


Coherence quantifier based on the quantum optimal transport cost

Xian Shi *

College of Information Science and Technology, Beijing University of Chemical Technology, Beijing 100029, China



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In this work, we present a coherence measure based on the quantum optimal transport cost in terms of the convex roof extended method. We also obtain the analytical solutions of the quantifier for pure states. We also propose an operational interpretation of the coherence measure for pure states.

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I. INTRODUCTION

Quantum coherence is one of the crucial resources in quantum information processing [1]. It springs from the state superposition principle, which distinguishes quantum information from classical high information. Coherence has high relations with other quantum resources, such as quantum entanglement [2], quantum nonlocality [3], and so on. Coherence also plays crucial roles in quantum algorithms [4–9], thermodynamical systems [10–12], transport theory [13], and biology [14,15].

One of the essential problems in quantum information theory is how to quantify coherence. In 2014, Baumgratz *et al.* proposed a framework with four postulates for a proper coherence measure [16]; the authors also presented proper coherence measures, the l_1 norm of coherence and the relative entropy of coherence. In 2015, Du *et al.* proposed a method to construct coherence measures and considered the optimal conversion for coherent states [17]. In 2016, Napoli *et al.* proposed the robustness of coherence and provided an operational interpretation of the coherence measure [18]. Subsequently, some coherence measures were offered based on the trace norm distance [19], the Tsallis q entropy [20], Fisher information [21,22], and so on. In addition, Yu and Yu proposed an alternative framework for quantifying coherence, and they also showed that the coherence measure in terms of 1-norm is improper based on the framework they proposed in Ref. [22]. Next, due to the relationship between coherence and entanglement, the approaches to building coherence measures based on entanglement monotone have been shown [23–26]. Subsequently, the methods to construct proper coherence measures based on other quantum correlations have been proposed [27,28]. Recently, Budiyo and Dipojono have proposed a way to quantify the coherence of a quantum system via the Kirkwood-Dirac quasiprobability [29].

The study of various distances between the states in quantum systems is important. Recently, the quantum Wasserstein distance, which originated from the quantum optimal transport, has attracted much attention as it has been shown helpful in quantum theory. In 2021, Palma *et al.* proposed

the Wasserstein distance of order 1 for n -qudit systems [30]. In 2022, Friedland *et al.* considered a quantum version of the Monge-Kantorovich optimal transport problem and conjectured that the 2-Wasserstein distance is monotone under partial traces [31]. Nevertheless, Müller-Hermes proved the conjecture is invalid and proposed a revised version of the quantum optimal transport cost [32]. Bistoń *et al.* [33] also considered the problem of the monotonicity of a quantum 2-Wasserstein distance. Then De Palma and Trevisan proposed a new quantum generalization of the Wasserstein distance, one-to-one correspondence with quantum channels [34]. Besides the applications on the quantum states, the authors of Refs. [35–37] also showed that distance is helpful for quantum algorithms. Recently, Tóth and Pitrik considered how to define the quantum Wasserstein distance by optimizing over bipartite separable states [38]. Then one can consider the following: is it possible to quantify the coherence by the quantum Wasserstein distance, and is a quantum task closely related to the quantifier?

Quantum speed limits are used to denote the lower bound on the minimal time required to evolve from an initial state to a target state [39,40]. Recently, researchers have paid much attention to the relationship between the quantum speed limit and quantum resource theories [41–47]. One of the most famous examples is the Mandelstam-Tamm bound [40,48] for pairs of orthogonal pure states, and the bound can be generalized to arbitrary pairs of pure states $|\phi_1\rangle$ and $|\phi_2\rangle$. The minimal time required, τ , to evolve from $|\phi_1\rangle$ into $|\phi_2\rangle$ is

$$\tau \geq \hbar \frac{\arccos(|\langle \phi_1 | \phi_2 \rangle|)}{\overline{\Delta H}}, \quad (1)$$

where $\overline{\Delta H} = \frac{1}{\tau} \int_0^\tau dt \sqrt{\langle \phi_t | H_t^2 | \phi_t \rangle - |\langle \phi_t | H_t | \phi_t \rangle|^2}$, H_t is Hamiltonian of the unitary evolution [49].

In this paper, due to the beautiful properties of the revised quantum optimal transport cost in Ref. [32], we present a quantifier for the coherence of the quantum systems based on the cost. We also deliver the analytical solutions of the quantifier for pure states of qudit systems. Besides, we showed the quantifier is not a coherence measure. To amend the flaw, we propose a coherence measure for mixed states based on the quantifier under the convex roof extended method. At last, we

*shixian01@gmail.com

show that this measure links to the quantum speed limits for pure states.

