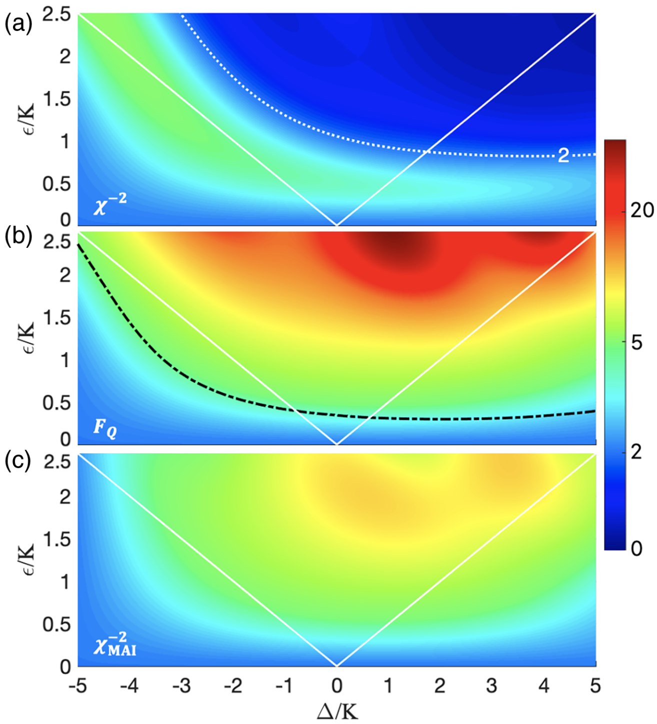

FIG. 3. Sensitivity comparison for the state $e^{-i\hat{H}t}|0\rangle$ with $Kt = 0.5$. (a) Squeezing χ^{-2}, where the white dotted lines represent the standard quantum limit $\chi^{-2}_{\text{SQL}} = 2$. (b) QFI F_Q, where the black dot-dashed line is a boundary $(F_Q - \chi^{-2})/F_Q = 0.05$. (c) Sensitivity for the MAI method χ^{-2}_{MAI}. White solid lines are for $\epsilon = |\Delta/2|$, and indicate the classical phase diagram for a squeezed Kerr oscillator [38].

These results shows us two things. First, for sensing displacements from quadrature measurements then $V_{\min}$ (i.e., the squeezing) is the correct figure of merit that needs to be optimized, but if the displacement is estimated from another type of measurement this might not be the case. Second, depending on the state ρ we are considering then the choice of performing a quadrature measurement might not be the optimal one saturating the quantum Cramér-Rao bound, and thus not achieving $\chi^{-2}[\rho, \hat{G}, \hat{M}] = F_Q[\rho, \hat{G}]$. This measurement choice is however proven to be optimal for sensing displacements with Gaussian states [see Secs. I–III of the Supplemental Material (SM) [44]].

To illustrate this last point for the scenario introduced in the previous section, we compute squeezing and QFI of the states $\rho = e^{-i\hat{H}t}|0\rangle$ prepared through Eq. (1) at $Kt = 0.5$. We plot in Figs. 3(a) and 3(b) the quantities $\chi^{-2} = 1/V_{\min}$ and $F_Q \equiv \max_\phi F_Q[\rho, \hat{G}(\phi)] = \max_\phi 4\text{Var}[\hat{G}(\phi)]$, respectively, for different values of Δ/K and ϵ/K. Figure 3(a) shows that when linear quadrature measurements are performed, then a quantum-enhanced sensitivity, i.e., $\chi^{-2} > 2$, is attained only for a limited set of states. In general, one can also have $\chi^{-2} < 2$, which indicates even worse sensitivity than the one achieved by coherent states [see also Fig. 2(a) for $t > 0.25$, when the colored lines show $V_{\min} > 1/2$]. When this is the case, since here we are dealing with pure states, it necessarily implies that the state is non-Gaussian. On the other hand, Fig. 3(b) shows that any state (besides the trivial case $\epsilon/K = 0$) has $F_Q > 2$ and can thus show a quantum-enhanced sensitivity to displacements if the correct measurement is performed.

