Generalized wave-particle-mixedness triality for *n*-path interferometers

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The wave-particle duality, one of the expressions of Bohr complementarity, is usually quantified by path predictability and the visibility of interference fringes. With the development of quantum resource theory, quantitative analysis of wave-particle duality is increasing, most of which are expressed in the form of specific functions. In this paper, we obtain the path-information measure for pure states by converting the coherence measure for pure states into a symmetric concave function. Then we prove the function as a path-information measure is also valid for mixed states. Furthermore, we also establish a generalized wave-particle-mixedness triality. Although the mixedness proposed in the text is not a complete mixedness measure, it also satisfies some conditions of the mixedness measure. From the perspective of resource theory, the path information we establish can be used as the measure of the resource of predictability, and the triality relationship we establish reveals the relationship among coherence, predictability, purity, and mixedness degree to a certain extent. According to our method, given either the coherence measure or path information, a particular form of wave-particle-mixedness triality can be established. This will play an important role in establishing connections between wave, particle, and other physical quantifiers.

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I. INTRODUCTION

Wave-particle duality is a fundamental property of quantum physics which plays a pivotal role in quantum mechanics and is ubiquitous in the microcosmic world. The earliest research on wave-particle duality can be traced back to the complementarity principle proposed by Bohr in 1928 [1], but it is just a qualitative idea. Wootters and Zurek were the first ones to quantitatively analyze wave-particle duality [2]. Later, Greenberger and Yasin proposed a complementarity relation; they analyzed the problem with the assumption that unequal beams in a two-path interferometer allow for predicting, to a certain degree, which of the two paths the quantum followed. They established the duality relation [3]

$$P^2 + W^2 \leqslant 1, \tag{1}$$

where P quantifies the particle feature (path information) and W quantifies the wave feature (interference) of the state in an interferometric setup [4].

As the theory progressed, Englert derived a new duality relation by including a path detector in the interferometer which was used to determine which path a specific quantum took [5]:

$$D^2 + V^2 \leqslant 1. \tag{2}$$

In this inequality, D is the distinguishability of the possible detector states, and V is a measure of the quality of the interference fringe. The equality holds if the particle state and the path detector state are pure. Investigations of quantitative measures of wave and particle properties in a multipath interferometer were initiated by Durr [6], whose criteria for generalized predictability and generalized visibility were later further pursued in [7–9].

In recent years, with the establishment of quantum resource theory, quantitative analysis of wave-particle duality has returned to the public's eye. These studies about the wave-particle duality have revealed close relationships and interplay among predictability, distinguishability, visibility [2,3,5–8], and some quantum informational concepts such as asymmetry [10], entropy [11–13], entanglement [14–19], coherence [20–32], and purity [33].

At the same time, people are trying to build a neat complementary relationship. In the pure case, the wave-particle duality is a strictly complementary relation, but attempts to generalize the equation to mixed states always turn it into an inequality. Through further analysis of the n-path interferometer with the addition of the path detector, Roy *et al.* proposed a coherence–path predictability–I concurrence triality [14,34]; I concurrence is a normalized entanglement measure with some functional relationship to the generalized concurrence.

By splitting the path distinguishability into path predictability and an entanglement measure, Basso and Maziero proposed a new triality relation [35]; this new triality formalizes entanglement as the third quantity.

Unlike the above authors, Fu and Luo did not adopt the interferometer analysis method but deduced a very concise triality relationship by combining the quantum uncertainty with coherence and path information [36].

In this paper, we analyze whether all coherence measures satisfy the neat complementarity relation according to the triality mentioned in [36]. We transform the coherence measure of pure states into a symmetric concave function

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related only to the main diagonal elements, and we find that all coherence measures can establish a neat complementarity relation; the function that is complementary to the symmetric concave function converted from the coherence measure perfectly meets the requirements of the particle property measure. Moreover, we can also obtain a triality relation in the mixed state; although the third quantity is not a complete measure of mixedness, it still satisfies some properties of the mixedness measure.

II. PRELIMINARIES

In this section, we introduce some basic notation we are going to use. In Sec. II A, we briefly introduce the quantum coherence resource theory and the conditions for the coherence measurement. We also show that the coherence measures for the pure state can be expressed as a symmetric concave function and the coherence measures for the mixed state can be constructed by convex tops. In Sec. II B, we discuss the basic knowledge of wave-particle duality and its recent development, which includes a brief introduction to the theoretical analysis of multipath interferometer experiments and the recently established wave-particle duality. Finally, we will discuss the conditions that must be satisfied for the measure of wave and particle feature in Sec. II C. We explore whether the measures of particles and waves we propose next meet these conditions.

A. Quantum coherence

Coherence is a fundamental feature of quantum physics that represents possible superposition between orthogonal quantum states. Similarly, it is widely believed that quantum superposition is a manifestation of the fluctuating nature of quantum particles. Therefore, the quantum coherence has a strong correspondence to the wave properties of quantum particles.

Based on Baumgratz *et al.*'s suggestion [37], any proper measure of coherence C must satisfy the following axiomatic postulates.

(1) The coherence measure vanishes in the set of incoherent states, $C(\rho) = 0$ for all $\rho \in I$.

(2a) Monotonicity exists under incoherent operation Φ, $C(\Phi(\rho)) \leqslant C(\rho)$.

(2b) Monotonicity exists under selective measurements on average, $\sum_n p_n C(\rho_n) \leqslant C(\rho)$, where $p_n = \mathrm{tr}(K_n \rho K_n^\dagger)$ and $\rho = \frac{1}{p_n} K_n \rho K_n^\dagger$ for all $\{K_n\}$, with $\sum_n K_n^\dagger K_n = I$ and

$$K_n \rho K_n^\dagger / \mathrm{Tr}(K_n \rho K_n^\dagger) \in I \tag{3}$$

for all $\rho \in I$.

(3) The coherence measure is nonincreasing under mixing of the quantum state (convexity),

$$C\left(\sum_n p_n \rho_n\right) \leqslant \sum_n p_n C(\rho_n) \tag{4}$$

for any ensemble $\{p_n, \rho_n\}$.

In general, the coherence monotone must satisfies conditions 1, 2a, and 3, and the coherence measure must meet all the conditions.