The paper is organized as follows, in Sec. II, we present the preliminary knowledge on the revised quantum optimal cost needed, and in Sec. III, we present our main results. First, we consider a coherence quantifier based on the revised quantum optimal cost and show some properties of the measure. We also obtain the analytical values of the quantity for pure states of the qudit system. Besides, we propose a coherence measure based on the quantum optimal cost under the convex roof extended method. At last, we obtain an operational interpretation of the coherence measure for pure states. In Sec. IV, we end with a summary.

II. PRELIMINARY KNOWLEDGE

In this article, we assume \mathcal{H} is a Hilbert space with finite dimension, let $\mathbb{H}(\mathcal{H}_d) = \{M|M = M^\dagger\}$ and $\mathcal{D}(\mathcal{H}_d) = \{\rho|\rho \geq 0, \text{tr}\rho = 1\}$. A pure state of \mathcal{H}_d is a vector $|\psi\rangle \in \mathcal{H}_d$ with norm 1.

In this section, we first recall the postulates for quantum coherence measures of states on finite-dimensional systems, then we review the knowledge needed on quantum transport cost.

A. Postulates for quantum coherence measures

Let \mathcal{H} be a d -dimensional Hilbert space. Let $\mathcal{E} = \{|i\rangle\}$ be the set consisting of a prescribed orthonormal basis in \mathcal{H} ; the set of incoherent states $I_{\mathcal{E}}$ is composed of all the states that are diagonal with respect to the basis $\{|i\rangle\}$, that is, any incoherent state $\delta \in I_{\mathcal{E}}$ can be written as

$$\delta = \sum_i \lambda_i |i\rangle\langle i|.$$

An important problem in coherence theory is how to quantify the coherence of a state. In Ref. [16], Baumgratz *et al.* proposed the following postulates for functions $C: \mathcal{D}(\mathcal{H}) \rightarrow \mathbb{R}^+$.

(B1) $C(\rho) \geq 0$ for any $\rho \in \mathcal{D}(\mathcal{H})$, and $C(\rho) = 0$ if and only if $\rho \in I_{\mathcal{E}}$;

(B2) monotonicity under incoherent completely positive and trace-preserving maps (ICPTP) Ψ , $C(\rho) \geq C(\Psi(\rho))$;

(B3) monotonicity for average coherence under subselection based on measurement outcomes: $C(\rho) \geq \sum_i p_i C(\rho_i)$, here $\rho_i = \frac{K_i \rho K_i^\dagger}{p_i}$, $p_i = \text{tr}(K_i \rho K_i^\dagger)$ for all K_i satisfying $\sum_i K_i^\dagger K_i = I$, and $K_i I_{\mathcal{E}} K_i^\dagger \subset I_{\mathcal{E}}$;

(B4) nonincreasing under mixing of quantum states, $\sum_i p_i C(\rho_i) \geq C(\sum_i p_i \rho_i)$ for any $\{\rho_i, p_i\}$.

The conditions (B1) and (B2) are the basic requirements for a quantity to be a coherence quantifier [1]. If C satisfies all four properties, it is a faithful coherence measure.

Later, Yu *et al.* proposed an alternative framework for quantifying coherence [50]. There conditions (B3) and (B4) are replaced by the following: (C3) $C(p_1 \rho_1 \oplus p_2 \rho_2) = p_1 C(\rho_1) + p_2 C(\rho_2)$ for block-diagonal states ρ in the incoherent basis. That is, when a quantity satisfies conditions (B1), (B2), and (C3), it also satisfies conditions (B1)–(B4).

Geometric measure of coherence is a commonly used coherence measure based on the fidelity, which is defined as [23]

$$C_g(\rho) = 1 - \max_{\delta \in I_{\mathcal{E}}} F(\rho, \delta), \quad (2)$$

where $F(\rho, \delta) = [\text{tr}(\sqrt{\sqrt{\rho}\delta\sqrt{\rho}})]^2$. Moreover, when $\rho = |\psi\rangle\langle\psi|$ is a pure state, then

$$C_g(|\psi\rangle) = 1 - \max_i \{\psi_{ii}\}, \quad (3)$$

where ψ_{ii} is the diagonal elements of ψ with respect to the basis $I_{\mathcal{E}}$.

B. Quantum transport cost

In this section, we first recall the definition of quantum coupling, then we review the knowledge on the quantum optimal transport cost.

Definition 1. Assume ρ_A and ρ_B are quantum states on \mathcal{H}_A and \mathcal{H}_B , respectively. A state ρ_{AB} on \mathcal{H}_{AB} is called a coupling matrix for (ρ_A, ρ_B) if

$$\text{tr}_A \rho_{AB} = \rho_B, \quad \text{tr}_B \rho_{AB} = \rho_A.$$

In this paper, we denote the set of all quantum couplings of (ρ_A, ρ_B) as $\mathcal{B}(\rho_A, \rho_B)$. For any two states ρ_A and ρ_B , $\mathcal{B}(\rho_A, \rho_B)$ cannot be empty; it at least contains $\rho_A \otimes \rho_B$.