Even if, strictly speaking, linear quadrature measurements are never optimal for $\epsilon/K > 0$, there are regimes in which they are very close to being optimal. This is indicated by the region below the black line in Fig. 3(b), which corresponds to states for which $(F_Q - \chi^{-2})/F_Q \leqslant 0.05$, and thus to states that are faithfully approximated by being Gaussian. Outside this region, different measurements are required to approach the maximum sensitivity, which is what we want to investigate in the next sections by considering two strategies that are experimentally relevant.

IV. NONLINEAR MEASUREMENTS

The first strategy consists of measuring higher-order moments of phase-space quadratures, and it can be studied systematically in the following way [22]. We define $\mathbf{M}^{(k)}$ to be a vector involving up to kth-order moments of the quadrature operators $\hat{X} = (\hat{a} + \hat{a}^\dagger)/\sqrt{2}$ and $\hat{P} = -i(\hat{a} - \hat{a}^\dagger)/\sqrt{2}$, such that, e.g., linear quadrature measurements are described by $\mathbf{M}^{(1)} = (\hat{X}, \hat{P})$, while second-order ones are described by $\mathbf{M}^{(2)} = (\hat{X}, \hat{P}, \hat{X}^2, \hat{P}^2, (\hat{X}\hat{P} + \hat{P}\hat{X})/2)$. Then, we introduce the nonlinear squeezing parameter as [22]

$$\chi^{-2}_{(k)} \equiv \max_{\phi, \vec{m}} \chi^{-2}[\rho, \hat{G}(\phi), \vec{m} \cdot \mathbf{M}^{(k)}]. \tag{3}$$

Crucially, this optimization task can be cast into an eigenvalue problem of easy solution (see Sec. I of the SM [44]). With higher-order moments involved, these parameters are capable of revealing quantum-enhanced sensitivities in a wider class of states, even beyond the Gaussian regime. Moreover, the hierarchy $\chi^{-2}_{(1)} \leqslant \chi^{-2}_{(2)} \leqslant \cdots \leqslant F_Q$ holds, showing that for sufficiently high k one can attain the maximum sensitivity.

In squeezed Kerr oscillators the nonlinearity can result in highly non-Gaussian states, whose metrological advantage is unlocked only for measurements of sufficiently high order. Observing $\chi^{-2}_{(1)} = \chi^{-2} < 2$ implies that linear quadrature measurements are not sufficient, and thus that $k > 1$ is necessary. Based on this observation, we computed $\chi^{-2}_{(2)}$ but, perhaps surprisingly, did not find any advantage compared to $\chi^{-2}_{(1)}$. A careful exploration of the terms involved shows that this is due to the fact that commutators between the generator and second moments of the quadrature give terms linear in the quadrature, whose expectation value is zero for the states we consider (see Sec. I of the SM [44]). This means that, in our scenario, considering $\mathbf{M}^{(2)}$ does not provide any advantage compared to $\mathbf{M}^{(1)}$. In order to see an advantage one would need to consider at least $\mathbf{M}^{(3)}$, which requires a massive increase in the measurement statistics and low detection noise. This can be of impractical implementation in several experimental situations, and it is thus viable to consider also alternative strategies.

V. MEASUREMENT-AFTER-INTERACTION PROTOCOL

The second approach we consider consists of preceding a linear quadrature measurement by a time-reversed evolution with Eq. (1), i.e., $e^{+i\hat{H}t}$. A similar idea has been investigated for spin states both theoretically [46–51] and experimentally

[52,53]. For CV systems, an experiment with Gaussian states and transformations has been presented in Ref. [54]. In this framework, reversing the evolution results in an amplification of the signal to be detected, but also of the quantum noise (see Fig. 1 of Refs. [47,54]). For this reason, an advantage is obtained only in the presence of detection noise limiting the measurement resolution, as the ratio between the signal and the quantum noise level remains constant (see Secs. IV and V of the SM [44]). In fact, linear quadrature measurements are already optimal for sensing displacements with Gaussian states, in the sense that they achieve $\chi^{-2} = F_Q$ (see Sec. III of the SM).