Corresponding to the functional form of the entanglement measure, the functional form of the coherence measure was given by Du *et al.* [38] as follows: Let $\Omega = \{\boldsymbol{x} = (x_1, x_2, \dots, x_d)^t \mid \sum_{i=1}^d x_i = 1 \quad \text{and} \quad x_i \geqslant 0\}$, where $(x_1, x_2, \dots, x_d)^t$ denotes the transpose of the row vector $(x_1, x_2, \dots, x_d)$. Let π be an arbitrary permutation of $\{1, 2, \dots, d\}$ and P_π be the permutation matrix corresponding to π that is obtained by permuting the rows of a $d \times d$ identity matrix according to π. Given any non-negative function $f : \Omega \mapsto R^+$ such that (1) the function is zero if only one of the terms is 1 and the rest are 0, i.e.,

$$f(P_\pi(1, 0, \dots, 0)^t) = 0 \tag{5}$$

for every permutation π, (2) it is invariant under any permutation transformation P_π, i.e.,

$$f(P_\pi \boldsymbol{x}) = f(\boldsymbol{x}) \tag{6}$$

for every $\boldsymbol{x} \in \Omega$, and (3) it is concave, i.e.,

$$f[\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y}] \geqslant \lambda f(\boldsymbol{x}) + (1-\lambda) f(\boldsymbol{y}) \tag{7}$$

for any $\lambda \in [0, 1]$ and $\boldsymbol{x}, \boldsymbol{y} \in \Omega$, then a coherence measure can be derived by defining it for pure states (normalized vectors $|\psi\rangle = (\psi_1, \psi_2, \dots, \psi_d)^t$ in the fixed basis $\{|i\rangle\}_{i=1}^d$) as

$$C_f(|\psi\rangle\langle\psi|) = f((|\psi_1|^2, |\psi_2|^2, \dots, |\psi_d|^2)^t). \tag{8}$$

The function f is extended to the whole set of density matrices by the form of the convex-roof structure:

$$C_f(\rho) = \min_{p_j, \rho_j} \sum_j p_j C_f(\rho_j), \tag{9}$$

where the minimization should be performed over all the pure-state ensembles of ρ, i.e., $\rho = \sum_j p_j \rho_j$.

For any pure state $|\psi\rangle$, all symmetric concave functions f satisfying the above conditions are coherence measures. In reverse, a symmetric concave function can be found for any coherence measure. Moreover, the above conditions also point out a way to construct a coherence measure of mixed states from a coherence measure of pure states. There are many examples, such as α entropy [39], the fidelity coherence measure [40], and so on.

B. Wave-particle duality

There are two main types of n-path interferometers, those with and without path detectors (see Fig. 1). A particle is fired from an emitter, travels through one of n paths as the manifestation of the property of the particle, and ends up hitting the screen to create a number of interference patterns as the manifestation of the property of the wave. With the development of coherence resource theory, interference fringes on the screen can be formulated as coherence, and which path the particle passes depends on the path information.

For an interferometer without path detectors, we cannot clearly determine which path the particle takes, so we can only predict the path with the greatest probability using the properties of the particle itself, which is called the path predictability of the particle. This in itself is guesswork, so there are many ways to characterize path-predictable rows, such as the classic single-bet protocol [4] and the multibet protocol [6,7].

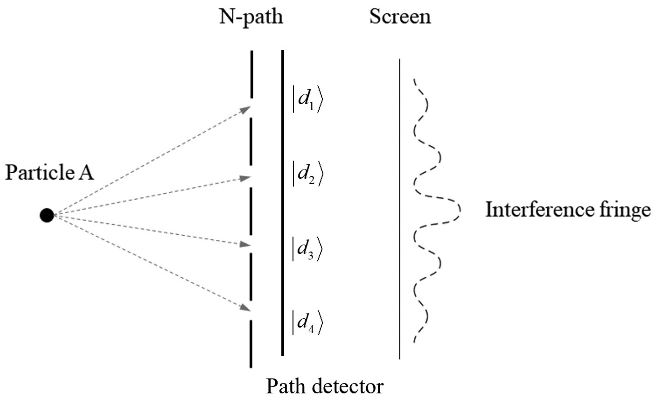

FIG. 1. Diagram of an n-path interferometer with the path detector; a detector device is installed on each path, and when a particle passes through the ith path, the detector changes from $|d_0\rangle$ to $|d_i\rangle$.

For an interferometer with path detectors, we can clearly determine which path the particle takes, so we do not have to guess, and instead, we judge the path, which is called the path distinguishability of the particle. The initial detector state is $|d_0\rangle$; when the particle $\rho = \sum_{ij} \rho_{ij} |\psi_i\rangle \langle\psi_j|$ passes through the ith path and becomes entangled with the detector, through the controlled unitary interaction, each of the paths is marked with a detector state $|d_i\rangle$, and the path distinguishability is equivalent to discriminating the detector states. In other words, if the quantum passes through the ith path, the resulting detector states becomes $|d_i\rangle$ with probability $|\rho_{ii}|$. Because detector states $|d_i\rangle$ are not mutually orthogonal in general, we have partial knowledge of the which-path information.

Although the above path predictability and path distinguishability are both types of path information, there are subtle differences between them. For the case of no path detector, the particle does not change when it passes through the path and finally hits the screen after being emitted from the emitter, remaining

$$\rho = \sum_{ij} \rho_{ij} |\psi_i\rangle \langle\psi_j| .$$

However, in the case of path detectors, the emitter fires a state $\rho = \sum_{ij} \rho_{ij} |\psi_i\rangle \langle\psi_j|$, and when it passes through the detector, it becomes entangled with the detector and becomes

$$\rho_{sd} = \sum_{i}^{n} \sum_{j}^{n} \rho_{ij} |\psi_i\rangle \langle\psi_j| \otimes |d_i\rangle \langle d_j| .$$

By tracing out the path-detector states in ρ_{sd}, one gets the reduced density matrix for the quanton

$$\rho_s = \sum_{i}^{n} \sum_{j}^{n} \rho_{ij} |\psi_i\rangle \langle\psi_j| \langle d_i| |d_j\rangle ;$$

this will be the state that finally hits the screen.

The first case involves only operations on a single system, while the second case involves two systems (state and path detectors). The path information is different in different cases (the path distinguishability is used in the case with detectors, and the path predictability is used in the case without detectors), and the corresponding coherence measure is also different (ρ_s is measured when there are detectors, and ρ is measured when there are no detectors). In both cases, the path information and coherence are complementary, which also satisfies Theorems 1 and 2 below.

C. Wave-particle-mixedness triality

Let's review some of the requirements for wave-particle duality in general [6–9]. Following [6–8], the quantification of the quantum wave feature [we denote it $W(\rho)$] should satisfy the following reasonable requirements:

(1) $W(\rho)$ reaches its global minimum if the state ρ is classical (i.e., diagonal in the computational basis).

(2) $W(\rho)$ reaches its global maximum if the state ρ is pure and a uniform superposition of the states in the computational basis (i.e., $\langle i|\rho|i\rangle = 1/n$ for all i).

(3) $W(\rho)$ is invariant under permutations of the diagonal elements $\langle i|\rho|i\rangle$ of ρ.

(4) $W(\rho)$ is convex.

In a dual fashion, the quantification of the quantum particle feature [we denote it $P(\rho)$] should have the following properties [6,8]:

(1) $P(\rho)$ reaches its global maximum if $\langle i|\rho|i\rangle = 1$ for some i.