Next assume a bipartite Hilbert space $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$, where \mathcal{H}_A and \mathcal{H}_B are two Hilbert spaces with the same dimension d . Let $I_{\mathcal{H}_{AB}}$ and $S = \sum_{i,j=0}^{d-1} |ij\rangle\langle ji|$ be the identity and SWAP operator of the space \mathcal{H}_{AB} , respectively. The symmetric subspace between \mathcal{H}_A and \mathcal{H}_B is defined as

$$\mathcal{H}_A \vee \mathcal{H}_B = \{|\psi\rangle \in \mathcal{H}_{AB} \mid S|\psi\rangle = |\psi\rangle\}, \quad (4)$$

and the antisymmetric subspace between \mathcal{H}_A and \mathcal{H}_B is defined as

$$\mathcal{H}_A \wedge \mathcal{H}_B = \{|\psi\rangle \in \mathcal{H}_{AB} \mid S|\psi\rangle = -|\psi\rangle\}. \quad (5)$$

Here we denote $P_s(d) = \frac{1}{2}(I + S)$ and $P_a(d) = \frac{1}{2}(I - S)$ as the projections onto the symmetric subspace and the antisymmetric subspace, respectively. Then the quantum optimal transport cost for the states $\rho_A \in \mathcal{H}_A$ and $\sigma_B \in \mathcal{H}_B$ is given by

$$T(\rho_A, \sigma_B) = \min_{\chi_{AB} \in \mathcal{B}(\rho_A, \sigma_B)} \text{tr}[\chi_{AB} P_a(d)], \quad (6)$$

where the minimum takes over all the elements in $\mathcal{B}(\rho_A, \sigma_B)$, which was studied in Ref. [31]. There Friedland *et al.* also showed that the cost satisfies the following monotonicity when $d = 2$,

$$T(\Psi(\rho), \Psi(\sigma)) \leq T(\rho, \sigma),$$

where ρ and σ take over all the states in $\mathcal{D}(\mathcal{H}_d)$ and all the channels $\Psi: \mathcal{H}_d \rightarrow \mathcal{H}_d$ when $d = 2$. However, the above inequality may be invalid when d is bigger than 2 [32]. In Ref. [32], Müller-Hermes proposed a revised cost $T_s(\rho_A, \sigma_B)$ of states ρ_A and σ_B , which is defined as

$$T_s(\rho_A, \sigma_B) = \inf_{\gamma} T(\rho \otimes \gamma, \sigma \otimes \gamma),$$

where the infimum takes over all quantum states $\gamma \in \mathcal{D}(\mathcal{H}_d)$ of any dimension d . There Müller-Hermes showed

that

$$\begin{aligned} T_s(\rho_A, \sigma_B) &= T\left(\rho_A \otimes \frac{I_2}{2}, \sigma_B \otimes \frac{I_2}{2}\right) \\ &= \min_{X_{AB}, Y_{AB}} \{\text{tr}[X_{AB}P_s(d) + Y_{AB}P_a(d)]\}, \\ \text{s.t. } X_A + Y_A &= \rho_A, \quad X_B + Y_B = \sigma_B, \\ X_{AB}, Y_{AB} &\geq 0. \end{aligned} \quad (7)$$

This quantity satisfies the unitary invariant and monotonicity. In the Appendix, we present the dual program of $T_s(\rho_A, \sigma_B)$.

III. MAIN RESULTS

In this section, we first propose a coherence quantifier based on the revised cost (7). Then we obtain the analytical formulas of pure states in terms of this coherence quantifier. Besides we propose a coherence measure based on the revised cost. At last, we present a relation between the measure and the quantum speed limits for pure states.

Assume $\rho \in \mathcal{D}(\mathcal{H})$, $\tilde{T}(\rho)$ is defined as

$$\tilde{T}(\rho) = \min_{\delta \in \mathcal{I}_{\mathcal{E}}} T_s(\rho, \delta),$$

where the minimum takes over all the states in $\mathcal{I}_{\mathcal{E}}$.

Theorem 2. For a finite-dimensional Hilbert space \mathcal{H} , $\tilde{T}(\cdot)$ satisfies (B1), (B2), and (B4). Moreover, it satisfies subadditivity for product states.

Proof. For property (B1), when ρ is a state, due to the definition of \tilde{T} , $\tilde{T}(\rho) \geq 0$. When $\rho = \sum_i p_i |i\rangle\langle i| \in \mathcal{I}_{\mathcal{E}}$, then we could choose $\delta = \rho$, and $X_{AB} = 0$, $Y_{AB} = \sum_i \sqrt{p_i} p_j |ii\rangle\langle jj|$, and

$$\begin{aligned} \tilde{T}(\rho) &\leq \text{tr}[X_{AB}P_s(d) + Y_{AB}P_a(d)] \\ &= \sum_{i,j} \sqrt{p_i p_j} \langle ii|P_a(d)|jj\rangle \\ &= 0. \end{aligned}$$

As $T_s(\rho, \sigma)$ is non-negative, then $\tilde{T}(\rho) = 0$.