In the following we will show that the measurement-after-interaction (MAI) protocol can provide tremendous sensitivity enhancements when considering CV non-Gaussian states and transformations. Necessary conditions for this protocol to be viable are (i) the capability of implementing a time-reversed evolution and (ii) low enough noise to tolerate a second time evolution of the state. For point (i), it is sufficient to be able to invert the sign of the parameters in the Hamiltonian. In the specific case of Eq. (1), Δ and ϵ are easily tuned by the parametric drives, while K can be tuned by changing the anharmonicity of a trapping potential (e.g., in the case of a trapped ion) or the coupling between the bosonic mode and a two-level system (e.g., in a circuit-QED setup).

Formally, the additional evolution that precedes the measurement can be absorbed into a redefinition of the measurement operator. In our case we have $\hat{M}_{\mathrm{MAI}}(\theta) = \hat{U}(ae^{-i\theta} + a^\dagger e^{i\theta})\hat{U}^\dagger/\sqrt{2}$, where $\hat{U} = e^{-iHt}$, from which we define

$$\chi_{\mathrm{MAI}}^{-2} \equiv \max_{\phi,\theta} \chi^{-2}[\rho, \hat{G}(\phi), \hat{M}_{\mathrm{MAI}}(\theta)]. \tag{4}$$

We plot this parameter in Fig. 3(c) for the states $e^{-iHt}|0\rangle$ prepared at $Kt = 0.5$. From a comparison with the F_Q shown in Fig. 3(b) we are able to conclude that the MAI protocol can attain a sensitivity close to optimal. In particular, this is true also for the non-Gaussian regime, where by looking at Fig. 3(a) we see that by performing linear quadrature without the time-reversed dynamics one would only get $\chi^{-2} < 2$. A more quantitative comparison will be discussed later in Fig. 4, while an analysis of the scaling with N can be found in Sec. VI of the SM [44].

VI. ROBUSTNESS TO LOSSES

To show that the MAI protocol gives an actual advantage in realistic scenarios we have to consider the effect of experimental imperfections. During both state preparation and time-reversal dynamics the evolution of the system is affected by inevitable losses and decoherence, which ultimately limit the maximum duration of the protocol. If these are too severe compared to the robustness of the MAI protocol, then no advantage is obtained.

To study this in detail, in our numerical simulations we replace the unitary time evolutions $\hat{U}$ and $\hat{U}^\dagger$ by evolutions according to a master equation. We focus on losses (i.e., energy relaxation), which pose a major limitation in CV systems. These can be described by a jump operator $\sqrt{\gamma}\hat{a}$, where $\gamma = 1/T_1$ is the energy relaxation rate.