(2) $P(\rho)$ reaches its global minimum if $\langle i|\rho|i\rangle = 1/n$ for all i.

(3) $P(\rho)$ is invariant under permutations of the diagonal elements $\langle i|\rho|i\rangle$ of ρ.

(4) $P(\rho)$ is convex.

All the above mathematical requirements are motivated by intuitive physical considerations [6–8], and all the wave properties and particle properties should satisfy the above conditions. (1) For a quantum state ρ, if it is classical, i.e., $\rho = \sum_i p_i |i\rangle \langle i|$, such states are formally equivalent to the set of free states of the coherence resource, $C(\rho) = 0$, based on condition 1 in Sec. II A. (2) The pure state with uniform superposition of the fixed basis is the maximum coherence state, $|\psi\rangle = \sum_i \frac{1}{\sqrt{n}} |i\rangle$, and $C(|\psi\rangle)$ reaches its maximum. (3) Any rearrangement of the main diagonal elements can be executed through a permutation operation that can be practically presented, so the positivity can be maintained. And the permutation operation is theoretically to switch only the order of the fixed basis without changing this basis, so it does not change the coherence, wave, and particle quantitative feature of the states. (4) Both wave and coherence are satisfied. Therefore, this shows that using coherence to quantify a wave not only perfectly works under a rigorous theoretical framework but is also reliable experimentally.

Next, we introduce a very important triality relation, and part of our results can be seen as a generalization of this relation. Reference [36], which, to some extent, has inspired our study, pointed out that coherence can be measured by the uncertainty of states [41], and the path traversed by particles can be quantified by the path certainty, that is, the certainty of the measurement; in this way, wave-particle-mixedness is proposed as follows:

$$P(\rho|\Pi) + W(\rho|\Pi) + M(\rho) = 1, \tag{10}$$

where $P(\rho|\Pi) = \Sigma_{i=1}^{n} \langle i|\rho|i\rangle^2$ is the measurement certainty, $W(\rho|\Pi) = \Sigma_{i\neq j} |\langle i|\rho|i\rangle|^2$ represents the state uncertainty, and $M(\rho) = 1 - tr\rho^2$ represents the mixedness of the particle, which is the uncertainty possessed by the particle itself and can be associated with other physical quantifiers under certain conditions.

III. WAVE-PARTICLE RELATION

Next, we present the main results of this article. In Sec. III A, we start our derivation from the basic case of pure states and obtain a generalized neat wave-particle complementary relation for pure states. In Sec. III B, we try to extend the complementary relation obtained in Sec. III A to the mixed states, and we finally get a generalized wave-particle complementary relation.

A. Generalized wave-particle relation for pure states

Reviewing the relations proposed in the literature, all of them are in a specific functional form; their special form ensures the complementarity of the wave and particle, and they can be related to other physical quantifiers (relative entropy and mutual information [21], measurement uncertainty and quantum uncertainty [36], etc.). However, not all coherence measures can find a perfect corresponding function to form the duality relation, and these coherence measures have good correlations with other physical quantifiers (such as fidelity [40]). So we wonder whether there is a relation that can be applied to all forms of coherence measures.

For formal unity, the coherence measures C_f we discuss next are all normalized measure functions, i.e., $C_f \in [0, 1]$; this does not cause the loss of the generality of C_f.

Based on [38], all coherence measures $C_f(|\psi\rangle)$ for pure states $|\psi\rangle = (\psi_1, \psi_2, \dots, \psi_n)^t$ can be transformed as functions with respect to the main diagonal elements of the matrix,

$$C_f(|\psi\rangle) = f(|\psi_1|^2, |\psi_2|^2, \dots, |\psi_n|^2). \tag{11}$$

It is easy to verify that all functional forms f satisfy the conditions regarding the quantification of the wave feature.

However, because the wave-particle duality relations that have been proposed can be neatly complementary in the pure state, out of intuition, we define

$$\begin{aligned} D_f(|\psi\rangle) &= 1 - C_f(|\psi\rangle) \\ &= 1 - f(|\psi_1|^2, |\psi_2|^2, \dots, |\psi_n|^2) \end{aligned} \tag{12}$$

as a measure of the properties of the corresponding particle, as well as path information.

Next, we will show that the function $D_f(|\psi\rangle)$ we have defined is perfectly consistent with the requirements of particle-feature quantification.

Theorem 1. For any given normalized coherence measure $C_f(|\psi\rangle)$ for pure state $|\psi\rangle$, there will always be corresponding path information $D_f(|\psi\rangle)$, satisfying $C_f(|\psi\rangle) + D_f(|\psi\rangle) = 1$.

Proof. From the definition of Eqs. (11) and (12), we can prove the following.

(1) Consider a state $|\psi\rangle = (0, \dots, 1, \dots, 0)^t$ with the form of a unit vector in which the ith term is 1 and the remaining term is 0. Based on Eq. (5), there is a permutation operation P_{1i} that satisfies $P_{1i}(1, 0, \dots, 0) = |\psi\rangle$, so we obtain $f(|\psi\rangle) = 0$; meanwhile, $D_f(|\psi\rangle)$ reaches its global maximum of 1.

(2) Set $\langle i| |\psi\rangle \langle\psi| |i\rangle = 1/n$ for all i; then $|\psi\rangle = \sum_{i=1}^{n} \frac{1}{\sqrt{n}} |i\rangle$ is a maximally coherent state. At this time, the coherence of state $|\psi\rangle$ reaches its maximum of 1, and the complementary path information to it reaches its global minimum of 0, $D_f(\psi) = 0$.

(3) From Eq. (6), $D_f(|\psi\rangle)$ is invariant under permutations of the diagonal elements of $|\psi\rangle\langle\psi|$.

(4) It is easy to see that $D_f = 1 - f$ is convex because f is concave. ■

In summary, $D_f(|\psi\rangle)$, which we have defined, satisfies the necessary conditions for the quantification of particle feature.

B. Generalized wave-particle relation

Now we extend the results we obtained to mixed states. Unlike in the pure state, the complementarity relations in the mixed state are mostly unneat; a third term is needed to complement it, and this third term is not the same for different analysis methods. In the interferometer with path detectors, this third term is formulated as a measure of entanglement [36], while in the interferometer without path detectors, this third term is formulated as the mixedness of the state [30,35,42]. This paper does not consider the case with detectors, and in fact, the analysis of the case including detectors is also pretty hard.