Next we show $\tilde{T}(\cdot)$ is faithful. Let ρ be a mixed state on the Hilbert space \mathcal{H} with $T_s(\rho) = 0$; next let σ be the optimal incoherent state for ρ in terms of $\tilde{T}(\cdot)$. Let X and Y be the optimal in terms of \tilde{T} for the couple (ρ, σ) , as $X, Y \geq 0$ and $\text{tr}[XP_s(d) + YP_a(d)] = 0$, then $\text{tr}[XP_s(d)] = \text{tr}[YP_a(d)] = 0$. Let

$$\begin{aligned} A &= \{|\phi_i\rangle\langle\phi_i| = |ii\rangle, i = 0, 1, \dots, d-1\}, \\ B &= \left\{ |\phi_{ij}^-\rangle\langle\phi_{ij}^-| = \frac{|ij\rangle - |ji\rangle}{\sqrt{2}}, 0 \leq i < j \leq d-1 \right\}, \\ C &= \left\{ |\phi_{ij}^+\rangle\langle\phi_{ij}^+| = \frac{|ij\rangle + |ji\rangle}{\sqrt{2}}, 0 \leq i < j \leq d-1 \right\}, \end{aligned}$$

then the sets $A \cup B \cup C$, $A \cup C$, and B constitute the orthonormal base of the space $\mathcal{H} \otimes \mathcal{H}$, $\text{Range}(P_s(d))$, and $\text{Range}(P_a(d))$, respectively. As $\text{tr}[XP_s(d)] = 0$,

$$\begin{aligned} X_{AB} &= \sum_{0 \leq i < j \leq d-1} \sum_{0 \leq k < l \leq d-1} \mu_{ijkl} |\phi_{ij}^-\rangle\langle\phi_{kl}^-|, \\ X_A &= X_B. \end{aligned}$$

Similarly, $Y_A = Y_B$. At last, due to the definition (7) of $T_s(\rho, \sigma)$, $\rho = \sigma$ is incoherent. We finish the proof of the faithfulness of $\tilde{T}(\cdot)$.

For property (B2), in Ref. [32], Müller-Hermes showed that for any pair of quantum states $\rho, \sigma \in \mathcal{D}(\mathcal{H})$, and any quantum channel Φ ,

$$T_s(\Phi(\rho), \Phi(\sigma)) \leq T_s(\rho, \sigma). \quad (8)$$

Assume Ψ is any ICPTP map, ρ is any quantum state, and σ is the optimal incoherent state in terms of $T_s(\rho)$,

$$\tilde{T}(\rho) = T_s(\rho, \sigma) \geq T(\Psi(\rho), \Psi(\sigma)) \geq \tilde{T}(\Psi(\rho)).$$

Here the first inequality is due to Eq (8). As σ is an incoherent state, and Ψ is an ICPTP map, $\Psi(\sigma)$ is incoherent, and then combining the definition of $\tilde{T}(\Psi(\rho))$, we finish the proof of property (B2).

For property (B4) and subadditivity for product states, we place their proof in Theorems 5 and 6, respectively, of the Appendix.

Next we show the analytical solutions of $\tilde{T}(\rho)$ when $\rho = |\phi\rangle\langle\phi|$ is a pure state.

Theorem 3. Assume $|\phi\rangle = \sum_{i=0}^{d-1} \lambda_i |i\rangle$ is a pure state, $|\lambda_0| \geq |\lambda_1| \geq \dots \geq |\lambda_{d-1}| \geq 0$ and $\sum_{i=0}^{d-1} |\lambda_i|^2 = 1$, then

$$\tilde{T}(|\phi\rangle) = \frac{1 - |\lambda_0|^2}{2}. \quad (9)$$

Moreover, it owns the following relationship with the geometric measure of coherence:

$$\tilde{T}(|\phi\rangle) = \frac{C_g(|\phi\rangle)}{2}. \quad (10)$$

Proof. Let $\sigma = \sum_k q_k |k\rangle\langle k|$ be the optimal incoherent state for ρ in terms of $\tilde{T}_s(\cdot)$. Let X_{AB} and Y_{AB} be the optimal in terms of Eq. (7) for the couple (ρ, σ) , that is,

$$T_s(\rho, \sigma) = \min_{\sigma \in \mathcal{I}_{\mathcal{E}}} \{\text{tr}[X_{AB}P_s(d) + Y_{AB}P_a(d)]\},$$

$$\begin{aligned} \text{such that } X_A + Y_A &= \rho, \quad X_B + Y_B = \sigma, \\ X_{AB}, Y_{AB} &\geq 0. \end{aligned}$$