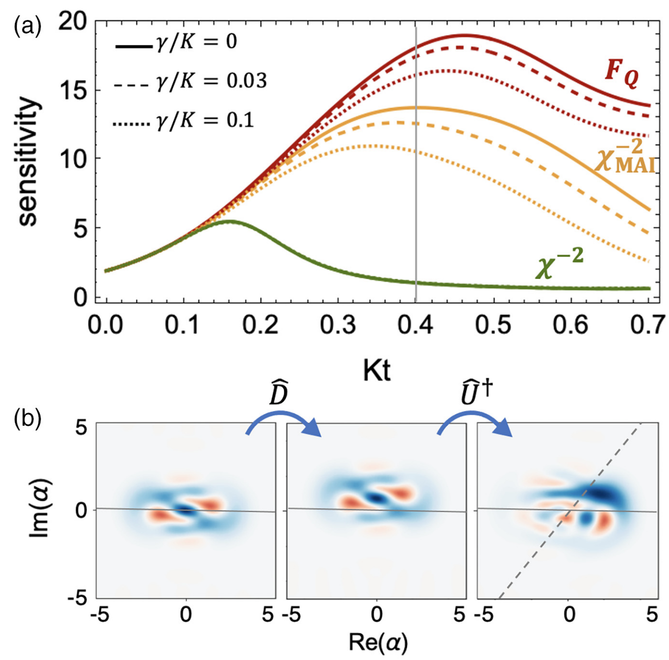

FIG. 4. Noise robustness. (a) Sensitivities F_Q, χ_{MAI}^{-2}, and χ^{-2} obtained for $\Delta/K = 0$, $\epsilon/K = 2$, and different levels of energy relaxation rates γ/K. (b) Wigner functions at different steps of the MAI protocol for $Kt = 0.4$ and $\gamma/K = 0.1$. From left to right: Prepared state, displacement, and time-reversed nonlinear evolution. Solid and dashed lines are optimal directions of the generator and linear quadrature measurement, respectively.

Since here we are dealing with mixed states and nonunitary evolutions, calculating the sensitivity becomes more tedious. The QFI for a general state ρ is computed as $F_Q[\rho, \hat{G}] = \sum_{kl} \frac{(\lambda_k-\lambda_l)^2}{(\lambda_k+\lambda_l)} |\langle k|\hat{G}|l\rangle|^2$, where λ_k and $|k\rangle$ are the eigenvalues and eigenstates of ρ, respectively. To calculate χ^{-2} and χ_{MAI}^{-2} we use the fact that $|\langle[\hat{G}, \hat{M}]\rangle|^2 = |\frac{\partial}{\partial d}\mathrm{Tr}[\hat{M}\rho_d]|_{d=0}^2$, where $\rho_d = e^{-id\hat{G}}\rho e^{id\hat{G}}$, and then discretize the derivative numerically by applying a small displacement to the state.

We show a comparison of the sensitivities in Fig. 4(a), for $\Delta/K = 0$, $\epsilon/K = 2$, and different loss rates γ/K (even stronger than the one observed in experiments [38]). The sensitivity χ^{-2} obtained from linear quadrature measurements is almost unaffected by losses, but as we have seen it approaches F_Q only for small times ($Kt \approx 0.1$). On the other hand, the sensitivity χ_{MAI}^{-2} obtained from the MAI protocol is always significantly larger than χ^{-2}, and it approaches F_Q for a longer time interval ($Kt \approx 0.25$). We thus conclude that the MAI protocol is robust, in the sense that a large amount of losses is required before having the maximum of χ_{MAI}^{-2} smaller than the maximum of χ^{-2} (see also Sec. VII of the SM [44]). Let us emphasize that the observed performance of the MAI protocol is remarkable, considering that doubling the time evolution significantly increases losses.

To have a better understanding of the MAI protocol, we plot in Fig. 4(b) the Wigner function of the state at different steps. First, a non-Gaussian state is prepared by evolving $|0\rangle$ according to Eq. (1) with $\Delta/K = 0$, $\epsilon/K = 2$, and $Kt = 0.4$. Then, the state is subject to the displacement we want to sense, followed by an evolution with $-\hat{H}$ for another $Kt = 0.4$. Finally, a linear quadrature measurement is performed on the state to estimate the displacement amplitude. Solid and dashed lines in Fig. 4(b) indicate the optimal generator and measurement directions, respectively.

VII. CONCLUSIONS

We addressed the problem of optimally using a squeezed Kerr oscillator for a metrological task. In particular, we focus on sensing displacements with the non-Gaussian states that result from the nonlinear evolution of this system, and investigate measurement strategies to approach the highest sensitivity. We show that, while a direct quadrature measurement requires us to access high-order moments, preceding the measurement by a time-reversed nonlinear evolution allows us to achieve high sensitivities even for first moments. Crucially, the protocol we propose is robust to noise, and can be implemented in current experiments for quantum-enhanced sensing of, e.g., electromagnetic fields [37,38] or forces [39]. Future works could address the interesting problem of multiparameter estimation for entangled nonlinear oscillators [55–58].

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