First, we analyze the path information for mixed states. From [12,13,36], we know that the path information is only related to the main diagonal elements of the state matrix, and pure and mixed states with the same main diagonal elements have the same path information. Therefore, for the sake of formal unity, we define the path information $D_f(\rho)$ of the mixed state ρ as the path information $D_f(|\rho\rangle\langle\rho|)$ of the pure state $|\rho\rangle = \sum_i \sqrt{\rho_{ii}} |i\rangle$ with the same main diagonal elements:

$$D_f(\rho) = D_f(|\rho\rangle) = 1 - f(\rho_{11}, \rho_{22}, \dots, \rho_{nn}). \tag{13}$$

It is not hard to see that this is essentially a function that only depends on the main diagonal elements.

For some coherence measure $C(\rho)$ which has a uniform form in the pure and mixed states, we call it a well-defined coherence measure. If the pure state is defined by a symmetric concave function, we here regard this symmetric concave function as a function of the elements of the matrix represented by a given basis without changing its form. That is, $C_{l_1}(\rho) = \frac{1}{d-1} \sum_{i \neq j} |\rho_{ij}|$ for the coherence measure defined by the l_1 norm, where $\rho_{ij} = \langle i|\rho|j\rangle$ and the coherence measure of pure states $C_{l_1}(|\psi\rangle) = \frac{1}{d-1} \sum_{i \neq j} |\psi_i||\psi_j|$. Well-defined coherence measures usually have special functional forms, and each wave-particle duality is established based on these special forms. However, like with fidelity, not all symmetric concave functions necessarily meet the requirements of the coherence measure; this is the part we are mainly interested in. We will discuss wave-particle duality and, furthermore, triality, with the addition of a third part, through a more general form of coherence measures. At this point we can construct a coherence measure suitable for mixed states using the convex-roof structure of $C_f(|\psi\rangle)$:

$$C_f(\rho) = \min_{p_j, |\psi_j\rangle} \sum_j p_j C_f(|\psi_j\rangle), \tag{14}$$

where $\rho = \sum_j p_j |\psi_j\rangle\langle\psi_j|$ is the minimization of all the pure-state ensembles of ρ.

Therefore, we get the following theorem combining the above analysis.

Theorem 2. For any mixed state ρ, $D_f(\rho)$ meets the basic requirements of the particle-feature measurement and satisfies $C_f(\rho) + D_f(\rho) \leqslant 1$.

Proof. First, we prove that $D_f(\rho)$ meets the basic requirements of the particle-feature measurement, like in the proof of Theorem 1.

(1) Considering the case $\rho_{ss} = 1$ for $1 \leqslant s \leqslant n$, the main diagonal elements of the quantum state ρ have the following form:

$$(\rho_{11}, \ldots, \rho_{ss}, \ldots, \rho_{nn}) = (0, \ldots, 1, \ldots, 0);$$

meanwhile, $D_f(\rho) = 1 - f(0, \ldots, 1, \ldots, 0) = 1$ reaches its global maximum.

(2) If the main diagonal elements of the quantum state ρ are

$$(\rho_{11}, \rho_{22}, \ldots, \rho_{nn}) = (1/n, 1/n, \ldots, 1/n),$$

then $D_f(\rho) = 1 - f(1/n, 1/n, \ldots, 1/n) = 1$ reaches its global minimum.

Conditions 3 and 4 are the same as in the proof of Theorem 1.

Next, we show that $C_f(\rho)$ and $D_f(\rho)$ satisfy the inequality $C_f(\rho) + D_f(\rho) \leqslant 1$. For any mixed state ρ, there are many pure-state decompositions, and we choose one of them at will,

$$\rho = \sum_j p_j |\psi_j\rangle\langle\psi_j| .$$

We can deduce that

$$\begin{aligned} & f(\rho_{11}, \rho_{22}, \ldots, \rho_{nn}) \\ &= f\left(\sum_{j=1} p_j|\psi_1^{(j)}|^2, \sum_{j=1} p_j|\psi_2^{(j)}|^2, \ldots, \sum_{j=1} p_j|\psi_n^{(j)}|^2\right) \\ &= f\left[\sum_{j=1} p_j\left(|\psi_1^{(j)}|^2, |\psi_2^{(j)}|^2, \ldots, |\psi_n^{(j)}|^2\right)\right] \\ &\geqslant \sum_{j=1} p_i f\left(|\psi_1^{(j)}|^2, |\psi_2^{(j)}|^2, \ldots, |\psi_n^{(j)}|^2\right) \\ &\geqslant \min_{p_j, |\psi_j\rangle} \sum_j p_j C_f(|\psi_j\rangle) \\ &= C_f(\rho), \end{aligned}$$

where $|\psi_j\rangle = \sum_i \psi_i^{(j)} |i\rangle$. So for a mixed state ρ, we get

$$D_f(\rho) + C_f(\rho) \leqslant 1. \quad \blacksquare$$

Thus, we extend the complementarity in the pure state to the mixed state. As we said earlier, for any given coherence measure, we can identify the specific form of the complementary relationship using Theorem 2. Moreover, if we can measure coherence by a physical quantity, we can also measure path information by that physical quantity; conversely, we can also apply physical quantities associated with path information to measure coherence (see Fig. 2).

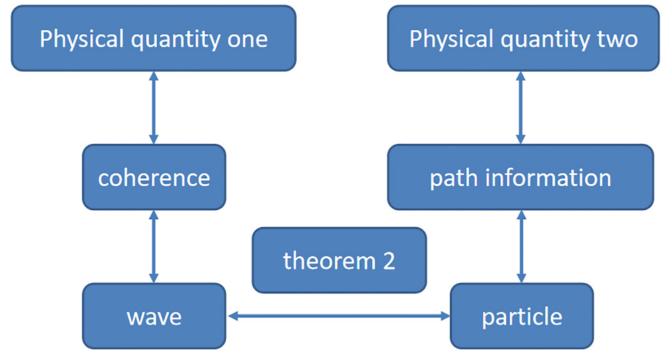

FIG. 2. If a physical quantity has a functional relationship with coherence, then Theorem 2 shows that the physical quantity also has a functional relationship with the path information and particle, and the two functions satisfy the duality relationship; the reverse is also true.

IV. A GENERALIZED TRIALITY

In the following, we analyze why $C_f(\rho)$ and $D_f(\rho)$ for mixed states are not neatly complementary. Note that the following analysis only applies to the absence of detectors, as the resource theory of path distinguishability has not been established. We define

$$M_f(\rho) = f(\rho_{11}, \rho_{22}, \ldots, \rho_{nn}) - \min_{p_j, |\psi_j\rangle} \sum_j p_j C_f(|\psi_j\rangle); \tag{15}$$

then we have

$$D_f(\rho) + C_f(\rho) + M_f(\rho) = 1,$$

which is exactly the triality that we are hoping for.

Recently, the resource theory of predictability was established, and in discussing the relation among the resource theories on predictability, coherence, and purity, the authors pointed out that complementarity relations can be used as purity measures [42]. Because purity and mixedness are two opposite concepts, taking our defined complementarity

$$C_f(\rho) + D_f(\rho) = P_f(\rho)$$

as a purity monotone, the complementary $M_f(\rho)$ is naturally a quantity associated with mixedness.