As ρ is a pure state, X_A and Y_A are semipositive definite with ranks 1, $X_A = m|\phi\rangle\langle\phi|$, $Y_A = n|\phi\rangle\langle\phi|$, $m, n \geq 0$, and $m + n = 1$. Next due to Lemma 1, $X_{AB} = m|\phi\rangle\langle\phi| \otimes X_B$, $Y_{AB} = n|\phi\rangle\langle\phi| \otimes Y_B$, then

$$\begin{aligned} &\text{tr}[X_{AB}P_s(d) + Y_{AB}P_a(d)] \\ &= \frac{\text{tr}[X_{AB} + Y_{AB} + F(X_{AB} - Y_{AB})]}{2} \\ &= \frac{\text{tr}[1 + F(X_{AB} - Y_{AB})]}{2} \\ &= \frac{\text{tr}\{1 + F[|\phi\rangle\langle\phi| \otimes (mX_B - nY_B)]\}}{2}, \\ &= \frac{1 + m\langle\phi|X_B|\phi\rangle - n\langle\phi|Y_B|\phi\rangle}{2}, \end{aligned}$$

and then

$$\begin{aligned}\tilde{T}(|\phi\rangle) &= \min \left[\frac{1 + m\langle\phi|X_B|\phi\rangle - n\langle\phi|Y_B|\phi\rangle}{2} \right], \\ &= \frac{1 - |\lambda_0|^2}{2}.\end{aligned}\quad (11)$$

Because $X_B, Y_B \geq 0$ and $m, n \geq 0$, we choose $X_B = 0, m = 0, n = 1$, and $Y_B = \sigma_B = |0\rangle\langle 0|$ in the second equality.

Next we have the following relations between $\tilde{T}(|\phi\rangle)$ and the geometric measure of coherence (2),

$$\tilde{T}(|\phi\rangle) = \frac{1 - |\lambda_0|^2}{2} = \frac{C_g(|\phi\rangle)}{2}, \quad (12)$$

and the second equality is due to the formula (3), that is, $C_g(|\phi\rangle) = 2\tilde{T}(|\phi\rangle)$.

Then we show that $\tilde{T}(\cdot)$ does not satisfy property (B3) by showing property (C3) is invalid for $\tilde{T}(\cdot)$. Assume $\rho_1 = \frac{1}{2}(|0\rangle + |1\rangle)(\langle 0| + \langle 1|)$, $\rho_2 = \frac{1}{3}(|2\rangle + |3\rangle + |4\rangle)(\langle 2| + \langle 3| + \langle 4|)$, and $\rho = \frac{1}{2}\rho_1 \oplus \frac{1}{2}\rho_2$. By Theorem 3, we have $\tilde{T}(\rho_1) = \frac{1}{4}$, $\tilde{T}(\rho_2) = \frac{1}{3}$, and $\frac{1}{2}\tilde{T}(\rho_1) + \frac{1}{2}\tilde{T}(\rho_2) = \frac{7}{24}$. Next, when we take

$$\begin{aligned}\delta &= \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|, \\ X_{AB} &= \frac{1}{2}|\psi_2\rangle\langle\psi_2| \otimes |\psi_1^-\rangle\langle\psi_1^-|, \\ Y_{AB} &= \frac{1}{2}|\psi_1^+\rangle\langle\psi_1^+| \otimes |\psi_1^+\rangle\langle\psi_1^+|, \\ |\psi_1^-\rangle &= \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle), \\ |\psi_1^+\rangle &= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \text{ and} \\ |\psi_2\rangle &= \frac{1}{\sqrt{3}}(|2\rangle + |3\rangle + |4\rangle), \\ \tilde{T}(\rho) \leq T_s(\rho, \delta) &\leq \frac{1 + \text{tr}[S(X - Y)]}{2} = \frac{1}{4} < \frac{7}{24}.\end{aligned}$$

Hence, $\tilde{T}(\rho) \neq \frac{1}{2}\tilde{T}(\rho_1) + \frac{1}{2}\tilde{T}(\rho_2)$, and $\tilde{T}(\cdot)$ does not satisfy property (B3).

To remedy the flaw, we propose a coherence measure based on the convex roof extended method [17] for mixed states; then its coherence measure based on the revised transport cost is

$$T(\rho) = \min_{\{p_i, |\phi_i\rangle\}} \sum_i p_i \tilde{T}(|\phi_i\rangle), \quad (13)$$

where the minimum takes over all the decompositions of $\rho = \sum_i p_i |\phi_i\rangle\langle\phi_i|$. Based on Theorem 3 and the result in Ref. [17], we have $T(\rho)$ satisfies properties (B1), (B3), and (B4), and properties (B3) and (B4) can lead to property (B2). Hence, $T(\cdot)$ is a proper coherence measure.

Finally, we show the relationship between the measure and the quantum speed limit. Here we consider a pair of pure states $|\psi\rangle$ and $|\phi\rangle$ with $\langle\phi|\psi\rangle \in \mathbb{R}$; this is always possible, as $|\psi\rangle$ and $e^{-i\theta}|\psi\rangle$ are equivalent. Recently, Rudnicki studied the relationship between quantum speed limits and quantum entanglement measures [42]. Rudnicki showed that when the

Hamiltonian H' is time independent,

$$H' = -i\hbar\omega(|\psi\rangle\langle\bar{\psi}| - |\bar{\psi}\rangle\langle\psi|), \quad (14)$$

where $|\bar{\psi}\rangle = \frac{|\phi\rangle - \langle\psi|\phi\rangle|\psi\rangle}{\sqrt{1 - |\langle\phi|\psi\rangle|^2}}$, which is orthogonal to $|\psi\rangle$. The bound (1) is saturated [42]. Moreover, when we denote $\tau(\psi, \phi) = \Delta t$,

$$\tau(\psi, \phi) = \frac{\arccos(\langle\phi|\psi\rangle)}{\omega}. \quad (15)$$

Here we can pose a question: for a given pure state $|\psi\rangle$, how much time $\tau(\psi)$ does a unitary evolution cost by turning $|\psi\rangle$ into an incoherent state? That is,

$$\tau(|\psi\rangle) = \frac{1}{\omega} \min_{\phi \in \mathcal{I}_\varepsilon} \arccos(|\langle\psi|\phi\rangle|).$$

By the above analysis, the bound (1) is saturated when taking $H = H'$. Next assume $|\psi\rangle = \sum_{i=0}^{d-1} \lambda_i |i\rangle$, $|\lambda_0| \geq |\lambda_1| \geq \dots \geq |\lambda_{d-1}| \geq 0$, and $\sum_{i=0}^{d-1} |\lambda_i|^2 = 1$, then

$$\begin{aligned}\min_{\phi \in \mathcal{I}_\varepsilon} \arccos(|\langle\psi|\phi\rangle|) &= \arccos(\max_{\phi \in \mathcal{I}_\varepsilon} \langle\psi|\phi\rangle) \\ &= \arccos(|\lambda_0|) \\ &= \arcsin(\sqrt{1 - |\lambda_0|^2}),\end{aligned}$$

that is,

$$\tau(|\psi\rangle) = \frac{1}{\omega} \arcsin(\sqrt{1 - |\lambda_0|^2}). \quad (16)$$

At last, by combing the formula (16) and Theorem 3, we obtain the following theorem.

Theorem 4. Assume $|\psi\rangle$ is a pure state, then the minimal time $\tau(|\psi\rangle)$ required to transform $|\psi\rangle$ into an incoherent state under a unitary evolution is

$$\tau(|\psi\rangle) = \frac{1}{\omega} \arcsin[\sqrt{2T(|\psi\rangle)}]. \quad (17)$$

When $|\psi\rangle$ is incoherent, $T(|\psi\rangle) = 0$ and $\tau(|\psi\rangle) = 0$; otherwise, due to the faithfulness of $T(\cdot)$, $\tau(|\psi\rangle)$ cannot be 0. Furthermore, as $T(\cdot)$ is monotone under the incoherent operations for pure states, $\tau(|\psi_1\rangle) \geq \tau(|\psi_2\rangle)$ when $|\psi_1\rangle$ can be turned into $|\psi_2\rangle$ under incoherent operations.

IV. CONCLUSION

In this article, we studied a coherence quantifier based on the quantum transport cost. First we showed the quantity satisfies properties (B1), (B2), and (B4), and it is faithful and subadditive for product states. Based on the analytical solutions of the quantifier for pure states, we proposed an example showing that $\tilde{T}(\cdot)$ does not satisfy property (B3). To remedy the flaw, we proposed a coherence measure based on the convex roof extended method for mixed states. At last, we obtained a close relation between the coherence measure and the minimal time necessary to evolve to an incoherent state for a pure state. Due to the importance of the study on the quantum coherence, our results can provide a reference for future work on the study of quantum coherence.