Although M_f is quite closely related to the mixedness, given that it is acquired from a fixed basis, we are not sure whether it is a significant measure of mixedness. This is because the mixedness of quantum states is independent of their eigenstates; from our definition of M_f we can intuit that it is related to a fixed basis. But in practical experiments and theoretical analysis, we always have to fix a set of bases in advance, so the following discussion of $M_f(\rho)$ is based on a fixed basis. From the special form of function f, we can show that M_f satisfies some properties with a degree of mixedness.

Next, we show that $M_f(\rho)$ satisfies several basic properties of mixedness and has a certain monotonicity.

Theorem 3. $M_f(\rho)$ meets the following properties:

(1) $M_f(\rho)$ reaches its global minimum if the state $\rho = |\psi\rangle\langle\psi|$ is a pure state.

(2) $M_f(\rho)$ reaches its global maximum if the state $\rho = \mathbb{1}/n$ is a maximum mixed state.

(3) $M_f(\rho)$ is concave.

(4) $M_f(\rho)$ is monotonic for quantum states that satisfy the condition $\sigma = p\rho + (1-p)\mathbb{1}/n$.

(5) $M_f(\rho)$ is monotonic for quantum states that satisfy the condition $\rho_1 = p\rho + (1-p)\triangle(\rho)$, where $\triangle(\rho) = \sum_i^n |i\rangle\langle i|\rho|i\rangle\langle i|$.

(6) $M_f(\rho)$ is monotonic for two-dimensional quantum states that satisfy the condition $\rho_2 = p\rho + (1-p)\{\mathbb{1}/2 + [\rho - \triangle(\rho)]\}$.

The first three terms in Theorem 3 are the basic properties of the degree of mixedness, and properties 4, 5, and 6 indicate that $M_f(\rho)$ has a certain monotonicity.

Proof. (1) Set $\rho = |\psi\rangle\langle\psi|$ as a pure state; at this point, quantum states do not have a degree of mixedness, and the $M_f(\rho)$ that we defined will not exist, i.e., $M_f(|\psi\rangle\langle\psi|) = 0$. The triality naturally transforms into the wave-particle relationship.

(2) Set $\rho = \mathbb{1}/n$ as the maximally mixed state; at this point the quantum state ρ has the greatest mixedness. Obviously, quantum state ρ has no coherence or path information because the maximally mixed state has no nonprimary diagonal elements and the main diagonal elements are all $1/n$, which means that $M_f(\rho)$ reaches its maximum $M_f(\rho) = 1$.

(3) For any decomposition of state $\rho = \sum_i p_i\rho_i$, $M_f(\rho)$ satisfies concavity,

$$\begin{aligned} M_f\left(\sum_i p_i\rho_i\right) &= 1 - C_f\left(\sum_i p_i\rho_i\right) - D_f\left(\sum_i p_i\rho_i\right) \\ &\geqslant 1 - \sum_i p_i C_f(\rho_i) - \sum_i p_i D_f(\rho_i) \\ &= \sum_i p_i[1 - C_f(\rho_i) - D_f(\rho_i)] \\ &= \sum_i p_i M(\rho_i), \end{aligned} \tag{16}$$

because C_f and D_f are convex.

(4) For any given state ρ, we find the state satisfies

$$\sigma = p\rho + (1-p)\mathbb{1}/n,$$

which is the common form of the mixed state. Obviously, the mixedness of σ is higher than the mixedness of ρ.

From properties 2 and 3, we can get

$$\begin{aligned} M_f(\sigma) &= M_f[p\rho + (1-p)\mathbb{1}/n] \\ &\geqslant pM_f(\rho) + (1-p)M_f(\mathbb{1}/n) \\ &\geqslant M_f(\rho). \end{aligned} \tag{17}$$

(5) For a given state ρ, if the relationship

$$\rho_1 = p\rho + (1-p)\triangle(\rho)$$

is satisfied, then states $\triangle\rho$ and ρ have the same principal diagonal elements, so the path information for both states is the same; meanwhile, $0 = C_f(\triangle\rho) \leqslant C_f(\rho)$. This implies that $M_f(\triangle\rho) \geqslant M_f(\rho)$. Correspondingly, we can get

$$\begin{aligned} M_f(\rho_1) &= M_f[p\rho + (1-p)\triangle(\rho)] \\ &\geqslant pM_f(\rho) + (1-p)M_f(\triangle\rho) \\ &\geqslant M_f(\rho). \end{aligned} \tag{18}$$

(6) For a given two-dimensional state ρ, $\rho_2 = p\rho + (1-p)\rho'$, where $\rho' = \mathbb{1}/2 + [\rho - \triangle(\rho)]$. Then, we have

$$\rho'_{12} = \rho_{12}, \qquad \rho_{11}\rho_{22} \leqslant \rho'_{11}\rho'_{22} = 1/4,$$

so the determinant of ρ' is greater than zero, which means that ρ' is a positive-definite matrix, that is,

$$|\rho'| = |\rho_{11}'\rho_{22}' - (\rho_{12}')^2| \geqslant |\rho_{11}\rho_{22} - (\rho_{12})^2| \geqslant 0.$$

We suppose that it has the minimum that provides the coherence measure of the given two-dimensional quantum state via a decomposition $\rho = q|\psi\rangle\langle\psi| + (1-q)|\phi\rangle\langle\phi|$, where

$$|\psi\rangle\langle\psi| = \begin{pmatrix} \psi & \psi_{12} \\ \psi_{21} & 1-\psi \end{pmatrix},$$

$$|\phi\rangle\langle\phi| = \begin{pmatrix} \phi & \phi_{12} \\ \phi_{21} & 1-\phi \end{pmatrix},$$

with $|\psi_{12}| = |\psi_{21}| = \sqrt{\psi(1-\psi)}$ and $|\phi_{12}| = |\phi_{21}| = \sqrt{\phi(1-\phi)}$, which means that

$$\begin{aligned} C_f(\rho) &= \min_{p_n, |\psi_n\rangle} \sum_{i=1}^{2} p_i C_f(|\psi_i\rangle) \\ &= qC_f(|\psi\rangle) + (1-q)C_f(|\phi\rangle). \end{aligned}$$

Then we set

$$\overline{\rho} = \begin{pmatrix} 1-\rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{11} \end{pmatrix}$$

and meet $\frac{1}{2}\rho + \frac{1}{2}\overline{\rho} = \rho' = \mathbb{1}/2 + [\rho - \triangle(\rho)]$, and there must be a set of pure-state decompositions $\overline{\rho} = q|\overline{\psi}\rangle\langle\overline{\psi}| + (1-q)|\overline{\phi}\rangle\langle\overline{\phi}|$, where

$$|\overline{\psi}\rangle\langle\overline{\psi}| = \begin{pmatrix} 1-\psi & \psi_{12} \\ \psi_{21} & \psi \end{pmatrix},$$

$$|\overline{\phi}\rangle\langle\overline{\phi}| = \begin{pmatrix} 1-\phi & \phi_{12} \\ \phi_{21} & \phi \end{pmatrix}.$$

Then $C_f(\overline{\rho}) = qC_f(|\overline{\psi}\rangle) + (1-q)C_f(|\overline{\phi}\rangle)$ because the difference between ρ and $\overline{\rho}$ is only that the order of the fixed bases has changed; for example, ρ is expanded under the basis $\{|0\rangle, |1\rangle\}$, and then $\overline{\rho}$ is expanded under the basis $\{|1\rangle, |0\rangle\}$. So their decomposition structure should be consistent. Also, from

$$\begin{aligned} C_f(\rho) &= qC_f(|\psi\rangle) + (1-q)C_f(|\phi\rangle) \\ &= qf(|\psi|, |1-\psi|) + (1-q)f(|\phi|, |1-\phi|) \\ &= C_f(\overline{\rho}), \end{aligned}$$

we have that

$$\begin{aligned} C_f(\rho') &= C_f\left(\tfrac{1}{2}\rho + \tfrac{1}{2}\overline{\rho}\right) \\ &\leqslant \tfrac{1}{2}C_f(\rho) + \tfrac{1}{2}C_f(\overline{\rho}) \\ &= C_f(\rho). \end{aligned}$$

And we know $0 = D_f(\rho') \leqslant D_f(\rho)$, which implies that $M_f(\rho') \geqslant M_f(\rho)$. Correspondingly, we can get

$$\begin{aligned} M_f(\rho_2) &= M_f[p\rho + (1-p)\rho'] \\ &\geqslant pM_f(\rho) + (1-p)M_f(\rho') \\ &\geqslant M_f(\rho). \end{aligned}$$

■

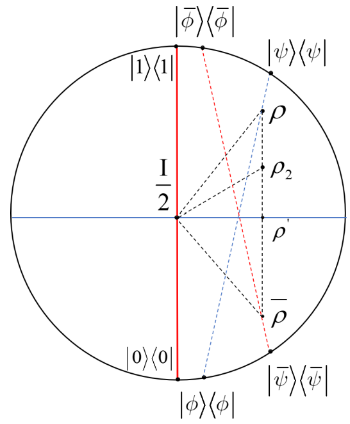

FIG. 3. The blue dashed line represents the pure-state decomposition of ρ, and the red dashed line represents the pure-state decomposition of $\overline{\rho}$.

We here visualize the proof of monotonicity (6) in Theorem 3 using the Bloch sphere. For a given state ρ, its XY-plane symmetry point is $\overline{\rho}$; visually, ρ and $\overline{\rho}$ have the same coherence (ρ and $\overline{\rho}$ are an equal perpendicular distance to the Z axis). For the pure-state decompositions of ρ, we can always find the corresponding pure-state decompositions of $\overline{\rho}$, and these decompositions are symmetric with respect to the XY-plane symmetry, as shown in the Fig. 3.

Furthermore, for ease of understanding, we use a more intuitive way to describe the mixedness measure $M_f(\rho)$ we defined and the properties expressed in Theorem 3. In an intuitive way, it is well known that two-dimensional states correspond one to one to points on the Bloch sphere; the pure state corresponds to a point on the sphere, and the mixed state corresponds to a point inside the sphere, as shown in Fig. 4. For a given state ρ, coherence measures are represented as the perpendicular line from point ρ to the Z axis (free-state set of coherence), and the path predictability is represented as the perpendicular line from point ρ to the XY plane (free-state set of path predictability) [42], which shows that the coherence and path predictability are only relevant for nonprimary diagonal elements and primary diagonal elements. Purity is expressed as the connection of point ρ to the center

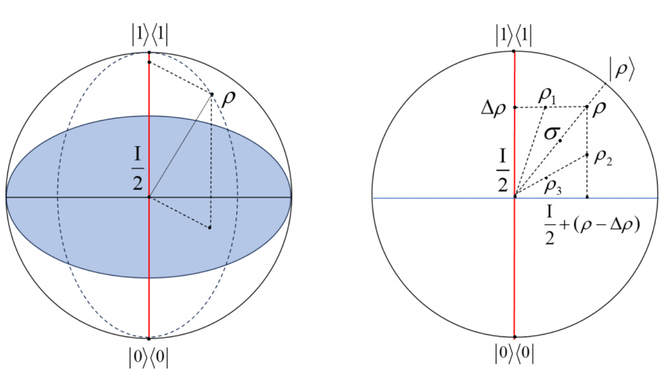

FIG. 4. A diagram of Bloch's sphere, where the set of free states $F_{\text{predictability}}$ is represented as a blue plane (XY plane), the set of free states $F_{\text{coherence}}$ is represented as a red line (Z axes), and the set of free states F_{purity} is represented as the center of the sphere.

of the sphere (free state of purity) [43], and conversely, the mixedness is expressed as the shortest distance from point ρ to the sphere. This is partly enough to help us understand the triality.

For a given state ρ, we make three perpendicular lines to each of the three following sets of free states: $F_{\text{purity}}(\rho)$ is the free-state set of purity, $F_{\text{coherence}}$ is the free-state set of coherence, and $F_{\text{predictability}}$ is the free-state set of path predictability. Since the resource theory of mixedness has not been established, because the free-state set of mixedness is a nonconvex set, we analyze mixedness through the resource theory of purity. The set of free states for each is then defined as follows:

$$F_{\text{purity}}(\rho) = \{\mathbb{1}/n\},$$

$$F_{\text{coherence}}(\rho) = \left\{\rho|\rho = \sum_i |i\rangle\langle i|\right\},$$

and

$$F_{\text{predictability}}(\rho) = \left\{\rho|\rho = (1-p)\frac{1}{n} + p\,|\psi_d\rangle\langle\psi_d|\right\},$$

where $|\psi_d\rangle = \frac{1}{\sqrt{d}}\sum_j e^{i\phi_j}|x_j\rangle$; then the main diagonal elements of states belonging to set F_{path} are $\frac{1}{n}$, and the nonprincipal diagonal elements can be any value [42]. So the points on all three lines satisfy the monotonicity of $M_f(\rho)$ from conditions 4, 5, and 6 in Theorem 3.

Now we are able to illustrate that in the two-dimensional case, $M_f(\rho)$ satisfies monotonicity for the points on the horizontal lines, vertical lines, and diagonal lines that cross the center of the sphere, as shown in Fig. 4. Moreover, for an arbitrary point on the black dashed line ρ_3, we can also show that monotonicity is satisfied by applying the monotonicity in Theorem 3, that is,

$$M_f(\rho_3) \geqslant M_f(\rho_2) \geqslant M_f(\rho).$$

V. EXAMPLE

Next, we give two examples to illustrate our proposed theorems.