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APPENDIX

Assume ρ_A and σ_B are two quantum states in $\mathcal{D}(\mathcal{H}_A)$ and $\mathcal{D}(\mathcal{H}_B)$, respectively, then we can write the revised definition as follows:

$$\begin{aligned} T_s(\rho_A, \sigma_B) &= \sup \operatorname{tr}(\rho_A H_1 + \sigma_B H_2), \quad (\text{A1}) \\ \text{such that } P_s - H_1 \otimes I - I \otimes H_2 &\geq 0, \\ P_{as} - H_1 \otimes I - I \otimes H_2 &\geq 0, \\ H_1 \in \mathbb{H}(\mathcal{H}_A), H_2 \in \mathbb{H}(\mathcal{H}_B). \end{aligned}$$

Proof. Here the Lagrangian can be written as

$$\begin{aligned} &\operatorname{tr}[X_{AB}P_s(d) + Y_{AB}P_{as}(d)] \\ &+ \operatorname{tr}[(\rho - X_A - Y_A)H_1] + \operatorname{tr}[(\sigma - X_B - Y_B)H_2] \\ &= \operatorname{tr}\{X_{AB}[P_s(d) - H_1 \otimes I - I \otimes H_2]\} \\ &+ \operatorname{tr}\{Y_{AB}[P_{as} - H_1 \otimes I - I \otimes H_2]\} \\ &+ \operatorname{tr}[\rho H_1 + \sigma H_2]. \end{aligned}$$

Hence, the dual SDP is

$$\begin{aligned} &\sup \operatorname{tr}[\rho H_1 + \sigma H_2], \\ \text{such that } P_s - H_1 \otimes I - I \otimes H_2 &\geq 0, \\ P_{as} - H_1 \otimes I - I \otimes H_2 &\geq 0. \end{aligned}$$

Then we show strong duality holds for the semidefinite program (7). Note that $(X_{AB} = 0, Y_{AB} = \rho \otimes \sigma)$ is a feasible solution to the primal program (A1). For the dual program (7), when $H_1 = H_2 = -I$, then $P_s - H_1 \otimes I - I \otimes H_2 = P_s + 2I_A \otimes I_B$, as for any $|v\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$, $\langle v|(P_s + 2I_A \otimes I_B)|v\rangle = 2 + \langle v|P_s|v\rangle \geq 2 > 0$, that is, $P_s - H_1 \otimes I - I \otimes H_2 \geq 0$, and similarly, $P_{as} - H_1 \otimes I - I \otimes H_2 \geq 0$. Since a strictly feasible solution exists to the dual program and the primal feasible set is nonempty, the Slater's conditions are satisfied; hence, we finish the proof [51].

Theorem 5. Assume $\rho = \sum_{i=1}^d p_i \rho_i$, then

$$\tilde{T}(\rho) \leq \sum_{i=1}^d p_i \tilde{T}(\rho_i).$$

Proof. Assume σ_i and σ are the optimal incoherent states for ρ_i and ρ , respectively, in terms of $\tilde{T}(\cdot)$, and $X_{AB}^{(i)}$ and $Y_{AB}^{(i)}$ are the optimal positive semidefinite matrices for $T_s(\rho_i, \sigma_i)$ in terms of Eq. (7), then

$$\begin{aligned} \sum_i p_i \tilde{T}(\rho_i) &= \sum_i p_i \operatorname{tr}[X_{AB}^{(i)}P_s(d) + Y_{AB}^{(i)}P_{as}(d)] \\ &\geq T_s\left(\sum_i p_i \rho_i, \sum_i p_i \sigma_i\right) \\ &\geq \tilde{T}(\rho), \quad (\text{A2}) \end{aligned}$$

where the first inequality is due to the properties of $X_{AB}^{(i)}$ and $Y_{AB}^{(i)}$ and the formula (7). Specifically,

$$\begin{aligned} X_A^{(i)} + Y_A^{(i)} &= \rho_i, \quad X_B^{(i)} + Y_B^{(i)} = \sigma_i \\ \Rightarrow \sum_i p_i (X_A^{(i)} + Y_A^{(i)}) &= \sum_i p_i \rho_i, \\ \sum_i p_i (X_B^{(i)} + Y_B^{(i)}) &= \sum_i p_i \sigma_i, \end{aligned}$$

then $\sum_i p_i X_{AB}^{(i)}$ and $\sum_i p_i Y_{AB}^{(i)}$ satisfy the properties of Eq. (7). The second inequality is due to the definition of $\tilde{T}(\cdot)$.

Theorem 6. Assume ρ and σ are two quantum states on the system \mathcal{H} , then

$$\tilde{T}(\rho \otimes \sigma) \leq \tilde{T}(\rho) + \tilde{T}(\sigma).$$

Proof. Let ρ and σ be two density matrices on the system \mathcal{H} , then we denote δ_1 and δ_2 as two optimal incoherent states for ρ and σ , respectively, in terms of \tilde{T} , that is,

$$\tilde{T}(\rho) = \frac{1}{2} \min_{X_{AB}, Y_{AB}} \{\operatorname{tr}[X_{AB}P_s(d) + Y_{AB}P_{as}(d)]\},$$