For some well-defined measures of coherence, the prevailing method to construct the wave-particle duality is to find suitable path information based on its special form. However, this approach has the disadvantage of being effective only for specific coherence measures, and it is difficult to present a generalized form that characterizes the triality involving a third quantity for mixed states. Differently, when the measure of coherence for the mixed states is defined by the convex-roof construction of an arbitrary symmetric concave function, the theorems we have established provide extensive and important, although not perfect, evidence for adopting the third quantity we have proposed to characterize the degree of mixedness of quantum states. The following is an example of a coherence measure $C_{l_1}(\rho) = \frac{1}{d-1}\sum_{i\neq j}|\rho_{ij}|$ defined by the l_1 norm. The wave-particle duality satisfying $C_{l_1}(\rho) + D_{l_1}(\rho) \leqslant 1$ through waves quantified by C_{l_1} is presented in [20]. What should be noted here is that the form of C_{l_1} for the pure state is the form of a symmetric concave function $C_{l_1}(|\psi\rangle) = \frac{1}{d-1}\sum_{i\neq j}|\psi_i||\psi_j|$, with $|\psi\rangle = \sum_i \psi_i|i\rangle$.

When defining the three measures as

$$C'_{l_1}(\rho) = \frac{1}{d-1} \min_{p_n, |\psi_n\rangle} \sum_n \sum_{i \neq j} |\psi_i^{(n)}||\psi_j^{(n)}|,$$

$$D_{l_1}(\rho) = 1 - \frac{1}{d-1} \sum_{i \neq j} \sqrt{|\rho_{ii}||\rho_{jj}|},$$

and

$$M_{l_1}(\rho) = \frac{1}{d-1}\left(\sum_{i \neq j} \sqrt{|\rho_{ii}||\rho_{jj}|} - \min_{p_n, |\psi_n\rangle} \sum_n \sum_{i \neq j} |\psi_i^{(n)}||\psi_j^{(n)}| \right),$$

we can get the triality relationship $C'_{l_1}(\rho) + D_{l_1}(\rho) + M_{l_1}(\rho) = 1$ using Theorem 2. Thus, we can interpret the duality presented by several well-defined coherence measures by extending it to triality through the general form of coherence measures presented. Although we cannot give a complete interpretation of the third part, Theorem 3 shows that it is deeply related to the mixedness of quantum states.

Next, we introduce how a triality relation can be established via a coherence measure derived from the convex-roof construction of another symmetric concave function. Fidelity, an important physical quantity, plays a vital role in quantum information theory; however, as mentioned above, the coherence measure defined by fidelity does not have a good form, and the coherence measure can only be established in the mixed state using the method of convex-roof construction [40]. The coherence measure defined by fidelity is

$$C_F(|\psi\rangle) = \min_{\sigma \in I} \sqrt{1 - F(|\psi\rangle, \sigma)},$$

where $F(\rho, \sigma) = (\mathrm{Tr}\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}})^2$. If we set $|\psi\rangle = \sum_i \psi_i |i\rangle$, we find that $C_F(|\psi\rangle) = \sqrt{1 - |\psi_i|^2}$, so according to Eqs. (13), (14), and (15), we set the following measures:

$$C_F(\rho) = \min_{p_n, |\psi_n\rangle} \sum_n p_n \sqrt{1 - |\psi_i^{(n)}|^2},$$

$$D_F(\rho) = 1 - \sqrt{1 - |\rho_{ii}|},$$

and

$$M_F(\rho) = \sqrt{1 - |\rho_{ii}|} - \min_{p_n, |\psi_n\rangle} \sum_n p_n \sqrt{1 - |\psi_i^{(n)}|^2}.$$

It is easy to verify that D_F can be used as a predictability measure [42], and for interferometers with or without detectors, the wave-particle relationships $C_F(\rho) + D_F(\rho) \leqslant 1$ and $C_F(\rho_s) + D_F(\rho_s) \leqslant 1$ hold. Here, ρ_s represents the quantum state after the detector. In addition, for an interferometer without detectors, we can also establish that the triality relation $C_F(\rho) + D_F(\rho) + M_F(\rho) = 1$, where M_F is a representation of mixedness, satisfies the properties in Theorem 3.

Thus, we have shown using two examples that a wave-particle-mixedness triality can be established for any form of coherence measure. D_f established in this way can also be used as a measure of predictable resources; the third term, which is closely related to mixedness, can be regarded as a representation of mixedness. The triality relation we proposed in this paper still requires some supplementary interpretation of the third quantity, but its wide application is possible.

VI. CONCLUSION

We ended up with a generalized wave-particle-mixedness triality which is applicable to any form of coherence measure and path information. We started with the pure-state case for any form of coherence measure by transforming the given coherence measure into a functional form related to the main diagonal elements; the corresponding path information was found in a functional form, and we proved that the defined path information meets the requirement of particle-feature quantification. Then we extended this form to the mixed-state case and established a generalized wave-particle-mixedness triality. Although the characterization of mixedness defined by us is not a complete measure of mixedness, it meets some basic requirements for mixedness measurement and has a certain monotonicity for some states that satisfy certain relationships. From the perspective of resource theory, the form of the path information D_f proposed in Theorem 2 can be used as a measure of predictability [42]. The triality we established also links resource theories such as coherence, predictability, purity, and mixedness.

Another use of the proposed theorem is as a coherence measure that is associated with other physical quantifiers; we can establish the relationship between the physical quantifier and the wave feature, and the relation between the physical quantifier and the particle feature can be established through our method now. Conversely, the same is true for path information associated with other physical quantities.

Finally, we gave two examples of how we can use our theorem to establish a triality relation for a given coherence measure, whether the coherence measure is well defined or not, and we declare that the theorem we proposed is applicable to all coherence measure that can be transformed into symmetric concave functions in the pure state.

At the same time, we look forward to applying this method to other analytical methods and building new triality relations, such as a general triality relationship for an interferometer with detectors. We look forward to applying our theorem to other concrete coherence measures and analyzing whether the third term has more complete properties, such as the recently proposed coherence measure based on Fisher information [44].

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[1] N. Bohr, The quantum postulate and the recent development of atomic theory, Nature (London) **121**, 580 (1928).

[2] W. K. Wootters and W. H. Zurek, Complementarity in the double-slit experiment: Quantum nonseparability and a

quantitative statement of Bohr's principle, Phys. Rev. D **19**, 473 (1979).

[3] D. M. Greenberger and A. Yasin, Simultaneous wave and particle knowledge in a neutron interferometer, Phys. Lett. A **128**, 391 (1988).

[4] G. Jaeger, A. Shimony, and L. Vaidman, Two interferometric complementarities, Phys. Rev. A **51**, 54 (1995).

[5] B. G. Englert, Fringe visibility and which-way information: An inequality, Phys. Rev. Lett. **77**, 2154 (1996).

[6] S. Dürr, Quantitative wave-particle duality in multibeam interferometers, Phys. Rev. A **64**, 042113 (2001).

[7] X. Peng, X. Zhu, D. Suter, J. Du, M. Liu, and K. Gao, Quantification of complementarity in multiqubit systems, Phys. Rev. A **72**, 052109 (2005).

[8] B. G. Englert, D. Kaszlikowsk, L. C. Kwek, and W. H. Chee, Wave-particle duality in multi-path interferometers: General concepts and three-path interferometers, Int. J. Quantum Inf. **6**, 129 (2008).

[9] X. Lü, Quantitative wave-particle duality as quantum state discrimination, Phys. Rev. A **102**, 022201 (2020).

[10] J. A. Vaccaro, Particle-wave duality: A dichotomy between symmetry and asymmetry, Proc. R. Soc. A **468**, 1065 (2012).

[11] P. J. Coles, J. Kaniewski, and S. Wehner, Equivalence of wave-particle duality to entropic uncertainty, Nat. Commun. **5**, 5814 (2014).

[12] R. M. Angelo and A. D. Ribeiro, Wave-particle duality: An information-based approach, Found. Phys. **45**, 1407 (2015).

[13] P. J. Coles, Entropic framework for wave-particle duality in multipath interferometers, Phys. Rev. A **93**, 062111 (2016).

[14] M. Jakob and J. A. Bergou, Complementarity and entanglement in bipartite qubit systems, Phys. Rev. A **76**, 052107 (2007).

[15] M. Jakob and J. A. Bergou, Quantitative complementarity relations in bipartite systems: Entanglement as a physical reality, Opt. Commun. **283**, 827 (2010).

[16] X. F. Qian, A. N. Vamivakas, and J. H. Eberly, Entanglement limits duality and vice versa, Optica **5**, 942 (2018).

[17] F. De Zela, Optical approach to concurrence and polarization, Opt. Lett. **43**, 2603 (2018).

[18] X. F. Qian, K. Konthasinghe, S. K. Manikandan, D. Spiecker, A. N. Vamivakas, and J. H. Eberly, Turning off quantum duality, Phys. Rev. Res. **2**, 012016(R) (2020).

[19] T. Qureshi, Predictability, distinguishability and entanglement, Opt. Lett. **46**, 492 (2021).

[20] M. N. Bera, T. Qureshi, M. A. Siddiqui, and A. K. Pati, Duality of quantum coherence and path distinguishability, Phys. Rev. A **92**, 012118 (2015).

[21] E. Bagan, J. A. Bergou, S. S. Cottrell, and M. Hillery, Relations between coherence and path information, Phys. Rev. Lett. **116**, 160406 (2016).

[22] L. Chang, S. Luo, and Y. Sun, Superposition quantification, Commun. Theor. Phys. **68**, 565 (2017).

[23] T. Biswas, M. G. Diaz, and A. Winter, Interferometric visibility and coherence, Proc. R. Soc. A **473**, 20170170 (2017).

[24] S. Luo and Y. Sun, Quantum coherence versus quantum uncertainty, Phys. Rev. A **96**, 022130 (2017).

[25] S. Luo and Y. Sun, Coherence and complementarity in state channel interaction, Phys. Rev. A **98**, 012113 (2018).

[26] M. L. Hu, X. Hu, J. Wang, Y. Peng, Y. R. Zhang, and H. Fan, Quantum coherence and geometric quantum discord, Phys. Rep. **762**, 1 (2018).

[27] C. Xiong, A. Kumar, and J. Wu, Family of coherence measures and duality between quantum coherence and path distinguishability, Phys. Rev. A **98**, 032324 (2018).

[28] K. Bu, L. Li, J. Wu, and S. M. Fei, Duality relation between coherence and path information in the presence of quantum memory, J. Phys. A **51**, 085304 (2018).

[29] E. Bagan, J. Calsamiglia, J. A. Bergou, and M. Hillery, Duality games and operational duality relations, Phys. Rev. Lett. **120**, 050402 (2018).

[30] P. Roy and T. Qureshi, Path predictability and quantum coherence in multi-slit interference, Phys. Scr. **94**, 095004 (2019).

[31] Y. Sun and S. Luo, Quantifying interference via coherence, Ann. Phys. (Berlin, Ger.) **533**, 2100303 (2021).

[32] M. L. W. Basso and J. Maziero, An uncertainty view on complementarity and a complementarity view on uncertainty, Quantum Inf. Process. **20**, 201 (2021).

[33] X. F. Qian and G. S. Agarwal, Quantum duality: A source point of view, Phys. Rev. Res. **2**, 012031(R) (2020).

[34] A. K. Roy, N. Pathania, N. K. Chandra, P. K. Panigrahi, and T. Qureshi, Coherence, path predictability, and I concurrence: A triality, Phys. Rev. A **105**, 032209 (2022).

[35] M. L. W. Basso and J. Maziero, Entanglement monotones connect distinguishability and predictability, Phys. Lett. A **425**, 127875 (2022).

[36] S. Fu and S. Luo, From wave-particle duality to wave-particle-mixedness triality: An uncertainty approach, Commun. Theor. Phys. **74**, 035103 (2022).

[37] T. Baumgratz, M. Cramer, and M. B. Plenio, Quantifying coherence, Phys. Rev. Lett. **113**, 140401 (2014).

[38] S. P. Du, Z. F. Bai, and X. F. Qi, Coherence measures and optimal conversion for coherent states, Quantum Inf. Comput. **15**, 1307 (2015).

[39] A. E. Rastegin, Quantum-coherence quantifiers based on the Tsallis relative α entropies, Phys. Rev. A **93**, 032136 (2016).

[40] C. L. Liu, D.-J. Zhang, X.-D. Yu, Q.-M. Ding, and L. Liu, A new coherence measure based on fidelity, Quantum Inf. Process. **16**, 198 (2017).

[41] Y. Sun and S. Luo, Coherence as uncertainty, Phys. Rev. A **103**, 042423 (2021).

[42] M. L. W. Basso and J. Maziero, Predictability as a quantum resource, Quantum Inf. Process. **21**, 187 (2022).

[43] A. Streltsov, H. Kampermann, S. Wolk, M. Gessner, and D. Bruß, Maximal coherence and the resource theory of purity, New J. Phys. **20**, 053058 (2018).

[44] Y. Sun, N. Li, and S. Luo, Coherence in multipath interference via quantum Fisher information, Phys. Rev. A **108**, 062416 (2023).