$$\text{such that } X_A + Y_A = \rho, \quad X_B + Y_B = \delta_1,$$

$$X_{AB}, Y_{AB} \geq 0, \quad (\text{A3})$$

$$\tilde{T}(\sigma) = \frac{1}{2} \min_{M_{AB}, N_{AB}} \{\operatorname{tr}[M_{AB}P_s(d) + N_{AB}P_{as}(d)]\},$$

$$\text{such that } M_A + N_A = \sigma, \quad M_B + N_B = \delta_2,$$

$$M_{AB}, N_{AB} \geq 0. \quad (\text{A4})$$

Here we denote (X_{AB}, Y_{AB}) and (M_{AB}, N_{AB}) are the optimal couples for (ρ, δ_1) and (σ, δ_2) , respectively. Next we compute

$$\begin{aligned} &\frac{1}{2} \{\operatorname{tr}[M + X + N + Y + (M + X - N - Y)S] \\ &- \operatorname{tr}[(X + Y) \otimes (M + N)] \\ &+ (M \otimes X - N \otimes X - M \otimes Y + N \otimes Y)S \otimes S\} \\ &= \frac{1}{2} [1 + \operatorname{tr}(M - N)S + \operatorname{tr}(X - Y)S \\ &+ \operatorname{tr}(M - N)S \otimes (X - Y)S] \\ &= \frac{1}{2} \{[1 + \operatorname{tr}(M - N)S][1 + \operatorname{tr}(X - Y)S]\} \geq 0. \quad (\text{A5}) \end{aligned}$$

As δ_1 and δ_2 are incoherent states, so it is with $\delta_1 \otimes \delta_2$. Next as (X_{AB}, Y_{AB}) and (M_{AB}, N_{AB}) satisfy Eqs. (A3) and (A4), respectively, we have

$$(X_A + Y_A) \otimes (M_A + N_A) = \rho \otimes \sigma,$$

$$(X_B + Y_B) \otimes (M_B + N_B) = \delta_1 \otimes \delta_2,$$

and hence,

$$\begin{aligned} \tilde{T}(\rho \otimes \delta) &\leq T_s(\rho \otimes \sigma, \delta_1 \otimes \delta_2), \\ &\leq \frac{1}{2} \operatorname{tr}[(X + Y) \otimes (M + N) \\ &+ (M \otimes X - N \otimes X - M \otimes Y + N \otimes Y)S \otimes S], \quad (\text{A6}) \end{aligned}$$

then by combining Eqs. (A5) and (A6), we have

$$\tilde{T}(\rho) + \tilde{T}(\sigma) \geq \tilde{T}(\rho \otimes \sigma). \quad (\text{A7})$$

Lemma 1. Assume X_{AB} is a bipartite substate on $\mathcal{H}_d \otimes \mathcal{H}_d$, i.e., $X_{AB} \geq 0$, and $\text{tr} X_{AB} \leq 1$, if $\text{rank}(X_B) = 1$, or $\text{rank}(X_A) = 1$, then $X_{AB} = X_A \otimes X_B$.

Proof. When $\text{rank}(X_B) = 1$, then $X_B = m|\phi\rangle\langle\phi|$, $m \in (0, 1]$. Next let

$$X_{AB} = \begin{pmatrix} X_{11} & X_{12} & \cdots & X_{1d} \\ X_{21} & X_{22} & \cdots & X_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ X_{d1} & X_{d2} & \cdots & X_{dd} \end{pmatrix}, \quad (\text{A8})$$

where X_{ij} are the block matrices of X_{AB} , $i, j = 1, 2, \dots, d$; $X_{kk} \geq 0, k = 1, 2, \dots, d$; and $X_B = X_{11} + X_{22} + \cdots + X_{dd} = m|\phi\rangle\langle\phi|$, that is, $X_{kk} = l_{kk}|\phi\rangle\langle\phi|$, $\sum_k l_{kk} = m$. Next as X_{AB} is

semidefinite positive, then

$$\begin{pmatrix} X_{ii} & X_{ij} \\ X_{ji} & X_{kk} \end{pmatrix} \geq 0,$$

and hence $X_{ij} = l_{ij}|\phi\rangle\langle\phi|$, that is,

$$X_{AB} = \begin{pmatrix} l_{11}\phi & l_{12}\phi & \cdots & l_{1d}\phi \\ l_{21}\phi & l_{22}\phi & \cdots & l_{2d}\phi \\ \vdots & \vdots & \ddots & \vdots \\ l_{d1}\phi & l_{d2}\phi & \cdots & l_{dd}\phi \end{pmatrix} = M \otimes \phi.$$

If $\text{rank}(X_A) = 1$, let $Y_{AB} = SX_{AB}S^\dagger$, then $\text{rank}(Y_B) = 1$, and due to the above proof, Y_{AB} can be written as the product form. Hence, X_{AB} can be written as the product form.

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